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# Spatial Periodic Adaptive Control For Rotary Machine Systems

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**Abstract:** A spatial periodic adaptive control (SPAC) approach is developed to deal with rotary machine systems performing speed tracking tasks. Since the angular displacement is periodic when rotating by $2\pi$ radians, most rotary machine systems present certain cyclic behaviors with a fixed periodicity which is either a fraction or multiple of $2\pi$. As a consequence, unknown system parameters and disturbances that characterize the system behaviors are also periodic in nature. By utilizing the spatial periodicity, the SPAC aims at improving the system performance. In the SPAC design, the dynamics of the rotary machine systems is first converted from the temporal to spatial coordinates. Then the new adaptive controller updates the parameters and the control signal periodically in a pointwise between two consecutive spatial cycles. Using a Lyapunov-Krasovskii functional, the convergence property of the SPAC can be analyzed for high-order rotary systems and the periodic adaptation can be applied to rotary systems with pseudo-periodic parametric uncertainties.

## 1. INTRODUCTION

Rotary machine systems are widely used in industries. Two representative classes of rotary machines are the electrical motor drives and vehicle engines. A fundamental property of any rotary machine systems is the spatial periodicity in terms of angular displacement, that is, the angular displacement will back to the same angular position after rotating certain degrees. This spatial periodicity is independent of the speed of rotational machines. On the other hand, a large class of system uncertainties in rotary machines are related to the angular position. In Burkov et al. (1999), the unknown engine crankshaft speed pulsation was expressed as Fourier series of the angular position. In Hull et al. (2000), the external disturbance of the satellite was modelled as a function of the position. In general, this class of uncertainties can be modeled as either periodic unknown parameters or periodic unknown disturbances with respect to (w.r.t.) the angular displacement.

Adaptive and learning control approaches were proposed to deal with the position or state-dependent periodic uncertainties. In Han et al. (1998), learning control was used when the position-dependent disturbance torque is presented in servo motor under velocity control. In Carlos et al. (2000), adaptive compensation was developed to reject oscillatory position-dependent unknown disturbance in a sinusoidal form. In Ahn et al. (2006), periodic adaptation was developed to handle the unknown position-dependent periodic disturbance. In this work, we extend the spatial periodic adaptive control (SPAC) approach to more generic classes of control problems.

The first extension is to high-order rotary systems. The extension of the SPAC to high-order systems is not straightforward, even if the original high-order system is in the canonical form in the time domain. In SPAC, the system dynamics is converted from the time domain to the spatial domain. The objective of the temporal-spatial conversion is to capture and fully utilize the spatial periodic characteristic of the process uncertainties, so that the controller and the spatial periodic adaptation can be designed in the spatial domain. The extra difficulty arises when the temporal-spatial conversion is carried out. A canonical dynamics in the time domain is no longer canonical in the spatial domain. In this work, to address this issue, a feedback linearization is proposed such that both the process dynamics and the reference model can be strictly linearized into the canonical form.

The second extension is to high-order systems with multiple periods or pseudo-periods. In the presence of multiple periods which are rational numbers, the periodic adaptation can be carried out according to the lowest common multiple. However, the use of the common period will make the periodic adaptation inefficient. For example, suppose a period is 3 and another is 100. The lowest common multiple is 300. As a result, the periodic adaptation for the period of 3 has been delayed by 100 cycles. If possible, the periodic adaptation should be conducted according to individual periods. In pseudo-periodic circumstances where periods are mixed with rational and irrational numbers such as 3 and $\sqrt{3}$, or irrational numbers such as $\sqrt{3}$ and $\pi$, there does not exist a common period. To address this issue, we develop a SPAC which can conduct periodic adaptation in parallel for all parameters with different periods.

## 2. PRELIMINARIES

*Definition 1.* When analyzing a vector valued function $\mathbf{f}(s)$, an important quantity is the integral over an interval of length $L$, namely $\int_{s-L}^{s} \|f(\tau)\|^2 d\tau$, where $\|\cdot\|$ is the 2-norm. $\mathbf{f}(s)$ is $L$-bounded if $\sup_{s\geq L} \int_{s-L}^{s} \|\mathbf{f}(\tau)\|^2 d\tau$ is finite, and that $\mathbf{f}(s)$ is $L$-convergent if $\lim_{s\to\infty}\sup_{s\geq L}\int_{s-L}^{s}\|\mathbf{f}(\tau)\|^2 d\tau = 0$.

*Definition 2.* A matrix-valued function $\Gamma(s,L) = diag\{\gamma_1(s,L),\cdots,\gamma_m(s,L)\}$ is defined in the interval $[0,\infty)$, satisfying

$$\Gamma(s,L) = \begin{cases} 0, & s = 0, \\ \mathcal{A}(s), & 0 \leq s < L, \\ \mathcal{B}, & s \geq L, \end{cases} \tag{1}$$

where $\mathcal{A} = diag\{\alpha_1(s),\cdots,\alpha_m(s)\}$ and $\mathcal{B} = diag\{\beta_1,\cdots,\beta_m\}$ are diagonal matrices, $\alpha_i(s)$ is a strictly increasing function for $s \in [0,L]$ with $\alpha_i(0) = 0$, $\alpha_i(L) = \beta_i$, and $\beta_i > 0$ is a constant.

To facilitate the convergence analysis of SPAC, the following Lyapunov-Krasovskii functional (LKF) is adopted

$$V(s,\mathbf{e},\boldsymbol{\phi}) = \frac{1}{2}\mathbf{e}^T\mathbf{e} + \frac{1}{2}\int_{\max\{0,s-L\}}^{s}\boldsymbol{\phi}^T(\tau)\mathcal{B}^{-1}\boldsymbol{\phi}(\tau)d\tau, \tag{2}$$

where $\mathbf{e} \in R^n$, $\boldsymbol{\phi} \in R^m$. For simplicity denote $V(s,\mathbf{e},\boldsymbol{\phi})$ by $V(s)$ in subsequent context. The $L$-convergence property associated with the LKF in (2) can be derived as shown the *Proposition* 1. Define a differential operator $\nabla = d/ds$ where $s$ is a coordinate.

*Proposition 1.* For the LKF defined in (2), if $\nabla V \leq -\|\mathbf{e}\|^2$ for $s \in [L,\infty)$ and the LKF is finite at $s = L$ for any constant $L > 0$, that is, $V(L) < \infty$, then

$$\lim_{s\to\infty}\int_{s-L}^{s}\|\mathbf{e}\|^2 d\tau = 0. \tag{3}$$

*Proof*: see Xu et al. (2006). ∎

Next we derive the boundedness of the LKF in the interval $[0,L]$ under certain conditions. Denote $\boldsymbol{\phi}(s) = \mathbf{a}(s) - \hat{\mathbf{a}}(s)$, where $\mathbf{a},\hat{\mathbf{a}} \in R^m$ and $\mathbf{a}$ has a vector valued upper bound $\bar{\mathbf{a}}$.

*Proposition 2.* For $s \in [0,L]$, $V(s)$ is bounded if the following equality holds

$$\nabla V(s) = -\lambda\|\mathbf{e}\|^2 + \boldsymbol{\phi}^T\mathcal{A}^{-1}(s)\hat{\mathbf{a}} + \frac{1}{2}\boldsymbol{\phi}^T\mathcal{B}^{-1}\boldsymbol{\phi}. \tag{4}$$

*Proof*: (4) can be rewritten as

$$\nabla V(s) = -\lambda\|\mathbf{e}\|^2 - \boldsymbol{\phi}^T(\mathcal{A}^{-1} - \mathcal{B}^{-1}/2)\boldsymbol{\phi} + \boldsymbol{\phi}^T\mathcal{A}^{-1}\mathbf{a}, \tag{5}$$

where $\mathcal{A}^{-1} - \mathcal{B}^{-1}/2 > 0$ because $\alpha_i(s)$ is strictly increasing with the upper limit $\beta_i$. Using Young's inequality, we have for any $C = diag\{c_1,\cdots,c_m\} > 0$

$$\boldsymbol{\phi}^T\mathcal{A}^{-1}\mathbf{a} \leq \boldsymbol{\phi}^T C\mathcal{A}^{-1}\boldsymbol{\phi} + \frac{1}{4}\mathbf{a}^T C\mathcal{A}^{-1}\mathbf{a}. \tag{6}$$

Choose $C$ such that $\mathcal{A}^{-1} - \mathcal{B}^{-1}/2 - C > 0$. Substituting (6) into (5) yields $\nabla V(s) = -\lambda\|\mathbf{e}\|^2 - \boldsymbol{\phi}^T(\mathcal{A}^{-1} - \mathcal{B}^{-1}/2 - C)\boldsymbol{\phi} + \frac{1}{4}\mathbf{a}^T C\mathcal{A}^{-1}\mathbf{a}$. Accordingly $\nabla V(s)$ for $s \in [0,L]$ is negative definite outside the region

$$\left\{(\|\mathbf{e}\|,\|\boldsymbol{\phi}\|) \in \mathcal{R}^2 \ : \ \lambda\|\mathbf{e}\|^2 + \lambda_1\|\boldsymbol{\phi}\|^2 \leq \lambda_2\|\bar{\mathbf{a}}\|^2\right\}, \tag{7}$$

where $\lambda_1 > 0$ is the minimum eigenvalue of the matrix $\mathcal{A}^{-1} - \mathcal{B}^{-1}/2 - C$, and $\lambda_2 < \infty$ is the maximum eigenvalue of the matrix $C\mathcal{A}^{-1}/4$. From (7) we conclude the boundedness of $V(s)$ in the interval $[0,L]$. ∎

Now investigate the relationship between the spatial and temporal coordinates. Denote $t$ the time axis, $s$ the angular displacement of rotary systems, and $x_1 = ds/dt$ is the angular speed. The spatial differentiator, or the $\nabla$-operator, is defined below and linked to the temporal differentiator

$$\nabla = \frac{d}{ds} = \frac{d}{dt}\left(\frac{ds}{dt}\right)^{-1} = \frac{1}{x_1}\frac{d}{dt}. \tag{8}$$

Let us further explore the relationship between the temporal coordinate $t$ and spatial coordinate $s$, so as to facilitate the conversion between $t$ and $s$. From $ds = x_1 dt$ we have

$$s = \int_0^t x_1(\tau)d\tau \triangleq f(t). \tag{9}$$

When the angular speed $x_1 > 0$, $s$ is a strictly increasing function of $t$, hence the relationship between $t$ and $s$ is bijective. The function $s = f(t)$ is analytic and the inverse function $t = f^{-1}(s)$ exists globally. Therefore a variable, $x(t)$, which is a temporal function, can also be expressed as a spatial function $x(f^{-1}(s))$ .

Throughout this paper, we make the following assumption.

*Assumption 1.* The rotary system under consideration is evolving in one direction, and the speed of the rotary system is strictly above zero, that is $x_1 > 0$ $\forall t$.

To facilitate the analysis of SPAC, the algebraic relationship

$$\begin{aligned}(\mathbf{a}-\mathbf{b})^T\mathcal{B}(\mathbf{a}-\mathbf{b}) - (\mathbf{a}-\mathbf{c})^T\mathcal{B}(\mathbf{a}-\mathbf{c}) \\ = (\mathbf{c}-\mathbf{b})^T\mathcal{B}[2(\mathbf{a}-\mathbf{b}) + (\mathbf{b}-\mathbf{c})],\end{aligned} \tag{10}$$

is introduced, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vectors with same dimensions and $\mathcal{B}$ is the diagonal gain matrix. For simplicity, we omit all the arguments from a function where no confusion arises, e.g. denote $f(\cdot,\cdot)$ by $f$.

## 3. SPAC FOR HIGH-ORDER SYSTEMS WITH PERIODIC PARAMETERS

Consider the canonical system

$$\begin{cases} \dfrac{ds}{dt} = x_1, \\ \dfrac{dx_i}{dt} = x_{i+1}, \qquad i = 1,2,\cdots,n-1, \\ \dfrac{dx_n}{dt} = \mathbf{a}^T(s)\boldsymbol{\zeta}^0(\mathbf{x}) + b(s)u, \end{cases} \tag{11}$$

where $\mathbf{x} = [s,x_1,\cdots,x_n]^T$, $\mathbf{a} = [a_1,\cdots,a_m]^T$ are unknown continuous $s$-periodic parameters, $\boldsymbol{\zeta}^0 = [\zeta_1^0,\cdots,\zeta_m^0]^T$ is a known vector valued local Lipschitz and continuously differentiable function w.r.t. arguments $\mathbf{x}$, and $b(s) \in \mathcal{C}^1[0,\infty)$ is an unknown $s$-periodic gain of the system input. The prior information about $b(s)$ is that $b(s)$ is positive for all $s$. Unknown parameters $a_i(s)$ may have different periods $L_i$ for $i = 1, \cdots, m$ and $b(s)$ has a period $L_{m+1}$. In this section we assume that all the periods are rational numbers. In such circumstances, there exists a lowest common multiple $L$ for all unknown coefficients $a_i(s)$ and $b(s)$. We can use the common period $L$ as the updating period.

### 3.1 State Transformation for High-order Systems by Feedback Linearization

Applying the $\nabla$-operator to convert the system (11) from $t$ domain to $s$ domain, we have

$$\begin{cases} \nabla x_i &= \dfrac{x_{i+1}}{x_1}, \qquad i = 1, \cdots, n-1, \\ \nabla x_n &= \mathbf{a}^T(s)\boldsymbol{\zeta}(\mathbf{x}) + b(s)x_1^{-1}u, \end{cases} \tag{12}$$

where $\boldsymbol{\zeta} = \boldsymbol{\zeta}^0/x_1$. Note that (12) is not in canonical form. To facilitate the SPAC design, we can apply feedback linearization to transform the system (12) into a canonical form.

First define a state transformation $\mathbf{z} = \mathcal{T}(x_1, \cdots, x_n)$ as

$$z_1 = x_1, \qquad z_2 = \nabla x_1, \quad \cdots, \quad z_n = \nabla^{n-1}x_1, \tag{13}$$

where $\mathbf{z} = [z_1, \cdots, z_n]^T$, $\nabla^k = \nabla \cdot \nabla^{k-1}$. For a scalar function $h$ with the arguments $x_1, \cdots, x_n$, denote the Lie derivative of $h$ with respect to a vector $\mathbf{f} = [f_1, \cdots, f_n]^T$ as

$$L_{\mathbf{f}}h = L_{[f_1,\cdots,f_n]}h = \left[\frac{\partial h}{\partial x_1}, \cdots, \frac{\partial h}{\partial x_n}\right]\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}. \tag{14}$$

The property of $\mathcal{T}$ is summarized in *Proposition* 3. The proof is given in *Appendix*.

*Proposition 3.* The transformation (13), which is a diffeomorphism, transforms the system (12) to the canonical form

$$\begin{cases} \nabla z_i &= z_{i+1}, \qquad i = 1, \cdots, n-1, \\ \nabla z_n &= \mathbf{a}^T(s)\boldsymbol{\xi}^0(\mathbf{z}) + \rho(\mathbf{z}) + b(s)\eta(\mathbf{z})u, \end{cases} \tag{15}$$

where $\eta$, $\boldsymbol{\xi}^0$, $\rho$ are generated by substituting $\mathbf{x} = \mathcal{T}^{-1}(\mathbf{z})$ into functions $\eta_x(\mathbf{x})$, $\boldsymbol{\xi}_x^0(\mathbf{x})$ and $\rho_x(\mathbf{x})$ defined below

$$\eta_x = \frac{1}{x_1^n}, \quad \boldsymbol{\xi}_x^0(\mathbf{x}) = x_1\eta_x\boldsymbol{\zeta}(\mathbf{x}),$$

$$\rho_x = \frac{x_1 L_{[x_2,\cdots,x_n]}N_{n-1} - (2n-3)x_2N_{n-1}}{x_1^{2n-1}},$$

$N_{n-1}$ is a polynomial with arguments $x_1, \cdots, x_n$ and its recursive form is given in the proof of the proposition. ■

In speed regulation problems, $x_1$ is to track a given speed $x_{1,r}(t)$ which is generated by a reference model

$$\begin{cases} \dfrac{dx_{i,r}}{dt} &= x_{i+1,r}, \qquad i = 1, 2, \cdots, n-1, \\ \dfrac{dx_{n,r}}{dt} &= w^0(\mathbf{x}_r, r), \end{cases} \tag{16}$$

where $\mathbf{x}_r = [x_{1,r}, \cdots, x_{n,r}]^T$ is a vector of states, $r$ is a constant reference input. To facilitate the controller design in the $s$ domain, $\nabla$-operator is applied to the reference model

$$\begin{cases} \nabla x_{i,r} &= \dfrac{x_{i+1,r}}{x_1}, \qquad i = 1, \cdots, n-1, \\ \nabla x_{n,r} &= \dfrac{w^0(\mathbf{x}_r, r)}{x_1}. \end{cases} \tag{17}$$

Define $\mathbf{z}_r = [z_{1,r}, \cdots, z_{n,r}]^T$ and the new state transformation $\mathbf{z}_r = \mathcal{T}_r(x_{1,r}, \cdots, x_{n,r})$

$$z_{1,r} = x_{1,r}, \; z_{2,r} = \nabla x_{1,r}, \; \cdots, \; z_{n,r} = \nabla^{n-1}x_{1,r}. \tag{18}$$

Analogous to the derivation procedure shown in *Proposition* 3, the state transformation (18) is a diffeomorphism and the inverse $\mathbf{x}_r = \mathcal{T}_r^{-1}(\mathbf{z}_r, \mathbf{z})$ exists. The reference model (17) is transformed into a new canonical model

$$\begin{cases} \nabla z_{i,r} &= z_{i+1,r}, \qquad i = 1, \cdots, n-1, \\ \nabla z_{n,r} &= w(\mathbf{z}_r, \mathbf{z}, r), \end{cases} \tag{19}$$

where $w(\mathbf{z}_r, \mathbf{z}, r)$ can be derived recursively in a similar way as $N_{n-1}$ in *Proposition* 3.

### 3.2 Periodic Adaptation and Convergence Analysis

Define the tracking error to be $\mathbf{e} = [e_1, \cdots, e_n]^T = \mathbf{z} - \mathbf{z}_r$. From the system (15) and the reference model (19), the error dynamics is

$$\begin{cases} \nabla e_i &= e_{i+1}, \qquad i = 1, \cdots, n-1, \\ \nabla e_n &= \mathbf{a}^T(s)\boldsymbol{\xi}^0(\mathbf{z}) + \rho(\mathbf{z}) + b(s)\eta(\mathbf{z})u - w(\mathbf{x}_r, \mathbf{z}, r), \end{cases} \tag{20}$$

or simply

$$\nabla\mathbf{e} = A\mathbf{e} + \mathbf{b}[\mathbf{a}^T\boldsymbol{\xi}^0 + (\sigma + \rho - w) + b\eta u], \tag{21}$$

where $\mathbf{b} = [0, \cdots, 0, 1]^T$, $\sigma = \mathbf{ce}$ and $\mathbf{c} = [c_1, \cdots, c_{n-1}, 1]$ is chosen such that

$$A \triangleq \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & -1 \end{pmatrix} \tag{22}$$

is asymptotically stable. The periodic adaptive control mechanism is constructed below

$$u(s) = -\frac{1}{\eta(\mathbf{z})}\left[k\sigma(s) + \hat{\boldsymbol{\theta}}^T(s)\boldsymbol{\xi}(s, \mathbf{e})\right], \tag{23}$$

where $k > 0$ is a constant feedback gain, $\hat{\boldsymbol{\theta}} = [\hat{\theta}_1, \cdots, \hat{\theta}_{m+2}]^T$ is the estimate of the extended parametric vector

$$\boldsymbol{\theta}(s) = \left[\frac{1}{b}\mathbf{a}^T, \; \frac{1}{b}, \; \frac{1}{b^2}\nabla b\right]^T$$

and

$$\boldsymbol{\xi}(s, \mathbf{e}) = \left[(\boldsymbol{\xi}^0)^T, \; \mathbf{c}_1\mathbf{e} + \rho - w, \; -\frac{\sigma}{2}\right]^T$$

where $\mathbf{c}_1 = [0, c_1, \cdots, c_{n-1}]$. Note that $\boldsymbol{\theta} \in \mathcal{C}^1([0, \infty); \mathcal{R}^{m+2})$.

The parametric updating law is

$$\begin{aligned} \hat{\boldsymbol{\theta}}(s) &= \hat{\boldsymbol{\theta}}(s - L) + \Gamma(s, L)\boldsymbol{\xi}(s)\sigma(s), \\ \hat{\boldsymbol{\theta}}(s) &= 0, \qquad \forall s \in [-L, 0], \end{aligned} \tag{24}$$

where $\Gamma > 0$ is the learning gain matrix defined in (1).

*Theorem 1.* For the system (15) and the reference model (19), the control law (23) and the periodic adaptation law (24) achieve the $L$-convergence of the tracking error $\mathbf{e}$.

*Proof.* Substituting the learning control law (23) into (21) yields the closed-loop error dynamics

$$\nabla \mathbf{e} = A\mathbf{e} + \mathbf{b}\left[-bk\sigma + \mathbf{a}^T\boldsymbol{\xi}^0 + (\sigma + \rho - w) - b\hat{\boldsymbol{\theta}}^T\boldsymbol{\xi}\right] \quad (25)$$

The error dynamics (25) and the parametric updating law (24) form a set of differential and continuous-space difference equations of neutral type. The existence of solution for this class of systems has been discussed in Xu et al. (2006). Thus we focus on the convergence property.

Notice the facts $\mathbf{cb} = 1$, $\mathbf{c}A + \mathbf{c} = \mathbf{c}_1$ and $\sigma = \mathbf{ce}$, multiplying $\mathbf{c}$ on both sides of (25) yields

$$\mathbf{c}\nabla\mathbf{e} = -bk\sigma + \mathbf{a}^T\boldsymbol{\xi}^0 + (\mathbf{c}_1\mathbf{e} + \rho - w) - b\hat{\boldsymbol{\theta}}^T\boldsymbol{\xi}. \quad (26)$$

First prove the convergence for $s \geq L$ by using the LKF

$$V(s) = \frac{1}{2b}\sigma^2 + \frac{1}{2}\int_{s-L}^{s} \boldsymbol{\phi}^T(\tau)\mathcal{B}^{-1}\boldsymbol{\phi}(\tau)d\tau, \quad (27)$$

where $\boldsymbol{\phi}(\tau) = \boldsymbol{\theta}(\tau) - \hat{\boldsymbol{\theta}}(\tau)$. The upper right hand derivative of $V$ w.r.t. $s$ is

$$\begin{aligned}\nabla V = &\frac{1}{b}\sigma\mathbf{c}\nabla\mathbf{e} - \frac{1}{2b^2}\nabla b \cdot \sigma^2\\ &+\frac{1}{2}\left[\boldsymbol{\phi}^T\mathcal{B}^{-1}\boldsymbol{\phi} - \boldsymbol{\phi}^T(s-L)\mathcal{B}^{-1}\boldsymbol{\phi}(s-L)\right] \quad (28)\end{aligned}$$

where $\mathbf{c}\nabla\mathbf{e} = \nabla\sigma$. Substituting the dynamics (26) into the first two terms on the right hand side of (28) gives

$$\frac{1}{b}\sigma\mathbf{c}\nabla\mathbf{e} - \frac{1}{2b^2}\nabla b \cdot \sigma^2 = -k\sigma^2 + \sigma\boldsymbol{\phi}^T\boldsymbol{\xi}. \quad (29)$$

Applying the parametric adaptation law (24) where $\Gamma = \mathcal{B}$ for $s \geq L$, the periodic property $\boldsymbol{\theta}(s) = \boldsymbol{\theta}(s-L)$, and the algebraic relationship (10), the third term on the right-hand side of (28) is

$$\begin{aligned}&\frac{1}{2}\left[\boldsymbol{\phi}^T\mathcal{B}^{-1}\boldsymbol{\phi} - \boldsymbol{\phi}^T(s-L)\mathcal{B}^{-1}\boldsymbol{\phi}(s-L)\right]\\ &= -\boldsymbol{\phi}^T\boldsymbol{\xi}\sigma - \frac{1}{2}\boldsymbol{\xi}^T\mathcal{B}\boldsymbol{\xi}\sigma^2. \quad (30)\end{aligned}$$

It can be seen that the system uncertainty $\boldsymbol{\phi}^T\boldsymbol{\xi}\sigma$ appears on (29) and (30) with opposite signs. Thus by substituting (29) and (30) into (28), the upper right hand derivative of $V$ is

$$\nabla V = -k\sigma^2 - \frac{1}{2}\boldsymbol{\xi}^T\mathcal{B}\boldsymbol{\xi}\sigma^2 \leq -k\sigma^2, \quad (31)$$

that is, $\nabla V$ is negative semi-definite for $s \in [L, \infty)$. From *Proposition* 1, we can derive the boundedness of $\sigma$ and the $L$-convergence property $\lim_{s\to\infty}\int_{s-L}^{s}\sigma^2(\tau)d\tau = 0$ when $V(L)$ is finite. Notice the relationship

$$\sigma = \mathbf{ce} = (\nabla^{n-1} + c_{n-1}\nabla^{n-2} + \cdots + c_2\nabla + c_1)e_1,$$

where $\nabla^{n-1} + c_{n-1}\nabla^{n-2} + \cdots + c_2\nabla + c_1$ is a stable polynomial of the differential operator $\nabla$. Therefore, the boundedness of $\sigma$ implies the boundedness of $\mathbf{e}$, and the $L$-convergence of $\sigma$ implies the $L$-convergence of $\mathbf{e}$.

Next prove the finiteness of $V(L)$. According to *Proposition* 2, we need only to prove the boundedness of $V(s)$ during the interval $[L_1, L)$, where $0 < L_1 \leq L$. From (24) and the definition of the gain matrix (1), the adaptation law is

$$\hat{\boldsymbol{\theta}}(s) = \mathcal{A}(s)\boldsymbol{\xi}(s)\sigma(s) \quad (32)$$

or $\boldsymbol{\xi}\sigma = \mathcal{A}^{-1}\hat{\boldsymbol{\theta}}$. Define $V(s) = \frac{\sigma^2}{2b} + \frac{1}{2}\int_0^s \boldsymbol{\phi}^T(\tau)\mathcal{B}^{-1}\boldsymbol{\phi}(\tau)d\tau$. Using the relationship (29), the upper right hand derivative of $V$ is

$$\nabla V = -k\sigma^2 + \boldsymbol{\phi}^T\mathcal{A}^{-1}\hat{\boldsymbol{\theta}} + \frac{1}{2}\boldsymbol{\phi}^T\mathcal{B}^{-1}\boldsymbol{\phi}. \quad (33)$$

By virtue of the analogy between (4) and (33), the boundedness of $V(L)$ is immediately obvious from *Proposition* 2. ■

## 4. SPAC FOR SYSTEMS WITH PSEUDO-PERIODIC PARAMETERS

In this section the parallel parametric adaptation is explored. The unknown parameter $b$ is assumed to a unknown positive constant.

From (21), the dynamics of the tracking error $\mathbf{e}$ is

$$\nabla\mathbf{e} = A\mathbf{e} + b\mathbf{b}\left[\boldsymbol{\theta}^T\boldsymbol{\xi} + \mu(\sigma + \rho - w) + \eta u\right], \quad (34)$$

where $\boldsymbol{\theta} = b^{-1}\mathbf{a}$, $\boldsymbol{\xi} = \boldsymbol{\xi}^0$, $\mu = b^{-1}$. Using Lyapunov stability theory for LTI systems, for a given positive definite matrix $Q \in \mathcal{R}^{n\times n}$, there exists a unique positive definite matrix $P \in \mathcal{R}^{n\times n}$ satisfying the Lyapunov equation

$$A^TP + PA = -Q.$$

Denote $\lambda_Q$ the minimum eigenvalue of the matrix $Q$ such that $-\mathbf{x}^TQ\mathbf{x} \leq -\lambda_Q\mathbf{x}^T\mathbf{x}$ for any $\mathbf{x} \in \mathcal{R}^n$. The spatial control mechanism is constructed as

$$u(s) = -\frac{1}{\eta}[\hat{\boldsymbol{\theta}}(s)^T\boldsymbol{\xi}^0 + \hat{\mu}(\sigma + \rho - w)], \quad (35)$$

where $\hat{\boldsymbol{\theta}}(s) = [\hat{\theta}_1, \cdots, \hat{\theta}_m]^T$ is the parameter estimate of $\boldsymbol{\theta}$ and $\hat{\mu}$ is the parameter estimate of $\mu$. Note that we have periodic parameters $\boldsymbol{\theta}$ and time invariant parameter $\mu$, hence use mixed periodic adaption and differential adaption laws

$$\begin{aligned}\hat{\theta}_i(s) &= \hat{\theta}_i(s - L_i) + \gamma_i(s, L_i)\xi_i(s)v(s),\\ \hat{\theta}_i(s) &= 0, \qquad \forall s \in [-L_i, 0], \quad i = 1, \cdots, m, \quad (36)\\ \nabla\hat{\mu}(s) &= \gamma[\sigma + \rho - w]v(s),\end{aligned}$$

where $v(s) = \mathbf{b}^TP\mathbf{e}(s)$, $L_i$ denotes the period of the unknown parameter $a_i$ or $\theta_i$, the adaptation gain $\gamma_i(s, L_i)$ is defined in (1), $\gamma > 0$ is a constant gain, $\xi_i$ is the $i$-th entry of vector $\boldsymbol{\xi}$.

*Theorem 2.* For the system (34), the spatial control mechanism (35) and (36) ensures the $L$-convergence of the tracking error $\mathbf{e}$.

*Proof.* Substituting the spatial control law (35) into the dynamics (34) yields the closed-loop error dynamics

$$\nabla \mathbf{e} = A\mathbf{e} + b\mathbf{b}[\boldsymbol{\phi}^T \boldsymbol{\xi} + \psi(\sigma + \rho - w)], \qquad (37)$$

where $\boldsymbol{\phi} = [\phi_1, \cdots, \phi_m]^T = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$, $\psi = \mu - \hat{\mu}$. Define the LKF

$$V(s) = \frac{1}{2b}\mathbf{e}^T P\mathbf{e} + \frac{1}{2}\sum_{i=1}^{m}\frac{1}{\beta_i}\int_{\max\{0,s-L_i\}}^{s} \phi_i^2 d\tau + \frac{1}{2\gamma}\psi^2. (38)$$

First consider the interval $[L, \infty)$, where $L = \max\{L_1, \cdots, L_m\}$. The upper right hand derivative of $V$ w.r.t. $s$ is

$$\nabla V = \frac{1}{2b}\left(\nabla \mathbf{e}^T P\mathbf{e} + \mathbf{e}^T P\nabla \mathbf{e}\right) + \frac{1}{2}\sum_{i=1}^{m}\frac{1}{\beta_i}\left[\phi_i^2(s) - \phi_i^2(s - L_i)\right] - \frac{1}{\gamma}\psi\nabla\hat{\mu}. \quad (39)$$

Substituting the dynamics (37) into the first term on the right-hand side of equation (39) gives

$$\begin{aligned}&\frac{1}{2b}(\nabla \mathbf{e}^T P\mathbf{e} + \mathbf{e}^T P\nabla \mathbf{e})\\ &= \frac{1}{2b}\mathbf{e}^T(A^T P + PA)\mathbf{e} + \boldsymbol{\phi}^T\boldsymbol{\xi}\mathbf{b}^T P\mathbf{e} + \psi(\sigma + \rho - w)\mathbf{b}^T P\mathbf{e}\\ &\le -\frac{\lambda_Q}{2b}\|\mathbf{e}\|^2 + \boldsymbol{\phi}^T\boldsymbol{\xi}v + \psi(\sigma + \rho - w)v. \qquad (40)\end{aligned}$$

The second term on the right-hand side of equation (39), by substituting the parametric updating law (36), the spatial periodicity $\theta_i(s) = \theta_i(s - L_i)$, and the algebraic relationship (10), is

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{m}\frac{1}{\gamma_i}\left[\phi_i^2 - \phi_i^2(s - L_i)\right]\\ &= -\boldsymbol{\phi}^T\boldsymbol{\xi}v - \frac{1}{2}\sum_{i=1}^{m}\gamma_i(\xi_i v)^2. \qquad (41)\end{aligned}$$

The third term on the right-hand side of equation (39), by substituting the parametric updating law (36), becomes

$$-\frac{1}{\gamma}\psi\nabla\hat{\mu} = -\psi(\sigma + \rho - w)v. \qquad (42)$$

The parametric uncertainties in (41) and (42) appear in (40) with opposite signs. As a result, substituting (40), (41) and (42) into (39) yields $\nabla V \le -\frac{\lambda_Q}{2b}\|\mathbf{e}\|^2 \le 0$, that is, $\nabla V$ is negative semi-definite for $s \ge L$. From *Proposition* 1, we can derive the boundedness of $\mathbf{e}$ and the $L$-convergence property $\lim_{s\to\infty}\int_{s-L}^{s}\|\mathbf{e}\|^2(\tau)d\tau = 0$ when $V(L)$ is finite.

The remaining is to prove the boundedness of $V(s)$ for $s \in [0, L]$. Without the loss of generality, assume the periods satisfy the relationship

$$L_1 < L_2 < \cdots < L_m = L,$$

and the interval $[0, L]$ is divided into $m$ different subintervals $[L_j, L_{j+1}]$. Suppose $s \in [L_j, L_{j+1}]$, the LKF (38) renders to

$$\begin{aligned}&V(s)\\ &= \frac{1}{2b}\mathbf{e}^T P\mathbf{e} + \frac{1}{2}\sum_{i=1}^{j-1}\frac{1}{\beta_i}\int_{s-L_i}^{s}[\theta_i(\tau) - \hat{\theta}_i(\tau)]^2 d\tau\\ &+ \frac{1}{2}\sum_{i=j}^{m}\frac{1}{\beta_i}\int_{0}^{s}[\theta_i(\tau) - \hat{\theta}_i(\tau)]^2 d\tau + \frac{1}{2\gamma}(\mu - \hat{\mu})^2. \quad (43)\end{aligned}$$

The upper right hand derivative of the functional $V$ w.r.t. $s$ is

$$\begin{aligned}\nabla V = &\frac{1}{2b}\left(\nabla \mathbf{e}^T P\mathbf{e} + \mathbf{e}^T P\nabla \mathbf{e}\right)\\ &+ \frac{1}{2}\sum_{i=1}^{j-1}\frac{1}{\beta_i}\left[\phi_i^2(s) - \phi_i^2(s - L_i)\right]\\ &+ \frac{1}{2}\sum_{i=j}^{m}\frac{1}{\beta_i}\phi_i^2(s) - \psi\nabla\hat{\mu}. \qquad (44)\end{aligned}$$

The differential adaptation law for the constant parameter $\mu$ is the same for the entire time horizon $[0, \infty)$. On the other hand, the periodic parameter adaption (36) can be divided into two groups

$$\hat{\theta}_i(s) = \hat{\theta}_i(s - L_i) + \beta_i\xi_i(s)v(s) \quad i = 1, \cdots, j - 1 \quad (45)$$

and

$$\hat{\theta}_i(s) = \alpha_i(s)\xi_i(s)v(s) \quad i = j, \cdots, m. \qquad (46)$$

Since the parameter estimates and the LKF associated with parameters $\mu$ and $\theta_i$, $i = 1, \cdots, j - 1$, are the same as the preceding circumstance $s \ge L$, they appear in $\nabla V$ with negative semi-definite results. Thus we need only focus on the parameters $\theta_i$, $i = j, \cdots, m$ and the upper right hand derivative of the LKF (43) is

$$\nabla V \le -\frac{\lambda_Q}{2b}\|\mathbf{e}\|^2 + \sum_{i=j}^{m}\phi_i\xi_i v + \frac{1}{2}\sum_{i=j}^{m}\frac{1}{\beta_i}\phi_i^2(s). \quad (47)$$

For the $i$th term in (47), by substituting the adaptation law (46) we have

$$\phi_i\xi_i v + \frac{1}{\beta_i}\phi_i^2 = \frac{1}{\alpha_i}\phi_i\hat{\theta}_i + \frac{1}{\beta_i}\phi_i^2,$$

which is analogous to (4) as the scalar case. Therefore by applying *Proposition* 2 we can directly conclude the boundedness of $\nabla V$ in (47), in the sequel the finiteness of $V(s)$ in $[0, L_{j+1})$. ■

## 5. CONCLUSION

In this work a SPAC approach was proposed. The SPAC can achieve $L$-convergence for any rotary machines systems that have spatially periodic parameter uncertainties or disturbances. The spatial periodic adaptation mechanism can work well even through the spatially periodic parameters may be aperiodic along the time axis. The main contributions of this work were to provide a feedback linearization method for high-order rotary systems, and extend the periodic adaptation to pseudo-periodic parameters without a common period.

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## Appendix A. THE PROOF OF PROPOSITION 3

First we apply the principle of induction to prove the relationship

$$z_{j+1} = \frac{N_j(x_1, x_2, \cdots, x_{j+1})}{x_1^{2j-1}}, \tag{A.1}$$

where $N_j$ is a polynomial of $x_1, x_2, \cdots, x_{j+1}$.

When $j = 1$, $\nabla z_1 = \nabla x_1 = x_2/x_1$. From the state transformation (13), $z_2 = \nabla x_1 = N_1(x_1, x_2)/x_1$ and $N_1 = x_2$. When $j = 2$,

$$\nabla z_2 = \nabla^2 x_1 = \nabla\left(\frac{N_1(x_1, x_2)}{x_1}\right) = \frac{x_1 \nabla N_1 - N_1 \nabla x_1}{x_1^2}. \tag{A.2}$$

Note that $\nabla x_j = x_{j+1}/x_1$, $\nabla N_1 = L_{[\nabla x_1,\, \nabla x_2]} N_1 = L_{[\frac{x_2}{x_1},\, \frac{x_3}{x_1}]} N_1 = \frac{1}{x_1} L_{[x_2,\, x_3]} N_1$. Substituting $\nabla N_1$ into (A.2) yields

$$z_3 = \nabla^2 x_1 = \frac{x_1 L_{[x_2,\, x_3]} N_1 - x_2 N_1}{x_1^3} = \frac{N_2}{x_1^3},$$
$$N_2 \stackrel{\triangle}{=} x_1 L_{[x_2,\, x_3]} N_1 - x_2 N_1. \tag{A.3}$$

It can be seen that $N_1$ and $N_2$ are polynomials. The expressions of $z_2$ and $z_3$ are consistent with (A.1).

Now assume $z_j = \frac{N_{j-1}(x_1, x_2, \cdots, x_j)}{x_1^{2j-3}}$. Our objective is to prove (A.1). Note that $N_{j-1}$ is a function of the arguments $x_1, x_2, \cdots, x_j$, by differentiation we have

$$z_{j+1} = \frac{x_1^{2j-3} \nabla N_{j-1} - N_{j-1} \nabla (x_1^{2j-3})}{x_1^{2(2j-3)}}. \tag{A.4}$$

Analogous to the preceding derivation,

$$\nabla N_{j-1} = L_{[\nabla x_1, \cdots, \nabla x_j]} N_{j-1} = \frac{1}{x_1} L_{[x_2, \cdots, x_{j+1}]} N_{j-1},$$

as well as $\nabla(x_1^{2j-3}) = (2j-3) x_1^{2j-4} \nabla x_1 = (2j-3) x_1^{2j-5} x_2$. Substituting the above relations into (A.4) yields

$$z_{j+1} = \frac{x_1 L_{[x_2, \cdots, x_{j+1}]} N_{j-1} - (2j-3) x_2 N_{j-1}}{x_1^{2j-1}} \tag{A.5}$$

which is consistent with (A.1).

Next we derive the dynamics of $\nabla z_n$ satisfying

$$\nabla z_n = \nabla\left(\frac{N_{n-1}}{x_1^{2n-3}}\right). \tag{A.6}$$

The differentiation of $N_{n-1}$ is

$$\nabla N_{n-1} = L_{[\nabla x_1, \cdots, \nabla x_n]} N_{j-1} = \frac{1}{x_1} L_{[x_2, \cdots, x_n]} N_{n-1} + \frac{\partial N_{n-1}}{\partial x_n} \nabla x_n$$

and $\nabla x_n = \mathbf{a}(s)^T \boldsymbol{\zeta}(\mathbf{x}) + b(s) x_1^{-1} u$. Substituting the above relations into (A.6) we obtain

$$\nabla z_n = \mathbf{a}(s)^T \boldsymbol{\xi}_x^0(\mathbf{x}) + \rho_x(\mathbf{x}) + b(s) \eta_x(\mathbf{x}) u, \tag{A.7}$$

where $\eta_x = \frac{1}{x_1^{2n-2}} \frac{\partial N_{n-1}}{\partial x_n}$, $\boldsymbol{\xi}_x^0(\mathbf{x}) = x_1 \eta_x(\mathbf{x}) \boldsymbol{\zeta}^T(\mathbf{x})$, $\rho_x = \frac{x_1 L_{[x_2, \cdots, x_n]} N_{n-1} - (2n-3) x_2 N_{n-1}}{x_1^{2n-1}}$.

Finally we prove the transformation $\mathbf{z} = \mathcal{T}(x_1, \cdots, x_n)$ is diffeomorphism, i.e., its inverse transformation exists and is smooth. Again we apply the principle of induction to prove a general relationship

$$x_{j+1} = z_1^j z_{j+1} + \frac{f_{j+1}(z_1, \cdots, z_j)}{z_1^{l_{j+1}}}, \quad \frac{\partial N_j}{\partial x_{j+1}} = x_1^{j-1} > 0, \tag{A.8}$$

where $f_{j+1}$ is a polynomial of $z_1, z_2, \cdots, z_j$, and $l_{j+1}$ is a non-negative integer.

First, we have $x_1 = z_1$, $x_2 = z_1 z_2$ and $\partial N_1/\partial x_2 = 1$ which are consistent with (A.8). Next assume

$$x_i = z_1^{i-1} z_i + \frac{f_i(z_1, \cdots, z_{i-1})}{z_1^{l_i}}, \quad \frac{\partial N_{i-1}}{\partial x_i} = x_1^{i-2}, \tag{A.9}$$

hold for $i = 1, \cdots, j$. From (A.5) and using the relationship (A.9)

$$z_{j+1} = \frac{x_1 L_{[x_2, \cdots, x_j]} N_{j-1} - (2j-3) x_2 N_{j-1}}{x_1^{2j-1}} + \frac{x_{j+1}}{x_1^j}. \tag{A.10}$$

Since $x_1 = z_1$, from (A.10) solving for $x_{j+1}$ yields

$$x_{j+1} = z_1^j z_{j+1} + \frac{x_1 L_{[x_2, \cdots, x_j]} N_{j-1} - (2j-3) x_2 N_{j-1}}{x_1^{j-1}}. \tag{A.11}$$

The polynomial $N_{j-1}$ consists of $x_1, \cdots, x_j$. By substituting $x_i$ in the second term on the right hand side of (A.11) with the first equality in (A.9),

$$\frac{x_1 L_{[x_2, \cdots, x_j]} N_{j-1} - (2j-3) x_2 N_{j-1}}{x_1^{j-1}}$$

becomes a function of $z_1, \cdots, z_j$, and the denominator consists of $z_1$ only. As a result, the relationship (A.8) holds.

In terms of (A.8), the relations between $x_j$ and $z_j$ can be summarized in a matrix form

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & z_1 & 0 & \cdots & 0 & 0 \\ \star & \star & z_1^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & \star & \cdots & z_1^{n-2} & 0 \\ \star & \star & \star & \cdots & \star & z_1^{n-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \tag{A.12}$$

where elements denoted by $\star$ at the $j$th row are fractions of $f_j(z_1, \cdots, z_{j-1})/z_1^{l_j}$ and therefore continuous and nonsingular when $x_1 > 0$. (A.12) shows that the inverse transformation $\mathbf{x} = \mathcal{T}^{-1}(\mathbf{z})$ exists and is smooth.