Special correspondences and Chow traces of Landweber-Novikov operations

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1 Introduction

In paper [Vi07] A. Vishik using the techniques of symmetric operations in algebraic cobordism proved that changing the base field by the function field of a smooth projective quadric doesn't change the property of being rational for cycles of small codimension. This fact which he calls Main Tool Lemma plays the crucial role in his construction of fields with u-invariant $2^r + 1$.

In the present paper we prove M.T.L. for the class of varieties introduced by M. Rost in the context of the proof of Bloch-Kato conjecture. Namely, for varieties which possess a special correspondence (see [Ro06, Definition 5.1]). As in the Vishik's proof the main technical tools are algebraic cobordism of M. Levine and F. Morel [LM07], generalized Rost degree formula and divisibility of Chow traces of certain Landweber-Novikov operations. Therefore, we always assume that our base field k has characteristic 0.

We use the following notation. By $\overline{k}$ we denote the algebraic closure of k and by $X_{\overline{k}}$ the respective base change $X \times_k \overline{k}$ of a variety X. By K we denote the function field of X. Given a prime p by $\overline{\mathrm{Ch}}^*(X)$ we denote the Chow ring of X modulo p-torsion taken with $\mathbb{Z}/p$-coefficients, i.e., $\overline{\mathrm{Ch}}^*(X) = \overline{\mathrm{CH}}^*(X)/p$, where $\overline{\mathrm{CH}}^*(X) = \mathrm{CH}^*(X)/(p\text{-}tors)$. We say that a cycle from $\overline{\mathrm{Ch}}^*(X_{\overline{k}})$ is *defined over* k if it belongs to the image of the restriction map $\overline{\mathrm{Ch}}^*(X) \to \overline{\mathrm{Ch}}^*(X_{\overline{k}})$.

The following notion will be central in this paper

1.1 Definition. Let X be a smooth proper irreducible variety over a field k of dimension n and p be a prime. Assume that X has no zero-cycles of degree coprime to p. Let d be an integer $0 \le d \le n$. We say X is a *d-splitting variety mod p* if for any smooth quasi-projective variety Y over k and all integers m satisfying $0 \le m < d$, the following condition holds

$$\text{A cycle } y \in \overline{\mathrm{Ch}}^m(Y_{\overline{k}}) \text{ is defined over } k \Longleftrightarrow y \times_{\overline{k}} \overline{K} \text{ is defined over } K \quad (1)$$

1.2 Example. According to the results of A. Vishik

(a) any smooth projective quadric Q of dimension n is a $[\frac{n+1}{2}]$-splitting variety mod 2 (see [Vi07, Cor. 3.5.(1)]);

(b) a quadric Q is a n-splitting variety (mod 2) if and only if Q possesses a Rost projector (see [Vi07, p. 373]).

The main result of the paper is the following

1.3 Theorem. *Let X be a smooth proper irreducible variety over a field of characteristik 0. Assume that X has no zero-cycles of degree coprime to p. If X possesses a special correspondence in the sense of Rost, then X is a $\frac{n}{p-1}$-splitting variety and the value $\frac{n}{p-1}$ is optimal.*

As an application of the techniques used in the proof of 1.3, we provide a complete list of d-splitting projective homogeneous varieties of type F_4.

To do this we introduce the following notation. Let G be a simple linear algebraic group over k. We say a projective homogeneous G-variety X is of type $\mathcal{D}$, if the group $G_{\overline{k}}$ has root system of type $\mathcal{D}$. Moreover, if $X_{\overline{k}}$ is the variety of parabolic subgroups of $G_{\overline{k}}$ defined by the subset of simple roots S of $\mathcal{D}$, then we say that X is of type $\mathcal{D}/P_S$. In this notation $P_{\mathcal{D}}$ defines a Borel subgroup and P_i, $i \in \mathcal{D}$, defines a maximal parabolic subgroup.

1.4 Corollary. *Let X be a projective homogeneous variety of type F_4 and p be one of its torsion primes (2 or 3). Assume that X has no zero-cycles of degree coprime to p. Then depending on p we have*

$p=2$: *If X is of type F_4/P_4, then X is a $(\dim X)$-splitting variety. For all other types X is a 3-splitting variety and this value is optimal.*

$p=3$: *X is always a 4-splitting variety and this value is optimal.*

2 Mod-p operations

In the present section we introduce certain operations

$$\phi_p^{q(t)}\colon \Omega^* \to \overline{\mathrm{Ch}}^* \text{ parametrized by } q(t) \in \overline{\mathrm{Ch}}^*[[t]]$$

from the ring of algebraic cobordism Ω^* of Levine-Morel to the Chow group modulo p-torsion $\overline{\mathrm{Ch}}^*$, where p is a given prime. All the facts used here can be found in [LM07], [Me03] and [Vi06].

Let $\overline{\mathrm{CH}}^*$ and $\overline{\mathrm{Ch}}^*$ denote the respective Chow groups modulo p-torsion. Consider the commutative diagram

$$\begin{tikzcd}
\Omega^m \arrow[r, "S_{LN}^{k(p)}"] \arrow[d, "pr"] & \Omega^{m+(p-1)k} \arrow[d, "pr"] \\
\overline{\mathrm{CH}}^m \arrow[d] & \overline{\mathrm{CH}}^{m+(p-1)k} \arrow[d] \\
\overline{\mathrm{Ch}}^m \arrow[r, "S^k"] & \overline{\mathrm{Ch}}^{m+(p-1)k}
\end{tikzcd},$$

where $S_{LN}^{k(p)}$ is the Landweber-Novikov operation corresponding to the partition $k(p) = \underbrace{(p-1, p-1, \dots, p-1)}_{k-times}$, $pr\colon \Omega^* \to \mathrm{CH}^* \to \overline{\mathrm{CH}}^*$ is the natural surjection and S^k is the reduced p-th power operation. By the properties of reduced power operations, $S^k = 0$ if $k > m$.

Define (cf. [Vi07, Prop. 2.1])

$$\phi_p^{t^{(p-1)a}} = \frac{pr \circ S_{LN}^{k(p)}}{p}, \text{ where } a = k - m.$$

By commutativity of the diagram $\phi_p^{t^{(p-1)a}}$ is well-defined. If r is not divisible by $(p-1)$, then $pr \circ S_{LN}^{k(r)} = 0$ and we set $\phi_p^{t^r} = 0$. Hence, we have constructed an operation $\phi_p^{t^r}$, $r > 0$, which maps Ω^m to $\overline{\mathrm{Ch}}^{r+pm}$.

Finally, given a power series $q(t) \in \overline{\mathrm{Ch}}^*[[t]]$ we define

$$\phi_p^{q(t)} = \sum_{r \geq 0} q_r \phi_p^{t^r}, \text{ where } q(t) = \sum_{r \geq 0} q_r t^r.$$

By definition the operations $\phi_p^{q(t)}$ are additive and respect pull-backs.

Let E be a vector bundle over a smooth variety U, Consider its total Chern class $c(E) = \prod_j (t+\xi_j)$, where ξ_j are the roots. Define $c(E)^{(p)}(t) = \prod_j (t^{p-1} + \xi_j^{p-1})$. Let $E = T_U$ be the tangent bundle of rank l. Set $b_l(U) = \deg c(-T_U)^{(p)}$. According to [Me03, §6] this number is always divisible by p. Moreover, it is trivial if l is not divisible by $(p-1)$. We define the Rost number η_p as

$$\eta_p(U) = \frac{b_l(U)}{p} \in \mathbb{Z}/p.$$

We will extensively use the following property of the operation ϕ_p which follows from the multiplicativity of Landweber-Novikov operations

2.1 Lemma. *(cf. [Vi07, Prop.2.3]) Let U be a smooth projective variety of positive dimension l and $[U]$ be its class in the Lazard ring $\mathbb{L}$. Let $\beta \in \Omega^j(X)$. Then*

$$\phi_p^{t^r}([U]\cdot\beta) = \eta_p(U)\cdot S^k(pr(\beta)), \text{ where } r = (p-1)(k-j)+lp > 0.$$

Observe that $\phi_p^{t^r}([U]\cdot\beta) = 0$ if r is not divisible by $(p-1)$.

3 Construction of a cycle defined over k

In the present section we prove Theorem 1.3.

I. We proceed following the proof of [Vi07, Thm. 3.1]. Let Y be a smooth projective variety over k. Let $y \in \overline{\mathrm{Ch}}^m(Y_{\overline{k}})$ be such that $y_{\overline{K}}$ is defined over K. We want to show that y is defined over k for all $m < d$. Let Ω^* be the algebraic cobordism of Levine-Morel. There is a natural surjection $pr\colon \Omega^* \to \overline{\mathrm{Ch}}^*$. Consider the commutative diagram

$$\begin{array}{ccccccc}
\omega & \Omega^m(X\times Y) & \xrightarrow{pr} & \overline{\mathrm{Ch}}^m(X\times Y) & \xrightarrow{p^*} & \overline{\mathrm{Ch}}^m(Y_K) & u \\
\downarrow & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & \downarrow \\
\bar\omega & \Omega^m(X_{\overline{k}}\times Y_{\overline{k}}) & \xrightarrow{pr} & \overline{\mathrm{Ch}}^m(X_{\overline{k}}\times Y_{\overline{k}}) & \xrightarrow{\bar p^*} & \overline{\mathrm{Ch}}^m(Y_{\overline{K}}) & y_{\overline{K}}
\end{array}$$

where $p\colon \mathrm{Spec}(K)\times Y \to X\times Y$ and the pull-back p^* is surjective because of the localization sequence for Chow groups. By the hypothesis there exists a preimage u of $y_{\overline{K}}$ by means of res. By surjectivity of pr and p^*, there exists a preimage ω of u. Let $\bar\omega = \mathrm{res}(\omega)$.

II. Let X be a variety which possesses a special correspondence. By the results of [Ro06] this implies that

(a) $n = p^s - 1$,

(b) $\eta_p(X_{\overline{k}}) \neq 0 \mod p$

(c) the Chow motive of X contains an indecomposable summand M which over $\overline{k}$ splits as a direct sum of Tate motives shifted by the multiples of $d = \frac{p^s-1}{p-1}$

$$M_{\overline{k}} \simeq \bigoplus_{i=0}^{p-1} \mathbb{Z}/p\{di\}.$$

Let π be an idempotent defining M. We can choose ω in such a way that the realization $\rho = \pi_*(\bar{\omega}) \in \Omega^m(X_{\overline{k}} \times Y_{\overline{k}})$ will have the following form (cf. [Vi07, p.368])

$$\rho = x_n \times y_n + \sum_i x_i \times y_i + x_0 \times y_0, \tag{2}$$

where the sum in the middle is taken over all $i \in \{d, 2d, \ldots, (p-2)d\}$, $x_i \in \Omega_i(X_{\overline{k}})$, $y_i \in \Omega^{m-n+i}(Y_{\overline{k}})$ are certain cobordism classes, x_0 is the class of a rational point, $x_n = \pi_*(1)$ is the class given by a proper birational morphism $\hat{X} \to X_{\overline{k}}$ and $pr(y_n) = y$ (cf. [Vi07, Lemma 3.2]).

III. Consider the cycle $\phi_p^{t^r}(p_{Y*}(\rho)) \in \overline{\mathrm{Ch}}^*(Y_{\overline{k}})$ defined over k. It has codimension m, if $r = (dp - m)(p-1) > 0$.

3.1 Lemma. $\phi_p^{t^r}(p_{Y*}(\rho)) = \eta_p(X_{\overline{k}}) \cdot y + \phi_p^{t^r}(y_0)$.

Proof. Applying Lemma 2.1 to each summand $x_i \times y_i$ in the middle of (2) we obtain

$$\phi_p^{t^r}(p_{Y*}(x_i \times y_i)) = \eta_p(x_i) \cdot S^{(n-i)/(p-1)}(pr(y_i)).$$

Since $pr(y_i) \in \overline{\mathrm{Ch}}^{m-n+i}(Y_{\overline{k}})$ and $m-n+i \leq m-n+(p-2)d = m-d < 0$, it must be trivial. Hence, only the very right and left summands of (2) survive.

By Rost Degree Formula [LM07, Cor. 13.8] we have

$$x_n = 1 + \sum_{u_Z \in \mathbb{L}_{>0}} u_Z \cdot [Z \to X_{\overline{k}}]. \tag{3}$$

Since π is an idempotent, $\pi_* \circ \pi_*(1) = \pi_*(1)$. The latter implies that

$$\pi_*(\sum u_Z \cdot [Z \to X_{\overline{k}}]) = \sum u_Z \cdot \pi_*([Z \to X_{\overline{k}}]) = 0.$$

Hence, applying η_p we get

$$0 = \sum_{u_Z \in \mathbb{L}_{>0}} \eta_p\Big(u_Z \cdot \pi_*([Z \to X_{\overline{k}}])\Big) = u_{pt} + \sum_{u_Z \in \mathbb{L}_{<n}} u_Z \cdot \pi_*([Z \to X_{\overline{k}}]).$$

Since $\eta_p(u_Z \cdot \pi_*([Z \to X_{\overline{k}}]))$ is divisible by p if $u_Z \in \mathbb{L}_{<n}$, we obtain that $\eta_p(u_{pt}) = 0 \mod p$.

Therefore, for the very left summand of (2) we get

$$\phi_p^{t^r}(p_{Y*}(x_n \times y_n)) = \eta_p\Big([X_{\overline{k}}] + \sum_{u_Z \in \mathbb{L}_{<n}} u_Z \cdot [Z] + u_{pt}\Big) \cdot pr(y_n) = \eta_p(X_{\overline{k}}) \cdot y.$$

Observe that $\eta_p(u_Z \cdot [Z])$ is divisible by p if $u_Z \in \mathbb{L}_{<n}$. □

Consider the cycle $p_{Y*}(\phi_p^{t^{r'}}(\rho)) \in \overline{\mathrm{Ch}}^*(Y_{\overline{k}})$ defined over k. It has codimension m, if $r' = (d-m)(p-1) > 0$.

3.2 Lemma. $p_{Y*}(\phi_p^{t^{r'}}(\rho)) = \phi^{t^r}(y_0)$.

Proof. By definition $p_{Y*}(\phi_p^{t^{r'}}(x_i \times y_i)) = p_{Y*}(\frac{1}{p} \cdot pr(S_{LN}^{d(p)}(x_i \times y_i)))$. By multiplicativity of S_{LN}^* the latter can be written as

$$p_{Y*}(\tfrac{1}{p} \cdot pr(S_{LN}^{d(p)}(x_i \times y_i))) = p_{Y*}(\tfrac{1}{p} \cdot \sum_{\alpha+\beta=n} pr(S_{LN}^{\alpha}(x_i)) \times pr(S_{LN}^{\beta}(y_i))) =$$

$$= \sum_{\alpha+\beta=d(p)} \tfrac{1}{p} \cdot \deg\big(pr(S_{LN}^{\alpha}(x_i))\big) \cdot pr(S_{LN}^{\beta}(y_i)) =$$

Since $m < d$, y_i has negative codimension for all $i < n$, therefore, $pr(S_{LN}^{\beta}(y_i))$ is divisible by p for all $i < n$.

On the other hand, we have $S^* = S_* \cdot c(-T_{X_{\overline{k}}})^{(p)}$, where S_* (resp. S^*) are the (co-)homological operations. Hence,

$$\deg\big(pr(S_{LN}^{\alpha}(x_i))\big) = \deg\big(S_*(pr(x_i)) \cdot c(-T_{X_{\overline{k}}})^{(p)}\big).$$

Since all Chern classes of the tangent bundle of $X_{\overline{k}}$ are defined over k, and X possesses a special correspondence, according to [Ro06, Lemma 9.3] we obtain that $\deg\big(S_*(pr(x_i))\cdot c(-T_{X_{\overline{k}}})^{(p)}\big)$ is congruent to $\deg(S_*(pr(x_i)))$ mod p. Since S_* respect push-forwards, $\deg(S_*(pr(x_i)))$ is trivial mod p for all $i>0$. Hence, $\deg\big(pr(S^{\alpha}_{LN}(x_i))\big)$ is divisible by p for all $i>0$.

Combining all together we get

$$p_{Y*}(\phi_p^{t^{r'}}(\rho)) = \sum_{|\alpha|=n} \tfrac{1}{p}\deg\big(pr(S^{\alpha}_{LN}(x_n))\big)y + \phi_p^{t^r}(y_0) = \phi_p^{t^r}(y_0),$$

since $S^{\alpha}_{LN}(x_n)=0$. $\square$

By Lemmas 3.1 and 3.2 the following cycle is defined over k

$$\phi_p^{t^r}(p_{Y*}(\rho)) - p_{Y_*}(\phi_p^{t^{r'}}(\rho)) = \eta_p(X_{\overline{k}})\cdot y.$$

Since $\eta_p(X_{\overline{k}})\neq 0 \mod p$, the cycle y is defined over k, therefore, X is a d-splitting variety.

To see that $d=\frac{n}{p-1}$ is an optimal value take $Y=X$ and consider the cycle $y\in\overline{\mathrm{Ch}}^d(X_{\overline{k}})$ defining the Tate motive $\mathbb{Z}/p\{n-d\}$ in the decomposition of the Rost motive M over $\overline{k}$. Since M splits over K, $y_{\overline{K}}$ is defined over K. By condition (1) it implies that y is defined over k, i.e., M splits over k which contradicts to the indecomposability of M. The theorem is proven.

4 F_4-varieties

In the present section we apply our methods to describe all d-splitting varieties of type F_4.

I. Let X be a smooth proper irreducible variety over a field k. As in the beginning of the proof of 1.3 given a cycle $y\in\overline{\mathrm{Ch}}^m(Y_{\overline{k}})$ we construct a cobordism class $\bar\omega\in\Omega^m(X_{\overline{k}}\times Y_{\overline{k}})$ defined over k.

Assume that the motive of X contains a motive $M=(X,\pi)$ such that the idempotent π can be written as

$$\pi_{\overline{k}} = \gamma\times\gamma^* + \sum_i x_i\times x_i^*,$$

where $\gamma\in\Omega_{(p-1)d}(X_{\overline{k}})$ is defined over k, γ^* denotes its Poincare dual, i.e., $pr(\gamma\cdot\gamma^*)=pt$ and $x_i\in\Omega_i(X_{\overline{k}})$, $i\in\{0,d,2d,\dots,(p-2)d\}$.

Then the realization $\rho = \pi_*(p_X^*(\gamma)\cdot\bar{\omega}) \in \Omega^{m+n-g}(X_{\overline{k}} \times Y_{\overline{k}})$, $g=(p-1)d$, can be written as (cf. (2))

$$\rho = \gamma \times y_g + \sum_i x_i \times y_i + x_0 \times y_0,$$

where $x_i \in \Omega_i(X_{\overline{k}})$, $y_i \in \Omega^{m-g+i}(Y_{\overline{k}})$, $i \in \{d, 2d, \ldots, (p-2)d\}$.

As in the proof of Lemma (3.1) we obtain that for $r=(dp-m)(p-1)>0$ and any $m<d$ the following cycle in $\overline{\mathrm{Ch}}^m(Y_{\overline{k}})$ is defined over k

$$\phi_p^{t^r}(p_{Y*}(\rho)) = \eta_p(\gamma)\cdot y + \phi_p^{t^r}(y_0).$$

The transposed cycle π^t defines a direct summand $M^t=(X,\pi^t)$ of the motive of X (the one which over $\overline{k}$ contains the generic point of X). The realization $\rho' = \pi_*^t(\bar{\omega}) \in \Omega^m(X_{\overline{k}} \times Y_{\overline{k}})$ can be written as

$$\rho' = \sum_i x^{(i)} \times y^{(i)} + \gamma^* \times y^{(g)},$$

where $x^{(i)} \in \Omega^i(X_{\overline{k}})$, $i \in \{0, d, 2d, \ldots, (p-2)d\}$, $y^{(g)} = y_0$.

Consider the cycle $\delta = p_{Y*}(p_X^*(\gamma)\cdot \phi_p^{t^{r'}}(\rho'))$. It is defined over k and for $r'=(d-m)(p-1)>0$ belongs to $\overline{\mathrm{Ch}}^m(Y_{\overline{k}})$. Assume that $\delta = \phi_p^{t^r}(y_0)$, then subtracting it from $\phi_p^{t^r}(p_{Y*}(\rho))$ we obtain the cycle $\eta_p(\gamma)\cdot y$. Hence, to prove that X is a d-splitting variety it is enough to show that $\eta_p(\gamma) \neq 0 \mod p$ and $\delta = \phi_p^{t^r}(y_0)$.

II. Observe that for $p=2$ we have $\delta = \phi_p^{t^r}(y_0)$ by simple dimension reasons. Indeed, in this case the cycle ρ' consists only of two terms

$$\rho' = x^{(0)} \times y^{(0)} + \gamma^* \times y^{(g)},$$

where the first summand vanishes in δ, since $S_{LN}^{\alpha}(x^{(0)}) = 0$ if $|\alpha|>0$ and the second summand gives the required cycle $\phi_p^{t^r}(y_0)$.

For $p=3$ the cycle ρ' consists of three terms

$$\rho' = x^{(0)} \times y^{(0)} + x^{(d)} \times y^{(d)} + \gamma^* \times y^{(g)},$$

where again the first summand vanishes, the last gives $\phi^{t^r}(y_0)$ and the middle gives

$$\sum_{\alpha+\beta=d} \deg\left(pr(\gamma\cdot S_{LN}^{\alpha}(x^{(d)}))\right)\cdot \tfrac{1}{p} pr(S_{LN}^{\beta}(y^{(d)})), \text{ where } |\alpha| = \tfrac{p-2}{p-1}d. \quad (4)$$

III. Given an F_4-variety X we provide a cycle γ satisfying the conditions above as follows

$p = 2$**:** If X is generically split, then we may assume that X is of type F_4/P_1. By the main result of [PSZ] the cycle

$$\gamma = H \cdot c_4(T_{X_{\overline{k}}}) \cdot c_7(T_{X_{\overline{k}}}),$$

where H is a generator of the Picard group of $X_{\overline{k}}$ defined over k, is the generic point of a Rost motive sitting inside the motive of X. Observe that $[\gamma] = [Q_3]$ is represented by the class of a 3-dimensional quadric. Hence, $\eta_2(Q_3) = 1 \mod 2$ and X is a d-splitting variety with $d = 3$.

If X is not generically split, i.e., X is of the type F_4/P_4, then $M(X)$ contains a Rost motive with $\gamma = [X]$, i.e., X is a variety which possesses a special correspondence. Indeed, one can show that X is a norm variety corresponding to the cohomological invariant f_5. Hence, X is a n-splitting variety (here $n = 15$).

$p = 3$ In this case all F_4-varieties are generically split. Hence, by the results of [PSZ] the motive of X contains a generalized Rost motive with the generic point given by the cycle

$$\gamma = H^7.$$

The direct computation using the Adjunction formula shows that $\eta_3([H^7]) \neq 0 \mod 3$. Moreover, in this case the expression (4) will have the following form

$$\deg\left(\gamma S^2(pr(x^{(4)}))\right) \cdot \phi_p^{t^{12-2m}}(y^{(4)}).$$

which is equal to zero since $S^2(pr(x^{(4)}))$ is equal to zero. Hence, X is a d-splitting variety with $d = 4$.

4.1 Remark. Let X be a d-splitting geometrically cellular variety. As an immediate consequence of the definition of a canonical p-dimension (see [KM06]) we obtain the following inequality

$$cd_p(X) \geq d.$$

In the case of a variety of type F_4/P_4 it gives $cd_2(X) = \dim X = 15$.

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