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Special Functions of Mathematical Physics

A Unified Introduction with Applications

Translated from the Russian by Ralph P. Boas

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Table of Contents

Preface to the American edition

With students of Physics chiefly in mind, we have collected the material on special functions that is most important in mathematical physics and quantum mechanics. We have not attempted to provide the most extensive collection possible of information about special functions, but have set ourselves the task of finding an exposition which, based on a unified approach, ensures the possibility of applying the theory in other natural sciences, since it provides a simple and effective method for the independent solution of problems that arise in practice in physics, engineering and mathematics.

For the American edition we have been able to improve a number of proofs; in particular, we have given a new proof of the basic theorem (§3). This is the fundamental theorem of the book; it has now been extended to cover difference equations of hypergeometric type (§§12, 13). Several sections have been simplified and contain new material.

We believe that this is the first time that the theory of classical orthogonal polynomials of a discrete variable on both uniform and nonuniform lattices has been given such a coherent presentation, together with its various applications in physics.

Acknowledgements

The authors are grateful to Professor Boas for his skillful and lucid translation. As a result of this work, some portions of the book seem to be more clear-cut and precise in English than they appear in Russian. We thank all those who have contributed to the production of this edition.

Foreword to the Russian edition

Interest in special functions has greatly increased as a result of the extensive development of numerical methods and the growing role of computer simulation.

There are two reasons for this trend. In the first place, for many physical processes a mathematical description based on "first principles" leads to differential, integral, or integro-differential equations of rather complex form. Consequently the original problem usually has to be considerably simplified in order to clarify its most important qualitative features and to understand the relative roles played by various factors. If a solution of the simplified problem can be obtained in an explicit mathematical form that can easily be analyzed, one may be able to obtain a qualitative picture, without much expenditure of time and effort (but possibly with aid of a computer). Then one can analyze how the behavior of the solution depends on the parameters of the problem.

In the second place, in the solution of complicated problems on a computer it is convenient to make use of simplified problems in order to select reliable and economical numerical algorithms. Here it is seldom possible to restrict one's self to problems that lead to elementary functions. Moreover, a knowledge of special functions is essential for understanding many important problems of theoretical and mathematical physics.

The most commonly encountered special functions are those that are known as the "special functions of mathematical physics": the classical orthogonal polynomials (Jacobi, Laguerre, Hermite), spherical harmonics, and the Bessel and hypergeometric functions. Much basic research has been devoted to the theory of these functions and their applications. Unfortunately this research involves rather cumbersome mathematical techniques and many special devices. Consequently there long been a need for a theory of special functions based on general but simple ideas.

The authors of the present book have discovered an easily comprehended way of presenting the theory of special functions, based on a generalization of the Rodrigues formula for the classical orthogonal polynomials. Their approach makes it possible to obtain explicit integral representations of all the special functions of mathematical physics and to derive their basic properties. In particular, this method can be used to solve the second-order linear differential equations that are usually solved by Laplace's method. The construction of the theory of special functions uses a minimal amount of mathematical apparatus: it requires only the elements of the theory of ordinary differential equations and of complex analysis. This is a significant advantage, since it is well known that the large amount of essential mathematical knowledge, in particular that involving special functions, is a fundamental obstacle to the study of theoretical and mathematical physics.

In the process of working through the book the reader will gain experience in the development of asymptotic formulas, expansions in series, recursion formulas, estimates of various kinds, and computational formulas, and will come to see the intrinsic logical connections among special functions that at first sight seem completely different.

The book discusses connections with other branches of mathematics and physics. Considerable attention is paid to applications in quantum mechanics. The main interest here is in the study of problems on the determination of discrete energy spectra and the corresponding wave functions in problems that can be solved by means of the classical orthogonal polynomials. The authors have succeeded in presenting these problems without the traditional use of generalized power series. Hence they have been able, in Chapter V, to give elegant and easy solutions of such fundamental problems of quantum mechanics as the problems of the harmonic oscillator and of the motion of particles in a central field, and solutions of the Schrödinger, Dirac and Klein-Gordon equations for the Coulomb potential. We also call attention to the presentation, based on the method of V.A. Steklov, of the Wentzel-Kramers-Brillouin method.

The authors discuss the addition theorems for spherical harmonics and Bessel functions, which are widely applied in the theory of atomic spectra, in scattering theory, and in the design of nuclear reactors. In the study of generalized spherical harmonics the authors really come to grips with the theory of representations of the rotation group and the general theory of angular momentum. Later on, readers will be able to deepen their knowledge of special functions by consulting books in which special functions are studied by group-theoretical methods. The classical orthogonal polynomials of a discrete variable are of interest in the theory of difference methods. From the point of view of numerical calculation, it is instructive to apply quadrature formulas of Gaussian type for calculating sums and constructing approximate formulas for special functions. We note that this book presents a number of problems that are needed in applications but are touched on only lightly, or not at all, in textbooks.

The authors are specialists in mathematical physics and quantum mechanics. The book originated in the course of their work on a current problem of plasma physics in the M.V. Keldysh Institute of Applied Mathematics of the Academy of Sciences of the USSR.

The book contains a large amount of material, presented concisely in a lucid and well-organized way. It is certain to be useful to a wide circle of readers — to both undergraduate and graduate students, and to workers in mathematical and theoretical physics.

A.A. Samarskiĭ

Member of the Academy of Sciences of the USSR

Preface to the Russian edition

In solving many problems of theoretical and mathematical physics one is led to use various special functions. Such problems arise, for example, in connection with heat conduction, the interaction between radiation and matter, the propagation of electromagnetic or acoustic waves, the theory of nuclear reactors, and the internal structure of stars.

In practice, special functions usually arise as solutions of differential equations. Consequently the natural approach for mathematical physics is to deduce the properties of the functions directly from the differential equations that arise in natural mathematical formulations of physical problems. For this reason the authors have developed a method which makes it possible to present the theory of special functions by starting from a differential equation of the form

$$u'' + \frac{\tilde{\tau}(z)}{\sigma(z)}u' + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}u = 0 , \tag{1}$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials, at most of second degree, and $\tilde{\tau}(z)$ is a polynomial, at most of first degree. The differential equations of most of the special functions that occur in mathematical physics and quantum mechanics are particular cases of (1).

The book is organized as follows. The first chapter discusses a class of transformations $u = \phi(z)y$ by means of which, for special choices of $\phi(z)$, equation (1) is transformed into an equation of the same type. We can select transformations that carry (1) into an equation of the simpler form

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0 , \tag{2}$$

where $\tau(z)$ is a polynomial of at most the first degree and λ is a constant.

We shall call (2) an *equation of hypergeometric type*;* and its solutions, *functions of hypergeometric type*. The theory of these functions is developed in the following stages. First we show that the derivatives of functions of hypergeometric type are again functions of hypergeometric type. This property lets us construct a family of particular solutions of (2) corresponding to particular values of λ, starting from the obvious solution of (2), namely $y(z)$ =const. for $\lambda = 0$. Such solutions are polynomials in z; they may be written in explicit form by means of the Rodrigues formula. A natural generalization of the integral representation of these polynomials follows from the Rodrigues formula and makes it possible to obtain an integral representation of all functions of hypergeometric type corresponding to arbitrary values of λ. By using this integral representation and the transformation of (2) into other equations of the same type, we can obtain all the basic properties of the functions in question: power series expansions, asymptotic representations, recursion relations and functional equations. This approach lets us obtain the complete family of solutions of (1).

The second, third and fourth chapters are devoted to carrying out this program for particular functions of hypergeometric type: the classical orthogonal polynomials, spherical harmonics, Bessel functions, and hypergeometric functions. Consequently chapters II–IV can be read in any order after chapter I.

Chapter II, devoted to classical orthogonal polynomials, has been extended to include the theory of classical orthogonal polynomials of a discrete variable. Here our methodology of the theory of the classical orthogonal polynomials is applied to a difference equation instead of to a differential equation. There then arise various families of classical orthogonal polynomials of a discrete variable on both uniform and nonuniform lattices. It is interesting to observe that the study of classical orthogonal polynomials of a discrete variable was initiated by Chebyshev as early as the middle of the nineteenth century; it was then continued by many eminent investigators. However, no books have developed the theory of these polynomials as solutions of a difference equation. It has not even been clear until recently which polynomials, among those introduced by various authors and arising from various considerations, belong to the class described above.

Chapter V is devoted to applications. It should be noticed that we have discussed practically all the basic problems of quantum mechanics that can be solved in explicit form, and have constructed their solutions by a unified method. Physicists will be interested in our presentation of the remarkably simple connection between the Clebsch-Gordan coefficients, so extensively

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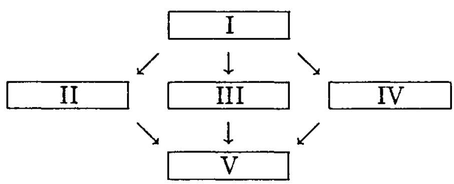
* This name is used because the particular solutions of equation (2) are hypergeometric functions when $\sigma(z)=z(1-z)$, and confluent hypergeometric functions when $\sigma(z)=z$ (see Chapter IV).

used in quantum mechanics, and orthogonal polynomials of a discrete variable — the Hahn polynomials. We also discuss the connection between the Racah coefficients and the classical orthogonal polynomials of a discrete variable, which are orthogonal on a discrete lattice.

Since familiarity with the properties of Euler's gamma function is a necessary prerequisite for the study of special functions, we present the theory of the gamma function in an appendix. There we also discuss the properties of Laplace integrals, which are used to obtain analytic continuations and asymptotic representations of special functions. At the end of the book we have provided a list of the basic formulas. If a reader requires more detailed information, we recommend the three volumes of the Bateman Project ([E2]), which contain all the formulas from the theory of special functions up to the middle 1940's, and also [A1] and [O1]. A more detailed idea of the contents of the book can be obtained from the table of contents and the following diagram of the connections among the chapters:



The method of studying special functions presented here is a further development of the method followed in the authors' book [N2]. In particular, it enables the reader to form a rather good idea of the theory of special functions after having studied only the first three sections of the book.

The basic material of the book was presented in a course of lectures on methods of mathematical physics given for several years in the Faculty of Theoretical and Experimental Physics of the Moscow Institute of Engineering Physics, and also in special courses in the Physical and Chemical Faculties and the Faculty of Numerical Mathematics and Cybernetics of the Moscow State University.

The authors thank T.T.Tsirulis, V.Ya. Arsenin, B.L. Rozhdestvenskiĭ and S.K. Suslov, as well as the staff of the Department of Theoretical and Nuclear Physics of the Moscow Institute of Engineering Physics, for helpful comments on the content of the book.

A.F. Nikiforov, V.B. Uvarov

Translator's preface

The book is divided, in the conventional Russian manner, into glavy (chapters), paragrafy (sections, abbreviated §), and punkty (abbreviated p. in Russian; I have used "part" for these). Sections (§§) are numbered consecutively through the book; the numbering of formulas begins again with each section. I have retained the authors' references to Russian sources, but I have added references to the corresponding English versions whenever I could find them, and otherwise to similar English books. References are cited in the form [Ln], where L is a letter and n is a numeral. When Russian terminology differs markedly from that customarily used in American English, I have silently adopted the latter (for example, "Bessel functions" instead of "cylinder functions").

For this translation, the authors have substantially rearranged and amplified the material, particularly in §13, where they discuss recent work on the connection between the Hahn polynomials and the Clebsch-Gordan coefficients, and between the Wigner $6j$-symbols and the Racah polynomials. They have also added (§27, parts 2 and 3) applications of orthogonal polynomials of a discrete variable to the compression of information, and of Bessel functions to laser sounding of the atmosphere.

I am indebted to Professor Mary L. Boas of the DePaul University Physics Department for help with the physics vocabulary.

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