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Research Article

Special Half Lightlike Submanifolds of an Indefinite Cosymplectic Manifold

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We study the geometry of half lightlike submanifolds M of an indefinite cosymplectic manifold $\overline{M}$. First, we construct two types of half lightlike submanifolds according to the form of the structure vector field of $\overline{M}$, named by tangential and ascreen half lightlike submanifolds. Next, we characterize the lightlike geometries of such two types of half lightlike submanifolds.

1. Introduction

The class of codimension 2 lightlike submanifolds of a semi-Riemannian manifold is composed entirely of two classes by virtue of the rank of its radical distribution, called half lightlike and coisotropic submanifolds [1–4]. Half lightlike submanifold is a special case of r-lightlike submanifold [5, 6] such that $r = 1$ and its geometry is more general than that of coisotropic submanifold. Moreover much of the works on half lightlike submanifolds will be immediately generalized in a formal way to arbitrary r-lightlike submanifolds. Recently several authors studied the geometry of lightlike submanifolds of indefinite cosymplectic manifolds. Much of them have studied so-called CR-types (CR, SCR, GCR, QCR, etc) lightlike submanifolds of indefinite cosymplectic manifolds. Unfortunately, an intrinsic study of lightlike submanifolds of indefinite cosymplectic manifolds is slight as yet. Only there are some limited papers on particular subcases recently studied [7–9].

The objective of this paper is to study the geometry of half lightlike submanifolds M of an indefinite cosymplectic manifold $\overline{M}$. There are many different types of half lightlike submanifolds of an indefinite cosymplectic manifold $\overline{M}$ according to the form of the structure vector field of $\overline{M}$. We study two types of them here: tangential and ascreen half lightlike submanifolds. We provide several new results on each types by using the structure of M induced by the contact metric structure of $\overline{M}$.

2. Half Lightlike Submanifolds

An odd dimensional smooth manifold $(\overline{M}, \overline{g})$ is called a contact metric manifold if there exists a contact metric structure $(J, \theta, \zeta, \overline{g})$, where J is a $(1,1)$-type tensor field, ζ a vector field which is called the structure vector field of $\overline{M}$ and θ a 1-form satisfying

$$\begin{gathered} J^2X = -X + \theta(X)\zeta, \qquad J\zeta = 0, \qquad \theta \circ J = 0, \qquad \theta(\zeta) = 1, \\ \overline{g}(\zeta, \zeta) = 1, \qquad \overline{g}(JX, JY) = \overline{g}(X, Y) - \theta(X)\theta(Y), \\ \theta(X) = \overline{g}(\zeta, X), \qquad d\theta(X, Y) = \overline{g}(JX, Y), \end{gathered} \tag{2.1}$$

for any vector fields X, Y on $\overline{M}$. We say that $\overline{M}$ has a normal contact structure if $N_J + d\theta \otimes \zeta = 0$, where N_J is the Nijenhuis tensor field of J. A normal contact metric manifold is called a cosymplectic [10, 11] for which we have

$$\overline{\nabla}_X \theta = 0, \qquad \overline{\nabla}_X J = 0, \tag{2.2}$$

for any vector field X on $\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{M}$. A cosymplectic manifold $\overline{M} = (\overline{M}, J, \zeta, \theta, \overline{g})$ is called an *indefinite cosymplectic manifold* [7–9] if $(\overline{M}, \overline{g})$ is a semi-Riemannian manifold of index $\mu(> 0)$.

For any indefinite cosymplectic manifold $\overline{M}$, applying $\overline{\nabla}_X$ to $J\zeta = 0$ and using (2.2), we have $J(\overline{\nabla}_X \zeta) = 0$. Applying J to this and using the fact $\theta(\overline{\nabla}_X \zeta) = 0$, we get

$$\overline{\nabla}_X \zeta = 0. \tag{2.3}$$

A submanifold M of a semi-Riemannian manifold $\overline{M}$ of codimension 2 is called a *half lightlike submanifold* if the rank of the radical distribution $\mathrm{Rad}(TM) = TM \cap TM^\perp$ is 1, where TM and $TM^\perp$ are the tangent and normal bundles of M, respectively. Then there exist complementary nondegenerate distributions $S(TM)$ and $S(TM^\perp)$ of $\mathrm{Rad}(TM)$ in TM and $TM^\perp$, respectively, which are called the *screen* and *coscreen distribution* on M:

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S(TM), \qquad TM^\perp = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S\left(TM^\perp\right), \tag{2.4}$$

where the symbol $\oplus_{\mathrm{orth}}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Choose L as a unit vector field of $S(TM^\perp)$ such that $\overline{g}(L, L) = \pm 1$. In this paper we may assume that $\overline{g}(L, L) = 1$ without loss of generality. Consider the orthogonal complementary distribution $S(TM)^\perp$ to

$S(TM)$ in $T\overline{M}$. Certainly $\mathrm{Rad}(TM)$ and $S(TM^{\perp})$ are vector subbundles of $S(TM)^{\perp}$. Thus we have the following orthogonal decomposition:

$$S(TM)^{\perp} = S\left(TM^{\perp}\right) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right)^{\perp}, \tag{2.5}$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. It is well-known [1, 2] that, for any null section ξ of $\mathrm{Rad}(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(\mathrm{ltr}(TM))$ satisfying

$$\overline{g}(\xi, N) = 1, \qquad \overline{g}(N, N) = \overline{g}(N, X) = \overline{g}(N, L) = 0, \quad \forall X \in \Gamma(S(TM)). \tag{2.6}$$

Let $\mathrm{tr}(TM) = S(TM^{\perp}) \oplus_{\mathrm{orth}} \mathrm{ltr}(TM)$. We say that N, $\mathrm{ltr}(TM)$ and $\mathrm{tr}(TM)$ are the lightlike transversal vector field, lightlike transversal vector bundle and transversal vector bundle of M with respect to $S(TM)$, respectively. Therefore $T\overline{M}$ is decomposed as

$$\begin{aligned} T\overline{M} &= TM \oplus \mathrm{tr}(TM) = \{\mathrm{Rad}(TM) \oplus \mathrm{tr}(TM)\} \oplus_{\mathrm{orth}} S(TM) \\ &= \{\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)\} \oplus_{\mathrm{orth}} S(TM) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right). \end{aligned} \tag{2.7}$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.4). The local Gauss and Weingarten formulas for M and $S(TM)$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{2.8}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.9}$$

$$\overline{\nabla}_X L = -A_L X + \phi(X)N; \tag{2.10}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.11}$$

$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \tag{2.12}$$

for all X, $Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$, respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_{ξ}, and A_L are linear operators on TM and τ, ρ, and ϕ are 1-forms on TM. Since $\overline{\nabla}$ is torsion-free, ∇ is also torsion-free, and B and D are symmetric. From the facts $B(X, Y) = \overline{g}(\overline{\nabla}_X Y, \xi)$ and $D(X, Y) = \overline{g}(\overline{\nabla}_X Y, L)$, we know that B and D are independent of the choice of the screen distribution $S(TM)$ and

$$B(X, \xi) = 0, \quad D(X, \xi) = -\phi(X), \quad \forall X \in \Gamma(TM). \tag{2.13}$$

We say that $h(X, Y) = B(X, Y)N + D(X, Y)L$ is the *second fundamental tensor* of M. The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{2.14}$$

for all $X, Y, Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \overline{g}(X, N), \quad \forall X \in \Gamma(TM). \tag{2.15}$$

But the connection ∇^* on $S(TM)$ is metric. The above three local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$B(X, Y) = g\left(A^*_\xi X, Y\right), \qquad \overline{g}\left(A^*_\xi X, N\right) = 0, \tag{2.16}$$

$$C(X, PY) = g(A_N X, PY), \qquad \overline{g}(A_N X, N) = 0, \tag{2.17}$$

$$D(X, PY) = g(A_L X, PY), \qquad \overline{g}(A_L X, N) = \rho(X), \tag{2.18}$$

$$D(X, Y) = g(A_L X, Y) - \phi(X)\eta(Y), \quad \forall X, Y \in \Gamma(TM). \tag{2.19}$$

By (2.16) and (2.17), we show that A^*_ξ and A_N are $\Gamma(S(TM))$-valued shape operators related to B and C, respectively, and A^*_ξ is self-adjoint on TM and

$$A^*_\xi \xi = 0. \tag{2.20}$$

Replacing Y by ξ to (2.8) and using (2.12) and (2.13), we have

$$\overline{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi - \phi(X)L, \quad \forall X \in \Gamma(TM). \tag{2.21}$$

3. Tangential Half Lightlike Submanifolds

Let M be a half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. In general the structure vector field ζ of $\overline{M}$, defined by (2.1), belongs to $T\overline{M}$. Thus, from the decomposition (2.7) of $T\overline{M}$, the structure vector field ζ is decomposed as follows:

$$\zeta = \omega + a\xi + bN + eL, \tag{3.1}$$

where ω is a smooth vector field on $S(TM)$, and $a = \theta(N)$, $b = \theta(\xi)$, and $e = \theta(L)$ are smooth functions on $\overline{M}$. First of all, we introduce the following result.

Proposition 3.1 (see [3]). *Let M be a half lightlike submanifold of an indefinite almost contact metric manifold $\overline{M}$. Then there exists a screen distribution $S(TM)$ such that*

$$J\left(S(TM)^\perp\right) \subset S(TM). \tag{3.2}$$

Note 1. Although, in general, $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^* = TM/\mathrm{Rad}(TM)$ considered by Kupeli [12]. Thus all screen distributions are mutually isomorphic. For this reason, we consider only half lightlike submanifold M equipped with a screen distribution $S(TM)$ such that $J(S(TM)^\perp) \subset S(TM)$, such a screen distribution $S(TM)$ is called a *generic screen distribution* [8] of M.

Proposition 3.2. *Let M be a half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then the structure vector field ζ does not belong to* $\mathrm{Rad}(TM)$ *and* $\mathrm{ltr}(TM)$.

Proof. Assume that ζ belongs to $\mathrm{Rad}(TM)$ (or $\mathrm{ltr}(TM)$). Then (3.1) deduces to $\zeta = a\xi$ and $a \neq 0$ (or $\zeta = bN$ and $b \neq 0$). From this, we have

$$1 = \overline{g}(\zeta,\zeta) = a^2\overline{g}(\xi,\xi) = 0 \qquad \left[\text{or } 1 = \overline{g}(\zeta,\zeta) = b^2\overline{g}(N,N) = 0\right]. \tag{3.3}$$

It is a contradiction. Thus ζ does not belong to $\mathrm{Rad}(TM)$ and $\mathrm{ltr}(TM)$. □

Note 2. If the structure vector field ζ is tangent to M, that is, $b = e = 0$, then ζ does not belong to $\mathrm{Rad}(TM)$ by Proposition 3.2. This enables one to choose a screen distribution $S(TM)$ which contains ζ. This implies that *if ζ is tangent to M, then it belongs to $S(TM)$*. Călin [13] also proved this result which we assume in this section.

Definition 3.3. A half lightlike submanifold M of an indefinite cosymplectic manifold $\overline{M}$ is said to be a *tangential half lightlike submanifold* [4] of $\overline{M}$ if ζ is tangent to M.

For any tangential half lightlike submanifold M, we show that ζ belongs to $S(TM)$, that is, $a = b = e = 0$ by Note 2. Then there exists a nondegenerate almost complex distribution H_o on M with respect to J, that is, $J(H_o) = H_o$, such that

$$S(TM) = \{J(\mathrm{Rad}(TM)) \oplus J(\mathrm{ltr}(TM))\} \oplus_{\mathrm{orth}} J\left(S\left(TM^{\perp}\right)\right) \oplus_{\mathrm{orth}} H_o. \tag{3.4}$$

Thus the general decompositions (2.4) and (2.7) reduce, respectively, to

$$TM = H \oplus H', \qquad T\overline{M} = H \oplus H' \oplus \mathrm{tr}(TM), \tag{3.5}$$

where H and H' are 2- and 1-lightlike distributions on M such that

$$\begin{gathered} H = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} J(\mathrm{Rad}(TM)) \oplus_{\mathrm{orth}} H_o, \\ H' = J(\mathrm{ltr}(TM)) \oplus_{\mathrm{orth}} J\left(S\left(TM^{\perp}\right)\right). \end{gathered} \tag{3.6}$$

H is an almost complex distribution of M with respect to J. Consider a pair of local null vector fields $\{U, V\}$ and a local nonnull vector field W on $S(TM)$ defined by

$$U = -JN, \qquad V = -J\xi, \qquad W = -JL. \tag{3.7}$$

Denote by S the projection morphism of TM on H with respect to the decomposition (3.5). Then any vector field X on M and its action JX by J are expressed as follows:

$$X = SX + u(X)U + w(X)W, \qquad JX = FX + u(X)N + w(X)L, \tag{3.8}$$

where u, v, and w are 1-forms locally defined on M by

$$u(X) = g(X,V), \qquad v(X) = g(X,U), \qquad w(X) = g(X,W) \tag{3.9}$$

and F is a tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Applying the operator $\overline{\nabla}_X$ to (3.7) and the second equation of (3.8) (denote $(3.8)_2$) and using (2.2), (2.8), (2.9), (2.10), (2.21), (3.7), (3.8) and (3.9), for all $X, Y \in \Gamma(TM)$, we have

$$B(X,U) = C(X,V), \qquad B(X,W) = D(X,V), \qquad C(X,W) = D(X,U), \tag{3.10}$$

$$\nabla_X U = F(A_N X) + \tau(X)U + \rho(X)W, \tag{3.11}$$

$$\nabla_X V = F\left(A^*_\xi X\right) - \tau(X)V - \phi(X)W, \tag{3.12}$$

$$\nabla_X W = F(A_L X) + \phi(X)U, \tag{3.13}$$

$$(\nabla_X F)(Y) = u(Y)A_N X + w(Y)A_L X - B(X,Y)U - D(X,Y)W. \tag{3.14}$$

Note 3. From now on, $\overline{M} = (R^{2m+1}_q, J, \zeta, \theta, \overline{g})$ will denote the semi-Euclidean manifold R^{2m+1}_q equipped with its usual cosymplectic structure given by

$$\begin{gathered} \theta = dz, \qquad \zeta = \partial z, \\ \overline{g} = \theta \otimes \theta - \sum_{i=1}^{q/2} (dx_i \otimes dx_i + dy_i \otimes dy_i) + \sum_{i=q+1}^{m} (dx_i \otimes dx_i + dy_i \otimes dy_i), \\ J\left(\sum_{i=1}^{m} (X_i \partial x_i + Y_i \partial y_i) + Z \partial z\right) = \sum_{i=1}^{m} (Y_i \partial x_i - X_i \partial y_i), \end{gathered} \tag{3.15}$$

where (x_i, y_i, z) are the Cartesian coordinates and $\overline{g}$ is a semi-Euclidean metric of signature $(-,+,\dots,+;\ -,+,\dots,+;\ +)$ with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \dots, \partial x_m;\ \partial y_1, \partial y_2, \dots, \partial y_m;\ \partial z\}. \tag{3.16}$$

This construction will help in understanding how the indefinite cosymplectic structure is recovered in examples of this paper.

Example 3.4. Consider a submanifold M of $\overline{M} = (R^9_2, J, \zeta, \theta, \overline{g})$ given by the equations

$$x_1 = y_4, \qquad x_2 = \sqrt{1 - y_2^2}, \quad y_2 \neq \pm 1. \tag{3.17}$$

Then a local frame fields of TM are given by

$$\begin{gathered}
\xi = \partial x_1 + \partial y_4, \qquad U_1 = \partial x_4 - \partial y_1, \\
U_2 = \partial x_3, \qquad U_3 = \partial y_3, \qquad U_4 = -\frac{y_2}{x_2}\partial x_2 + \partial y_2, \\
U_5 = \partial x_4 + \partial y_1, \qquad U_6 = \zeta = \partial z.
\end{gathered} \tag{3.18}$$

This implies $\mathrm{Rad}(TM) = \mathrm{Span}\{\xi\}$, $J\xi = U_1$, and $\mathrm{Rad}(TM) \cap J(\mathrm{Rad}(TM)) = \{0\}$. Next, $JU_2 = -U_3$ implies that $H_o = \{U_2, U_3\}$ invariant with respect to the almost contact structure tensor J. By direct calculations, we have

$$S\left(TM^{\perp}\right) = \mathrm{Span}\left\{L = \partial x_2 + \frac{y_2}{x_2}\partial y_2\right\}, \qquad \mathrm{ltr}(TM) = \mathrm{Span}\left\{N = \frac{1}{2}(-\partial x_1 + \partial y_4)\right\}. \tag{3.19}$$

We show that $JL = -U_4$, $JN = (1/2)U_5$, $J\zeta = 0$ and $\overline{\nabla}_X\zeta = 0$ for all $X \in \Gamma(TM)$. Therefore M is a tangential half-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$.

Theorem 3.5. *Let M be a tangential half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then the structure vector field ζ is parallel with respect to the connections ∇ and ∇^*. Furthermore, ζ is conjugate to any vector field of M with respect to h and C.*

Proof. Replacing Y by ζ to (2.8) and using (2.3) and the fact $\zeta \in \Gamma(TM)$, we get

$$\nabla_X\zeta + B(X,\zeta)N + D(X,\zeta)L = 0, \quad \forall X \in \Gamma(TM). \tag{3.20}$$

Taking the scalar product with ξ and L to this equation by turns, we have

$$\nabla_X\zeta = 0, \quad B(X,\zeta) = 0, \quad D(X,\zeta) = 0, \quad \forall X \in \Gamma(TM). \tag{3.21}$$

From $(3.21)_1$, we see that ζ is parallel with respect to the induced connection ∇. $(3.21)_{2,3}$ implies that ζ is conjugate to any vector field on M with respect to the second fundamental form h. Replacing PY by ζ to (2.11) and using $(3.21)_1$ and the fact $\zeta \in \Gamma(S(TM))$, we get

$$\nabla^*_X\zeta + C(X,\zeta)\xi = 0, \quad \forall X \in \Gamma(TM). \tag{3.22}$$

Taking the scalar product with N to this equation, we have

$$\nabla^*_X\zeta = 0, \quad C(X,\zeta) = 0, \quad \forall X \in \Gamma(TM). \tag{3.23}$$

Thus ζ is also parallel with respect to the lieasr connection ∇^* and conjugate to any vector field on M with respect to C. Thus we have our assertions. $\square$

Definition 3.6. A half lightlike submanifold M of $\overline{M}$ is *totally umbilical* [5] if there is a smooth vector field $\mathscr{H}$ on $\mathrm{tr}(TM)$ on any coordinate neighborhood $\mathcal{U}$ such that

$$h(X,Y) = \mathscr{H}g(X,Y), \quad \forall X,Y \in \Gamma(TM). \tag{3.24}$$

In case $\mathscr{H} = 0$, that is, $h = 0$ on $\mathcal{U}$, we say that M is *totally geodesic*.

It is easy to see that M is totally umbilical if and only if there exist smooth functions β and δ on each coordinate neighborhood $\mathcal{U}$ such that

$$B(X,Y) = \beta g(X,Y), \quad D(X,Y) = \delta g(X,Y), \quad \forall X,Y \in \Gamma(TM). \tag{3.25}$$

Theorem 3.7. *Any totally umbilical tangential half lightlike submanifold M of an indefinite cosymplectic manifold $\overline{M}$ is totally geodesic.*

Proof. Assume that M is totally umbilical. From (3.21) and (3.25), we have

$$\beta g(X,\zeta) = 0, \quad \delta g(X,\zeta) = 0, \quad \forall X \in \Gamma(TM). \tag{3.26}$$

Replacing X by ζ in this equations and using the fact $g(\zeta,\zeta) = 1$, we have $\beta = \delta = 0$, that is, $B = D = 0$. Thus we have $h = 0$ and M is totally geodesic. □

Definition 3.8. Ascreen distribution $S(TM)$ is called *totally umbilical* [5] (in M) if there is a smooth function γ on any coordinate neighborhood $\mathcal{U}$ in M such that

$$C(X,PY) = \gamma g(X,Y), \quad \forall X,Y \in \Gamma(TM). \tag{3.27}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that $S(TM)$ is *totally geodesic* (in M).

Theorem 3.9. *Let M be a tangential half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$ such that $S(TM)$ is totally umbilical. Then $S(TM)$ is totally geodesic.*

Proof. Assume that $S(TM)$ is totally umbilical in M. Replacing Y by ζ to (3.27) and using (3.23), we have $\gamma g(X,\zeta) = 0$ for all $X \in \Gamma(TM)$. Replacing X by ζ to this equation and using the fact $g(\zeta,\zeta) = 1$, we obtain $\gamma = 0$. Thus $S(TM)$ is totally geodesic in M. □

Theorem 3.10. *Let M be a tangential half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then H is an integrable distribution on M if and only if*

$$h(X,FY) = h(FX,Y), \quad \forall X,Y \in \Gamma(H). \tag{3.28}$$

Moreover, if M is totally umbilical, then H is a parallel distribution on M.

Proof. Taking $Y \in \Gamma(H)$, we show that $FY = JY \in \Gamma(H)$. Applying $\overline{\nabla}_X$ to $FY = JY$ and using (2.3), (2.8), (3.7), $(3.8)_2$, and (3.9), we have

$$B(X, FY) = g(\nabla_X Y, V), \qquad D(X, FY) = g(\nabla_X Y, W), \tag{3.29}$$

$$(\nabla_X F)(Y) = -B(X, Y)U - D(X, Y)W, \quad \forall X \in \Gamma(TM). \tag{3.30}$$

By direct calculations from two equations of (3.29), we have

$$h(X, FY) - h(FX, Y) = g([X, Y], V)N + g([X, Y], W)L. \tag{3.31}$$

If H is integrable, then $[X, Y] \in \Gamma(H)$ for any X, $Y \in \Gamma(H)$. This implies $g([X, Y], V) = g([X, Y], W) = 0$. Thus we get $h(X, FY) = h(FX, Y)$ for all X, $Y \in \Gamma(H)$. Conversely if $h(X, FY) = h(FX, Y)$ for all X, $Y \in \Gamma(H)$, then we have $g([X, Y], V) = g([X, Y], W) = 0$. This imply $[X, Y] \in \Gamma(H)$ for all $X, Y \in \Gamma(H)$. Thus H is an integrable distribution of M.

If M is totally umbilical, from Theorem 3.7 and (3.29), we have

$$g(\nabla_X Y, V) = g(\nabla_X Y, W) = 0, \quad \forall X \in \Gamma(TM), \ \forall Y \in \Gamma(H). \tag{3.32}$$

This imply $\nabla_X Y \in \Gamma(H)$ for all X, $Y \in \Gamma(H)$, that is, H is a parallel distribution on M. □

Theorem 3.11. *Let M be a tangential half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then F is parallel on H with respect to the connection ∇ if and only if H is a parallel distribution on M.*

Proof. Assume that F is parallel on H with respect to ∇. For any X, $Y \in \Gamma(H)$, we have $(\nabla_X F)Y = 0$. Taking the scalar product with V and W to (3.30) with $(\nabla_X F)Y = 0$, we have $B(X, Y) = 0$ and $D(X, Y) = 0$ for all X, $Y \in \Gamma(H)$, respectively. From (3.29), we have $g(\nabla_X Y, V) = 0$ and $g(\nabla_X Y, W) = 0$. This imply $\nabla_X Y \in \Gamma(H)$ for all $X, Y \in \Gamma(H)$. Thus H is a parallel distribution on M.

Conversely if H is a parallel distribution on M, from (3.29) we have

$$B(X, FY) = 0, \qquad D(X, FY) = 0, \quad \forall X, Y \in \Gamma(H). \tag{3.33}$$

For any $Y \in \Gamma(H)$, we show that $F^2 Y = J^2 Y = -Y + \theta(Y)\zeta$. Replacing Y by FY to (3.33) and using (3.21), we have $B(X, Y) = 0$ and $D(X, Y) = 0$ for any $X, Y \in \Gamma(H)$. Thus F is parallel on H with respect to ∇ by (3.30). □

Theorem 3.12. *Let M be a tangential half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. If F is parallel with respect to the induced connection ∇, then H is a parallel distribution on M and M is locally a product manifold $L_U \times L_W \times M^T$, where L_U and L_W are null curves tangent to $J(\mathrm{ltr}(TM))$ and $J(S(TM^{\perp}))$, respectively, and M^T is a leaf of H.*

Proof. Assume that F is parallel on TM with respect to ∇. Then F is parallel on H with respect to ∇. By Theorem 3.11, H is a parallel distribution on M. Applying the operator F to (3.14) with $(\nabla_X F)Y = 0$, we have

$$u(Y)F(A_N X) + w(Y)F(A_L X) = 0, \quad \forall X, Y \in \Gamma(TM), \tag{3.34}$$

due to $FU = FW = 0$. Replacing Y by U and W to this equation by turns and using (3.9), we have $F(A_N X) = 0$ and $F(A_L X) = 0$. Taking the scalar product with W and N to (3.14) with $(\overline{\nabla}_X F)Y = 0$ by turns, we have

$$D(X, Y) = u(Y)w(A_N X) + w(Y)w(A_L X), \tag{3.35}$$

$$w(Y)g(A_L X, N) = 0, \quad \forall X, Y \in \Gamma(TM). \tag{3.36}$$

Replacing Y by ξ to (3.35), we get $\phi = 0$ due to $(2.13)_2$. Also replacing Y by W to (3.36), we have $\rho = 0$ due to $(2.18)_2$. From thess results, (3.11) and (3.13), we get $\nabla_X U = \tau(X)U$ and $\nabla_X W = 0$ for all $X \in \Gamma(H')$. Thus $J(\mathrm{ltr}(TM))$ and $J(S(TM^{\perp}))$ are also parallel distributions on M. By the decomposition theorem of de Rham [14], we show that $M = L_U \times L_W \times M^T$, where L_U and L_W are null curves tangent to $J(\mathrm{ltr}(TM))$ and $J(S(TM^{\perp}))$, respectively, and M^T is a leaf of H. □

Definition 3.13. A half lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be *irrotational* [12] if $\overline{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$.

Note 4. From (2.21) we see that a necessary and sufficient condition for M to be irrotational is $D(X, \xi) = 0 = \phi(X)$ for all $X \in \Gamma(TM)$.

Theorem 3.14. *Let M be a tangential half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then one has the following assertions.*

(i) *If V is parallel with respect to ∇, then M is irrotational, $\tau = 0$ and*

$$A^*_\xi X = u\left(A^*_\xi X\right)U + w\left(A^*_\xi X\right)W, \quad \forall X \in \Gamma(TM). \tag{3.37}$$

(ii) *If U is parallel with respect to ∇, then one has $\tau = \rho = 0$ and*

$$A_N X = u(A_N X)U + w(A_N X)W, \quad \forall X \in \Gamma(TM). \tag{3.38}$$

(iii) *If W is parallel with respect to ∇, then M is irrotational and*

$$A_L X = u(A_L X)U + w(A_L X)W, \quad \forall X \in \Gamma(TM). \tag{3.39}$$

Moreover, if all of V, U, and W are parallel on TM with respect to ∇, then $S(TM)$ is totally geodesic in M and $\tau = \phi = \rho = 0$ on $\Gamma(TM)$. In this case, the null transversal vector field N of M is a constant on M.

Proof. If V is parallel with respect to ∇, then, taking the scalar product with U and W to (3.12) by turns, we have $\tau = 0$ and $\phi = 0$ (M is irrotational), respectively. Thus we have $F(A^*_\xi X) = 0$ for all $X \in \Gamma(TM)$. From this result and (3.8), we obtain $J(A^*_\xi X) = u(A^*_\xi X)N + w(A^*_\xi X)L$. Applying J to this equation and using $\theta(A^*_\xi X) = 0$, we obtain (i). In a similar way, by using (3.11), (3.13), (3.21), and (3.23), we have (ii) and (iii).

Assume that all of V, U, and W are parallel on TM with respect to ∇. Substituting the equation of (i) into (3.10)-1, we have

$$u(A_N X) = v\left(A^*_\xi X\right) = g\left(A^*_\xi X, U\right) = 0, \quad \forall X \in \Gamma(TM). \tag{3.40}$$

Also, substituting the equation of (iii) into (3.10)-3, we have

$$w(A_N X) = v(A_L X) = g(A_L X, U) = 0, \quad \forall X \in \Gamma(TM). \tag{3.41}$$

From the last two equations and the equation of (ii), we see that $A_N = 0$. Thus $S(TM)$ is totally geodesic in M and the 1-forms τ, ϕ, and ρ, defined by (2.9) and (2.10), satisfy $\tau = \phi = \rho = 0$ on $\Gamma(TM)$. Using this results, we see that N is a constant on M. □

Theorem 3.15. *Let M be a totally umbilical tangential half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$ such that $S(TM)$ is totally umbilical. Then M is locally a product manifold $M = M_4 \times M^{T_o}$, where M_4 and M^{T_o} are leaves of $H_o^\perp$ and H_o, respectively.*

Proof. By Theorem 3.10, H is a parallel distribution M. Thus, for all $X, Y \in \Gamma(H_o)$, we have $\nabla_X Y \in \Gamma(H)$. From (2.11) and (3.30), we have

$$\begin{aligned} C(X, FY) &= g(\nabla_X FY, N) = g((\nabla_X F)Y + F(\nabla_X Y), N) \\ &= g(F(\nabla_X Y), N) = -g(\nabla_X Y, JN) = g(\nabla_X Y, U), \end{aligned} \tag{3.42}$$

due to $FY \in \Gamma(H_o)$. If $S(TM)$ is totally umbilical in M, then we have $C = 0$ due to Theorem 3.7. By (2.11) and (3.42), we get

$$g(\nabla_X Y, N) = 0, \quad g(\nabla_X Y, U) = 0, \quad \forall X \in \Gamma(TM),\ \forall Y \in \Gamma(H_o). \tag{3.43}$$

These results and (3.29) imply $\nabla_X Y \in \Gamma(H_o)$ for all $X, Y \in \Gamma(H_o)$. Thus H_o is a parallel distribution on M and $TM = H_o \oplus_{\text{orth}} H_o^\perp$, where $H_o^\perp = \operatorname{Span}\{\xi, V, U, W\}$. By Theorems 3.5 and 3.7, we have $B = D = A_N = \phi = 0$ and $A_L X = \rho(X)\xi$. Thus (2.12) and (3.11)~(3.13) deduce, respectively, to

$$\begin{aligned} &\nabla_X \xi = -\tau(X)\xi, \qquad \nabla_X U = \tau(X)U + \rho(X)W, \\ &\nabla_X V = -\tau(X)V, \qquad \nabla_X W = -\rho(X)V, \quad \forall X \in \Gamma\left(H_o^\perp\right). \end{aligned} \tag{3.44}$$

Thus $H_o^\perp$ is also a parallel distribution on M. Thus we have $M = M_4 \times M^{T_o}$, where M_4 is a leaf of $H_o^\perp$ and M^{T_o} is a leaf of H_o. □

4. Ascreen Half Lightlike Submanifolds

Definition 4.1. A half lightlike submanifold M of an indefinite cosymplectic manifold $\overline{M}$ is said to be an *ascreen half lightlike submanifold* [4] of $\overline{M}$ if the structure vector field ζ of $\overline{M}$ belongs to the distribution $\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)$.

For any ascreen half lightlike submanifold M, the vector field ζ is decomposed as

$$\zeta = a\xi + bN. \tag{4.1}$$

In this case, we show that $a \neq 0$ and $b \neq 0$ by Proposition 3.2.

Definition 4.2. A half lightlike submanifold M is called *screen conformal* [2, 3] if there exists a nonvanishing smooth function φ such that $A_N = \varphi A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi B(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.2}$$

Theorem 4.3. *Let M be an ascreen half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then M is screen conformal.*

Proof. Applying $\overline{\nabla}_X$ to (4.1) and using (2.3), (2.9), and (2.21), we have

$$aA^*_\xi X + bA_N X = \{Xa - a\tau(X)\}\xi + \{Xb + b\tau(X)\}N + \{b\rho(X) - a\phi(X)\}L. \tag{4.3}$$

Taking the product with ξ, N, and L by turns and using $(2.16)_2$ and $(2.17)_2$, we get

$$A_N X = \varphi A^*_\xi X, \qquad Xa = a\tau(X), \qquad Xb = -b\tau(X), \qquad a\phi(X) = b\rho(X), \tag{4.4}$$

for all $X \in \Gamma(TM)$, where we set $\varphi = -a/b$. Thus M is screen conformal. □

Substituting (4.1) into $g(\zeta, \zeta) = 1$, we have $2ab = 1$. Consider the local unit timelike vector field V^* on M and its 1-form v^* defined by

$$V^* = -\, b^{-1} J\xi, \qquad v^*(X) = -\, g(X, V^*), \quad \forall X \in \Gamma(TM). \tag{4.5}$$

Let $U^* = -\, a^{-1} JN$. Then U^* is a unit timelike vector field on $S(TM)$ such that $g(V^*, U^*) = 1$. Applying J to (4.1) and using (2.1) and $2ab = 1$, we have

$$0 = aJ\xi + bJN = -\frac{V^* + U^*}{2} \quad \text{i.e.,}\ U^* = -\, V^*. \tag{4.6}$$

From this we show that $J(\mathrm{Rad}(TM)) = J(\mathrm{ltr}(TM))$. Using this and Proposition 3.1, the tangent bundle TM of M is decomposed as follows:

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} \left\{ J(\mathrm{Rad}(TM)) \oplus_{\mathrm{orth}} J\left(S\left(TM^\perp\right)\right) \oplus_{\mathrm{orth}} H^* \right\}, \tag{4.7}$$

where H^* is a nondegenerate and almost complex distribution on M with respect to the indefinite cosymplectic structure tensor J, otherwise $S(TM)$ is degenerate. Consider the local unit spacelike vector field W^* on $S(TM)$ and its 1-form w^* defined by

$$W^* = -JL, \qquad w^*(X) = g(X, W^*), \quad \forall X \in \Gamma(TM). \tag{4.8}$$

Denote by S^* the projection morphism of TM on H^*. Using (4.7), for any vector field X on M, the vector field JX is expressed as follows:

$$JX = fX + av^*(X)\xi - b\eta(X)V^* - bv^*(X)N + w^*(X)L, \tag{4.9}$$

because $JV^* = a\xi - bN$, where f is a tensor field of type (1, 1) defined by

$$fX = JS^*X, \quad \forall X \in \Gamma(TM). \tag{4.10}$$

Applying J to (2.10) and (2.21) and using (2.2), (2.8) and (4.4)~(4.9), we get

$$b\nabla_X V^* = f\left(A^*_\xi X\right) - aB(X, V^*)\xi - \phi(X)W^*, \tag{4.11}$$

$$\nabla_X W^* = f(A_{\,L} X) - aD(X, V^*)\xi - 2a\phi(X)V^*, \tag{4.12}$$

$$bD(X, V^*) = B(X, W^*), \quad \forall X \in \Gamma(TM). \tag{4.13}$$

Example 4.4. Consider a submanifold M of $\overline{M} = (R^5_2, J, \zeta, \theta, \overline{g})$ given by the equation

$$X(u_1, u_2, u_3) = \left(u_1, u_2, u_3, u_2, \frac{1}{\sqrt{2}}(u_1 + u_3)\right). \tag{4.14}$$

By direct calculations we easily check that

$$\begin{gathered} TM = \mathrm{Span}\left\{\xi = \partial x_1 + \partial y_1 + \sqrt{2}\partial z,\ U = \partial x_1 - \partial y_1,\ V = \partial x_2 + \partial y_2\right\}, \\ TM^\perp = \mathrm{Span}\{\xi, L = \partial x_2 - \partial y_2\}, \qquad \mathrm{Rad}(TM) = \mathrm{Span}\{\xi\}. \end{gathered} \tag{4.15}$$

We obtain the lightlike transversal and transversal vector bundles

$$\mathrm{ltr}(TM) = \mathrm{Span}\left\{N = \frac{1}{4}\left(-\partial x_1 - \partial y_1 + \sqrt{2}\partial z\right)\right\}, \qquad \mathrm{tr}(TM) = \mathrm{Span}\{N, L\}. \tag{4.16}$$

From this results, we show that $J\xi = U$, $\mathrm{Rad}(TM) \cap J(\mathrm{Rad}(TM)) = \{0\}$, $JN = -(1/4)U$, $JL = -V$, $JN = -(1/4)J\xi$ and $J(\mathrm{Rad}(TM) = J(\mathrm{ltr}(TM)$, $\zeta = (1/2\sqrt{2})\xi + \sqrt{2}N$ and $J\zeta = 0$. Thus M is an ascreen half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$.

Theorem 4.5. *Let M be an ascreen half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. If M is totally umbilical, then M and $S(TM)$ are totally geodesic.*

Proof. Assume that M is totally umbilical. From (3.25) and (4.13), we have

$$b\delta g(X,V^*) = \beta g(X,W^*), \quad \forall X \in \Gamma(TM). \tag{4.17}$$

Replacing X by W^* and V^* to this equation by turns, we have $\beta = 0$ and $\delta = 0$, respectively. Thus we have $h = 0$ and M is totally geodesic. By (4.2), we also have $C = 0$. Thus $S(TM)$ is also totally geodesic in M. □

Taking $Y \in \Gamma(H^*)$. Then we have $fY = JY \in \Gamma(H^*)$ due to (4.9). Applying $\overline{\nabla}_X$ to $JY = fY$ and using (2.2), (2.8), (4.2), (4.5), and (4.9), we have

$$B(X,fY) = bg(\nabla_X Y,V^*), \qquad D(X,fY) = g(\nabla_X Y,W^*), \tag{4.18}$$

$$(\nabla_X f)Y = -ag(\nabla_X Y,V^*)\xi + 2aB(X,Y)V^* - D(X,Y)W^*, \tag{4.19}$$

for all $X \in \Gamma(TM)$. By the procedure same as the proofs of Theorem 3.10 and Theorem 3.11 and by using (4.18) and (4.19), instead of (3.29) and (3.30), and that $S(TM)$ is integrable due to (4.2), the following two theorems hold.

Theorem 4.6. *Let M be an ascreen half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. H^* is an integrable distribution on M if and only if one has*

$$h(X,fY) = h(fX,Y), \quad \forall X,Y \in \Gamma(H^*). \tag{4.20}$$

Moreover, if M is totally umbilical, then H^* is a parallel distribution on M.

Theorem 4.7. *Let M be an ascreen half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then f is parallel on H^* with respect to the induced connection ∇ if and only if H^* is a parallel distribution on M.*

Theorem 4.8. *Let M be an ascreen half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. If M is totally umbilical, then M is locally a product manifold $L_\xi \times L_{V^*} \times L_{W^*} \times M^*$, where L_ξ, L_{V*}, and L_{W^*} are null, timelike, and spacelike curves tangent to $\operatorname{Rad}(TM)$, $J(\operatorname{Rad}(TM))$, and $J(S(TM^\perp))$, respectively, and M^* is a leaf of H^*.*

Proof. If M is totally umbilical, then H^* is a parallel distribution on M by Theorem 4.6 and we have $B = D = A^*_\xi = \phi = 0$; $A_L X = \rho(X)\xi$ by Theorem 4.5. From $(4.4)_1$, we also have $A_N = 0$. Using (2.12), (4.11), and (4.12), we have $\nabla_X\xi = -\tau(X)\xi$ and $\nabla_X V^* = \nabla_X W^* = 0$ due to $f\xi = 0$. This implies that all of the distributions $\operatorname{Rad}(TM)$, $J(\operatorname{Rad}(TM))$, and $J(S(TM^\perp))$ are parallel on M. Thus we have $M = L_\xi \times L_{V^*} \times L_{W^*} \times M^*$, where L_ξ, L_{V*}, and L_{W^*} are null, timelike, and spacelike curves tangent to $\operatorname{Rad}(TM)$, $J(\operatorname{Rad}(TM))$ and $J(S(TM^\perp))$, respectively, and M^* is a leaf of H^*. □

By straightforward calculations from (4.11) and (4.12) and the same method as the proof of Theorem 3.14, the following theorem holds.

Theorem 4.9. *Let M be an ascreen half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then one has the following assertions.*

(i) *If V^* is parallel with respect to ∇ on M, then M is irrotational and*

$$A_{\xi}^{*}X = B(X, W^*)W^*, \quad B(X, V^*) = 0, \quad \rho(X) = 0, \quad \forall X \in \Gamma(TM). \tag{4.21}$$

(ii) *If W^* is parallel with respect to ∇ on M, then M is irrotational and*

$$A_L X = D(X, W^*)W^*, \quad D(X, V^*) = 0, \quad \rho(X) = 0, \quad \forall X \in \Gamma(TM). \tag{4.22}$$

Moreover, if V^* and W^* are parallel with respect to ∇, then one sees that $A_{\xi}^{*} = 0$ and the screen distribution $S(TM)$ is totally geodesic in M.

Theorem 4.10. *Let M be an ascreen half lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. If V^* and W^* are parallel with respect to ∇, then M is locally a product manifold $L_{\xi} \times L_{V^*} \times L_{W^*} \times M^*$, where L_{ξ}, L_{V^*}, and L_{W^*} are null, timelike, and spacelike curves tangent to $\operatorname{Rad}(TM)$, $J(\operatorname{Rad}(TM))$, and $J(S(TM^{\perp}))$, respectively, and M^* is a leaf of H^*.*

Proof. If V^* is parallel with respect to ∇, for any $Y \in \Gamma(H^*)$, we have

$$B(X, Y) = g\left(A_{\xi}^{*}X, Y\right) = B(X, W^*)g(Y, W^*) = 0, \quad \forall X \in \Gamma(TM). \tag{4.23}$$

Thus we get $g(\nabla_X Y, V^*) = b^{-1}B(X, fY) = 0$ because $fY \in \Gamma(H^*)$. Also if W^* is parallel with respect to ∇, then, for any $Y \in \Gamma(H^*)$, we have

$$D(X, Y) = g(A_L X, Y) = D(X, W^*)g(Y, W^*) = 0, \quad \forall X \in \Gamma(TM). \tag{4.24}$$

From these results and (4.19), we show that f is parallel on H^* with respect to ∇. Thus, by Theorem 4.7, we see that H^* is a parallel distribution on M. As V^* and W^* are parallel with respect to ∇ and $\nabla_X \xi = -\tau(X)\xi$ due to $A_{\xi}^{*} = 0$, we have our theorem. □

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