Special linear systems on curves lying on a K3 surface

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Introduction

In this article C will aways denote a nonsingular curve of genus g lying on a K3 surface X. By a g^1_r I understand a linear system of degree r and dimension 1 which is without fixed points and complete. The g^1_r is said to be *separable* if the associated map to $\mathbb{P}^1$ is, and this is obviously equivalent to the g^1_r containing a divisor $P_1 + \cdots + P_r$ made up of distinct points P_i.

My aim is to prove the following result.

Theorem 1 *Suppose that $|d|$ is a separable g^1_r on C, and that*

$$g > \frac{1}{4}r^2 + r + 2;$$

then $|d|$ is cut out on C by an elliptic pencil $|E|$ on X.

Since a K3 surface has only a discrete (at most countable) collection of elliptic pencils, Theorem 1 has the following consequence.

Corollary 2 *Let C be a curve of genus ≥ 11, having a 2-to-1 map $C \to E$ to an elliptic curve E; then C does not lie as a nonsingular curve on any K3 surface.*

The existence of curves not lying on any K3 surface follows from an easy dimension count, and was known to Severi; this is possibly the first explicit example.

The proof of Theorem 1 uses the techniques of Saint-Donat's thesis [1]; it should be noted that the cases $r = 2$ and $r = 3$ of the theorem are contained implicitly in [1].

A counter-example shows that the function $f(r) = \frac{1}{4}r^2 + r + 2$ occurring in Theorem 1 cannot be improved if one wants the linear system $|d|$ on C

to be cut out by an elliptic pencil on X; however, I have partial results saying that if g is fairly large ($\geq 2r$ at least), then one should expect that our linear system g^1_r is contained in a linear system g^{1+s}_{r+t} with $t \leq 2s$, and s small. Thus for example, if

$$g > \frac{1}{8}r^2 + r + 3;$$

then our g^1_r $|d|$ is *either* cut out by an elliptic pencil $|E|$ of X, or belongs to a g^2_{r+t} with $t = 1$ or 2, and this g^2_{r+t} is cut out by an irreducible linear system $|B|$ of X (with $B^2 = 2$).

Note finally that the method also gives the following result for a nonsingular curve lying on any regular surface.

Theorem $1'$ *Let C be a nonsingular curve lying on a surface X, with $H^1(X, \mathcal{O}_X) = 0$; suppose that*

(i) $h^0(X, \mathcal{O}_X(C)) \geq 3$;

(ii) the genus of C satisfies

$$g > \frac{1}{2}r^2 + r + 2 - \frac{1}{2}(CK + K^2).$$

Then $|d|$ is cut out on C by a pencil $|E|$ of curves on X.

Unfortunately, for (i) we need to know that C^2 is greater than CK, whereas for the case of a K3 surface this was obvious.

The proof of Theorem 1

The curve C lying on the K3 surface X belongs to a linear system $|C|$ without fixed points, which defines a morphism

$$\phi_C\colon X \to \mathbb{P}^g;$$

the restriction of ϕ_C to C is just the canonical map of C, and if C is nonhyperelliptic, then ϕ_C is birational onto a surface $\overline{X}$, and C can be considered as the nonsingular hyperplane section $\mathbb{P}^{g-1} \cap \overline{X}$ of $\overline{X}$.

To say that r points $P_1, \dots, P_r$ of C form a g^1_r without fixed points is precisely to assert that the images of $P_1, \dots, P_r$ under the canonical map of C are linearly dependent, whereas any $r - 1$ of them are not. Since the canonical map of C is just ϕ_C, this is equivalent to

$$\left.\begin{aligned} \dim \operatorname{coker}\Big[H^0(X,\mathcal{O}_X(C)) \to \bigoplus_j k_{P_j}\Big] &= 1, \\ \text{and} \qquad H^0(X,\mathcal{O}_X(C)) \to \bigoplus_{j\neq i} k_{P_j} \quad &\text{is onto.} \end{aligned}\right\} \qquad \text{(i)}$$

Then

$$\left.\begin{aligned} h^1(X,\mathcal{O}_X(C)\cdot I_{P_1}\cdots I_{P_r}) &= 1, \\ \text{and} \qquad H^1(X,\mathcal{O}_X(C)\cdot I_{P_1}\cdots \widehat{I_{P_i}}\cdots I_{P_r}) &= 0, \end{aligned}\right\} \qquad \text{(ii)}$$

I_P denoting the ideal defining P in X.

Now let $f\colon \widetilde{X} \to X$ be the blowing up of $P_1,\dots,P_r$ in X, and let l_i be the exceptional curve of f above P_i. Since X is a K3 surface, $K_{\widetilde{X}} = \sum l_i$, and (ii) is equivalent to

$$\left.\begin{aligned} h^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}(f^*C - K_{\widetilde{X}})) &= 1, \\ \text{and} \qquad H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}(f^*C - K_{\widetilde{X}} + l_i)) &= 0; \end{aligned}\right\} \qquad \text{(iii)}$$

then by Serre duality (iii) is equivalent to

$$\left.\begin{aligned} h^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-f^*C + 2K_{\widetilde{X}})) &= 1, \\ \text{and} \qquad H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}(-f^*C + 2K_{\widetilde{X}} - l_i)) &= 0. \end{aligned}\right\} \qquad \text{(iv)}$$

Suppose now that $|f^*C - 2\sum l_j|$ contains a positive divisor D. Then by the cohomology sequence associated to

$$0 \to \mathcal{O}_{\widetilde{X}}(-D) \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_D \to 0,$$

(iv) is equivalent to

$$\left.\begin{aligned} h^0(\mathcal{O}_D) &= 2, \\ \text{and} \qquad h^0(\mathcal{O}_{D+l_i}) &= 1, \end{aligned}\right\} \qquad \text{(v)}$$

I now want to make a technical digression to improve slightly C.P. Ramanujan's result on numerically connected divisors. First some definitions:

Definition 1 Let D_1 and D_2 be positive divisors on a surface F; D_2 is said to be *effectively disconnected* from D_1 if the line bundle $\mathcal{O}_{D_1}(-D_2)$ is generated outside a subset of codimension 1 by its global sections.

(Note that the definition is unsymmetric.)

Definition 2 D_1 and D_2 are said to be *effectively disjoint* if both

$$\mathcal{O}_{D_1}(-D_2) \cong \mathcal{O}_{D_1} \quad \text{and} \quad \mathcal{O}_{D_2}(-D_1) \cong \mathcal{O}_{D_2}.$$

Note that if D_2 is effectively disconnected from D_1, then we have the numerical assertion

$$D_2 \cdot \theta \leq 0 \quad \text{for every component } \theta \text{ of } D_1;$$

similarly if D_1 and D_2 are effectively disjoint, we have

$$D_1 \cdot \theta_2 = D_2 \cdot \theta_1 = 0 \quad \text{for every component } \theta_i \text{ of } D_i.$$

Example Let $|E|$ be a pencil of curves on a surface X, and suppose that $|E|$ is without fixed points. Let E_0 be a reducible fibre, and A a component of E_0; then E_0 is effectively disconnected from A, since $\mathcal{O}_A(-E_0) = \mathcal{O}_A(-E) = \mathcal{O}_A$. However, it is not true that A is effectively disconnected from E_0, and even the numerical assertion usually fails – for if A is not some submultiple of E_0, it will meet some other component B of E_0, and then $A \cdot B > 0$. In this case $H^0(\mathcal{O}_{E_0+A})$ is the ring $k[\varepsilon]$ with $\varepsilon^2 = 0$.

Lemma 1 *Let D be a divisor on a (complete) surface X; then*

(i) if $\operatorname{Supp} D$ *is connected, then $H^0(\mathcal{O}_D)$ is an Artinian local ring;*

(ii) if $h^0(\mathcal{O}_D) > 1$ then there is a decomposition $D = D_1 + D_2$ for which

either

(a) $\operatorname{Supp} D_1$ *and* $\operatorname{Supp} D_2$ *are disjoint,*

or

(b) D_2 is effectively disconnected from D_1, and $D_1 < D_2$.

Proof If $H^0(\mathcal{O}_D)$ is not local, then there exists a nontrivial decomposition

$$1 = e + f$$

of $1 \in H^0(\mathcal{O}_D)$ as the sum of two orthogonal idempotents; now the image of e and f under the map $H^0(\mathcal{O}_D) \to H^0(\mathcal{O}_{D_{\text{red}}})$ defines a similar decomposition

$$1 = \overline{e} + \overline{f}$$

of $1 \in H^0(\mathcal{O}_{D_{\text{red}}})$; this is a nontrivial decomposition, since if $\overline{e} = 0$ then e would be a nilpotent section of $\mathcal{O}_D$. But now $\overline{e} = 0$ and $\overline{f} = 0$ define two disjoint open and closed subsets of $\operatorname{Supp} D$.

Similarly, if $H^0(\mathcal{O}_D)$ is local, and $h^0(\mathcal{O}_D) > 1$, then $H^0(\mathcal{O}_D)$ contains an element $e \neq 0$ with $e^2 = 0$. Let $Z_2 \subset D$ be the subscheme defined by the $\mathcal{O}_D$-ideal $e\mathcal{O}_D$, and $D_2 \subset Z_2$ the greatest divisor contained in Z_2. (Thus D_2 and Z_2 only differ at the "embedded points" of Z_2, at which Z_2 fails to be Cohen–Macaulay.) D_2 is nonzero, since Z_2 is defined by a nilpotent ideal, and so contains at least D_{red}. The inclusion $D_2 \subset D$ gives rise to a decomposition

$$D = D_1 + D_2$$

and hence to an exact sequence

$$0 \to \mathcal{O}_{D_1}(-D_2) \to \mathcal{O}_D \to \mathcal{O}_{D_2} \to 0,$$

identifying $\mathcal{O}_{D_1}(-D_2)$ as the ideal of $\mathcal{O}_D$ defining D_2. This is generated outside a finite set by the section e by construction.

To get $D_1 < D_2$, note that D_2 is defined outside a finite set by e, and $e^2 = 0$; hence, for some dense open set U of $\operatorname{Supp} D$ we have $D_{|U} < 2D_{2|U}$, and hence $D < 2D_2$, and $D_1 < D_2$.

To return to the proof of Theorem 1, let $|d|$ be a separable g^1_r on C, C lying on the K3 surface X. Note that as soon as $g \geq 3r$ there will exist a divisor $D \in |f^*C - 2\sum l_i|$; Lemma 1 transforms (v) into

> There is a decomposition $D = D_1 + D_2$ such that either $\operatorname{Supp} D_1$ and $\operatorname{Supp} D_2$ are disjoint, or $D_1 < D_2$ and D_2 is effectively disconnected from D_1. Furthermore, there is no such decomposition for $D + l_i$. (vi)

In either case we can write

$$D_1 = f^*E_1 - \sum(1 + \varepsilon_i)l_i$$
$$D_2 = f^*E_2 - \sum(1 - \varepsilon_i)l_i$$

with E_1 and E_2 divisors on X such that $E_1 + E_2 \sim C$, and ε_i are integers.

Lemma 2 *(a) If* $\operatorname{Supp} D_1$ *and* $\operatorname{Supp} D_2$ *are disjoint, then for all* i, $\varepsilon_i = 0$ *and* E_1 *and* E_2 *meet transversally at* P_i.

(b) if $D_1 < D_2$, *then* $E_1 < E_2$ *and* $\varepsilon_i \geq 0$.

Proof In either case $D_2 \cdot (\text{any component of } D_1) \leq 0$; thus if $\varepsilon_i < 0$, $D_2 \cdot l_i > 0$, so that l_i cannot be a component of D_1. Thus $\varepsilon_i = -1$, and E_1 does not pass through P_i; this contradicts the final clause of (vi) – trivially in case (a), since we can just add l_i to D_2; in case (b), the argument is as follows: if E_1 does not pass through P_i, then $\mathcal{O}_{D_1}(-l_i) = \mathcal{O}_{D_1}$, so that $\mathcal{O}_{D_1}(-D_2 - l_i) = \mathcal{O}_{D_1}(-D_2)$ is generated outside a finite set by its global sections.

In case (a) of the lemma, $\varepsilon_i = 0$ now follows by symmetry, and the transversality of E_1 and E_2 at P_i is obvious.

The proof of Theorem 1 is now straightforward; let us first establish the following numerical version:

Lemma 3 *Under the above conditions, suppose that* $g > \frac{1}{4}r^2 + r + 2$; *then (after interchanging* E_1 *and* E_2 *if necessary in case (a) of Lemma 2), we have*

$$E_1^2 = 0 \quad \textit{and} \quad E_1 C = r.$$

Proof In case (b) of Lemma 2 we have $E_1^2 \leq E_2^2$, since $E_1^2 + E_1 E_2 = E_1 C$, and $E_2^2 + E_1 E_2 = E_2 C$, and $E_1 < E_2$; in case (a) we can assume $E_1^2 \leq E_2^2$ by symmetry.

Now since $D_1 D_2 \leq 0$ it follows that $E_1 E_2 \leq r$; on the other hand,

$$(E_1 + E_2)^2 = C^2 > 0,$$

so that the Index Theorem may be written in the form

$$E_1^2 E_2^2 - (E_1 E_2)^2 = \det \begin{vmatrix} E_1^2 & E_1 E_2 \\ E_1 E_2 & E_2^2 \end{vmatrix} \leq 0;$$

hence $E_1^2 E_2^2 \leq r^2$. If $E_1^2 > 0$ then $E_1^2 \geq 2$, so that $E_2^2 \leq \frac{1}{2}r^2$; then

$$g = 1 + \frac{1}{2}(E_1 + E_2)^2 \leq \frac{1}{4}r^2 + r + 2;$$

thus $E_1^2 \leq 0$.

But now from $D_1 D_2$ we also get the assertion that $E_1 E_2 + \sum \varepsilon_i^2 \leq r$; on the other hand, $E_1 E_2 + E_1^2 = E_1 C \geq r + \sum \varepsilon_i$ (since E_1 has intersection number at least $1+\varepsilon_i$ with C at P_i). Hence $E_1^2 \geq \sum(\varepsilon_i^2+\varepsilon_i)$. We conclude that $E_1^2 = 0$, and that the ε_i are also zero. $E_1C = r$ then follows.

Now the mobile part of $|E_1|$ is an elliptic pencil, which cuts out the g^1_r $|P_1 + \cdots + P_r|$ on C. Theorem 1 is proved.

Note that we get an easy counterexample to any improvement of the function $f(r) = \frac{1}{4}r^2 + r + 2$ occurring in the statement of Theorem 1 as follows: let X be a K3 surface, and $|B|$ an irreducible linear system with $B^2 = 2$; let $C \sim (m+1)B$, for some $m \geq 2$. It is clear that C can be chosen such that the double covering morphism

$$\phi_B\colon X \to \mathbb{P}^2$$

takes C birationally into a curve $\overline{C}$ of degree $2m+2$ having a certain number of ordinary double points P_i; the lines of $\mathbb{P}^2$ passing through one of the P_i cut out a g^1_{2m} on C, which can only be realised on X as being cut out residually by the sublinear system $|B|_{P_i} \subset |B|$ consisting of the curves of $|B|$ passing through the points of X lying over P_i; however,

$$g(C) = 1 + (m+1)^2 = \frac{1}{4}(2m)^2 + 2m + 2.$$

References

[1] B. Saint-Donat, Projective models of K3 surfaces, Amer. J. Math **96** (1974), 602–639

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