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SPECIAL SYMPLECTIC CONNECTIONS

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Abstract

By a special symplectic connection we mean a torsion free connection which is either the Levi-Civita connection of a Bochner-Kähler metric of arbitrary signature, a Bochner-bi-Lagrangian connection, a connection of Ricci type or a connection with special symplectic holonomy. A manifold or orbifold with such a connection is called special symplectic.

We show that the symplectic reduction of (an open cell of) a parabolic contact manifold by a symmetry vector field is special symplectic in a canonical way. Moreover, we show that any special symplectic manifold or orbifold is locally equivalent to one of these symplectic reductions.

As a consequence, we are able to prove a number of global properties, including a classification in the compact simply connected case.

1. Introduction

Among the basic objects of interest in differential geometry are connections on a differentiable manifold M which are compatible with a given geometric structure, and the relation between the local invariants of such connections and the geometric and topological features of M. For example, in Riemannian geometry, the Levi-Civita connection of the metric is uniquely determined, hence every feature of the connection reflects a property of the metric structure.

In contrast, for a symplectic manifold (M, ω), there are many symplectic connections, where we call a connection on M symplectic if it is torsion free and ω is parallel. Indeed, the space of symplectic connections on M is an affine space whose linear part is given by the sections in $S^3(TM)$. Thus, in order to investigate 'meaningful' symplectic connections, we have to impose further conditions.

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In this article, we shall introduce the notion of a *special symplectic connection* which is defined as a symplectic connection on a manifold of dimension at least 4 which belongs to one of the following classes.

1) **Bochner-Kähler and Bochner-bi-Lagrangian connections**
   If the symplectic form is the Kähler form of a (pseudo-)Kähler metric, then its curvature decomposes into the Ricci curvature and the Bochner curvature [**Bo**]. If the latter vanishes, then (the Levi-Civita connection of) this metric is called Bochner-Kähler.
   Similarly, if the manifold is equipped with a bi-Lagrangian structure, i.e. two complementary Lagrangian distributions, then the curvature of a symplectic connection for which both distributions are parallel decomposes into the Ricci curvature and the Bochner curvature. Such a connection is called Bochner-bi-Lagrangian if its Bochner curvature vanishes.
   For results on Bochner-Kähler and Bochner-bi-Lagrangian connections, see [**Br2**] and [**K**] and the references cited therein.
2) **Connections of Ricci type**
   Under the action of the symplectic group, the curvature of a symplectic connection decomposes into two irreducible summands, namely the Ricci curvature and a Ricci flat component. If the latter component vanishes, then the connection is said to be of Ricci type.
   Connections of Ricci type are critical points of a certain functional on the moduli space of symplectic connections [**BC1**]. Furthermore, the canonical almost complex structure on the twistor space induced by a symplectic connection is integrable iff the connection is of Ricci type [**BR**], [**V**]. For further properties see also [**CGR**], [**CGHR**], [**BC2**], [**CGS**].
3) **Connections with special symplectic holonomy**
   A symplectic connection is said to have *special symplectic holonomy* if its holonomy is contained in a proper absolutely irreducible subgroup of the symplectic group.
   The special symplectic holonomies have been classified in [**MS**] and further investigated in [**Br1**], [**CMS**], [**S1**], [**S2**], [**S3**].

We can consider all of these conditions also in the complex case, i.e. for complex manifolds of complex dimension at least 4 with a holomorphic symplectic form and a holomorphic connection.

At first, it may seem unmotivated to collect all these structures in one definition, but we shall provide ample justification for doing so. Indeed, our main results show that there is a beautiful link between special symplectic connections and parabolic contact geometry.

For this, consider a (real or complex) simple Lie group G with Lie algebra $\mathfrak{g}$. We say that $\mathfrak{g}$ is 2-gradable, if $\mathfrak{g}$ contains the root space of a long root. In this case, the projectivization of the adjoint orbit of a maximal root vector $\mathcal{C} \subset \mathbb{P}^o(\mathfrak{g})$ carries a canonical G-invariant contact structure. Here, $\mathbb{P}^o(V)$ denotes the set of oriented lines through 0 of a vector space V, so that $\mathbb{P}^o(V)$ is a sphere if V is real and a complex projective space if V is complex. Each $a \in \mathfrak{g}$ induces an action field a^* on $\mathcal{C}$ with flow $\mathrm{T}_a := \exp(\mathbb{F}a) \subset \mathrm{G}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, which hence preserves the contact structure on $\mathcal{C}$. Let $\mathcal{C}_a \subset \mathcal{C}$ be the open subset on which a^* is positively transversal to the contact distribution. We can cover $\mathcal{C}_a$ by open sets U such that the local quotient $M_U := \mathrm{T}_a^{loc}\backslash U$, i.e. the quotient of U by a sufficiently small neighborhood of the identity in T_a, is a manifold. Then M_U inherits a canonical symplectic structure. Our first main result is the following

Theorem A. *Let* $\mathfrak{g}$ *be a simple* 2*-gradable Lie algebra with* $\dim \mathfrak{g} \geq 14$*, and let* $\mathcal{C} \subset \mathbb{P}^o(\mathfrak{g})$ *be the projectivization of the adjoint orbit of a maximal root vector. Let* $a \in \mathfrak{g}$ *be such that* $\mathcal{C}_a \subset \mathcal{C}$ *is nonempty, and let* $\mathrm{T}_a = \exp(\mathbb{F}a) \subset \mathrm{G}$*. If for an open subset* $U \subset \mathcal{C}_a$ *the local quotient* $M_U = \mathrm{T}_a^{loc}\backslash U$ *is a manifold, then* M_U *carries a special symplectic connection.*

The dimension restriction on $\mathfrak{g}$ guarantees that $\dim M_U \geq 4$ and rules out the Lie algebras of type A_1, A_2 and B_2.

The type of special symplectic connection on M_U is determined by the Lie algebra $\mathfrak{g}$. In fact, there is a one-to-one correspondence between the various conditions for special symplectic connections and simple 2-gradable Lie algebras. More specifically, if the Lie algebra $\mathfrak{g}$ is of type A_n, then the connections in Theorem A are Bochner-Kähler of signature (p, q) if $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$ or Bochner-bi-Lagrangian if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{F})$; if $\mathfrak{g}$ is of type C_n, then $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{F})$ and these connections are of Ricci type; if $\mathfrak{g}$ is a 2-gradable Lie algebra of one of the remaining types, then the holonomy of M_U is contained in one of the special symplectic holonomy groups. Also, for two elements $a, a' \in \mathfrak{g}$ for which $\mathcal{C}_a, \mathcal{C}_{a'} \subset \mathcal{C}$ are nonempty, the corresponding connections from Theorem A are equivalent iff a' is G-conjugate to a positive multiple of a.

If $\mathrm{T}_a \cong S^1$ then $\mathrm{T}_a\backslash\mathcal{C}_a$ is an orbifold which carries a special symplectic orbifold connection by Theorem A. Hence it may be viewed as the "standard orbifold model" for (the adjoint orbit of) $a \in \mathfrak{g}$. For example, in the case of positive definite Bochner-Kähler metrics, we have $\mathcal{C} \cong S^{2n+1}$, and for connections of Ricci type, we have $\mathcal{C} \cong \mathbb{RP}^{2n+1}$. Thus, in both cases the orbifolds $\mathrm{T}_a\backslash\mathcal{C}$ are weighted projective spaces if $\mathrm{T}_a \cong S^1$, hence the standard orbifold models $\mathrm{T}_a\backslash\mathcal{C}_a \subset \mathrm{T}_a\backslash\mathcal{C}$ are open subsets of weighted projective spaces.

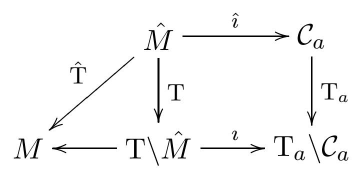
Surprisingly, the connections from Theorem A exhaust *all* special symplectic connections, at least locally. Namely we have the following

Theorem B. *Let (M, ω) be a (real or complex) symplectic manifold with a special symplectic connection of class C^4, and let $\mathfrak{g}$ be the Lie algebra associated to the special symplectic condition as above.*

1) *Then there is a principal $\hat{\mathrm{T}}$-bundle $\hat{M} \to M$, where $\hat{\mathrm{T}}$ is a one dimensional Lie group which is not necessarily connected, and this bundle carries a principal connection with curvature ω.*
2) *Let $\mathrm{T} \subset \hat{\mathrm{T}}$ be the identity component. Then there is an $a \in \mathfrak{g}$ such that $\mathrm{T} \cong \mathrm{T}_a \subset \mathrm{G}$, and a T_a-equivariant local diffeomorphism $\hat{\imath} : \hat{M} \to \mathcal{C}_a$ which for each sufficiently small open subset $V \subset \hat{M}$ induces a connection preserving diffeomorphism $\imath : \mathrm{T}^{loc}\backslash V \to \mathrm{T}_a^{loc}\backslash U = M_U$, where $U := \hat{\imath}(V) \subset \mathcal{C}_a$ and M_U carries the connection from Theorem A.*

The situation in Theorem B can be illustrated by the following commutative diagram, where the vertical maps are quotients by the indicated Lie groups, and $\mathrm{T}\backslash\hat{M} \to M$ is a regular covering.

(1)



In fact, one might be tempted to summarize Theorems A and B by saying that for each $a \in \mathfrak{g}$, the quotient $\mathrm{T}_a\backslash\mathcal{C}_a$ carries a canonical special symplectic connection, and the map $\imath : \mathrm{T}\backslash\hat{M} \to \mathrm{T}_a\backslash\mathcal{C}_a$ is a connection preserving local diffeomorphism. If $\mathrm{T}_a\backslash\mathcal{C}_a$ is a manifold or an orbifold, then this is indeed correct. In general, however, $\mathrm{T}_a\backslash\mathcal{C}_a$ may be neither Hausdorff nor locally Euclidean, hence one has to formulate these results more carefully.

As consequences, we obtain the following

Corollary C. *All special symplectic connections of C^4-regularity are analytic, and the local moduli space of these connections is finite dimensional, in the sense that the germ of the connection at one point up to 3rd order determines the connection entirely. In fact, the generic special symplectic connection associated to the Lie algebra $\mathfrak{g}$ depends on $(\mathrm{rk}(\mathfrak{g}) - 1)$ parameters.*

Moreover, the Lie algebra $\mathfrak{s}$ of vector fields on M whose flow preserves the connection is isomorphic to $stab(a)/(\mathbb{F}a)$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, with $a \in \mathfrak{g}$ from Theorem B, where $stab(a) = \{x \in \mathfrak{g} \mid [x, a] = 0\}$. In particular, $\dim \mathfrak{s} \geq \mathrm{rk}(\mathfrak{g}) - 1$ with equality implying that $\mathfrak{s}$ is abelian.

When counting the parameters in the above corollary, we regard homothetic special symplectic connections as equal, i.e. (M, ω, ∇) is considered equivalent to $(M, e^{t_0}\omega, \nabla)$ for all $t_0 \in \mathbb{F}$.

We can generalize Theorem B and Corollary C easily to orbifolds. Indeed, if M is an orbifold with a special symplectic connection, then we can write $M = \hat{\mathrm{T}} \backslash \hat{M}$ where $\hat{M}$ is a manifold and $\hat{\mathrm{T}}$ is a one dimensional Lie group acting properly and locally freely on $\hat{M}$, and there is a local diffeomorphism $\hat{\imath} : \hat{M} \to \mathcal{C}_a$ with the properties stated in Theorem B.

While the analyticity of the connection and the determinedness by the 3rd order germ at a point has been known in the Bochner-Kähler and Bochner-bi-Lagrangian case ([**Br2**] (The C^4-regularity of the connection is equivalent to the C^5-regularity of the Bochner-Kähler metric)) and for connections with special symplectic holonomies (e.g. [**CMS**], [**MS**]), it was unclear what the maximal analytic continuations of these structures look like and in which cases they are regular. This question is now answered in principle. Furthermore, the inequality $\dim \mathfrak{s} \geq \mathrm{rk}(\mathfrak{g}) - 1$ was known for the Bochner cases [**Br2**], whereas for the special symplectic holonomies, it was only known that $\mathfrak{s} \neq 0$ [**S3**].

We also address the question of the existence of compact manifolds with special symplectic connections. In the simply connected case, compactness already implies that the connection is hermitian symmetric. More specifically, we have the following

Theorem D. *Let M be a compact simply connected manifold with a special symplectic connection of class C^4. Then M is equivalent to one of the following hermitian symmetric spaces.*

1) $M \cong (\mathbb{CP}^p \times \mathbb{CP}^q, ((q+1)g_0, -(p+1)g_0))$, *where g_0 is the Fubini-Study metric. These are Bochner-Kähler metrics of signature (p, q). Moreover, $M \cong (\mathbb{CP}^n, g_0)$ is also of Ricci type.*
2) $M \cong \mathrm{SO}(n+2)/(\mathrm{SO}(2) \cdot \mathrm{SO}(n))$, *whose holonomy is contained in the special symplectic holonomy group* $\mathrm{SL}(2, \mathbb{R}) \cdot \mathrm{SO}(n) \subset \mathrm{Aut}(\mathbb{R}^2 \otimes \mathbb{R}^n)$.
3) $M \cong \mathrm{SU}(2n+2)/\mathrm{S}(\mathrm{U}(2) \cdot \mathrm{U}(2n))$, *whose holonomy is contained in the special symplectic holonomy group* $\mathrm{Sp}(1) \cdot \mathrm{SO}(n, \mathbb{H}) \subset \mathrm{Aut}(\mathbb{H}^n)$.
4) $M \cong \mathrm{SO}(10)/\mathrm{U}(5)$, *whose holonomy is contained in the special symplectic holonomy group* $\mathrm{SU}(1, 5) \subset \mathrm{GL}(20, \mathbb{R})$.
5) $M \cong \mathrm{E}_6/(\mathrm{U}(1) \cdot \mathrm{Spin}(10))$, *whose holonomy is contained in the special symplectic holonomy group* $\mathrm{Spin}(2, 10) \subset \mathrm{GL}(32, \mathbb{R})$.

In particular, there are no compact simply connected manifolds with any of the remaining types of special symplectic connections, i.e. M can be neither complex with a holomorphic connection, nor Bochner-bi-Lagrangian, nor can the holonomy of M be contained in any of the remaining special symplectic holonomies.

The only case for which Theorem D was previously known are the positive definite Kähler metrics. In fact, it is shown in [**Br2**] that a compact positive definite Bochner-Kähler manifold must be a quotient of $\mathbb{CP}^r \times (\mathbb{CP}^s)^*$, where the asterisk denotes the non-compact dual.

Following this introduction, we first develop the algebraic formulas needed to describe the curvature conditions for special symplectic connections uniformly. In section 3, we construct the special symplectic connections on the local quotients $\mathrm{T}_a^{loc}\backslash\mathcal{C}_a$ and hence prove Theorem A, and in section 4, we investigate the structure equations of special symplectic connections and derive results which culminate in Theorem B. Finally, in the last section we show the existence of connection preserving vector fields and Corollary C, and the rigidity result from Theorem D.

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2. Algebraic preliminaries

2.1. A brief review of representation theory. In this section, we shall give a brief outline of standard facts of representation theory of complex semi-simple Lie algebras. For a more detailed exposition, see e.g. [**FH**], [**Hu**] or [**OV**].

Let $\mathfrak{g}_\mathbb{C}$ be a semi-simple complex Lie algebra, and let $\mathfrak{t} \subset \mathfrak{g}_\mathbb{C}$ be a *Cartan subalgebra*, i.e. a maximal abelian self-normalizing subalgebra. The *rank of* $\mathfrak{g}_\mathbb{C}$ is by definition $\mathrm{rk}(\mathfrak{g}_\mathbb{C}) := \dim \mathfrak{t}$.

If $\rho : \mathfrak{g}_\mathbb{C} \to \mathrm{End}(V)$ is a representation of $\mathfrak{g}_\mathbb{C}$ on a complex vector space V, then for any $\lambda \in \mathfrak{t}^*$ we define the *weight space* V_λ by

$$V_\lambda = \{v \in V \mid \rho(h)v = \lambda(h)v \text{ for all } h \in \mathfrak{t}\}.$$

An element $\lambda \in \mathfrak{t}^*$ is called a *weight* of V if $V_\lambda \neq 0$. We let $\Phi \subset \mathfrak{t}^*$ be the set of weights of ρ, and thus have the decomposition

$$V = \bigoplus_{\lambda \in \Phi} V_\lambda.$$

In particular, if $V = \mathfrak{g}_\mathbb{C}$ and ρ is the adjoint representation, then we get the *root decomposition*

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

i.e. $\mathfrak{t}$ is the weight space of weight 0, and $\Delta \subset \mathfrak{t}^*$ is the set of non-zero weights. Δ is called the *set of roots* or the *root system* of $\mathfrak{g}_\mathbb{C}$. It is well known that $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$. For any root $\alpha \in \Delta$, there is a unique element $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t}$ such that $\alpha(H_\alpha) = 2$.

There is an $\mathrm{ad}(\mathfrak{g}_\mathbb{C})$-invariant non-degenerate symmetric bilinear form B on $\mathfrak{g}_\mathbb{C}$, the so-called *Killing form*, which is given by $B(x, y) := \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$ for all $x, y \in \mathfrak{g}_\mathbb{C}$. We shall use it to identify $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{g}_\mathbb{C}^*$. The restriction of B to $\mathfrak{t}$ is non-degenerate as well, and $B(H_\alpha, H_\alpha) \in \mathbb{Z}^+$ for all $\alpha \in \Delta$. In fact, there are at most two possible values for $B(H_\alpha, H_\alpha)$ for $\alpha \in \Delta$ which allows us to speak of *long* and *short* roots, respectively.

Given an element $\lambda \in \mathfrak{t}^*$ and a root α, we let

$$\langle \lambda, \alpha \rangle := \lambda(H_\alpha), \qquad \text{so that} \qquad \langle \lambda, \alpha \rangle = \frac{2B(\lambda, \alpha)}{B(\alpha, \alpha)}. \tag{2}$$

Note that $\langle \ , \ \rangle$ is linear in the first entry only. We define the *weight lattice* $\Lambda \subset \mathfrak{t}^*$ as the set of elements $\lambda \in \mathfrak{t}^*$ such that $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Delta$. Then $\Phi \subset \Lambda$ for any representation ρ.

For $\lambda \in \Phi$, the significance of $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ is the following. If λ occurs as the weight of an irreducible representation of $\mathfrak{g}_\mathbb{C}$ and $\langle \lambda, \alpha \rangle > 0$ ($\langle \lambda, \alpha \rangle < 0$, respectively) then $\lambda - k\alpha$ ($\lambda + k\alpha$, respectively) is also a weight of that representation for $k = 1, \ldots, |\langle \lambda, \alpha \rangle|$.

For any root $\alpha \in \Delta$, denote by σ_α the orthogonal reflection of $\mathfrak{t}^*$ in the hyperplane perpendicular to α. The *Weyl group* W of $\mathfrak{g}_\mathbb{C}$ is the group generated by all σ_α. W is always finite. If $\mathfrak{g}_\mathbb{C}$ is simple then W acts irreducibly on $\mathfrak{t}^*$ and transitively on the set of roots of equal length. The set of weights Φ of any representation is W-invariant.

If $\mathfrak{g}_\mathbb{C}$ is simple, then the adjoint representation $\rho : \mathfrak{g}_\mathbb{C} \to \mathrm{End}(\mathfrak{g}_\mathbb{C})$ is irreducible. Also, $|\langle \alpha, \beta \rangle| \leq 3$ for all $\alpha, \beta \in \Delta$, and if α is long and β short, then either $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = 0$, or $|\langle \alpha, \beta \rangle| > 1$ and $|\langle \beta, \alpha \rangle| = 1$. Moreover, if α is long then $|\langle \beta, \alpha \rangle| \leq 2$, and $\langle \beta, \alpha \rangle = \pm 2$ iff $\beta = \pm\alpha$.

2.2. Special symplectic representations. Let $\mathfrak{g}_\mathbb{C}$ be a complex simple Lie algebra and let $\mathrm{G}_\mathbb{C}$ be a connected complex Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$. Choose a root decomposition of $\mathfrak{g}$ as in the preceding section, and fix a long root α and an element $0 \neq x \in \mathfrak{g}_\alpha$. Then the orbit of x under the adjoint action of $\mathrm{G}_\mathbb{C}$ is called the *root cone of* $\mathfrak{g}_\mathbb{C}$. Evidently, the root cone is well defined, independently of the choice of root decomposition. Elements of the root cone are called *maximal root elements*.

Definition 2.1. Let $\mathfrak{g}$ be a simple real or complex Lie algebra. We say that $\mathfrak{g}$ is 2-*gradable* if either $\mathfrak{g}$ is complex, or $\mathfrak{g}$ is real and contains a maximal root element of the simple complex Lie algebra $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes \mathbb{C}$.

We shall justify this terminology in (4) below. If $\mathfrak{g}$ is 2-gradable and G is a Lie group with Lie algebra $\mathfrak{g}$, then we write

$$\hat{\mathcal{C}} := \mathrm{Ad}_G x \subset \mathfrak{g}, \tag{3}$$

where $x \in \mathfrak{g}$ is a maximal root element. Given $x \in \hat{\mathcal{C}}$, there is a $y \in \hat{\mathcal{C}}$ with $B(x, y) \neq 0$, and we can choose a root decomposition of $\mathfrak{g}$ such that $x \in \mathfrak{g}_{\alpha_0}$ and $y \in \mathfrak{g}_{-\alpha_0}$, where α_0 is a long root. Hence $H_{\alpha_0} \in \mathbb{F}[x, y] \subset \mathfrak{t}$, so that $\mathfrak{g}$ contains the Lie subalgebra $\mathfrak{sl}_{\alpha_0} := span < \mathfrak{g}_{\alpha_0}, \mathfrak{g}_{-\alpha_0}, H_{\alpha_0} >$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then $\mathrm{ad}(H_{\alpha_0})|_{\mathfrak{g}_\beta} = \langle \beta, \alpha_0 \rangle \, Id_{\mathfrak{g}_\beta}$, and since $\alpha_0 \in \Delta$ is a long root, the eigenvalues of $\mathrm{ad}(H_{\alpha_0})$ are $\{0, \pm 1, \pm 2\}$, so that we get the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{1} \oplus \mathfrak{g}^{2}, \quad \text{and} \quad [\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}, \tag{4}$$

where $\mathfrak{g}^i = \bigoplus_{\{\beta \in \Delta | \langle \beta, \alpha_0 \rangle = i\}} \mathfrak{g}_\beta$ for $i \neq 0$ and $\mathfrak{g}^0 = \mathfrak{t} \oplus \bigoplus_{\{\beta \in \Delta | \langle \beta, \alpha_0 \rangle = 0\}} \mathfrak{g}_\beta$. In particular, $\mathfrak{g}^{\pm 2} = \mathfrak{g}_{\pm \alpha_0}$, and $\mathfrak{g}^0 = \mathbb{F} H_{\alpha_0} \oplus \mathfrak{h}$, where the Lie algebra $\mathfrak{h}$ is characterized by $[\mathfrak{h}, \mathfrak{sl}_{\alpha_0}] = 0$. Observe that $\mathfrak{g}^0$ and hence $\mathfrak{h}$ are reductive. Thus, as a Lie algebra,

$$\mathfrak{g}^{ev} := \mathfrak{g}^{-2} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{2} \cong \mathfrak{sl}_{\alpha_0} \oplus \mathfrak{h} \quad \text{and} \quad \mathfrak{g}^{odd} := \mathfrak{g}^{-1} \oplus \mathfrak{g}^{1} \cong \mathbb{F}^2 \otimes V,$$

where $\mathfrak{h}$ acts effectively on $V \cong \mathfrak{g}^{\pm 1}$. Identifying $\mathfrak{h}$ with its image under this representation, we may regard it as a subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$, and hence we have the decomposition

$$\mathfrak{g} = \mathfrak{g}^{ev} \oplus \mathfrak{g}^{odd} \cong (\mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h}) \oplus (\mathbb{F}^2 \otimes V), \tag{5}$$

where this notation indicates the representation $\mathrm{ad} : \mathfrak{g}^{ev} \to \mathrm{End}(\mathfrak{g}^{odd})$.

We fix a non-zero $\mathbb{F}$-bilinear area form $a \in \Lambda^2(\mathbb{F}^2)^*$. There is a canonical $\mathfrak{sl}(2, \mathbb{F})$-equivariant isomorphim

$$\begin{aligned} S^2(\mathbb{F}^2) &\longrightarrow \mathfrak{sl}(2, \mathbb{F}), \\ (ef) \cdot g &:= a(e, g) f + a(f, g) e \quad \text{for all } e, f, g \in \mathbb{F}^2, \end{aligned} \tag{6}$$

and under this isomorphism, the Lie bracket on $\mathfrak{sl}(2, \mathbb{F})$ is given by

$$[ef, gh] = a(e, g) fh + a(e, h) fg + a(f, g) eh + a(f, h) eg. \tag{7}$$

Thus, if we fix a basis $e_+, e_- \in \mathbb{F}^2$ with $a(e_+, e_-) = 1$, then we have the identifications

$$H_{\alpha_0} = -e_+ e_-, \quad \mathfrak{g}^{\pm 2} = \mathbb{F} e_\pm^2, \quad \mathfrak{g}^{\pm 1} = e_\pm \otimes V.$$

Proposition 2.2. *Let $\mathfrak{g}$ be a 2-gradable simple Lie algebra, and consider the decompositions (4) and (5). Then there is an $\mathfrak{h}$-invariant symplectic form $\omega \in \Lambda^2 V^*$ and an $\mathfrak{h}$-equivariant product $\circ : S^2(V) \to \mathfrak{h}$ such that*

$$[\ ,\ ] : \Lambda^2(\mathfrak{g}^{odd}) \longrightarrow \mathfrak{g}^{ev} \cong \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h}$$

is given as

$$[e \otimes x, f \otimes y] = \omega(x, y) ef + a(e, f) x \circ y \quad \textit{for } e, f \in \mathbb{F}^2 \textit{ and } x, y \in V, \tag{8}$$

using the identification $S^2(\mathbb{F}^2) \cong \mathfrak{sl}(2, \mathbb{F}) \subset \mathfrak{g}^{ev}$ from (6). Moreover, the symmetric bilinear form $(\ ,\ )$ on $\mathfrak{g}$ defined by

$$(u, v) := -\frac{1}{2(\dim V + 4)} B(u, v), \quad \textit{for all } u, v \in \mathfrak{g}, \tag{9}$$

where B is the Killing form of $\mathfrak{g}$, satisfies the following:

1) $(\mathfrak{g}^i, \mathfrak{g}^j) = 0$ *if* $i + j \neq 0$,
2) $(ef, gh) = a(e, g)a(f, h) + a(e, h)a(f, g)$ *for all* $e, f, g, h \in \mathbb{F}^2$,
3) $B(u, v) = 2\ tr_V(uv) + B_{\mathfrak{h}}(u, v)$ *for all* $u, v \in \mathfrak{h} \subset \mathfrak{g}$, *where* $B_{\mathfrak{h}}$ *denotes the Killing form of* $\mathfrak{h}$.
4) $(e \otimes x, f \otimes y) = a(e, f)\omega(x, y)$, *for all* $e, f \in \mathbb{F}^2$ *and* $x, y \in V$, *using the identification* $\mathfrak{g}^{odd} \cong \mathbb{F}^2 \otimes V$,
5) *For all* $x, y, z \in V$ *and* $h \in \mathfrak{h}$, *we have*

$$(h, x \circ y) = \omega(hx, y) = \omega(hy, x) \tag{10}$$
$$(x \circ y)z - (x \circ z)y = 2\ \omega(y, z)x - \omega(x, y)z + \omega(x, z)y.$$

Proof. By (4) the bracket $[\ ,\ ] : \Lambda^2\mathfrak{g}^{odd} \to \mathfrak{g}^{ev}$ is well-defined and must be $\mathfrak{g}^{ev}$-equivariant by the Jacobi identity. We decompose $\Lambda^2\mathfrak{g}^{odd} = \Lambda^2(\mathbb{F}^2 \otimes V) = S^2(\mathbb{F}^2) \otimes \Lambda^2 V \oplus S^2(V)$, so that any $\mathfrak{g}^{ev}$-equivariant map $\Lambda^2\mathfrak{g}^{odd} \to \mathfrak{g}^{ev}$ must be of the form (8) for some $\mathfrak{h}$-invariant $\omega \in \Lambda^2 V^*$ and $\circ : S^2(V) \to \mathfrak{h}$.

Since $(\ ,\ )$ is $\mathrm{ad}_{\mathfrak{g}}$-invariant, i.e. it satisfies the identity $([u, v], w) = (u, [v, w])$ for all $u, v, w \in \mathfrak{g}$, we have for $u_i \in \mathfrak{g}^i$ and $u_j \in \mathfrak{g}^j$

$$\begin{aligned} 0 &= ([H_{\alpha_0}, u_i], u_j) + (u_i, [H_{\alpha_0}, u_j]) \\ &= (i\ u_i, u_j) + (u_i, j\ u_j) = (i + j)(u_i, u_j), \end{aligned}$$

which shows 1.

To show the second equation, note that the inner product on $S^2(\mathbb{F}^2) \cong \mathfrak{sl}(2, \mathbb{F})$ given by the right hand side of this equation is $\mathrm{ad}_{\mathfrak{sl}(2,\mathbb{F})}$-invariant and hence must be a multiple of the restriction of the Killing form B to $\mathfrak{sl}(2, \mathbb{F})$. Thus, it suffices to verify the second equation for $e = g = e_+$ and $f = h = e_-$. In this case, the right hand side equals -1, whereas the left hand side equals $(e_+e_-, e_+e_-) = (H_{\alpha_0}, H_{\alpha_0})$. But $B(H_{\alpha_0}, H_{\alpha_0}) = tr(\mathrm{ad}(H_{\alpha_0})^2)$ and since $\mathrm{ad}(H_{\alpha_0})|_{\mathfrak{g}^i} = iId_{\mathfrak{g}^i}$, we conclude that $(e_+e_-, e_+e_-) = -1$ by the choice of the scaling factor in (9). This implies 2. Likewise, if $u, v \in \mathfrak{h}$, then $\mathrm{ad}(u)|_{\mathfrak{sl}(2,\mathbb{F})} = \mathrm{ad}(v)|_{\mathfrak{sl}(2,\mathbb{F})} = 0$, from which 3. follows as well.

For 4. note that $(e_\pm \otimes x, e_\pm \otimes y) \in (\mathfrak{g}^{\pm 1}, \mathfrak{g}^{\pm 1}) = 0$ by 1., and from 2. and the $\mathrm{ad}_{\mathfrak{g}}$-invariance, we get

$$\begin{aligned} (e_+ \otimes x, e_- \otimes y) &= -\frac{1}{2}\left(e_+ \otimes x, [e_-^2, e_+ \otimes y]\right) \\ &= \frac{1}{2}\left([e_+ \otimes x, e_+ \otimes y], e_-^2\right) = \frac{1}{2}\omega(x, y)\left(e_+^2, e_-^2\right) \\ &= \omega(x, y). \end{aligned}$$

This also implies that ω is symplectic; indeed, if $\omega(x, V) = 0$ for some $x \in V$, then by 1. and 4. it follows that $(e_+ \otimes x, \mathfrak{g}) = 0$ so that $x = 0$.

To show the first equation in (10), we note that $(\mathfrak{h}, \mathfrak{sl}(2, \mathbb{F})) = 0$ so that for $h \in \mathfrak{h}$ and $x, y \in V$ we have

$$\begin{aligned}(h, x \circ y) &= (h, [e_+ \otimes x, e_- \otimes y]) = ([h, e_+ \otimes x], e_- \otimes y) \\ &= (e_+ \otimes (hx), e_- \otimes y) = \omega(hx, y),\end{aligned}$$

where the last identity follows from 4.

Finally, the second equation in (10) follows when applying the Jacobi identity to the elements $e_+ \otimes x$, $e_- \otimes y$ and $e_- \otimes z$. q.e.d.

In general, given a (real or complex) symplectic vector space (V, ω), i.e. $\omega \in \Lambda^2 V^*$ is non-degenerate, we define the *symplectic group* $\mathrm{Sp}(V, \omega)$ and the *symplectic Lie algebra* $\mathfrak{sp}(V, \omega)$ by

$$\mathrm{Sp}(V, \omega) := \{g \in \mathrm{Aut}(V) \mid \omega(gx, gy) = \omega(x, y) \text{ for all } x, y \in V\},$$

$$\mathfrak{sp}(V, \omega) := \{h \in \mathrm{End}(V) \mid \omega(hx, y) + \omega(x, hy) = 0 \text{ for all } x, y \in V\}.$$

Then $\mathrm{Sp}(V, \omega)$ is a Lie group with Lie algebra $\mathfrak{sp}(V, \omega)$.

Definition 2.3. Let (V, ω) be a symplectic vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$ be a subalgebra for which there exists an $\mathfrak{h}$-equivariant map $\circ : S^2(V) \to \mathfrak{h}$ and an $\mathrm{ad}_{\mathfrak{h}}$-invariant inner product $(\ , \ )$ on $\mathfrak{h}$ for which the identities (10) hold. Then we call $\mathfrak{h}$ a *special symplectic subalgebra*. Moreover, we call the connected subgroup $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ with Lie algebra $\mathfrak{h}$ a *special symplectic subgroup*.

Thus, by Proposition 2.2, each (real or complex) 2-gradable simple Lie algebra yields a (real or complex) special symplectic subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$. The converse is also true. Namely, we have

Proposition 2.4. *Let (V, ω) be a symplectic vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$ be a special symplectic subalgebra. Then there exists a unique 2-gradable simple Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, which admits the decompositions (4) and (5), and the Lie bracket of $\mathfrak{g}$ is given by (8).*

Proof. Given the special symplectic Lie algebra $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$, we define the $(\mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h})$-equivariant map $R : \Lambda^2(\mathbb{F}^2 \otimes V) \to \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h}$ by (8) and verify that R satisfies the Jacobi identity by the property of $\circ$.

Thus, R defines a Lie algebra structure on $\mathfrak{g} := \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h} \oplus \mathbb{F}^2 \otimes V$ which makes $(\mathfrak{g}, \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h})$ into a symmetric pair. Choose a basis $e_\pm$ of $\mathbb{F}^2$ with $a(e_+, e_-) = 1$ and let $\mathfrak{g}^0 := \mathbb{F}e_+e_- \oplus \mathfrak{h}$, $\mathfrak{g}^{\pm 1} := e_\pm \otimes V$ and $\mathfrak{g}^{\pm 2} := \mathbb{F}e_\pm^2$. Then $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ follows from the definition of the bracket, so that (4) and (5) hold.

Let $\mathfrak{g}' \subset \mathfrak{g}$ be an ideal. Since e_+e_- is a grading element, it follows that $\mathfrak{g}' = \bigoplus_{i=-2}^{2} (\mathfrak{g}' \cap \mathfrak{g}^i)$. Moreover, $\mathfrak{g}' \cap \mathfrak{sl}(2, \mathbb{F}) \subset \mathfrak{sl}(2, \mathbb{F})$ is an ideal, hence either $\mathfrak{g}' \cap \mathfrak{sl}(2, \mathbb{F}) = 0$ or $\mathfrak{sl}(2, \mathbb{F}) \subset \mathfrak{g}'$.

First, suppose that $\mathfrak{g}' \cap \mathfrak{sl}(2, \mathbb{F}) = 0$ so that $\mathfrak{g}' \cap \mathfrak{g}^{\pm 2} = 0$. If $e_\pm \otimes x \in \mathfrak{g}' \cap \mathfrak{g}^{\pm 1}$, then for all $y \in V$, we have $[e_\pm \otimes x, e_\pm \otimes y] = \omega(x, y) e_\pm^2 \in \mathfrak{g}' \cap \mathfrak{g}^{\pm 2} = 0$

so that $\omega(x,y) = 0$ for all $y \in V$, i.e. $x = 0$, hence $\mathfrak{g}' \cap \mathfrak{g}^{\pm 1} = 0$, whence $\mathfrak{g}' \subset \mathfrak{g}_0$. Next, $[\mathfrak{g}', \mathfrak{g}^{\pm 2}] \subset \mathfrak{g}' \cap [\mathfrak{g}^0, \mathfrak{g}^{\pm 2}] = \mathfrak{g}' \cap \mathfrak{g}^{\pm 2} = 0$, so that $\mathfrak{g}' \subset \mathfrak{h}$. Finally, if $h \in \mathfrak{g}' \subset \mathfrak{h}$, then for all $x \in V$, $[h, e_\pm \otimes x] = e_\pm \otimes (hx) \in \mathfrak{g}' \cap \mathfrak{g}^{\pm 1} = 0$, i.e. $hx = 0$ for all $x \in V$, hence $h = 0$, i.e. $\mathfrak{g}' = 0$.

On the other hand, if $\mathfrak{sl}(2,\mathbb{F}) \subset \mathfrak{g}'$, then $e_+e_- \in \mathfrak{g}'$ so that $\mathfrak{g}^i = [e_+e_-, \mathfrak{g}^i] \subset \mathfrak{g}'$ for all $i \neq 0$. Moreover, $[\mathfrak{g}^1, \mathfrak{g}^{-1}] \subset \mathfrak{g}'$, so that $x \circ y \in \mathfrak{g}'$ for all $x, y \in V$. By the first identity of (10), we have $V \circ V = \mathfrak{h}$, so that $\mathfrak{h} \subset \mathfrak{g}'$ and hence $\mathfrak{g}' = \mathfrak{g}$.

We conclude that $\mathfrak{g}$ is simple, and since $\mathrm{ad}(e_+e_-)$ is diagonalizable, we can choose the Cartan subalgebra $\mathfrak{t}$ such that $e_+e_- \in \mathfrak{t}$. Then $\mathfrak{t} = \mathbb{F}e_+e_- \oplus (\mathfrak{t} \cap \mathfrak{h})$, and hence $[\mathfrak{t}, \mathfrak{g}^{\pm 2}] = \mathfrak{g}^{\pm 2}$, so that $\mathfrak{g}^{\pm 2} = \mathfrak{g}_{\pm \alpha_0}$ are root spaces and $H_{\alpha_0} = -e_+e_-$. Recall that $\mathrm{ad}(H_{\alpha_0})|_{\mathfrak{g}_\beta} = \langle \beta, \alpha_0 \rangle \, Id_{\mathfrak{g}_\beta}$ which implies that $|\langle \beta, \alpha_0 \rangle| \leq 1$ for all roots $\beta \neq \pm\alpha_0$, hence α_0 is a long root. q.e.d.

From this proposition, we obtain a complete classification of special symplectic subalgebras by considering all complex simple Lie algebras and their 2-gradable real forms [**OV**].

Corollary 2.5. *Table 1 on page* 240 *yields the complete list of special symplectic subgroups* $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$.

It is worth pointing out that in the case $\mathfrak{h} = \mathfrak{sp}(V, \omega)$ the map $\circ : S^2(V) \to \mathfrak{h}$ is an isomorphism which is given explicitly by

$$(x \circ y)z = \omega(x,z)y + \omega(y,z)x \quad \text{for all } x, y, z \in V. \tag{11}$$

Namely, by Proposition 2.4 it suffices to show that this product is well defined, $\mathfrak{h}$-equivariant and satisfies (10), and all of this is easily verified.

Definition 2.6. Let $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$ be a special symplectic Lie algebra, and let $\mathfrak{g}$ be the (unique) simple Lie algebra from Proposition 2.4. Then we say that $\mathfrak{h}$ is *associated to* $\mathfrak{g}$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then we say that the special symplectic group $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ is *associated to* G.

Proposition 2.7. *Let* $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$ *be a special symplectic Lie algebra and* $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ *be the corresponding special symplectic Lie subgroup. Then* $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ *is closed and reductive, and*

$$\mathfrak{h} = \{h \in \mathfrak{sp}(V, \omega) \mid [h, x \circ y] = (hx) \circ y + x \circ (hy) \textit{ for all } x, y \in V\}. \tag{12}$$

Moreover, let $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h} \oplus \mathbb{F}^2 \otimes V$ *be the simple Lie algebra from Proposition* 2.4 *and* G *the corresponding simply connected Lie group from Definition* 2.6*. Then the Lie subgroup*

$$\tilde{\mathrm{H}} := \left\{ g \in G \mid \mathrm{Ad}_g|_{\mathfrak{g}^{-2} \oplus \mathfrak{g}^2} = Id_{\mathfrak{g}^{-2} \oplus \mathfrak{g}^2} \right\} \subset G \tag{13}$$

is generated by H *and the center* $Z(\mathrm{G})$*.*

Table 1: Special symplectic subgroups

Notation: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

	Type of Δ	G	H	V
(i)	A_k, $k \geq 2$	$\mathrm{SL}(n+2, \mathbb{F})$, $n \geq 1$	$\mathrm{GL}(n, \mathbb{F})$	$W \oplus W^*$ with $W \cong \mathbb{F}^n$
(ii)		$\mathrm{SU}(p+1, q+1)$, $p+q \geq 1$	$\mathrm{U}(p, q)$	$\mathbb{C}^{p+q}$
(iii)	C_k, $k \geq 2$	$\mathrm{Sp}(n+1, \mathbb{F})$	$\mathrm{Sp}(n, \mathbb{F})$	$\mathbb{F}^{2n}$
(iv)	B_k, D_{k+1}, $k \geq 3$	$\mathrm{SO}(n+4, \mathbb{C})$, $n \geq 3$	$\mathrm{SL}(2, \mathbb{C}) \cdot \mathrm{SO}(n, \mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n$
(v)		$\mathrm{SO}(p+2, q+2)$, $p+q \geq 3$	$\mathrm{SL}(2, \mathbb{R}) \cdot \mathrm{SO}(p, q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p+q}$
(vi)		$\mathrm{SO}(n+2, \mathbb{H})$, $n \geq 2$	$\mathrm{Sp}(1) \cdot \mathrm{SO}(n, \mathbb{H})$	$\mathbb{H}^n$
(vii)	G_2	G_2', $\mathrm{G}_2^{\mathbb{C}}$	$\mathrm{SL}(2, \mathbb{F})$	$S^3(\mathbb{F}^2)$
(viii)	F_4	$\mathrm{F}_4^{(1)}$, $\mathrm{F}_4^{\mathbb{C}}$	$\mathrm{Sp}(3, \mathbb{F})$	$\mathbb{F}^{14} \subset \Lambda^3 \mathbb{F}^6$
(ix)	E_6	$\mathrm{E}_6^{\mathbb{F}}$	$\mathrm{SL}(6, \mathbb{F})$	$\Lambda^3 \mathbb{F}^6$
(x)		$\mathrm{E}_6^{(2)}$	$\mathrm{SU}(1, 5)$	$\mathbb{R}^{20} \subset \Lambda^3 \mathbb{C}^6$
(xi)		$\mathrm{E}_6^{(3)}$	$\mathrm{SU}(3, 3)$	$\mathbb{R}^{20} \subset \Lambda^3 \mathbb{C}^6$
(xii)	E_7	$\mathrm{E}_7^{\mathbb{C}}$	$\mathrm{Spin}(12, \mathbb{C})$	$\Delta^{\mathbb{C}} \cong \mathbb{C}^{32}$
(xiii)		$\mathrm{E}_7^{(5)}$	$\mathrm{Spin}(6, 6)$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xiv)		$\mathrm{E}_7^{(6)}$	$\mathrm{Spin}(6, \mathbb{H})$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xv)		$\mathrm{E}_7^{(7)}$	$\mathrm{Spin}(2, 10)$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xvi)	E_8	$\mathrm{E}_8^{\mathbb{C}}$	$\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{C}^{56}$
(xvii)		$\mathrm{E}_8^{(8)}$	$\mathrm{E}_7^{(5)}$	$\mathbb{R}^{56}$
(xviii)		$\mathrm{E}_8^{(9)}$	$\mathrm{E}_7^{(7)}$	$\mathbb{R}^{56}$

Proof. In principle, we could prove this theorem from Table 1 on page 240, but we prefer to give more conceptual arguments.

Let us suppose that $\mathfrak{h}$ and V are complex. Then, by Proposition 2.4, we can find a complex simple Lie algebra $\mathfrak{g}$ for which (4) holds. Thus, $\mathfrak{g}^0 = \mathfrak{t} \oplus \bigoplus_{\{\beta\in\Delta | \langle\beta,\alpha_0\rangle=0\}} g_\beta$ where Δ is the set of roots of $\mathfrak{g}$. Then $\mathfrak{g}^0$ is evidently reductive, and since $\mathfrak{g}^0 \cong \mathbb{C} \oplus \mathfrak{h}$, it follows that $\mathfrak{h}$ is reductive as well, hence so is every real form of $\mathfrak{h}$. Thus, H is also reductive.

Let $\tilde{\mathfrak{h}}$ denote the right hand side of (12). Then the $\mathfrak{h}$-equivariance of $\circ$ implies that $\mathfrak{h} \subset \tilde{\mathfrak{h}}$. Also, $\mathfrak{h} = V \circ V$ by the first identity of (10) so that $\mathfrak{h}$ is an ideal of $\tilde{\mathfrak{h}}$. Therefore, if $\tilde{h} \in \tilde{\mathfrak{h}}$ then we define

$$\phi : \mathfrak{g} \to \mathfrak{g} \quad \text{by} \quad \phi(\mathfrak{sl}(2,\mathbb{F})) = 0, \quad \phi|_{\mathfrak{h}} = (\mathrm{ad}_{\tilde{h}})|_{\mathfrak{h}}, \quad \phi|_{\mathbb{F}^2\otimes V} := Id_{\mathbb{F}^2} \otimes \tilde{h}.$$

Since $\mathrm{ad}_{\tilde{h}}(\mathfrak{h}) \subset \mathfrak{h}$, this definition makes sense. Moreover, it is now straightforward to verify that ϕ is a derivation of $\mathfrak{g}$, and since $\mathfrak{g}$ is simple, it follows that $\phi = \mathrm{ad}_h$ for some $h \in \mathfrak{g}$. But $\phi(\mathfrak{sl}(2,\mathbb{F})) = 0$, so that $h \in \mathfrak{h}$, hence $e \otimes (hx) = \mathrm{ad}_h(e \otimes x) = \phi(e \otimes x) = e \otimes (\tilde{h}x)$ for all $e \in \mathbb{F}^2$ and $x \in V$, whence $\tilde{h} = h \in \mathfrak{h}$ which shows (12).

Now the subgroup $\{h \in \mathrm{Sp}(V,\omega) \mid \mathrm{Ad}_h(x \circ y) = (hx) \circ (hy)\} \subset \mathrm{Sp}(V,\omega)$ is closed and has $\mathfrak{h}$ as its Lie algebra by (12), thus H is its identity component and hence also closed.

For the last part, note that the Lie algebra of $\tilde{\mathrm{H}}$ equals $\{x \in \mathfrak{g} \mid [x, \mathfrak{g}^{\pm 2}] = 0\} = \mathfrak{h}$. As H is connected, this implies that $\mathrm{H} \subset \tilde{\mathrm{H}}$ is the identity component, and it thus suffices to show that every component of $\tilde{\mathrm{H}}$ contains an element of $Z(\mathrm{G})$.

Let $g \in \tilde{\mathrm{H}}$. Then $\mathfrak{h}$ is Ad_g-invariant, and if we let $\mathfrak{t}_\mathfrak{h} \subset \mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{h}$, so that $\mathfrak{t}_\mathfrak{g} := \mathfrak{t}_\mathfrak{h} \oplus \mathbb{F}e_+e_- \subset \mathfrak{g}^0$ is a Cartan subalgebra of $\mathfrak{g}$, then $\mathrm{Ad}_g(\mathfrak{t}_\mathfrak{h}) \subset \mathfrak{h}$ is another Cartan subalgebra. Since any two Cartan subalgebras are conjugate via an element of H, we may assume w.l.o.g. that $\mathrm{Ad}_g(\mathfrak{t}_\mathfrak{h}) = \mathfrak{t}_\mathfrak{h}$, and since $\mathrm{Ad}_g(e_+e_-) = e_+e_-$, it follows that $\mathrm{Ad}_g \in Norm(\mathfrak{t}_\mathfrak{g})$. Thus, Ad_g yields an inner automorphism of the root system of $\mathfrak{g}$ which stabilizes the root α_0, so that the restriction $(\mathrm{Ad}_g)|_{\mathfrak{t}_\mathfrak{h}}$ is an inner automorphism of the root system of $\mathfrak{h}$, hence after multiplying g by an element of $Norm(\mathfrak{t}_\mathfrak{h}) \subset \mathrm{H}$, we may assume that $(\mathrm{Ad}_g)|_{\mathfrak{t}_\mathfrak{g}} = Id_{\mathfrak{t}_\mathfrak{g}}$, so that $g \in \mathrm{T} = \exp(\mathfrak{t}_\mathfrak{g}) = \exp(\mathbb{F}e_+e_-)\exp(\mathfrak{t}_\mathfrak{h})$. Since $\exp(\mathfrak{t}_\mathfrak{h}) \subset \mathrm{H}$, we may further assume that $g = \exp(te_+e_-)$ for some $t \in \mathbb{F}$, hence $\mathrm{Ad}_g|_{\mathfrak{g}^i} = c^i Id_{\mathfrak{g}^i}$ with $c := \exp(-t)$. But $g \in \tilde{\mathrm{H}}$, so that we must have $c = \pm 1$.

If $c = 1$ then $\mathrm{Ad}_g = Id$, i.e. $g \in Z(\mathrm{G})$, so that we are done.

If $c = -1$ then $\mathbb{F} = \mathbb{C}$ and $\mathrm{Ad}_g|_{\mathfrak{g}^{\pm 1}} = -Id_{\mathfrak{g}^{\pm 1}}$, hence we are done if we can show that $-Id_V \in \mathrm{H}$, since then $g \cdot (-Id_V) \in Z(\mathrm{G})$.

If $\mathrm{H} = \mathrm{Sp}(V,\omega)$, then this is certainly the case, and if $\mathrm{H} \subsetneq \mathrm{Sp}(V,\omega)$ is a proper subgroup, then we shall see in Lemma 2.13, 5. that there is an $h \in \mathfrak{t}_\mathfrak{h}$ such that $\lambda(h)$ is an odd integer for all weights λ of V, hence $\exp(\sqrt{-1}\pi h) = -Id_V \in \mathrm{H}$. q.e.d.

In general, for a given Lie subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$ we define the space of *formal curvature maps* as

$$K(\mathfrak{h}) := \left\{ R \in \Lambda^2 V^* \otimes \mathfrak{h} \;\middle|\; \begin{array}{c} R(x,y)z + R(y,z)x + R(z,x)y = 0 \\ \text{for all } x,y,z \in V \end{array} \right\}.$$

This terminology is due to the fact that the curvature map of a torsion free connection always satisfies the first Bianchi identity, i.e. is contained in $K(\mathfrak{h})$ for an appropriate $\mathfrak{h}$. $K(\mathfrak{h})$ is an H-module in an obvious way.

There is a map $Ric : K(\mathfrak{h}) \to V^* \otimes V^*$, given by $Ric(R)(x,y) := tr(R(x,\_)y)$ for all $R \in K(\mathfrak{h})$ and $x, y \in V$. Note that $Ric(R)(x,y) - Ric(R)(y,x) = trR(x,y)$. Thus, if $\mathfrak{h} \subset \mathfrak{sl}(n,\mathbb{F})$, then $Ric(R) \in S^2(V^*)$.

Proposition 2.8. *Let $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ be a special symplectic subalgebra. Then there is an* H*-equivariant injective map $\mathfrak{h} \to K(\mathfrak{h})$, given by*

$$(14) \quad h \longmapsto R_h, \quad \textit{where} \quad R_h(x,y) := 2\,\omega(x,y)h + x \circ (hy) - y \circ (hx).$$

In fact, $Ric(R_h) = 0$ iff $h = 0$.

Proof. The fact that $R_h \in K(\mathfrak{h})$ follows immediately from (10), and the H-equivariance is evident. The injectivity will follow from the last statement. We begin with the following two lemmata.

Lemma 2.9. *Let $\mathfrak{g}$ be a (real or complex) semi-simple Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be simple. Then there is a $c \in (0,1]$ such that for the Killing forms of $\mathfrak{g}$ and $\mathfrak{h}$, the relation*

$$B_{\mathfrak{h}} = c(B_{\mathfrak{g}})|_{\mathfrak{h}}$$

holds. Moreover, $c = 1$ iff $\mathfrak{h} \lhd \mathfrak{g}$.

Proof. Since $\mathfrak{h}$ is simple and both $B_{\mathfrak{h}}$ and $(B_{\mathfrak{g}})|_{\mathfrak{h}}$ are $ad_{\mathfrak{h}}$-invariant, Schur's Lemma implies that this relation holds for some $0 \neq c \in \mathbb{F}$. Note that c remains unchanged if we replace $\mathfrak{h}$ and $\mathfrak{g}$ by their complexification or a real form. Thus, it suffices to show that $c \in (0,1]$ for compact Lie algebras $\mathfrak{h} \subset \mathfrak{g}$, i.e., for $B_{\mathfrak{h}}, B_{\mathfrak{g}} < 0$. For $0 \neq x \in \mathfrak{h}$, we have

$$B_{\mathfrak{g}}(x,x) = tr(ad_x^2) = B_{\mathfrak{h}}(x,x) + tr(ad_x^2|_{\mathfrak{h}^\perp}).$$

Since ad_x is skew symmetric w.r.t. the (positive definite) inner product $-B_{\mathfrak{g}}$, it follows that $ad_x^2|_{\mathfrak{h}^\perp}$ is negative semidefinite, so that $tr(ad_x^2|_{\mathfrak{h}^\perp}) \leq 0$ with equality iff $ad_x|_{\mathfrak{h}^\perp} = 0$, which implies the claim. q.e.d.

Lemma 2.10. *Let $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ be a symplectic subalgebra. Then $Ric(R)(x,y) = -\omega(R(\omega^{-1})x,y)$. In particular, $Ric(R) \in \mathfrak{h} \subset \mathfrak{sp}(V) \cong S^2(V)$.*

Proof. Let (e_i, f_i) be a basis of V such that, using the summation convention, $\omega^{-1} = e_i \wedge f_i$. Thus,

$$\begin{aligned} Ric(R)(x,y) &= tr(R(x,\_)y) = \omega(R(x,e_i)y,f_i) - \omega(R(x,f_i)y,e_i) \\ &= \omega(R(x,e_i)f_i,y) + \omega(R(f_i,x)e_i,y) \\ &= -\omega(R(e_i,f_i)x,y). \end{aligned}$$

q.e.d.

Let us now suppose that $Ric(R_h) = 0$. By the lemma, this is the case iff for all $u \in \mathfrak{h}$ we have

$$
\begin{aligned}
0 &= (R_h(e_i, f_i), u) = 2\omega(e_i, f_i)(h, u) + (e_i \circ (hf_i), u) - (f_i \circ (he_i), u) \\
&= \dim V(h, u) + \omega((uhf_i), e_i) - \omega((uhe_i), f_i) \\
&= \dim V(h, u) - tr_V(uh) \\
&= \dim V(h, u) - \frac{1}{2}(B(h, u) - B_{\mathfrak{h}}(h, u)) \\
&= \dim V(h, u) - \frac{1}{2}(-2(\dim V + 4)(h, u) - B_{\mathfrak{h}}(h, u)) \\
&= 2(\dim V + 2)(h, u) + \frac{1}{2}B_{\mathfrak{h}}(h, u).
\end{aligned}
$$

Here, we use repeatedly the identities from Proposition 2.2. Let

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$$

be the decomposition of $\mathfrak{h}$ with $\mathfrak{h}_0 := \mathfrak{z}(\mathfrak{h})$ and $\mathfrak{h}_r$ simple for $r \geq 1$. By simplicity of $\mathfrak{h}_r$ and by Lemma 2.9, there are constants $c_r \in [0, 1]$ such that $B_{\mathfrak{h}_r} = c_r B|_{\mathfrak{h}_r}$, where $c_0 = 0$ and $c_r \in (0, 1]$ for $r > 0$. Thus, if we decompose $h = h_0 + \ldots + h_k$ with $h_r \in \mathfrak{h}_r$, then $R_h = 0$ iff for all $u_r \in \mathfrak{h}_r$ we have

$$
\begin{aligned}
0 &= 2(\dim V + 2)(h_r, u_r) + \frac{1}{2}B_{\mathfrak{h}_r}(h_r, u_r) \\
&= 2(\dim V + 2)(h_r, u_r) + \frac{1}{2}c_r B(h_r, u_r) \\
&= (2(\dim V + 2) - c_r(\dim V + 4))(h_r, u_r),
\end{aligned}
$$

using again Proposition 2.2. But since $c_r \leq 1$ by Lemma 2.9, it follows that $2(\dim V + 2) - c_r(\dim V + 4) \geq \dim V > 0$, so that we must have $h_r = 0$ for all r which completes the proof. q.e.d.

For a special symplectic subalgebra $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$, we can now decompose its curvature space as an $\mathfrak{h}$-module into

$$K(\mathfrak{h}) = \mathcal{R}_{\mathfrak{h}} \oplus \mathcal{W}_{\mathfrak{h}}, \quad \text{where} \quad \mathcal{R}_{\mathfrak{h}} = \{R_h \mid h \in \mathfrak{h}\}. \tag{15}$$

By Proposition 2.8 and Lemma 2.10, it follows that $\mathcal{R}_{\mathfrak{h}} \cong \mathfrak{h}$ as an H-module and $\mathcal{W}_{\mathfrak{h}}$ is the kernel of the map $Ric : K(\mathfrak{h}) \to \mathfrak{h} \subset \mathfrak{sp}(V, \omega) \cong S^2(V^*)$, i.e. $\mathcal{W}_{\mathfrak{h}}$ consists of all *Ricci flat curvature maps.*

In fact, the curvature spaces $K(\mathfrak{h})$ have been calculated. Summarizing, we have the following

Theorem 2.11. *Let* $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ *be a special symplectic subgroup with Lie algebra* $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$ *listed in Table 1 on page* 240. *Then*

1) *For the representations corresponding to* (i) *and* (ii)*, we have* $\mathcal{W}_{\mathfrak{h}} = 0$ *if* $n = 1$ *(*$p + q = 1$*, respectively) and* $\mathcal{W}_{\mathfrak{h}} \neq 0$ *if* $n \geq 2$ *(*$p + q \geq 2$*, respectively).*
2) *For the representations corresponding to* (iii)*, we have* $\mathcal{W}_{\mathfrak{h}} = 0$ *for* $n = 1$ *whereas* $\mathcal{W}_{\mathfrak{h}} \neq 0$ *for* $n \geq 2$.
3) *For the representations corresponding to entries* (iv) – $(xviii)$*, we have* $K(\mathfrak{h}) = \mathcal{R}_{\mathfrak{h}}$ *and hence* $\mathcal{W}_{\mathfrak{h}} = 0$.

Proof. First of all, note that since $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{R}} \otimes \mathbb{C}$ and $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$, we also have $K(\mathfrak{h}_{\mathbb{C}}) = K(\mathfrak{h}_{\mathbb{R}}) \otimes \mathbb{C}$ and $\mathcal{R}_{\mathfrak{h}_{\mathbb{C}}} = \mathcal{R}_{\mathfrak{h}_{\mathbb{R}}} \otimes \mathbb{C}$ by complexification. Thus, it suffices to show the claim for the complex representations.

Therefore, to show the first part, it suffices to show that in case (i), $K(\mathfrak{h}) \cong S^2(W) \otimes S^2(W^*)$ as an $\mathfrak{h}$-module, so that the assertion follows by a dimension count. To see this, let $x, y \in W$ and $\overline{z}, \overline{w} \in W^*$. Then for any $R \in K(\mathfrak{h})$ we have $R(\overline{z}, x)y - R(\overline{z}, y)x = -R(x, y)\overline{z}$, and since the left hand side lies in W while the right hand side lies in W^*, it follows that both sides vanish.

The vanishing of the right hand side implies that $R(W, W) = 0$ since $x, y \in W$ and $\overline{z} \in W^*$ are arbitrary. Analogously, $R(W^*, W^*) = 0$. Moreover, the vanishing of the left hand side implies that $R(\overline{z}, x)y = R(\overline{z}, y)x$ and, analogously, $R(x, \overline{z})\overline{w} = R(x, \overline{w})\overline{z}$. Thus, if we define the tensor $\sigma_R \in W \otimes W \otimes W^* \otimes W^*$ by

$$\sigma_R(x, y, \overline{z}, \overline{w}) := \overline{w}(R(\overline{z}, x)y) = -(R(\overline{z}, x)\overline{w})y \tag{16}$$

for all $x, y \in W$ and $\overline{z}, \overline{w} \in W^*$, then σ_R is symmetric in x and y and in $\overline{z}$ and $\overline{w}$, i.e. $\sigma_R \in S^2(W) \otimes S^2(W^*)$.

Conversely, given $\sigma \in S^2(W) \otimes S^2(W^*)$, one verifies that the map $R_\sigma : \Lambda^2(V) \to \mathfrak{h}$ determined by $R(W, W) = R(W^*, W^*) = 0$ and (16) lies in $K(\mathfrak{h})$, showing the above equivalence.

For the second part, consider the Koszul exact sequence $\ldots \to \Lambda^k V^* \otimes S^l(V^*) \to \Lambda^{k+1} \otimes S^{l-1}(V^*) \to \ldots$ where the maps are given by skew symmetrization. One observes that under the identification $\mathfrak{sp}(V, \omega) \cong S^2(V^*)$ we may regard $K(\mathfrak{sp}(V))$ as the kernel of the map $\Lambda^2 V^* \otimes S^2 V^* \to \Lambda^3 V^* \otimes V^*$, hence $K(\mathfrak{sp}(V)) \cong (V^* \otimes S^3(V^*))/S^4(V^*)$, so that the statement follows by a dimension count (cf. [**BC1**]). The last part was shown in [**MS**]. q.e.d.

Now the *second Bianchi identity* of the covariant derivative of a torsion free connection motivates the following definition. We define the space of *covariant* $\mathcal{R}$*-derivations* by

$$\mathcal{R}_{\mathfrak{h}}^{(1)} := \left\{ \psi \in V^* \otimes \mathcal{R}_{\mathfrak{h}} \;\middle|\; \begin{array}{r} \psi(x)(y, z) + \psi(y)(z, x) \\ +\psi(z)(x, y) = 0 \\ \text{for all } x, y, z \in V \end{array} \right\}. \tag{17}$$

Again, $\mathcal{R}_{\mathfrak{h}}^{(1)}$ is an H-module in an obvious way.

Proposition 2.12. *Let $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ be a special symplectic subalgebra other than the subalgebra $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{F})$, $V = \mathbb{F}^2$. Then as an $\mathfrak{h}$-module, $\mathcal{R}_{\mathfrak{h}}^{(1)} \cong V$ with an explicit isomorphism given by*

$$u \longmapsto \psi_u, \quad \textit{where} \quad \psi_u(x) := R_{u \circ x} \in \mathcal{R}_{\mathfrak{h}} \quad \textit{for all } u, x \in V.$$

Proof. As in the proof of Theorem 2.11, it suffices to show the proposition in the complex case by complexifying $\mathfrak{h}$ and V.

Using (10), it is straightforward to verify that $\psi_u \in \mathcal{R}_{\mathfrak{h}}^{(1)}$ for all $u \in V$. Also, $\psi_u = 0$ iff $R_{u \circ V} = 0$ iff $u \circ V = 0$ by Proposition 2.8. But, again by (10), $u \circ V = 0$ iff $u = 0$, so that $\{\psi_u \mid u \in V\} \subset \mathcal{R}_{\mathfrak{h}}^{(1)}$ is isomorphic to V as an H-module.

If $\mathfrak{h} = \mathfrak{sp}(V,\omega)$ then $\circ : S^2(V) \to \mathfrak{h}$ is given in (11), and from there the statement follows for $\dim V > 2$ by a direct calculation [**BC1**]. On the other hand, if $\dim V = 2$ then evidently, $\mathcal{R}_{\mathfrak{h}}^{(1)} = V \otimes \mathfrak{h}$, and $\dim \mathfrak{h} \in \{1, 3\}$ as $\mathfrak{h} \subset \mathfrak{sl}(2,\mathbb{C})$. Thus, by a dimension count the statement follows if $\dim \mathfrak{h} = 1$ while it fails if $\dim \mathfrak{h} = 3$, i.e. if $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{C})$ and $V = \mathbb{C}^2$.

Thus, the major part of the proof is to show that the inclusion $\{\psi_u \mid u \in V\} \subset \mathcal{R}_{\mathfrak{h}}^{(1)}$ is an equality if $\mathfrak{h} \subsetneq \mathfrak{sp}(V,\omega)$ and $\dim V > 2$. For this, we begin with the following

Lemma 2.13. *(cf.* [**S2**]*) Let $\mathfrak{h} \subsetneq \mathfrak{sp}(V,\omega)$ be a special symplectic proper subalgebra, where $\mathfrak{h}$ and V are complex and* $\dim V > 2$. *Let $\mathfrak{t}_{\mathfrak{h}} \subset \mathfrak{h}$ be a Cartan subalgebra and $\Delta_{\mathfrak{h}}$ be the set of roots of $\mathfrak{h}$. Consider the decomposition $V = \bigoplus_{\lambda \in \Phi} V_\lambda$ where $\Phi \subset \mathfrak{t}_{\mathfrak{h}}^*$ is the set of weights. Then the following holds:*

1) *All weight spaces V_λ are one dimensional, and if $\lambda \in \Phi$ then $-\lambda \in \Phi$.*
2) *There are at most two possible length for the weights which allows to refer to* long *and* short *weights.*
3) *If $\lambda_0 \in \Phi$ is a long weight, then there is a disjoint decomposition*
$$\Phi = \Phi_{-3} \cup \Phi_{-1} \cup \Phi_1 \cup \Phi_3,$$
where $\Phi_{\pm 3} = \{\pm\lambda_0\}$ and $\Phi_{\pm 1} = \{\mu \in \Phi \mid \pm\lambda_0 - \mu \in \Delta_{\mathfrak{h}}\}$.
4) *Let $V_{\frac{i}{2}} := \bigoplus_{\lambda \in \Phi_i} V_\lambda$ for $i \in \{\pm 1, \pm 3\}$. Then there are decompositions*
$$\mathfrak{h} = \mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_1, \qquad V = V_{-\frac{3}{2}} \oplus V_{-\frac{1}{2}} \oplus V_{\frac{1}{2}} \oplus V_{\frac{3}{2}} \qquad \textit{with}$$
$$[\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_{i+j}, \qquad \mathfrak{h}_i V_r \subset V_{i+r},$$
$$V_r \circ V_s \subset \mathfrak{h}_{r+s}, \qquad \mathfrak{h}_i = \bigoplus\nolimits_{r+s=i} V_r \circ V_s.$$
5) *Let $v_\pm \in V_{\pm\frac{3}{2}}$, $w_r \in V_r$ and $h_i \in \mathfrak{h}_i$. Then $(v_+ \circ v_-)w_r = -2r\ \omega(v_+, v_-)w_r$ and $[v_+ \circ v_-, h_i] = -2i\ \omega(v_+, v_-)h_i$.*

Proof. Let $\mathfrak{g}$ be the simple Lie algebra associated to $\mathfrak{h}$ by Proposition 2.4, and let Δ be the root system of $\mathfrak{g}$. Note that $\mathfrak{t}_{\mathfrak{h}} = \mathfrak{t} \cap (H_{\alpha_0})^\perp$ where $\mathfrak{t}$ is the Cartan subalgebra of $\mathfrak{g}$. Moreover, $\Delta_{\mathfrak{h}} = \{\beta \in \Delta \mid \langle \beta, \alpha_0 \rangle = 0\} \subset \Delta$, and $V \cong \mathfrak{g}^1 = \bigoplus_{\{\beta \in \Delta \mid \langle \beta, \alpha_0 \rangle = 1\}} \mathfrak{g}_\beta$ as an $\mathfrak{h}$-module. It follows that

$$\Phi = \left\{ \lambda = \beta - \frac{1}{2}\alpha_0 \,\middle|\, \beta \in \Delta, \langle \beta, \alpha_0 \rangle = 1 \right\} \quad \text{and} \quad V_\lambda = \mathfrak{g}_\beta.$$

Thus, $\dim V_\lambda = 1$ as all root spaces are one dimensional. Moreover, if $\langle \beta, \alpha_0 \rangle = 1$, then $\gamma := \alpha_0 - \beta \in \Delta$ and $\langle \gamma, \alpha_0 \rangle = 1$, whence $-\lambda = -(\beta - \frac{1}{2}\alpha_0) = \gamma - \frac{1}{2}\alpha_0 \in \Phi$.

Next, $(\lambda, \lambda) = (\beta - \frac{1}{2}\alpha_0, \beta - \frac{1}{2}\alpha_0) = (\beta, \beta) - (\beta, \alpha_0) + \frac{1}{4}(\alpha_0, \alpha_0) = (\beta, \beta) - \frac{1}{4}(\alpha_0, \alpha_0)$ since $1 = \langle \beta, \alpha_0 \rangle = 2(\beta, \alpha_0)/(\alpha_0, \alpha_0)$ by (2). Thus, $(\lambda, \lambda) > 0$ is determined by (β, β), and for the latter there are at most two possible values.

To show the third property, pick a long weight $\lambda_0 \in \Phi$, i.e. $\lambda_0 = \beta_0 - \frac{1}{2}\alpha_0$ for some long root $\beta_0 \in \Delta$ with $\langle \beta_0, \alpha_0 \rangle = 1$. Since our hypothesis implies that Δ is not of type C_k, such a β_0 and hence such a λ_0 exists.

Let $\gamma \in \Delta$ with $\langle \gamma, \alpha_0 \rangle = 1$, and let $\mu := \gamma - \frac{1}{2}\alpha_0 \in \Phi$. Then $\gamma \neq -\beta_0$ so that $\langle \gamma, \beta_0 \rangle \in \{-1, 0, 1, 2\}$, and $\langle \gamma, \beta_0 \rangle = 2$ iff $\gamma = \beta_0$ iff $\mu = \lambda_0$.

If $\langle \gamma, \beta_0 \rangle = 1$ then $\beta_0 - \gamma \in \Delta$ with $\langle \beta_0 - \gamma, \alpha_0 \rangle = 0$, so that $\lambda_0 - \mu = \beta_0 - \gamma \in \Delta_{\mathfrak{h}}$.

If $\langle \gamma, \beta_0 \rangle \in \{0, -1\}$ then $\langle \gamma, \alpha_0 - \beta_0 \rangle = 1 - \langle \gamma, \beta_0 \rangle \in \{1, 2\}$, thus when replacing λ_0 by $-\lambda_0$ and hence β_0 by $\alpha_0 - \beta_0$, then we can reduce to the previous cases.

From this description, it also follows that $\Phi_i = \{\mu \in \Phi \mid \langle \mu, \beta_0 \rangle = \frac{i}{2}\}$

To show the fourth part, let $\Delta_{\mathfrak{h}}^i := \{\gamma \in \Delta_{\mathfrak{h}} \mid \langle \gamma, \beta_0 \rangle = i\}$. Since $\pm\beta_0 \notin \Delta_{\mathfrak{h}}$, it follows that $\Delta_{\mathfrak{h}} = \Delta_{\mathfrak{h}}^{-1} \cup \Delta_{\mathfrak{h}}^0 \cup \Delta_{\mathfrak{h}}^1$, and we let $\mathfrak{h}_{\pm 1} := \bigoplus_{\gamma \in \Delta_{\mathfrak{h}}^{\pm 1}} \mathfrak{g}_\gamma$ and $\mathfrak{h}_0 := \mathfrak{t}_{\mathfrak{h}} \oplus \bigoplus_{\gamma \in \Delta_{\mathfrak{h}}^0} \mathfrak{g}_\gamma$. Since $\Phi_i = \{\mu \in \Phi \mid \langle \mu, \beta_0 \rangle = \frac{i}{2}\}$, the claims follow.

Finally, for the last part, note that by (10),

$$(v_+ \circ v_-) w_r = (v_+ \circ w_r) v_- + 2\omega(v_-, w_r) v_+ + \omega(v_+, w_r) v_- - \omega(v_+, v_-) w_r.$$

Now if $r > 0$ then $v_+ \circ w_r \in \mathfrak{h}_{\frac{3}{2}+r} = 0$ and $\omega(v_+, w_r) = 0$. Also, $\omega(v_-, w_r) = 0$ for $r = 1/2$ showing the claim in this case, whereas for $r = 3/2$, w_r is a scalar multiple of v_+ so that $\omega(v_-, w_r) v_+ = \omega(v_-, v_+) w_r$ which implies the assertion in this case as well. The proof of the cases $r < 0$ follows analogously.

Note that then for $w_r \in V_r, w_s \in V_s$ we also have $[v_+ \circ v_-, w_r \circ w_s] = ((v_+ \circ v_-) w_r) \circ w_s + w_r \circ ((v_+ \circ v_-) w_s) = -2(r+s)\omega(v_+, v_-) w_r \circ w_s$, and the last assertion follows. q.e.d.

Let us now suppose that $\mathfrak{h} \subsetneq \mathfrak{sp}(V,\omega)$ and $\dim V > 2$, so that we have the decompositions from the lemma. Let $\psi \in \mathcal{R}^{(1)}_{\mathfrak{h}}$ be a weight element of weight $\mu \in \Phi$. Choose a long weight $\lambda_0 \in \Phi$, $\lambda_0 \neq \pm\mu$ so that – after replacing λ_0 by its negative if necessary – we may assume that $\mu \in \Phi_1$. Whence, $\psi(V_\lambda) \in \mathfrak{g}_{\lambda+\mu}$ implies that $\psi(V_r) \subset \mathfrak{h}_{r+\frac{1}{2}}$ and, in particular, $\psi(V_{\frac{3}{2}}) = 0$.

Note that $\mathfrak{g}_{-\lambda_0+\mu} = V_\mu \circ V_{-\lambda_0}$; namely, $\langle -\lambda_0, \lambda_0 - \mu\rangle < 0$ so that $\mathfrak{g}_{\lambda_0-\mu} V_{-\lambda_0} = V_{-\mu}$ as all weight spaces are one dimensional. Thus, $(\mathfrak{g}_{\lambda_0-\mu}, V_\mu \circ V_{-\lambda_0}) = \omega(\mathfrak{g}_{\lambda_0-\mu}V_{-\lambda_0}, V_\mu) = \omega(V_{-\mu}, V_\mu) \neq 0$ so that $0 \neq V_\mu \circ V_{-\lambda_0} \subset \mathfrak{g}_{-\lambda_0+\mu}$ and the latter is one dimensional.

Pick $0 \neq v_{-\lambda_0} \in V_{-\lambda_0}$. Since $\psi(v_{-\lambda_0}) \in \mathfrak{g}_{-\lambda_0+\mu}$, there is a $u \in V_\mu$ such that $\psi(v_{-\lambda_0}) = u \circ v_{-\lambda_0}$. Therefore, after replacing ψ by $\psi - \psi_u$, we may assume that $\psi(v_{-\lambda_0}) = 0$ and hence $\psi(V_{\pm\frac{3}{2}}) = 0$.

If we let $v_\pm \in V_{\pm\frac{3}{2}}$ with $\omega(v_+, v_-) \neq 0$ and $w_\pm \in V_{\pm\frac{1}{2}}$ then by (17) we must have

$$
\begin{aligned}
0 &= R_{\psi(w_\pm)}(v_+, v_-) \\
&= 2\omega(v_+, v_-)\psi(w_\pm) + v_+ \circ (\psi(w_\pm)v_-) - v_- \circ (\psi(w_\pm)v_+).
\end{aligned} \tag{18}
$$

Now $\psi(w_+) \in \mathfrak{h}_1$, hence $\psi(w_+)v_+ = 0$ and thus $v_+ \circ (\psi(w_+)v_-) = [\psi(w_+), v_+ \circ v_-] = 2\omega(v_+, v_-)\psi(w_+)$, where the last identity follows from the lemma. Then (18) implies that $\psi(w_+) = 0$.

On the other hand, $\psi(w_-) \in \mathfrak{h}_0$ so that $\psi(w_-)v_\pm \in V_{\pm\frac{3}{2}}$, hence (18) implies that $\psi(w_-) = c\, v_+ \circ v_-$ for some constant c. But then, $\psi(w_-)v_\pm = \mp 3c\, \omega(v_+, v_-)v_\pm$ by the lemma, and substituting into (18) yields $c = 0$, i.e. $\psi(w_-) = 0$, and hence, $\psi = 0$.

Let $W \subset \mathcal{R}^{(1)}_{\mathfrak{h}}$ be the H-invariant complement of $\{\psi_u \mid u \in V\} \subset \mathcal{R}^{(1)}_{\mathfrak{h}}$, and let Ψ be the set of weights of W. Since $W \subset \mathcal{R}^{(1)}_{\mathfrak{h}} \subset V \otimes K(\mathfrak{h}) \cong V \otimes \mathfrak{h}$, it follows that $\Psi \subset \Phi + \Delta_0$. Also, by what we have shown above, we must have $\Psi \cap \Phi = \emptyset$.

Let $\nu \in \Psi$, and write $\nu = \mu + \alpha$ with $\mu \in \Phi$ and $\alpha \in \Delta_0$. Since $\Psi \cap \Phi = \emptyset$, it follows that that $\alpha \neq 0$, i.e., $\alpha \in \Delta$. If $\langle \mu, \alpha\rangle < 0$, then $\nu = \mu + \alpha \in \Psi \cap \Phi$; if $\langle \nu, \alpha\rangle > 0$ then $\mu = \nu - \alpha \in \Psi \cap \Phi$, both of which are impossible. Thus, $2 = \langle \alpha, \alpha\rangle = \langle \nu, \alpha\rangle - \langle \mu, \alpha\rangle \leq 0$ which is a contradiction.

Thus, we must have $\Psi = \emptyset$ and hence $W = 0$. q.e.d.

Finally, we prove the following result which we shall need later on.

Lemma 2.14. *Let $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ be a special symplectic subalgebra, $\dim V \geq 4$, and let $\phi : V \to V$ be a linear map such that*

$$\phi(x) \circ y = \phi(y) \circ x \quad \text{for all } x, y \in V. \tag{19}$$

Then ϕ is a multiple of the identity.

Proof. By (10) we have

$$(\phi(x)\circ y)z - (\phi(x)\circ z)y = 2\omega(y,z)\phi(x) + \omega(\phi(x),z)y - \omega(\phi(x),y)z.$$

But (19) now implies that the cyclic sum in x, y, z of the left hand side vanishes, hence so does the cyclic sum of the right hand side, i.e.

$$\begin{aligned} & 2\left(\omega(x,y)\phi(z) + \omega(y,z)\phi(x) + \omega(z,x)\phi(y)\right) \\ & = (\omega(\phi(y),z) - \omega(\phi(z),y))x + (\omega(\phi(z),x) - \omega(\phi(x),z))y \\ & \quad +(\omega(\phi(x),y) - \omega(\phi(y),x))z. \end{aligned} \tag{20}$$

For each $x \in V$, we may choose vectors $y, z \in V$ with $\omega(x,y) = \omega(x,z) = 0$ and $\omega(y,z) \neq 0$ since $\dim V \geq 4$. Then (20) implies that $\phi(x) \in$ *span*(x,y,z) so that $\omega(\phi(x),x) = 0$. Polarization then implies that $\omega(\phi(x),y) + \omega(\phi(y),x) = 0$ for all $x, y \in V$.

Next, we take the symplectic form of (20) with x, and together with the preceding identity this yields

$$\omega(x,y)\omega(\phi(x),z) = \omega(x,z)\omega(\phi(x),y) \quad \text{ for all } x,y,z \in V.$$

Thus, $\omega(x,y)\phi(x) = \omega(\phi(x),y)x$ for all $x, y \in V$, and since for $0 \neq x \in V$ we can pick $y \in V$ such that $\omega(x,y) \neq 0$, this implies that $\phi(x)$ is a scalar multiple of x for all $x \in V$, whence ϕ is a multiple of the identity. q.e.d.

Key Definition 2.15. Let (M,ω) be a (real or complex) symplectic manifold of (real or complex) dimension at least 4, equipped with a symplectic connection ∇, i.e. a torsion free connection for which ω is parallel. We say that ∇ is a *special symplectic connection associated to the (simple) Lie group* G if there is a special symplectic subgroup $\mathrm{H} \subset \mathrm{Sp}(V,\omega)$ associated to G in the sense of Definition 2.6 such that the curvature of ∇ is contained in $\mathcal{R}_{\mathfrak{h}}$ (cf. (14) and (15)).

Definition 2.15 coincides with the definition of special symplectic connections from the introduction. Namely, note that by the Ambrose-Singer holonomy theorem, the (restricted) holonomy of a special symplectic connection is evidently contained in $\mathrm{H} \subset \mathrm{Sp}(V,\omega)$, so that we have an H-reduction $B \to M$ of the frame bundle of M which is compatible with the connection.

If $\mathrm{H} \subset \mathrm{Sp}(V,\omega)$ is one of the subgroups (*i*) or (*ii*), then either there are two complementary parallel Lagrangian foliations (case (*i*)), or the connection is the Levi-Civita connection of a pseudo-Kähler metric (case (*ii*)). In either case, the condition that the curvature lies in $\mathcal{R}_{\mathfrak{h}}$ is equivalent to the vanishing of the *Bochner curvature*, and such connections have been called *Bochner-bi-Lagrangian* in the first and *Bochner-Kähler* in the second case. For a detailed study of these connections, see [**Br2**].

If $\mathrm{H} = \mathrm{Sp}(V, \omega)$ as in (iii), then the condition that the curvature lies in $\mathcal{R}_{\mathfrak{h}}$ is equivalent to saying that the connection is a (real or holomorphic) *symplectic connection of Ricci type* in the sense of [**BC1**].

Finally, if $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ is one of the subgroups $(iv) - (xviii)$ in Table 1 on page 240, then, by Theorem 2.11, *any* torsion free connection on such an H-structure must be special. In fact, these subgroups H are precisely the absolutely irreducible proper subgroups of the symplectic group which can occur as the holonomy of a torsion free connection (cf. [**MS**], [**S1**], [**S3**]).

It shall be the aim of the following sections to study special symplectic connections using the general algebraic setup established here rather than dealing with each of the geometric structures separately.

3. Special symplectic connections and contact manifolds

We shall now recall some well known facts about contact manifolds and their symplectic reductions.

Definition 3.1. A *contact structure* on a real (complex, respectively) manifold $\mathcal{C}$ is a smooth (holomorphic, respectively) distribution $\mathcal{D} \subset T\mathcal{C}$ of codimension one such that the Lie bracket induces a non-degenerate map

$$\mathcal{D} \times \mathcal{D} \longrightarrow T\mathcal{C}/\mathcal{D} =: L.$$

The line bundle $L \to \mathcal{C}$ is called the *contact line bundle*, and its dual can be embedded as

$$L^* = \{\lambda \in T^*\mathcal{C} \mid \lambda(\mathcal{D}) = 0\} \subset T^*\mathcal{C}. \tag{21}$$

Notice that we can define the line bundles $L \to \mathcal{C}$ and $L^* \to \mathcal{C}$ for an arbitrary distribution $\mathcal{D} \subset T\mathcal{C}$ of codimension one. It is well known that such a distribution $\mathcal{D}$ yields a contact structure iff the restriction of the canonical symplectic form Ω on $T^*\mathcal{C}$ to $L^*\backslash 0$ is non-degenerate, so that in this case $L^*\backslash 0$ is a symplectic manifold in a canonical way.

We regard $p : L^*\backslash 0 \to \mathcal{C}$ as a principal $(\mathbb{R}\backslash 0)$-bundle ($\mathbb{C}^*$-bundle, respectively). In the real case, we may assume that $L^*\backslash 0$ has two components each of which is a principal $\mathbb{R}^+$-bundle, since this can always be achieved when replacing $\mathcal{C}$ by a double cover if necessary. Thus, we get the principal $\mathbb{R}^+$-bundle ($\mathbb{C}^*$-bundle, respectively)

$$p : \hat{\mathcal{C}} \longrightarrow \mathcal{C},$$

where $\hat{\mathcal{C}} \subset L^*\backslash 0$ is a connected component. The vector field $E_0 \in \mathfrak{X}(\hat{\mathcal{C}})$ which generates the principal action is called *Euler field*, so that the flow along E_0 is fiberwise scalar multiplication in $\hat{\mathcal{C}} \subset L^* \subset T^*\mathcal{C}$. Thus, the *Liouville form* on $T^*\mathcal{C}$ is given as $\lambda := E_0 \lrcorner \Omega$, and hence $\mathfrak{L}_{E_0}(\Omega) = \Omega$ and $\Omega = d\lambda$. This process can be reverted. Namely, we have the following

Proposition 3.2. *Let $p : \hat{\mathcal{C}} \to \mathcal{C}$ be a principal $\mathbb{R}^+$-bundle ($\mathbb{C}^*$-bundle, respectively) with a symplectic form Ω on $\hat{\mathcal{C}}$ such that $\mathfrak{L}_{E_0}\Omega = \Omega$ where $E_0 \in \mathfrak{X}(\hat{\mathcal{C}})$ generates the principal action. Then there is a unique contact structure $\mathcal{D}$ on $\mathcal{C}$ and an equivariant imbedding $\imath : \hat{\mathcal{C}} \hookrightarrow L^*\backslash 0 \subset T^*\mathcal{C}$ with L^* from (21) such that Ω is the pullback of the canonical symplectic form on $T^*\mathcal{C}$ to $\hat{\mathcal{C}}$.*

Proof. By hypothesis, $\Omega = d\lambda$ where $\lambda := (E_0 \lrcorner \Omega)$. Since $\lambda(E_0) = 0$, there is for each $x \in \hat{\mathcal{C}}$ a unique $\underline{\lambda}_x \in T^*_{p(x)}\mathcal{C}$ satisfying $p^*(\underline{\lambda}_x) = \lambda_x$. Moreover, $\mathfrak{L}_{E_0}(\lambda) = \lambda$, hence $\underline{\lambda}_{e^t x} = e^t \underline{\lambda}_x$ for all $t \in \mathbb{F}$, so that the codimension one distribution $\mathcal{D} := dp(\ker(\lambda)) \subset T\mathcal{C}$ is well defined, and the correspondence $x \mapsto \underline{\lambda}_x$ yields an equivariant imbedding $\hat{\mathcal{C}} \hookrightarrow L^*\backslash 0$ whose image is thus a connected component of $L^*\backslash 0$. Moreover, by construction, λ is the restriction of the Liouville form to $\hat{\mathcal{C}} \subset L^*\backslash 0 \subset T^*\mathcal{C}$. Since $\Omega = d\lambda$ is non-degenerate on $\hat{\mathcal{C}}$ by assumption, it follows that $\mathcal{D}$ is a contact structure. q.e.d.

Next, we define the fiber bundle

$$\mathfrak{R} := \left\{ (\lambda, \hat{\xi}) \in \hat{\mathcal{C}} \times T\hat{\mathcal{C}} \subset T^*\mathcal{C} \times T\hat{\mathcal{C}} \mid \lambda(dp(\hat{\xi})) = 1 \right\}.$$

Projection onto the first factor yields a fibration $\mathfrak{R} \to \hat{\mathcal{C}}$ whose fiber is an affine space.

We call a vector field ξ on $\mathcal{C}$ a *contact symmetry* if $\mathfrak{L}_\xi(\mathcal{D}) \subset \mathcal{D}$. This means that the flow along ξ preserves the contact structure $\mathcal{D}$. For each contact symmetry ξ on $\mathcal{C}$, there is a unique vector field $\hat{\xi} \in \mathfrak{X}(\hat{\mathcal{C}})$, called the *Hamiltonian lift of* ξ, satisfying $dp(\hat{\xi}) = \xi$ and $\mathfrak{L}_{\hat{\xi}}\lambda = 0$, so that $\mathfrak{L}_{\hat{\xi}}\Omega = 0$.

We call ξ a *transversal* contact symmetry if in addition $\xi \notin \mathcal{D}$ at all points. Equivalently, we have $\Omega(E_0, \hat{\xi}) \neq 0$ everywhere. In the real case, we say that ξ is *positively transversal* if $\Omega(E_0, \hat{\xi}) > 0$ everywhere, while in the complex case it is convenient to call *any* transversal vector field positively transversal.

Given a positively transversal contact symmetry ξ with Hamiltonian lift $\hat{\xi}$, there is a unique section λ of the bundle $p : \hat{\mathcal{C}} \to \mathcal{C}$ such that $\lambda(\xi) \equiv 1$, and hence we obtain a section of the bundle $\mathfrak{R} \to \hat{\mathcal{C}} \to \mathcal{C}$

$$\sigma_\xi : \mathcal{C} \longrightarrow \mathfrak{R}, \qquad \sigma_\xi := (\lambda, \hat{\xi}) \in \mathfrak{R}. \tag{22}$$

We call an open subset $U \subset \mathcal{C}$ *regular* w.r.t. the transversal contact symmetry ξ if there is a submersion $\pi_U : U \to M_U$ onto some manifold M_U whose fibers are connected lines tangent to ξ. Evidently, since ξ is pointwise non-vanishing, $\mathcal{C}$ can be covered by regular open subsets.

Since ξ is a contact symmetry, it follows that $\xi \lrcorner d\lambda = 0$ and $\mathfrak{L}_\xi\lambda = 0$. Thus, on each M_U there is a unique symplectic form ω such that

$$\pi_U^*\omega = -2d\lambda, \tag{23}$$

where the factor -2 only occurs to make this form coincide with one we shall construct later on.

To link all of this to our situation, let $\mathfrak{g}$ be a 2-gradable simple real or complex Lie algebra and let G be the corresponding connected Lie group with trivial center $Z(\mathrm{G}) = \{1\}$. Recall the decomposition

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{1} \oplus \mathfrak{g}^{2} \cong \mathbb{F}e_-^2 \oplus (e_- \otimes V) \oplus (\mathbb{F}e_+e_- \oplus \mathfrak{h}) \oplus (e_+ \otimes V) \oplus \mathbb{F}e_+^2$$

from (4). We let $\mu := g^{-1}dg$ be the left invariant Maurer-Cartan form on G, which we can decompose as

$$\mu = \sum_{i=-2}^{2} \mu_i, \qquad \mu_0 = \mu_{\mathfrak{h}} + \nu_0 e_+ e_- \tag{24}$$

where $\mu_i \in \Omega^1(\mathrm{G}) \otimes \mathfrak{g}^i$, $\mu_{\mathfrak{h}} \in \Omega^1(\mathrm{G}) \otimes \mathfrak{h}$ and $\nu_0 \in \Omega^1(\mathrm{G})$. Furthermore, we define the subalgebras

$$\mathfrak{p} := \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2, \qquad \text{and} \qquad \mathfrak{p}_0 := \mathfrak{h} \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2,$$

and we let $\mathrm{P}, \mathrm{P}_0 \subset \mathrm{G}$ be the corresponding connected subgroups. Using the bilinear form $(\ ,\ )$ from (9), we identify $\mathfrak{g}$ and $\mathfrak{g}^*$, and recall the root cone from (3) and its (oriented) projectivization

$$\hat{\mathcal{C}} := \mathrm{G} \cdot e_+^2 \subset \mathfrak{g} \cong \mathfrak{g}^*, \qquad \mathcal{C} := p(\hat{\mathcal{C}}) \subset \mathbb{P}^o(\mathfrak{g}) \cong \mathbb{P}^o(\mathfrak{g}^*), \tag{25}$$

where $\mathbb{P}^o(\mathfrak{g})$ is the set of *oriented* lines in $\mathfrak{g}$, i.e. $\mathbb{P}^o \cong S^d$ if $\mathbb{F} = \mathbb{R}$, and $\mathbb{P}^o \cong \mathbb{CP}^d$ if $\mathbb{F} = \mathbb{C}$, where $d = \dim \mathfrak{g} - 1$, and where $p : \mathfrak{g}\backslash 0 \to \mathbb{P}^o(\mathfrak{g})$ is the principal $\mathbb{R}^+$-bundle ($\mathbb{C}^*$-bundle, respectively) defined by the canonical projection. Thus, the restriction $p : \hat{\mathcal{C}} \to \mathcal{C}$ is a principal bundle as well.

Being a coadjoint orbit, $\hat{\mathcal{C}}$ carries a canonical G-invariant symplectic structure Ω. Moreover, the *Euler vector field* defined by

$$E_0 \in \mathfrak{X}(\hat{\mathcal{C}}), \qquad (E_0)_v := v$$

generates the principal action of p and satisfies $\mathfrak{L}_{E_0}(\Omega) = \Omega$, so that the distribution $\mathcal{D} = dp(E_0^{\perp_\Omega}) \subset T\mathcal{C}$ yields a G-invariant contact distribution on $\mathcal{C}$ by Proposition 3.2.

Lemma 3.3. *As homogeneous spaces, we have $\mathcal{C} = \mathrm{G}/\mathrm{P}$, $\hat{\mathcal{C}} = \mathrm{G}/\mathrm{P}_0$ and $\mathfrak{R} = \mathrm{G}/\mathrm{H}$. Moreover, the fiber bundles $\mathfrak{R} \to \hat{\mathcal{C}} \to \mathcal{C}$ from before are equivalent to the corresponding homogeneous fibrations.*

Proof. Using the pairing $(\ ,\ )$ to identify $\mathfrak{g}$ and $\mathfrak{g}^*$, it follows that the fiber of $\mathfrak{R}$ over $e_+^2 \in \hat{\mathcal{C}}$ can be identified with

$$\mathfrak{R}_{e_+^2} = \left\{ \frac{1}{2} e_-^2 + e_- \otimes v + t e_+ e_- + \mathfrak{p}_0 \mid v \in V, t \in \mathbb{F} \right\} \subset \mathfrak{g}/\mathfrak{p}_0 \cong T_{e_+^2}\hat{\mathcal{C}}.$$

Now it is straightforward to verify that $\mathrm{P}_0 = \exp(\mathfrak{p}_0)$ acts transitively on this set. Moreover, for all $p_0 \in \mathfrak{p}_0$ one calculates that $(\mathrm{ad}(\frac{1}{2}e_-^2 + p_0))^2(e_+^2) \in \mathbb{F}(\frac{1}{2}e_-^2 + p_0)$ iff $p_0 = 0$. Since $(\mathrm{ad}_x)^2(\mathfrak{g}) \subset \mathbb{F}x$ for all $x \in \hat{\mathcal{C}}$, it follows that $(\frac{1}{2}e_-^2 + \mathfrak{p}_0) \cap \hat{\mathcal{C}} = \frac{1}{2}e_-^2$, and hence each of the cosets $\{\frac{1}{2}e_-^2 + e_- \otimes v + te_+e_- + \mathfrak{p}_0\} \in \mathfrak{g}/\mathfrak{p}_0$ has a unique representative in $\hat{\mathcal{C}}$.

From all of this it now follows that G acts transitively on $\mathfrak{R}$, and the stabilizer of the pair $(e_+^2, \frac{1}{2}e_-^2 + \mathfrak{p}_0)$ equals the stabilizer of the pair $(e_+^2, \frac{1}{2}e_-^2)$ which is H by Proposition 2.7 as $Z(\mathrm{G}) = \{1\}$. Thus, $\mathfrak{R} = \mathrm{G}/\mathrm{H}$ as claimed.

The fibers of the homogeneous fibrations $\mathfrak{R} \to \hat{\mathcal{C}}$ and $\mathfrak{R} \to \mathcal{C}$ are connected, and since $\mathfrak{R} = \mathrm{G}/\mathrm{H}$ and H is connected, it follows that the stabilizers of $e_+^2 \in \hat{\mathcal{C}}$ and $[e_+^2] \in \mathcal{C}$ are connected as well. Since the Lie algebras of these stabilizers are evidently $\mathfrak{p}_0$ and $\mathfrak{p}$, respectively, the claim follows. q.e.d.

For each $a \in \mathfrak{g}$ we define the vector fields $a^* \in \mathfrak{X}(\mathcal{C})$ and $\hat{a}^* \in \mathfrak{X}(\hat{\mathcal{C}})$ corresponding to the infinitesimal action of a, i.e.

$$(26)\quad (a^*)_{[v]} := \left.\frac{d}{dt}\right|_{t=0} (\exp(ta)\cdot[v]) \quad \text{and} \quad (\hat{a}^*)_v := \left.\frac{d}{dt}\right|_{t=0} (\exp(ta)\cdot v).$$

Note that a^* is a contact symmetry and $\hat{a}^*$ is its Hamiltonian lift. Let

$$(27)\quad \begin{aligned} &\hat{\mathcal{C}}_a := \{\lambda \in \hat{\mathcal{C}} \mid \lambda(a^*) \in \mathbb{R}^+ (\in \mathbb{C}^*, \text{ respectively})\} \qquad \text{and} \\ &\mathcal{C}_a := p(\hat{\mathcal{C}}_a) \subset \mathcal{C}, \end{aligned}$$

so that $p : \hat{\mathcal{C}}_a \to \mathcal{C}_a$ is a principal $\mathbb{R}^+$-bundle ($\mathbb{C}^*$-bundle, respectively) and the restriction of a^* to $\mathcal{C}_a$ is a positively transversal contact symmetry. Therefore, we obtain the section $\sigma_a : \mathcal{C}_a \to \mathfrak{R} = \mathrm{G}/\mathrm{H}$ from (22).

Let $\pi : \mathrm{G} \to \mathrm{G}/\mathrm{H} = \mathfrak{R}$ be the canonical projection, and let $\Gamma_a := \pi^{-1}(\sigma_a(\mathcal{C}_a)) \subset \mathrm{G}$. Then evidently, the restriction $\pi : \Gamma_a \to \sigma_a(\mathcal{C}_a) \cong \mathcal{C}_a$ is a (right) principal H-bundle.

Theorem 3.4. *Let $a \in \mathfrak{g}$ be such that $\mathcal{C}_a \subset \mathcal{C}$ from (27) is non-empty, define $a^* \in \mathfrak{X}(\mathcal{C})$ and $\hat{a}^* \in \mathfrak{X}(\hat{\mathcal{C}})$ as in (26), and let $\pi : \Gamma_a \to \mathcal{C}_a$ with $\Gamma_a \subset \mathrm{G}$ be the principal* H*-bundle from above. Then there are functions $\rho : \Gamma_a \to \mathfrak{h}$, $u : \Gamma_a \to V$, $f : \Gamma_a \to \mathbb{F}$ such that*

$$(28)\qquad \mathrm{Ad}_{g^{-1}}(a) = \frac{1}{2}e_-^2 + \rho + e_+ \otimes u + \frac{1}{2} f e_+^2$$

for all $g \in \Gamma_a$. Moreover, the restriction of the components $\mu_{-2} + \mu_{-1} + \mu_{\mathfrak{h}}$ of the Maurer-Cartan form (24) to Γ_a yields a pointwise linear isomorphism $T\Gamma_a \to \mathfrak{h} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$, and if we decompose this coframe as

$$\mu_{-2} + \mu_{-1} + \mu_{\mathfrak{h}} = -2\kappa \left(\frac{1}{2}e_-^2 + \rho\right) + e_- \otimes \theta + \eta,$$

where $\kappa \in \Omega^1(\Gamma_a), \quad \theta \in \Omega^1(\Gamma_a) \otimes V, \quad \eta \in \Omega^1(\Gamma_a) \otimes \mathfrak{h},$

then $\kappa = -\frac{1}{2}\pi^*(\lambda)$ *where* $\lambda \in \Omega^1(\mathcal{C}_a)$ *is the contact form for which* $\sigma_a = (\lambda, \hat{a}^*)$. *Moreover, we have the structure equations*

$$d\kappa = \frac{1}{2}\omega(\theta \wedge \theta), \tag{29}$$

and

$$\begin{aligned} d\theta + \eta \wedge \theta &= 0, \\ d\eta + \tfrac{1}{2}[\eta, \eta] &= R_\rho(\theta \wedge \theta), \end{aligned} \qquad \begin{aligned} d\rho + [\eta, \rho] &= u \circ \theta \\ du + \eta \cdot u &= (\rho^2 + f) \cdot \theta \\ df + d(\rho, \rho) &= 0. \end{aligned} \tag{30}$$

Proof. According to the above identifications, we have $g \in \Gamma_a$ iff $(g \cdot e_+^2, g \cdot (\frac{1}{2}e_-^2 + \mathfrak{p}_0)) = \sigma_a([g \cdot e_+^2])$ iff $g \cdot (\frac{1}{2}e_-^2 + \mathfrak{p}_0) = (\hat{a}^*)_{g \cdot e_+^2}$ iff $(\mathrm{Ad}_{g^{-1}}(\hat{a}^*))_{e_+^2} = \frac{1}{2}e_-^2 \mod \mathfrak{p}_0$ iff $\mathrm{Ad}_{g^{-1}}(a) = \frac{1}{2}e_-^2 \mod \mathfrak{p}_0$, i.e.

$$\Gamma_a = \left\{ g \in \mathrm{G} \;\middle|\; \mathrm{Ad}_{g^{-1}}(a) \in Q \right\}, \text{where}$$

$$Q := \tfrac{1}{2}e_-^2 + \mathfrak{p}_0 = \left\{ \tfrac{1}{2}e_-^2 + \rho + e_+ \otimes u + \tfrac{1}{2} f e_+^2 \;\middle|\; \begin{array}{l} \rho \in \mathfrak{h}, \\ u \in V, \\ f \in \mathbb{F} \end{array} \right\}, \tag{31}$$

and from this (28) follows. Thus, if $dL_g v \in T_g\Gamma_a$ with $v \in \mathfrak{g}$, then we must have

$$\begin{aligned} \mathfrak{p}_0 \;\ni\; & \left.\frac{d}{dt}\right|_{t=0} \left(\mathrm{Ad}_{(g\exp(tv))^{-1}}(a)\right) = -[v, \mathrm{Ad}_{g^{-1}}(a)] \\ =\; & -\left[v, \frac{1}{2}e_-^2 + \rho + e_+ \otimes u + \frac{1}{2} f e_+^2\right], \end{aligned}$$

and from here it follows by a straightforward calculation that v must be contained in the space

$$\mathbb{F}\mathrm{Ad}_{g^{-1}} a \oplus \left\{ e_- \otimes x + e_+ \otimes \rho x + \frac{1}{2}\omega(u, x) e_+^2 \;\middle|\; x \in V \right\} \oplus \mathfrak{h}, \tag{32}$$

and since v was arbitrary, it follows that $\mu(T_g\Gamma_a)$ is contained in (32). In fact, a dimension count yields that $\dim(\mu(T_g\Gamma_a)) = \dim \Gamma_a = \dim \mathcal{C}_a + \dim \mathrm{H}$ coincides with the dimension of (32), hence (32) equals $\mu(T_g\Gamma)$, i.e. $\mu_{-2} + \mu_{-1} + \mu_{\mathfrak{h}} : T\Gamma_a \to \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{h}$ yields a pointwise isomorphism.

Let ξ_a denote the *right* invariant vector field on G characterized by $\mu(\xi_a) = \mathrm{Ad}_{g^{-1}}(a)$. Then the flow of ξ_a is *left* multiplication by $\exp(ta)$ and hence evidently leaves Γ_a invariant. Moreover, by (32) we have

$$\mathfrak{L}^*_{\xi_a}(\mu) = 0, \quad \text{and} \quad d\rho(\xi_a) = du(\xi_a) = df(\xi_a) = 0. \tag{33}$$

Let us write the components of the Maurer-Cartan form μ as

$$\mu_{\pm 2} := \kappa_\pm e_\pm^2, \quad \mu_{\pm 1} := e_\pm \otimes \alpha_\pm, \quad \mu_0 := \nu e_+ e_- + \mu_{\mathfrak{h}}$$

with $\kappa_\pm, \nu \in \Omega^1(G)$, $\alpha_\pm \in \Omega^1(G) \otimes V$ and $\mu_{\mathfrak{h}} \in \Omega^1(G) \otimes \mathfrak{h}$. Now $\mu(T\Gamma_a)$ is given by (32) so that by (28), the restriction of μ to Γ_a satisfies

$$\nu = 0, \quad \alpha_+ = \rho\alpha_- - 2u\kappa, \quad \kappa_+ = \tfrac{1}{2}\omega(u, \theta) - 2f\kappa \quad \mu_{\mathfrak{h}} = \eta - 2\kappa\rho,$$

where $\kappa := -\kappa_-$, $\theta := \alpha_-$ and $\eta := \mu_{\mathfrak{h}} + 2\kappa\rho$. Substituting this into the Maurer-Cartan equation $d\mu + \frac{1}{2}[\mu, \mu] = 0$, a straightforward calculations yields (29) and

(34)

$$\begin{aligned} d\theta + \eta \wedge \theta &= 0, \\ d\eta + \tfrac{1}{2}[\eta, \eta] - R_\rho(\theta \wedge \theta) &= -2\kappa \wedge (d\rho + [\eta, \rho] - u \circ \theta), \\ (d\rho + [\eta, \rho] - u \circ \theta) \wedge \theta &= -2\kappa \wedge (du + \eta \cdot u - (\rho^2 + f) \cdot \theta), \\ \omega(du + \eta \cdot u - (\rho^2 + f) \cdot \theta, \theta) &= -2\kappa \wedge (df + d(\rho, \rho)). \end{aligned}$$

By (33), we have $\theta(\xi_a) = \eta(\xi_a) = 0$, $\kappa(\xi_a) \equiv -\frac{1}{2}$, and $\xi_a \lrcorner d\theta = \xi_a \lrcorner d\eta = 0$. Thus, the contraction of the left hand sides of (34) with ξ_a vanishes, and from there, (30) follows.

Note that $dp(\xi_a) = \hat{a}^*$, where $p : \Gamma_a \to \hat{\mathcal{C}}$ is the canonical projection, and from (28) it follows that $\lambda(a^*) = -2\kappa(\tilde{a}^*) \equiv 1$, so that $(\lambda, \hat{a}^*) \in \mathfrak{R}$ which shows that $\kappa = -\frac{1}{2}\pi^*(\lambda)$. q.e.d.

With these structure equations, we are now ready to prove the following result which immediately implies Theorem A of the introduction.

Theorem 3.5. *Let $a \in \mathfrak{g}$ and $\mathcal{C}_a \subset \mathcal{C}$ as before. Let $U \subset \mathcal{C}_a$ be a regular open subset , i.e. the local quotient $M_U := \mathrm{T}_a^{loc} \backslash U$ is a manifold, where*

$$\mathrm{T}_a := \exp(\mathbb{F}a) \subset \mathrm{G}.$$

Let $\omega \in \Omega^2(M)$ be the symplectic form from (23). Then M_U carries a canonical special symplectic connection associated to $\mathfrak{g}$, and the (local) principal T_a-bundle $\pi : U \to M$ admits a connection $\kappa \in \Omega^1(U)$ whose curvature is given by $d\kappa = \pi^*(\omega)$.

Proof. Let us consider the commutative diagram

(35)
$$\begin{array}{ccc} \Gamma_a & \xrightarrow{\mathrm{T}_a} & \mathrm{T}_a \backslash \Gamma_a \\ {\scriptstyle \mathrm{H}}\downarrow & & {\scriptstyle \mathrm{H}}\downarrow \\ \mathcal{C}_a & \xrightarrow{\mathrm{T}_a} & \mathrm{T}_a \backslash \mathcal{C}_a \end{array}$$

where the maps $\pi : \Gamma_a \to \mathrm{T}_a \backslash \Gamma_a$ and $\Gamma_a \to \mathcal{C}_a$ are principal bundles with the indicated structure groups, whereas the arrows $\mathrm{T}_a \backslash \Gamma_a \to \mathrm{T}_a \backslash \mathcal{C}_a$ and $\mathcal{C}_a \to \mathrm{T}_a \backslash \mathcal{C}_a$ stand for fibrations with a locally free, but not necessarily free group action of the indicated structure group.

It follows now immediately from (29) and (30) that $\theta + \eta$ and κ are the pull backs of one forms on $\mathrm{T}_a \backslash \Gamma_a$ and $\mathcal{C}_a$, respectively, and we shall by abuse of notation denote these forms by the same symbols.

Let $U \subset \mathcal{C}_a$ be a regular open subset, let $\Gamma_U := \pi^{-1}(U) \subset \Gamma_a$ and $B := \mathrm{T}_a^{loc} \backslash \Gamma_U$ be the corresponding subsets. It follows then that the induced commutative diagram

$$\begin{array}{ccc} \Gamma_U & \xrightarrow{\mathrm{T}_a^{loc}} & B \\ {\scriptstyle \mathrm{H}}\downarrow & & \downarrow{\scriptstyle \mathrm{H}} \\ U & \xrightarrow{\mathrm{T}_a^{loc}} & M \end{array}$$

consists of *(local) principal* bundles, and B and U carry a $V \oplus \mathfrak{h}$-valued coframe $\theta + \eta$ and a one form κ, respectively, satisfying $d\kappa = \pi^*(\omega)$ and (30), where $\omega \in \Omega^2(M)$ is the canonically induced symplectic form from (23).

Standard arguments now show that $B \to M$ is an H-structure with tautological one form θ, and η defines a connection on M. By (30), this connection is torsion free and its curvature is given by $R_\rho(\theta \wedge \theta)$, i.e. this connection is special symplectic in the sense of Definition 2.15.

q.e.d.

Remark 3.6. *If we replace a by $a' := \mathrm{Ad}_{g_0}(a)$, then it is clear that in the above construction we have $\Gamma_{a'} = L_{g_0}\Gamma_a$. Thus, identifying Γ_a and $\Gamma_{a'}$ via L_{g_0}, the functions $\rho + \mu + f$ and the forms $\kappa + \theta + \omega$ will be canonically identified and hence both satisfy (*30*). Therefore, the connections from the preceding theorem only depend on the adjoint orbit of a.*

Also, let e^{t_0} with $t_0 \in \mathbb{F}$. Since $\mathcal{C}_a = \mathcal{C}_{e^{t_0}a}$ and $\mathrm{T}_a = \mathrm{T}_{e^{t_0}a}$, the above construction yields equivalent connections when replacing a by $e^{t_0}a$. In this case, however, the symplectic form ω on the quotient will be replaced by $e^{-t_0}\omega$.

4. The structure equations

In this section, we shall revert the process of the preceding section, showing that any special symplectic connection is equivalent to the ones given in Theorem 3.5 in a sense which is to be made precise. We begin by deriving the structure equations for special symplectic connections.

Proposition 4.1. *Let (M, ω, ∇) be a (real or complex) symplectic manifold of dimension ≥ 4 with a special symplectic connection of regularity C^4 associated to the Lie algebra $\mathfrak{g}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be as before. Then there is an associated $\tilde{\mathrm{H}}$-structure $\pi : B \to M$ on M which is compatible with ∇, where $\tilde{\mathrm{H}} \subset \mathrm{Sp}(V, \omega)$ is a Lie subgroup with Lie algebra $\mathfrak{h}$, and there are maps $\rho : B \to \mathfrak{h}$, $u : B \to V$ and $f : B \to \mathbb{F}$,* *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*, such that the tautological form* $\theta \in \Omega^1(B) \otimes V$*, the connection form* $\eta \in \Omega^1(B) \otimes \mathfrak{h}$ *and the functions* ρ, u *and* f *satisfy the structure equations (*30*).*

To slightly simplify our arguments, we shall assume that $\tilde{\mathrm{H}} = \mathrm{H}$ is connected, which can be achieved by passing to an appropriate covering of M. However, our results (and in particular Theorem B) also hold if $\tilde{\mathrm{H}}$ is *not* connected.

For clarification, we restate the structure equations (30) as follows. If for $h \in \mathfrak{h}$ and $x \in V$ we let the vector fields $\xi_h, \xi_x \in \mathfrak{X}(B)$ be the vector fields which are characterized by

$$\theta(\xi_h) \equiv 0, \quad \eta(\xi_h) \equiv h \quad \text{and} \quad \theta(\xi_x) \equiv x, \quad \eta(\xi_x) \equiv 0, \tag{36}$$

then for all $h, l \in \mathfrak{h}$ and $x, y \in V$,

$$\begin{aligned}
[\xi_h, \xi_l] &= \xi_{[h,l]}, & [\xi_h, \xi_x] &= \xi_{hx}, \\
[\xi_x, \xi_y] &= -2\omega(x,y)\xi_\rho - \xi_{x \circ \rho y} + \xi_{y \circ \rho x} && \\
\xi_h(\rho) &= -[h, \rho], & \xi_x(\rho) &= u \circ x, \\
\xi_h(u) &= -hu, & \xi_x(u) &= (\rho^2 + f)x, \\
\xi_h(f) &= 0, & \xi_x(f) &= -2\omega(\rho u, x)
\end{aligned} \tag{37}$$

The proof can be found e.g. in [**BC1**] for the case of connections of Ricci type, in [**S3**] for the case of the special symplectic holonomies and in [**Br2**] in the case of Bochner Kähler metrics. But for the sake of completeness (and since our notation here is slightly different) we restate it here.

Proof. Let F be the H-structure on the manifold M, and denote the tautological and the connection 1-form on F by θ and η, respectively. Since by hypothesis, the curvature maps are all contained in $\mathcal{R}_{\mathfrak{h}}$, it follows that there is an H-equivariant map $\rho : B \to \mathfrak{h}$ such that the curvature at each point is given by R_ρ with the notation from (14). Thus, we have the structure equations

$$\begin{aligned}
d\theta + \eta \wedge \theta &= 0 \\
d\eta + \tfrac{1}{2}[\eta, \eta] &= R_\rho \cdot (\theta \wedge \theta),
\end{aligned} \tag{38}$$

The H-equivariance of ρ yields that $\xi_h(\rho) = -[h, \rho]$ for all $h \in \mathfrak{h}$. Moreover, since the covariant derivative of the curvature is represented by $\xi_x(\rho)$ for all $x \in V$ and this must lie in $\mathcal{R}_{\mathfrak{h}}^{(1)}$, it follows by Proposition 2.12 that $\xi_x(\rho) = u \circ \rho$ for some H-equivariant map $u : B \to V$, which shows the asserted formula

$$d\rho + [\eta, \rho] = u \circ \theta. \tag{39}$$

Since u is H-equivariant, it follows that $\xi_h(u) = -hu$ for all $h \in \mathfrak{h}$. Also, differentiation of (39) yields that for all $x, y \in V$

$$\left(\xi_x u - \rho^2 x\right) \circ y = \left(\xi_y u - \rho^2 y\right) \circ x.$$

Thus, by Lemma 2.14 it follows that there is a smooth function $f : B \to \mathbb{F}$ for which $\xi_x u - \rho^2 x = fx$ for all $x \in V$ so that

$$du + \eta \cdot u = (\rho^2 + f)\theta. \tag{40}$$

Finally, taking the exterior derivative of (40) yields that $df + d(\rho, \rho) = 0$. q.e.d.

It is now our aim to construct the equivalent to the line bundle $\Gamma \to B$ from the preceding section. Motivated by (31) and (32), we define the following function A and one form σ

(41)
$$\begin{aligned} &A : B \longrightarrow Q \subset \mathfrak{g}, && A := \tfrac{1}{2}e_-^2 + \rho + e_+ \otimes u + \tfrac{1}{2}fe_+^2, \\ &\sigma \in \Omega^1(B) \otimes \mathfrak{g}, && \sigma := e_- \otimes \theta + \eta + e_+ \otimes (\rho\theta) + \tfrac{1}{2}\omega(u, \theta)e_+^2, \end{aligned}$$

where $Q := \frac{1}{2}e_-^2 + \mathfrak{p}_0 \subset \mathfrak{g}$ is the affine hyperplane from (31). It is then straightforward to verify that (30) is equivalent to

$$dA = -[\sigma, A] \qquad \text{and} \qquad d\sigma + \frac{1}{2}[\sigma, \sigma] = 2\pi^*(\omega)A. \tag{42}$$

Let us now enlarge the principal H-bundle $B \to M$ to the principal G-bundle

$$\mathbf{B} := B \times_{\mathrm{H}} \mathrm{G} \longrightarrow M,$$

where H acts on $B \times \mathrm{G}$ from the right by $(b, g) \cdot h := (b \cdot h, h^{-1}g)$, using the principal H-action on B in the first component. Evidently, the inclusion $B \times \mathrm{H} \hookrightarrow B \times \mathrm{G}$ induces an embedding $B \hookrightarrow \mathbf{B}$.

Proposition 4.2. *The function* $\mathbf{A}$ *and the one form* α *defined by*

$$\begin{aligned} &\mathbf{A} : \mathbf{B} \longrightarrow \mathfrak{g}, && \mathbf{A}[(b, g)] := \mathrm{Ad}_{g^{-1}}(A(b)), \\ &\alpha \in \Omega^1(\mathbf{B}) \otimes \mathfrak{g}, && \alpha_{[(b,g)]} := \mathrm{Ad}_{g^{-1}}\sigma_b + \mu, \end{aligned} \tag{43}$$

on $\mathbf{B}$ *are well defined, where* $\mu = g^{-1}dg \in \Omega^1(\mathrm{G}) \otimes \mathfrak{g}$ *is the left invariant Maurer-Cartan form on* G, *and the restriction of* $\mathbf{A}$ *to* $B \subset \mathbf{B}$ *coincides with* A. *Moreover,* α *yields a connection on the principal* G*-bundle* $\mathbf{B} \to M$ *which satisfies*

$$d\mathbf{A} = -[\alpha, \mathbf{A}] \qquad \textit{and} \qquad d\alpha + \frac{1}{2}[\alpha, \alpha] = 2\pi^*(\omega)\mathbf{A}. \tag{44}$$

Proof. First, note that $A : B \to \mathrm{H}$ and $\sigma \in \Omega^1(B) \otimes \mathfrak{g}$ are H-equivariant, i.e. $R_h^* A = \mathrm{Ad}_{h^{-1}} A$ and $R_h^* \sigma = \mathrm{Ad}_{h^{-1}} \sigma$. Thus, if we define the function $\hat{\mathbf{A}}$ and the one form $\hat{\alpha}$ by

$$\hat{\mathbf{A}} := \mathrm{Ad}_{g^{-1}}(A) : \quad B \times \mathrm{G} \longrightarrow \mathfrak{g}$$

$$\hat{\alpha} := \mathrm{Ad}_{g^{-1}} \sigma + \mu \quad \in \Omega^1(B \times \mathrm{G}) \otimes \mathfrak{g},$$

then $\hat{\mathbf{A}}(bh, h^{-1}g) = \hat{\mathbf{A}}(b, g)$, so that $\hat{\mathbf{A}}$ is the pull back of a well defined function $\mathbf{A} : \mathbf{B} \to \mathfrak{g}$. Also, $\hat{\alpha}$ is invariant under the right H-action from above, and for $h \in \mathfrak{h}$ we have

$$\begin{aligned} \hat{\alpha}(((\xi_h)_b, dR_g(-h)) &= \mathrm{Ad}_{g^{-1}}(\sigma_b(\xi_h)) - \mu(dR_g(h)) \\ &= \mathrm{Ad}_{g^{-1}}(h) - \mathrm{Ad}_{g^{-1}}(h) = 0, \end{aligned}$$

so that $\hat{\alpha}$ is indeed the pull back of a well defined form $\alpha \in \Omega^1(\mathbf{B}) \otimes \mathfrak{g}$. Moreover, $R_g^*(\hat{\alpha}) = \mathrm{Ad}_g^{-1} \hat{\alpha}$ is easily verified, and since $\hat{\alpha}$ coincides with μ on the fibers of the projection $B \times \mathrm{G} \to B$, it follows that the value of $\hat{\alpha}$ on each left invariant vector field on G is constant. Since the left invariant vector fields generate the principal right action of the bundle $B \times \mathrm{G} \to B$, it follows that $\hat{\alpha}$ is a connection on this bundle, hence so is α on the quotient $\mathbf{B} \to M$.

Finally, to show (44) it suffices to show the corresponding equations for $\hat{\alpha}$ and $\hat{\mathbf{A}}$. We have

$$\begin{aligned} d\hat{\mathbf{A}} &= -[\mu, \mathrm{Ad}_{g^{-1}}(A)] + \mathrm{Ad}_{g^{-1}}(dA) = -[\mu, \hat{\mathbf{A}}] - \mathrm{Ad}_{g^{-1}}([\sigma, A]) \\ &= -[\mu, \hat{\mathbf{A}}] - [\mathrm{Ad}_{g^{-1}}\sigma, \hat{\mathbf{A}}] = -[\hat{\alpha}, \hat{\mathbf{A}}] \end{aligned}$$

by (42), and

$$\begin{aligned} d\hat{\alpha} + \frac{1}{2}[\hat{\alpha}, \hat{\alpha}] &= (-[\mu, \mathrm{Ad}_{g^{-1}}\sigma] + \mathrm{Ad}_{g^{-1}} d\sigma + d\mu) \\ &\quad + \frac{1}{2}(\mathrm{Ad}_{g^{-1}}[\sigma, \sigma] + 2[\mu, \mathrm{Ad}_{g^{-1}}\sigma] + [\mu, \mu]) \\ &= \mathrm{Ad}_{g^{-1}}(d\sigma + \frac{1}{2}[\sigma, \sigma]) + d\mu + \frac{1}{2}[\mu, \mu] \\ &= \mathrm{Ad}_{g^{-1}}(2\pi^*(\omega) A) = 2\pi^*(\omega)\hat{\mathbf{A}}, \end{aligned}$$

where the second to last equation follows from the Maurer-Cartan equation and (42). q.e.d.

Proof of Theorem B. Let $\hat{M} \subset \mathbf{B}$ be a holonomy reduction of α, and let $\hat{\mathrm{T}} \subset \mathrm{G}$ be the holonomy group, so that the restriction $\hat{M} \to M$ becomes a principal $\hat{\mathrm{T}}$-bundle. By the first equation of (44), it follows that $\hat{M} \subset \mathbf{A}^{-1}(a)$ for some $a \in \mathfrak{g}$, and by choosing the holonomy reduction such that it contains an element of $B \subset \mathbf{B}$, we may assume w.l.o.g. that

$a \in Q$. We let

$$\hat{\mathrm{S}} := Stab(a) = \{g \in \mathrm{G} \mid \mathrm{Ad}_g a = a\} \subset \mathrm{G} \quad \text{and} \tag{45}$$
$$\hat{\mathfrak{s}} := \mathfrak{z}(a) = \{x \in \mathfrak{g} \mid [x,a] = 0\},$$

so that $\hat{\mathrm{S}} \subset \mathrm{G}$ is a closed Lie subgroup whose Lie algebra equals $\hat{\mathfrak{s}}$. Observe that the restriction $\mathbf{A}^{-1}(a) \to M$ is a principal $\hat{\mathrm{S}}$-bundle, hence we conclude that $\hat{\mathrm{T}} \subset \hat{\mathrm{S}}$. Moreover, on $\hat{M}$, we have

$$\hat{\alpha} = 2\kappa a$$

for some $\kappa \in \Omega^1(\hat{M})$ which by (44) satisfies $d\kappa = \pi^*(\omega)$. In particular, the *Ambrose-Singer Holonomy theorem* implies that $\mathrm{T}_a = \exp(\mathbb{F}a) \subset \mathrm{G}$ is the identity component of $\hat{\mathrm{T}}$ which is thus a one dimensional (possibly non-regular) subgroup of $\hat{\mathrm{S}}$, and κ yields the desired connection form on the principal $\hat{\mathrm{T}}$-bundle $\hat{M} \to M$ which shows the first part.

Define $\mathcal{C}_a \subset \mathcal{C}$ as in (27) and $\Gamma_a \subset \mathrm{G}$ and $Q \subset \mathfrak{g}$ as in (31), and let

$$\hat{B} := p^{-1}(\hat{M}) \subset B \times \mathrm{G}, \tag{46}$$

where $p : B \times \mathrm{G} \to B \times_{\mathrm{H}} \mathrm{G} = \mathbf{B}$ is the canonical projection. Then the restriction of the map

$$\bar{\imath} : B \times \mathrm{G} \longrightarrow \mathrm{G}, \qquad \bar{\imath}(b,g) := g^{-1}$$

satisfies $\bar{\imath}(\hat{B}) \subset \Gamma_a$; indeed, since $\mathbf{A}(\hat{M}) \equiv a$, it follows that $\mathrm{Ad}_{g^{-1}} A(b) = a$ for all $(b,g) \in \hat{B}$ and hence $\mathrm{Ad}_g a = A(b) \in Q$, so that $g^{-1} \in \Gamma_a$. Since $2\kappa a = \hat{\alpha} = \mathrm{Ad}_{g^{-1}}\sigma + \mu$, it follows by (41) that

$$\begin{aligned} \bar{\imath}^*(\mu) &= -\mathrm{Ad}_g \mu = -2\kappa \mathrm{Ad}_g a + \sigma \\ &= -2\kappa A + e_- \otimes \theta + \eta + e_+ \otimes (\rho\theta) + \frac{1}{2}\omega(u,\theta) e_+^2, \end{aligned}$$

and hence

$$\bar{\imath}^*(\mu) = -2\kappa \left(\frac{1}{2} e_-^2 + \rho \right) + e_- \otimes \theta + \eta \qquad \mathrm{mod}\ \mathfrak{g}^1 \oplus \mathfrak{g}^2.$$

Comparing this equation with the structure equations in Theorem 3.4, it follows that the induced map $\hat{\imath} : \hat{M} = \hat{B}/\mathrm{H} \to \mathcal{C}_a = \Gamma_a/\mathrm{H}$ is a local diffeomorphism and the induced map $\imath : \tilde{M} := \mathrm{T}\backslash\hat{M} \to \mathrm{T}_a\backslash\mathcal{C}_a$ is connection preserving, where $\mathrm{T}_a\backslash\mathcal{C}_a$ is (locally) equipped with the special symplectic connection from Theorem 3.5. q.e.d.

Remark 4.3. *The proof of Theorem B generalizes immediately to orbifolds. Namely, if M is an* orbifold*, then a special symplectic orbifold connection consists of an* almost principal H-bundle $B \to M$*, i.e.* H *acts locally freely and properly on B such that $M = B/\mathrm{H}$, and a coframing $\theta + \eta \in \Omega^1(B) \otimes (V \oplus \mathfrak{h})$ on B such that $\eta(\xi_h) \equiv h \in \mathfrak{h}$ and $\theta(\xi_h) \equiv 0$ for all infinitesimal generators ξ_h of the* H*-action, and such that the structure equations (*38*) hold for some function $\rho : B \to \mathfrak{h}$.*

Now the proofs of Propositions 4.1 *and* 4.2 *as well as the proof of Theorem B go through verbatim as we never used the freeness of the* H*-action on B. In particular, the holonomy reduction $\hat{M}$ is a* manifold *on which* $\hat{\mathrm{T}}$ *acts locally freely, and $M = \hat{\mathrm{T}}\backslash\hat{M}$ as an orbifold.*

5. Symmetries and compact special symplectic manifolds

Definition 5.1. Let (M, ∇) be a manifold with a connection. A *(local) symmetry* of the connection is a (local) diffeomorphism $\underline{\phi} : M \to M$ which preserves ∇, i.e. such that $\nabla_{d\underline{\phi}(X)} d\underline{\phi}(Y) = d\underline{\phi}(\nabla_X Y)$ for all vector fields X, Y on M. An *infinitesimal symmetry* of the connection is a vector field $\underline{\zeta}$ on M such that for all vector fields X, Y on M we have the relation

$$[\underline{\zeta}, \nabla_X Y] = \nabla_{[\underline{\zeta}, X]} Y + \nabla_X [\underline{\zeta}, Y].$$

Furthermore, let $\pi : B \to M$ be an H-structure compatible with ∇, and let θ, η denote the tautological and the connection form on B, respectively. A *(local) symmetry* on B is a (local) diffeomorphism $\phi : B \to B$ such that $\phi^*(\theta) = \theta$ and $\phi^*(\eta) = \eta$. An *infinitesimal symmetry* on B is a vector field ζ on B such that $\mathfrak{L}_\zeta(\theta) = \mathfrak{L}_\zeta(\eta) = 0$.

The ambiguity of the terminology above is justified by the one-to-one correspondence between (local or infinitesimal) symmetries on M and B. Namely, if $\underline{\phi} : M \to M$ is a (local) symmetry, then there is a unique (local) symmetry $\phi : B \to B$ with $\pi \circ \phi = \underline{\phi} \circ \pi$, and vice versa. Likewise, for any infinitesimal symmetry $\underline{\zeta}$ on M, there is a unique infinitesimal symmetry ζ on B such that $\underline{\zeta} = d\pi(\zeta)$.

The infinitesimal symmetries form the Lie algebra of the (local) group of (local) symmetries. We also observe that an infinitesimal symmetry on B is uniquely determined by its value at any point. (The corresponding statement fails for infinitesimal symmetries on M in general.)

Proof of Corollary C. The first part follows immediately from Theorem B since $\mathcal{C}_a \subset \mathcal{C}$ is an open subset of the analytic manifold $\mathcal{C}$, and the action of T_a on $\mathcal{C}_a$ is analytic as well. Also, the C^4-germ of the connection at a point determines uniquely the G-orbit of $a \in \mathfrak{g}$ by (30) and hence the connection by Theorem B.

Note that the generic element $a \in \mathfrak{g}$ is G-conjugate to an element in the Cartan subalgebra which is uniquely determined up to the action of the (finite) Weyl group. Since multiplying $a \in \mathfrak{g}$ by a scalar does not change the connection, it follows that the generic special symplectic connection associated to $\mathfrak{g}$ depends on $(\mathrm{rk}(\mathfrak{g}) - 1)$ parameters.

For the second part, by virtue of Theorem B it suffices to show the statement for manifolds of the form $M = M_U$ where $U \subset \mathcal{C}_a$ is a regular open subset for some $a \in \mathfrak{g}$. Let $\Gamma_U \subset \Gamma_a \subset \mathrm{G}$ be the H-invariant subset such that we have the principal H-bundle $\Gamma_U \to U$, and let $B_U := \mathrm{T}_a \backslash \Gamma_U$ so that $B_U \to M_U$ is the associated H-structure.

Let $x \in \hat{\mathfrak{s}}$, and denote by $\hat{\zeta}_x$ the right invariant vector field on G corresponding to $-x$, so that the map $x \mapsto \hat{\zeta}_x$ is a Lie algebra homomorphism. Then $\mathfrak{L}_{\hat{\zeta}_x}(\mu) = 0$ where μ denotes the Maurer-Cartan form. By (31), it follows that the restriction of $\hat{\zeta}_x$ to Γ_a is tangent, and since $\Gamma_U \subset \Gamma_a$ is open, we may regard $\hat{\zeta}_x$ as a vector field on Γ_U. Since $\hat{\zeta}_x$ commutes with the action of T_a, it follows that there is a related vector field ζ_x on the quotient $B_U = \mathrm{T}_a^{loc} \backslash \Gamma_U$, and since the tautological and curvature form of the induced connection on B_U pull back to components of μ, it follows that ζ_x is an infinitesimal symmetry on B_U.

Conversely, suppose that ζ is an infinitesimal symmetry on B_U. Since an infinitesimal symmetry must preserve the curvature and its covariant derivatives, we must have $\zeta(A) = 0$. But the tangent of the fiber of the map $A : \Gamma_U \to \mathfrak{g}$ is spanned by the vector fields ζ_x, $x \in \hat{\mathfrak{s}}$, and since infinitesimal symmetries are uniquely determined by their value at a point, it follows that $\zeta = \zeta_x$ for some $x \in \hat{\mathfrak{s}}$.

Finally, it is evident that $\zeta_x = 0$ iff $\hat{\zeta}_x$ is tangent to T_a iff $x \in \mathbb{F}a$, hence the claim follows. q.e.d.

The rest of this section shall be devoted to the study of *compact simply connected* manifolds with special symplectic connections. In fact, the main result which we aim to prove is the following

Theorem 5.2. *Let* $\mathfrak{g}$ *be a* 2*-gradable simple Lie algebra, let* G *be the connected Lie group with Lie algebra* $\mathfrak{g}$ *and trivial center, and let* $\hat{\mathrm{S}} \subset \mathrm{G}$ *be a maximal compact subgroup. Then* $\mathcal{C} = \hat{\mathrm{S}}/\mathrm{K}$ *for some compact subgroup* $\mathrm{K} \subset \hat{\mathrm{S}}$ *where* $\mathcal{C} \subset \mathbb{P}^o(\mathfrak{g})$ *is the root cone. Moreover, let* $\mathrm{T} \subset \hat{\mathrm{S}}$ *be the identity component of the center of* $\hat{\mathrm{S}}$*. Then the following are equivalent:*

1) *There is a compact simply connected symplectic manifold* M *with a special symplectic connection associated to the simple Lie algebra* $\mathfrak{g}$*.*
2) $\mathfrak{g}$ *is a* real *Lie algebra and* $\dim \mathrm{T} = 1$*, i.e.* $\mathrm{T} \cong S^1$*.*
3) $\mathfrak{g}$ *is a* real *Lie algebra and* $\mathrm{T} \neq \{e\}$*.*

If these conditions hold then $\mathrm{T}\backslash\mathcal{C} \cong \hat{\mathrm{S}}/(\mathrm{T}\cdot\mathrm{K})$ *is a compact hermitian symmetric space, and the map* $\imath : M \to \mathrm{T}\backslash\mathcal{C}$ *from Theorem B is a connection preserving covering. Thus,* M *is a hermitian symmetric space as well.*

This theorem allows us to classify all compact simply connected manifolds with special symplectic connections, as the maximal compact subgroups of semisimple Lie groups are fully classified (e.g. [**OV**]). Thus, we obtain Theorem D from the introduction as an immediate consequence.

The proof of Theorem 5.2 will be split up into several steps. First, we observe the following

Lemma 5.3. *If the connected Lie group* G *acts transitively on the compact manifold* X*, then so does any maximal compact subgroup* $\hat{\mathrm{S}} \subset \mathrm{G}$.

Thus, we can write the root cone as $\mathcal{C} = \hat{\mathrm{S}}/\mathrm{K}$ for some compact subgroup $\mathrm{K} \subset \hat{\mathrm{S}}$ as asserted in Theorem 5.2.

Proof. Let $X = \mathrm{G}/\mathrm{H}$ as a homogeneous space, and let $\mathrm{K} \subset \mathrm{H}$ be a maximal compact Lie subgroup. Then there is a maximal compact Lie subgroup $\hat{\mathrm{S}} \subset \mathrm{G}$ which contains K. Since the inclusions $\hat{\mathrm{S}} \hookrightarrow \mathrm{G}$ and $\mathrm{K} \hookrightarrow \mathrm{H}$ are homotopy equivalences, standard homotopy arguments imply that the inclusion $\hat{\mathrm{S}}/\mathrm{K} \hookrightarrow X$ is also a homotopy equivalence. In particular, since both spaces are compact, they have equal dimension, so that $\hat{\mathrm{S}}/\mathrm{K} = \mathrm{G}/\mathrm{H}$. q.e.d.

Let us now suppose that M is *real*. The proof that the first condition in Theorem 5.2 implies the second and that in this case M is the universal cover of the hermitian symmetric space $\mathrm{T}\backslash\mathcal{C}$ is pursued in Lemmas 5.4 through Proposition 5.12.

Lemma 5.4. *Let* M *be a compact real simply connected manifold with a special symplectic connection associated to the real Lie algebra* $\mathfrak{g}$*, and let* $a \in \mathfrak{g}$*,* $\mathrm{T}_a \subset \mathrm{G}$ *and* $\mathcal{C}_a \subset \mathcal{C}$ *as in Theorem* 3.5*. Then* $\mathrm{T}_a \cong S^1$ *and* $\mathcal{C}_a = \mathcal{C}$*. Moreover,* T_a *acts freely on the universal cover* $\tilde{\mathcal{C}}$ *of* $\mathcal{C}$*, and* $M = \mathrm{T}_a\backslash\tilde{\mathcal{C}}$.

Proof. If M is simply connected, then by Theorem B from the introduction there is an $a \in \mathfrak{g}$ and a principal T_a-bundle $\pi : \hat{M} \to M$ with a connection form κ whose curvature equals $d\kappa = \pi^*(\omega)$. Thus, $\pi^*(\omega)$ is exact, while ω cannot be exact if M is compact. This implies that π cannot be a homotopy equivalence, i.e. T_a cannot be contractible, hence $\mathrm{T}_a \cong S^1$. Thus, $\hat{M}$ is also compact, hence the local diffeomorphism $\hat{\imath} : \hat{M} \to \mathcal{C}$ from (1) must be surjective and a finite T_a-equivariant covering. In particular, $\mathcal{C}_a = \mathcal{C}$.

Therefore, there is a T_a-equivariant covering $\tilde{\mathcal{C}} \to \hat{M}$, where $\tilde{\mathcal{C}}$ is the universal cover of $\mathcal{C}$, and since T_a acts freely on $\hat{M}$, it acts also freely on $\tilde{\mathcal{C}}$. Thus, the induced map $\mathrm{T}_a\backslash\tilde{\mathcal{C}} \to \mathrm{T}_a\backslash\hat{M} = M$ must also be a covering, hence a diffeomorphism as M is simply connected. q.e.d.

We continue with the investigation of two special classes of examples.

Proposition 5.5. *For* $\mathfrak{g} := \mathfrak{su}(p+1, q+1)$ *with* $p + q \geq 1$*, there are two orbits of maximal root vectors which are negatives of each other and are hence denoted by* $\mathcal{C}$ *and* $-\mathcal{C}$*. Moreover, these orbits are simply connected.*

Let $a \in \mathfrak{g}$ be such that $\mathrm{T}_a \cong S^1$, $\mathcal{C}_a = \mathcal{C}$ and the action of T_a on $\mathcal{C}_a$ is free. Then a is conjugate to a scalar multiple of $diag((q+1)i, \dots, (q+1)i, -(p+1)i, \dots, -(p+1)i)$. In particular, $\mathrm{T}_a \backslash \mathcal{C} \cong \mathbb{CP}^p \times \mathbb{CP}^q$ with the hermitian symmetric connection as described in Theorem E.

Proof. We let $J : \mathbb{C}^{p+1,q+1} \to \mathbb{C}^{p+1,q+1}$ be the $\mathfrak{g}$-equivariant complex structure such that the metric $g(x, y) := \omega(Jx, y)$ has signature $(p+1, q+1)$. Now $\mathfrak{g}$ is a real form of $\mathfrak{sl}(p+q+2, \mathbb{C})$ whose maximal root cone consists of all traceless endomorphisms of (complex) rank 1, hence the same is true for $\mathfrak{g}$. The image of such an endomorphism must be a null line, so that the maximal root cone of $\mathfrak{g}$ consist of all endomorphisms of the form

$$\{\alpha_x \mid x \neq 0, g(x, x) = 0\} \dot{\cup} \{-\alpha_x \mid x \neq 0, g(x, x) = 0\} =: \hat{\mathcal{C}} \dot{\cup} (-\hat{\mathcal{C}}),$$

where

$$\alpha_x(v) := g(v, x)Jx - g(v, Jx)x.$$

Observe that $\alpha_{\lambda x} = |\lambda| \alpha_x$ for all $\lambda \in \mathbb{C}^*$, hence the projectivizations $\pm\mathcal{C}$ of $\pm\hat{\mathcal{C}}$ consist of all null lines in $\mathbb{C}^{p+1,q+1}$.

Decomposing $\mathbb{C}^{p+1,q+1} = \mathbb{C}^{p+1,0} \oplus \mathbb{C}^{0,q+1} =: \mathbb{C}^+ \oplus \mathbb{C}^-$, each null vector can be written as $x = x_+ + x_-$ with $x_\pm \in \mathbb{C}^\pm$ and $||x_+|| = ||x_-||$. In particular, $\mathcal{C} = (S^{2p+1} \times S^{2q+1})/diag(S^1)$, and a glance at the homotopy exact sequence now implies that $\mathcal{C}$ is simply connected for $p + q \geq 1$.

Let $a \in \mathfrak{g}$ be such that $\mathrm{T}_a \cong S^1$. Then a is conjugate to an element of the form $diag(i\theta_0, \dots i\theta_p, i\psi_0, \dots, i\psi_q)$. If we denote the standard basis of $\mathbb{C}^{p+1,q+1}$ by $e_0, \dots, e_p, f_0, \dots, f_q$, then $x = e_r + f_s$ is a null vector and $(a, x \circ x) = \theta_r - \psi_s$. Since $x \circ x \in \hat{\mathcal{C}}$, this implies that $\theta_r > \psi_s$ for all r, s.

Consider $T := \exp(2\pi/(\theta_r - \psi_s)a) \in \mathrm{T}_a$. We have $T(e_r + f_s) = \exp(2\pi i\theta_r/(\theta_r - \psi_s))(e_r + f_s)$ so that T fixes $\mathbb{C}(e_r + f_s) \in \mathcal{C}$. Thus, since T_a acts freely on $\mathcal{C}$, it follows that $T = \exp(2\pi i\theta_r/(\theta_r - \psi_s))Id$, which implies that $\exp(2\pi i\theta_t/(\theta_r - \psi_s)) = \exp(2\pi i\psi_u/(\theta_r - \psi_s)) = \exp(2\pi i\theta_r/(\theta_r - \psi_s))$ for all t, u. Therefore, $(\theta_t - \psi_u)/(\theta_r - \psi_s) \in \mathbb{Z}$ for all r, s, t, u, and by switching (r, s) and (t, u) we conclude that $(\theta_t - \psi_u)/(\theta_r - \psi_s) = \pm 1$. But $\theta_t - \psi_u, \theta_r - \psi_s > 0$, whence this quotient must equal 1 for all r, s, t, u, so that $\theta_r = \theta_t$ and $\psi_s = \psi_u$ for all r, s, t, u, hence a must be of the asserted form, and the remaining statements now follow from the construction of the special symplectic connection. q.e.d.

Evidently, there is no need to consider both maximal root orbits $\mathcal{C}$ and $-\mathcal{C}$, since one is obtained from the other by replacing the symplectic form ω (or the complex structure J) by its negative which is irrelevant for our purposes. An analogous remark applies to the following case.

Proposition 5.6. *For $\mathfrak{g} := \mathfrak{sp}(n+1, \mathbb{R})$ with $n \geq 2$, there are two orbits of maximal root vectors which are negatives of each other and are* *hence denoted by $\mathcal{C}$ and $-\mathcal{C}$. Moreover, $\mathcal{C}$ and $-\mathcal{C}$ are diffeomorphic to $\mathbb{RP}^{2n+1}$ and therefore have fundamental group $\mathbb{Z}_2$.*

Let $a \in \mathfrak{g}$ be such that $\mathrm{T}_a \cong S^1$, $\mathcal{C}_a = \mathcal{C}$ and the action of T_a on the universal cover $\tilde{\mathcal{C}} \cong S^{2n+1}$ is free. Then $a = cJ$ for some $c > 0$, where J is a complex structure on $\mathbb{R}^{2n+2}$ such that $g(x, y) := \omega(Jx, y)$ is symmetric and positive definite. In particular, $\mathrm{T}_a \backslash \tilde{\mathcal{C}} = \mathrm{T}_a \backslash \mathcal{C} \cong \mathbb{CP}^n$ with the hermitian symmetric connection as stated in Theorem E.

Proof. Since the conjugatets of the maximal root vectors in $\mathfrak{sp}(n+1, \mathbb{R})$ are the elements of rank one, the maximal root cone can be written as $\{x \circ x \mid x \neq 0\} \dot\cup \{-x \circ x \mid x \neq 0\} =: \mathcal{C} \dot\cup - \mathcal{C}$, where the product $\circ : S^2(\mathbb{R}^{n+1}) \to \mathfrak{sp}(n+1, \mathbb{R})$ is given in (11). Thus, $\pm\mathcal{C} = \pm\hat{\mathcal{C}}/\mathbb{R}^+ \cong \mathbb{RP}^{2n+1}$.

Let $J \in \mathfrak{sp}(n+1, \mathbb{R})$ be a complex structure such that $\omega(Jx, x) > 0$ for all $x \neq 0$, and let $a \in \mathfrak{g}$ be such that $\mathrm{T} = \exp(\mathbb{R}a) \cong S^1$. Then a is conjugate to an element of the form $J diag(\theta_1, \ldots, \theta_{2n+2})$, and $\mathcal{C}_a = \mathcal{C}$ implies that $0 < (a, x \circ x) = \omega(ax, x)$ which is equivalent to $\theta_i > 0$ for all i.

Consider $T := \exp(\pi/\theta_i a) \in \mathrm{T}_a$. We have $T(e_i) = -e_i$ so that T (if we consider the action on $\mathcal{C}$) or T^2 (if we consider the action on $\tilde{\mathcal{C}}$) has a fixed point. Thus, it follows that $T(e_j) = \pm e_j$ for all j which implies that $\theta_j | \theta_i$ for all j, and switching the roles of i and j, it follows that $\theta_i = \pm\theta_j$. But since $\theta_i > 0$ for all i, we must have $\theta_i = \theta_j$, hence a is of the asserted form, and the remaining statements follow. q.e.d.

In order to work towards the general case, we continue with the following

Lemma 5.7. *Let $\mathfrak{g}$ be a real 2-gradable simple Lie algebra with the decomposition (4). Let $a \in \mathfrak{g}$ be such that $\mathrm{T}_a \cong S^1$ is a circle. Then a is conjugate to an element of the form*

$$\frac{c}{2}(e_+^2 + e_-^2) + \rho_0 \tag{47}$$

with $c \in \mathbb{R}$ and $\rho_0 \in \mathfrak{h}$.

Proof. Let $\mathrm{T}_G \subset \mathrm{G}$ be the maximal compact abelian subgroup containing the circle $\mathrm{SO}(2) := \exp(\mathbb{R}(e_+^2 + e_-^2))$. Since $stab(e_+^2 + e_-^2) = \mathbb{R}(e_+^2 + e_-^2) \oplus \mathfrak{h}$, it follows that $\mathrm{T}_G \subset \mathrm{SO}(2) \cdot \mathrm{H}$.

The lemma now follows since *any* subgroup of G isomorphic to S^1 is conjugate to a subgroup of T_G. q.e.d.

Lemma 5.8. *Let $\mathfrak{g}$ be a real 2-gradable simple Lie algebra with the decomposition (4), and let $\alpha_0 \in \Delta$ be the long root with $\mathfrak{g}^{\pm 2} = \mathfrak{g}_{\pm\alpha_0}$. Let $\beta \in \Delta$ be a root with $\langle \beta, \alpha_0 \rangle = 1$, and let $\overline{\beta}$ denote the conjugate root w.r.t. the real form $\mathfrak{g}$. If we define*

$$\mathfrak{g}_{<\beta>} := \mathfrak{g} \cap \left\langle \mathfrak{g}_{\pm\alpha_0} \oplus \mathfrak{g}_{\pm\beta} \oplus \mathfrak{g}_{\pm\overline{\beta}} \right\rangle,$$

where $< >$ denotes the generated Lie subalgebra, then $\mathfrak{g}_{<\beta>}$ *is isomorphic to either* $\mathfrak{sl}(3,\mathbb{R})$*,* $\mathfrak{sp}(2,\mathbb{R})$*,* $\mathfrak{g}_2'$*,* $\mathfrak{su}(1,2)$*, or* $\mathfrak{so}(2,4)\cong\mathfrak{su}(2,2)$*.*

Proof. Since α_0 is a real root, it follows that $\{\alpha_0,\beta,\overline{\beta}\}$ is invariant under conjugation, hence $\mathfrak{g}_{<\beta>}$ is a real form of the complex simple Lie algebra whose root system is generated by $\{\alpha_0,\beta,\overline{\beta}\}$. Since $\mathfrak{g}_{\pm\alpha_0}\subset\mathfrak{g}_{<\beta>}$, it follows that $\mathfrak{g}_{<\beta>}$ is also 2-gradable, and the decomposition (4) reads $\mathfrak{g}_{<\beta>}=\bigoplus_{i=-2}^{2}(\mathfrak{g}_{<\beta>}\cap\mathfrak{g}^i)$.

If $\beta=\overline{\beta}$ is a real root, then the root system generated by α_0,β is irreducible of rank two and contains only real roots, i.e. $\mathfrak{g}_{<\beta>}$ is the split real form of type A_2, B_2 or G_2 as listed above.

Therefore, for the rest of the proof we shall assume that $\beta\neq\overline{\beta}$. Since α_0 is real, it follows that $\langle\overline{\beta},\alpha_0\rangle=\langle\beta,\alpha_0\rangle=1$, hence $\overline{\beta}\neq-\beta$ and thus $\beta,\overline{\beta}$ are linearly independent roots of equal length, so that they generate a root system either of type A_2 or of type A_1+A_1. Since this root system is invariant under conjugation, it follows that there is a corresponding subalgebra $\hat{\mathfrak{g}}_{<\beta>}\subset\mathfrak{g}_{<\beta>}$ which is a real form of either $\mathfrak{sl}(3,\mathbb{C})$ or $\mathfrak{so}(4,\mathbb{C})$. This real form must contain roots which are neither real nor purely imaginary since $\beta\neq\pm\overline{\beta}$. In particular, it is neither split nor compact, and thus, the only real forms possible are $\mathfrak{su}(1,2)$ in the first and $\mathfrak{so}(1,3)$ in the second case.

If $\hat{\mathfrak{g}}_{<\beta>}\cong\mathfrak{su}(1,2)$, then $\hat{\mathfrak{g}}_{<\beta>}$ is 2-gradable, and hence the root system generated by $\beta,\overline{\beta}$ must contain a real root. This implies that $\beta+\overline{\beta}\in\Delta$, and since $\langle\beta+\overline{\beta},\alpha_0\rangle=2$, it follows that $\beta+\overline{\beta}=\alpha_0$, i.e. $\hat{\mathfrak{g}}_{<\beta>}=\mathfrak{g}_{<\beta>}\cong\mathfrak{su}(1,2)$.

Let us now suppose that $\hat{\mathfrak{g}}_{<\beta>}\cong\mathfrak{so}(1,3)$. We assert that in this case, β must be a long root. For if β and hence $\overline{\beta}$ are short, then $\langle\beta,\alpha_0\rangle=\langle\overline{\beta},\alpha_0\rangle=1$ implies that $\beta+\overline{\beta}=\alpha_0$ so that the root system generated by $\{\alpha_0,\beta,\overline{\beta}\}$ is irreducible of rank two with roots of different length, i.e. $\mathfrak{g}_{<\beta>}$ is a 2-gradable real form with root system B_2 or G_2. However, by Table 1 on page 240, the only 2-gradable real forms of these root systems are the split forms which have only real roots, contradicting that $\beta\neq\overline{\beta}$.

Thus, we are left with the case where $\hat{\mathfrak{g}}_{<\beta>}\cong\mathfrak{so}(1,3)$ and $\beta\in\Delta$ is a long root. Then $\langle\beta,\overline{\beta}\rangle=0$, and the intersections

$$W_\pm:=\mathfrak{g}\cap\left(\mathfrak{g}_{\pm\alpha_0}\oplus\mathfrak{g}_{\pm(\alpha_0-\beta)}\oplus\mathfrak{g}_{\pm(\alpha_0-\overline{\beta})}\oplus\mathfrak{g}_{\pm(\alpha_0-\beta-\overline{\beta})}\right)$$

are $\hat{\mathfrak{g}}_{<\beta>}$-modules. In fact, considering the weights of the action of $\hat{\mathfrak{g}}_{<\beta>}$ on $W_\pm$ implies that $W_\pm\cong\mathbb{R}^{1,3}$ as a $\hat{\mathfrak{g}}_{<\beta>}$-module, and one verifies that $[W_+,W_+]=[W_-,W_-]=0$, whereas $[W_+,W_-]\subset\hat{\mathfrak{g}}_{<\beta>}\oplus\mathbb{R}H_{\alpha_0}$. It follows now that $(\mathfrak{g}_{<\beta>},\hat{\mathfrak{g}}_{<\beta>}\oplus\mathbb{R}\mathrm{H}_{\alpha_0})$ is an irreducible symmetric pair whose isotropy representation coincides with that of the symmetric pair $(\mathfrak{so}(2,4),\mathfrak{so}(1,3)\oplus\mathfrak{so}(1,1))$, hence these symmetric pairs are isomorphic.

In particular, we have $\mathfrak{g}_{<\beta>} \cong \mathfrak{so}(2,4) \cong \mathfrak{su}(2,2)$ which completes the proof. q.e.d.

Lemma 5.9. *Let $\mathfrak{g}$ be one of the real Lie algebras from Lemma 5.8, and let $a \in \mathfrak{g}$ be such that $\mathrm{T}_a \cong S^1$. Define $\mathcal{C}_a \subset \mathcal{C}$ as in (27). Then*

1) *If $\mathfrak{g} \cong \mathfrak{sl}(3,\mathbb{R})$, $\mathfrak{g}_2'$ then $\mathcal{C}_a \subsetneq \mathcal{C}$ is a* proper *subset for any such $a \in \mathfrak{g}$.*
2) *Let $\mathfrak{g} \cong \mathfrak{sp}(2,\mathbb{R})$, $\mathfrak{su}(1,2)$, $\mathfrak{su}(2,2)$ and $\tilde{\mathcal{C}}$ be the universal cover of $\mathcal{C}$. If $\mathcal{C}_a = \mathcal{C}$ and the (lifted) action of T_a on $\tilde{\mathcal{C}}$ is free, then the action of T_a on $\mathcal{C}$ is free and a is conjugate to an element of the form (47) with $c > 0$, $\rho_0^2 = -c^2 Id_V$ and $\omega(\rho_0 x, x) > 0$ for all $0 \neq x \in V$.*

Proof. By Lemma 5.7, we may assume that a is of the form (47). Since $\mathfrak{sl}(3,\mathbb{R})$, $\mathfrak{g}_2'$ are split real forms, it follows that $\mathfrak{g}_\beta \subset V_1 \cap \mathcal{C}$ for all long roots β with $\langle \beta, \alpha_0 \rangle = 1$, hence $V_1 \cap \mathcal{C} \neq \emptyset$, whereas $(a, V_1 \cap \mathcal{C}) = 0$. Thus, $\mathcal{C}_a \neq \mathcal{C}$.

The second part now follows immediately from Propositions 5.5 and 5.6 where the explicit form of a was given. q.e.d.

This lemma now allows us to treat the general case. Namely we have

Lemma 5.10. *Let $\mathfrak{g}$ be a 2-gradable real Lie algebra, and let $a \in \mathfrak{g}$ be such that $\mathrm{T}_a \cong S^1$, $\mathcal{C}_a = \mathcal{C}$ and that the action of T_a on the universal cover of $\mathcal{C}$ is free. Then a is conjugate to an element of the form (47) with $c > 0$, $\rho_0^2 = -c^2 Id_V$ and $\omega(\rho_0 x, x) > 0$ for all $0 \neq x \in V$.*

Proof. Lemma 5.7 allows us to assume that a is of the form (47). Indeed, we may assume that ρ_0 is contained in the Cartan subalgebra of $\mathfrak{h}_{\mathbb{C}}$, so that the Lie subalgebras $\mathfrak{g}_{<\beta>} \subset \mathfrak{g}$ from Lemma 5.8 are T_a-invariant.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\mathrm{G}_{<\beta>} \subset \mathrm{G}$ be the connected Lie subgroup with Lie algebra $\mathfrak{g}_{<\beta>} \subset \mathfrak{g}$. Then $\mathcal{C}_\beta := \mathrm{G}_{<\beta>} \cdot e_+^2 \subset \mathcal{C}$ is T_a-invariant, and $(\mathcal{C}_\beta)_a = \mathcal{C}_\beta \cap \mathcal{C}_a = \mathcal{C}_\beta$ as $\mathcal{C}_a = \mathcal{C}$. Thus, since $\mathcal{C}_\beta$ is the cone of maximal roots of $\mathfrak{g}_{<\beta>}$, by Lemma 5.8 and the first part of Lemma 5.9 we conclude that $\mathfrak{g}_{<\beta>} \cong \mathfrak{sp}(2,\mathbb{R}), \mathfrak{su}(1,2)$ or $\mathfrak{su}(2,2)$.

The inverse image $\hat{M}_{<\beta>} := \hat{\imath}^{-1}(\mathcal{C}_\beta) \subset \hat{M}$ with the covering $\hat{\imath} : \hat{M} \to \mathcal{C}$ from (1) must also be T_a-invariant, and every connected component of $\hat{M}_{<\beta>}$ is a covering of $\mathcal{C}_\beta$. Since T_a acts freely on $\hat{M}_{<\beta>} \subset \hat{M}$, it follows from the second part of Lemma 5.9 that $c > 0$, $\rho_0^2|_{V_{<\beta>}} = -c^2 Id$ and $\omega(\rho_0 x, x) > 0$ for all $0 \neq x \in V_{<\beta>}$, where $V_{<\beta>} \subset V$ is defined by the relation $e_\pm \otimes V_{<\beta>} = \mathfrak{g}_{<\beta>} \cap \mathfrak{g}^{\pm 1}$.

The claim now follows since V is the direct sum of the $V_{<\beta>}$, and for all $\beta, \gamma \in \Delta$ with $\langle \beta, \alpha_0 \rangle = \langle \gamma, \alpha_0 \rangle = 1$ and $V_{<\beta>} \cap V_{<\gamma>} = 0$ we have $\omega(V_{<\beta>}, V_{<\gamma>}) = 0$. q.e.d.

Lemma 5.11. *Let $\mathfrak{g}$ be a 2-gradable real Lie algebra, and let $a \in \mathfrak{g}$ be of the form (47) with $c > 0$, $\rho_0^2 = -c^2 Id_V$ and $\omega(\rho_0 x, x) > 0$ for all $0 \neq x \in V$. Then*

$$\begin{aligned} \hat{\mathfrak{s}} = stab(a) &:= \{x \in \mathfrak{g} \mid [x, a] = 0\} \\ &= \mathbb{R}a \oplus \mathfrak{k} \oplus \{c(e_+ \otimes x) - (e_- \otimes \rho_0 x) \mid x \in V\}, \end{aligned} \tag{48}$$

where $\mathfrak{k} := \{h \in \mathfrak{h} \mid [h, \rho_0] = 0\}$. Moreover, $\hat{\mathfrak{s}} \cong \mathbb{R}a \oplus \mathfrak{s}$, where $\mathfrak{s}$ is a compact semisimple Lie algebra, and $(\hat{\mathfrak{s}}, \mathbb{R}a \oplus \mathfrak{k})$ is a hermitian symmetric pair. Also, $\hat{\mathfrak{s}} \subset \mathfrak{g}$ is a maximal Lie subalgebra.

Proof. It is straightforward to verify (48) and $\mathfrak{z}(\hat{\mathfrak{s}}) = \mathbb{R}a$, and that $(\hat{\mathfrak{s}}, \mathbb{R}a \oplus \mathfrak{k})$ is a hermitian symmetric pair. Also, note that $\mathfrak{k}$ is the Lie algebra of the compact group $\mathrm{K} = \mathrm{H} \cap \mathrm{U}(V, 1/c\ \rho_0)$. Thus, there is a positive definite $\mathrm{ad}_{\mathfrak{k}}$-invariant metric on $\mathfrak{g}$, so that $\mathrm{ad}_h^2 : \mathfrak{g} \to \mathfrak{g}$ is negative semidefinite for all $h \in \mathfrak{k}$ and hence $B(h, h) = tr(\mathrm{ad}_h^2) \leq 0$ with equality iff $\mathrm{ad}_h = 0$ iff $h = 0$ since $\mathfrak{g}$ is simple and hence has trivial center. Thus, $(h, h) > 0$ for all $0 \neq h \in \mathfrak{k}$ by (9). Also, $((e_+ \otimes x) - (e_- \otimes \rho_0 x), (e_+ \otimes x) - (e_- \otimes \rho_0 x)) = 2\omega(\rho_0 x, x) > 0$ for all $0 \neq x \in V$, and $(e_+^2 + e_-^2, e_+^2 + e_-^2) = 4 > 0$. Since $e_+^2 + e_-^2$, $\mathfrak{k}$ and $\{c(e_+ \otimes x) - (e_- \otimes \rho_0 x) \mid x \in V\}$ are orthogonal w.r.t. $(\ ,\ )$, it follows that $(\ ,\ )$ is positive definite and ad-invariant on $\hat{\mathfrak{s}}$. Thus, $\mathrm{ad}_x : \hat{\mathfrak{s}} \to \hat{\mathfrak{s}}$ is skew symmetric w.r.t. $(\ ,\ )$ for all $x \in \hat{\mathfrak{s}}$, so that $B_{\hat{\mathfrak{s}}}(x, x) = tr(\mathrm{ad}_x^2) \leq 0$ with equality iff $x \in \mathfrak{z}(\hat{\mathfrak{s}})$, hence $\hat{\mathfrak{s}} = \mathfrak{z}(\hat{\mathfrak{s}}) \oplus \mathfrak{s}$ for a compact semisimple Lie algebra $\mathfrak{s}$ as asserted.

To see that $\hat{\mathfrak{s}} \subset \mathfrak{g}$ is a maximal subalgebra, let $\hat{\mathfrak{s}} \subset \mathfrak{g}' \subsetneq \mathfrak{g}$ be a subalgebra. Considering the eigenspaces of $\mathrm{ad}(e_+^2 + e_-)^2$, it follows that $\mathfrak{g}' = (\mathfrak{g}' \cap \mathfrak{sl}(2, \mathbb{R})) \oplus (\mathfrak{g}' \cap \mathfrak{h}) \oplus (\mathfrak{g}' \cap \mathbb{R}^2 \otimes V)$.

But $\mathfrak{sl}(2, \mathbb{R})$ and $\hat{\mathfrak{s}}$ generate $\mathfrak{g}$, so it follows that $\mathfrak{g}' \cap \mathfrak{sl}(2, \mathbb{R}) = \mathbb{R}(e_+^2 + e_-^2)$. Also, if $e_+ \otimes x \in \mathfrak{g}'$, then $[e_+ \otimes x, e_+ \otimes y - e_- \otimes \rho_0 y] \in \mathfrak{g}'$ implies $\omega(x, y) = 0$, as one sees by looking at the $\mathfrak{sl}(2, \mathbb{R})$-component. Since this is the case for all $y \in V$, it follows that $\mathfrak{g}' \cap \mathbb{R}^2 \otimes V = \hat{\mathfrak{s}} \cap \mathbb{R}^2 \otimes V$. Finally, if $h \in \mathfrak{g}' \cap \mathfrak{h}$ then $[h, e_+ \otimes x - e_- \otimes \rho_0 x] \in \mathfrak{g}' \cap \mathbb{R}^2 \otimes V \subset \hat{\mathfrak{s}}$, and from here it follows that $h \in \mathfrak{k}$, so that $\mathfrak{g}' = \hat{\mathfrak{s}}$ as claimed. q.e.d.

Now we are ready to prove that in the real case, the first condition in Theorem 5.2 implies the second, and that in this case M is hermitian symmetric.

Proposition 5.12. *Let M be a real compact simply connected manifold with a special symplectic connection, and let $a \in \mathfrak{g}$ be from Theorem B. Then $\mathrm{T}_a \backslash \mathcal{C}$ is a hermitian symmetric space, and the map $\imath : M \to \mathrm{T}_a \backslash \mathcal{C}$ is a connection preserving covering. Moreover, T_a is the connected component of the center of $\hat{\mathrm{S}} \subset \mathrm{G}$, where $\hat{\mathrm{S}}$ is a maximal compact subgroup of G.*

Proof. By Lemma 5.11, it follows that the connected Lie subgroup $\hat{\mathrm{S}} \subset \mathrm{G}$ with Lie subalgebra $\hat{\mathfrak{s}}$ must be compact as $\mathrm{T}_a \cong S^1$ is compact.

Indeed, it is a maximal compact subgroup as $\hat{\mathfrak{s}} \subset \mathfrak{g}$ is maximal, and $\mathrm{T}_a \subset \hat{\mathrm{S}}$ is the connected component of its center.

Thus, if we write $\mathcal{C} = \hat{\mathrm{S}}/\mathrm{K}$ by Lemma 5.3, then K has $\mathfrak{k} = \hat{\mathfrak{s}} \cap \mathfrak{p}$ as its Lie algebra by (48), and hence $\mathrm{T}_a \backslash \mathcal{C} = \hat{\mathrm{S}}/(\mathrm{T}_a \cdot \mathrm{K})$ is a hermitian symmetric space by Lemma 5.11, and the covering $\imath : M \to \mathrm{T}_a \backslash \mathcal{C}$ is connection preserving. q.e.d.

Evidently, the second condition in Theorem 5.2 implies the third, hence the real case will be finished with the following

Lemma 5.13. *Let* G *be a real simple connected Lie group with* 2-*gradable Lie algebra* $\mathfrak{g}$ *and trivial center, and let* $\hat{\mathrm{S}} \subset \mathrm{G}$ *be a maximal compact Lie subgroup whose center contains* $\mathrm{T}_a = \exp(\mathbb{R}a)$*, some* $0 \neq a \in \mathfrak{g}$*. Then - after changing* a *to its negative if necessary - we have* $\mathcal{C}_a = \mathcal{C}$*, and the action of* T_a *on* $\mathcal{C}$ *is free. Moreover,* $\mathrm{T}_a \backslash \mathcal{C}$ *has finite fundamental group.*

By Theorem 3.5, it then follows that $\mathrm{T}_a \backslash \mathcal{C}$ carries a special symplectic connection associated to $\mathfrak{g}$, hence so does its universal cover $M := (\mathrm{T}_a \backslash \mathcal{C})\tilde{}$. Since $\mathrm{T}_a \backslash \mathcal{C}$ is compact and has finite fundamental group, M is compact as well. Thus, the lemma shows that the third condition in Theorem 5.2 implies the first.

Proof. Since G acts transitively on $\mathcal{C}$, so does $\hat{\mathrm{S}}$ by Lemma 5.3, hence we can write $\mathcal{C} = \hat{\mathrm{S}}/\mathrm{K}$ for some compact subgroup $\mathrm{K} \subset \hat{\mathrm{S}}$. Let $a \in \mathfrak{g}$ be such that $\mathrm{T} = \mathrm{T}_a$ and consider the corresponding contact symmetry a^* from (26). We assert that a^* is *transversal.* For if there is a $p \in \mathcal{C}$ with $(a^*)_p \in \mathcal{D}_p$, then $dL_g((a^*)_p) \in dL_g(\mathcal{D}_p) = \mathcal{D}_{g \cdot p}$ for all $g \in \hat{\mathrm{S}}$ by the $\hat{\mathrm{S}}$-equivariance of the contact structure. On the other hand,

$$dL_g((a^*)_p) = \left.\frac{d}{dt}\right|_{t=0} g \cdot \exp(ta) \cdot p = \left.\frac{d}{dt}\right|_{t=0} \exp(t\mathrm{Ad}_g(a)) \cdot g \cdot p = (a^*)_{g \cdot p},$$

since $\mathrm{Ad}_g(a) = a$ for $g \in \hat{\mathrm{S}}$. Thus, $(a^*)_{g \cdot p} \in \mathcal{D}_{g \cdot p}$, and since $\hat{\mathrm{S}}$ acts transitively on $\mathcal{C}$, it follows that $(a^*)_q \in \mathcal{D}_q$ for *all* $q \in \mathcal{C}$. But a^* is a contact symmetry, hence this implies that $a^* \equiv 0$ which is a contradiction.

Thus, $\lambda(a^*) \neq 0$ for all $\lambda \in \hat{\mathcal{C}}$ and - after replacing a by its negative if necessary - we may assume that $\lambda(a^*) > 0$ for all $\lambda \in \hat{\mathcal{C}}$, so that $\mathcal{C}_a = \mathcal{C}$. Since T_a lies in the center of $\hat{\mathrm{S}}$ and G acts effectively on $\mathcal{C} = \hat{\mathrm{S}}/\mathrm{K}$ as G has trivial center, it follows that T_a acts freely on $\mathcal{C}$.

It now follows from Lemmas 5.10 and 5.11 that $\mathrm{T}_a \subset \hat{\mathrm{S}}$ is the connected component of the center, hence the inclusion $\mathrm{T}_a \cdot \mathrm{K} \hookrightarrow \hat{\mathrm{S}}$ induces a map with finite cokernel between the fundamental groups. Now the homotopy exact sequence of the fibration $\mathrm{T}_a \cdot \mathrm{K} \hookrightarrow \hat{\mathrm{S}} \to \hat{\mathrm{S}}/(\mathrm{T}_a \cdot \mathrm{K}) = \mathrm{T}_a \backslash \mathcal{C}$ implies that $\mathrm{T}_a \backslash \mathcal{C}$ has finite fundamental group as claimed. q.e.d.

Finally, we need to deal with the *complex* case which we do in the following

Proposition 5.14. *There are no compact simply connected complex manifolds M with a special symplectic connection associated to a complex simple Lie algebra $\mathfrak{g}$.*

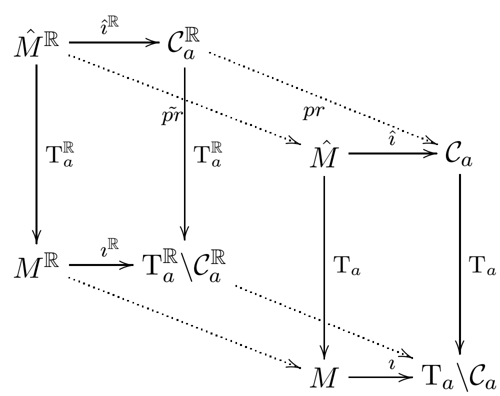
Proof. If M is such a manifold, then as in the proof of Lemma 5.4, we conclude that the fibration $\hat{M} \to M$ cannot be a homotopy equivalence, so that $\mathrm{T}_a \cong \mathbb{C}^*$ and hence $a \in \mathfrak{g}$ is semisimple. This means that the eigenvalues of ad_a are all linearly dependent over $\mathbb{Q}$, so that – after replacing a by a suitable non-zero multiple – we may assume that all these eigenvalues are integers. Thus, we may choose the split real form $\mathfrak{g}_\mathbb{R} \subset \mathfrak{g}$ such that $a \in \mathfrak{g}_\mathbb{R}$ and $\mathrm{T}_a^\mathbb{R} := \exp(\mathbb{R}a) \subset \mathfrak{g}_\mathbb{R}$ is isomorphic to $\mathbb{R}$.

Let $\mathcal{C}^\mathbb{R} \subset \mathbb{P}^0(\mathfrak{g}_\mathbb{R})$ be the projectivization of the root cone of $\mathfrak{g}_\mathbb{R}$, and consider the *Hopf fibration* $pr : \mathbb{P}^0(\mathfrak{g}_\mathbb{R}) \to \mathbb{P}^0(\mathfrak{g})$ which maps each real line to the corresponding complex one. Then $pr(\mathcal{C}^\mathbb{R}) \subset \mathcal{C}$ is a regular submanifold which is diffeomorphic to either $\mathcal{C}^\mathbb{R}$ or $\mathcal{C}^\mathbb{R}/\mathbb{Z}_2$. In particular, the restriction $pr : \mathcal{C}^\mathbb{R} \to pr(\mathcal{C}^\mathbb{R})$ is a regular covering. Also, as the distribution $\mathcal{D}$ consists of *complex* subspaces, it follows that

$$pr\left(\mathcal{C}_a^\mathbb{R}\right) = pr(\mathcal{C}^\mathbb{R}) \cap \mathcal{C}_a,$$

so that the restriction $pr : \mathcal{C}_a^\mathbb{R} \to pr(\mathcal{C}^\mathbb{R}) \cap \mathcal{C}_a$ is also a regular covering. In particular, $pr(\mathcal{C}_a^\mathbb{R}) \subset \mathcal{C}_a$ is a regular closed submanifold.

Recall the covering map $\hat{\imath} : \hat{M} \to \mathcal{C}_a$ from (1). Standard homotopy arguments show that there is a manifold $\hat{M}^\mathbb{R}$ and regular coverings $\hat{\imath}^\mathbb{R} : \hat{M}^\mathbb{R} \to \mathcal{C}_a^\mathbb{R}$ and $\tilde{pr} : \hat{M}^\mathbb{R} \to \hat{\imath}^{-1}(pr(\mathcal{C}^\mathbb{R}) \cap \mathcal{C}_a)$ where $\hat{\imath}^\mathbb{R}$ is equivariant w.r.t. the action of $\mathrm{T}_a^\mathbb{R} \subset \mathrm{T}_a$. Note that $\mathrm{T}_a^\mathbb{R}$ acts freely and properly discontinuously on $\hat{M}$ and hence also on $\hat{M}^\mathbb{R}$, so that $M^\mathbb{R} := \mathrm{T}_a^\mathbb{R} \backslash \hat{M}^\mathbb{R}$ is a manifold. Hence, we obtain the following commutative diagram, where the dotted lines indicate immersions which are regular covers of their images with a deck group of order at most 2:



Thus, Theorem B implies that $M^\mathbb{R}$ carries a special symplectic connection associated to $\mathfrak{g}_\mathbb{R}$, and the principal $\mathrm{T}_a^\mathbb{R}$-bundle $\hat{M}^\mathbb{R} \to M^\mathbb{R}$ coincides with the one given in that theorem.

But the image of the covering $M^{\mathbb{R}} \to M$ is a closed submanifold, and since we assume that M is compact, it follows that $M^{\mathbb{R}}$ is compact. Thus, as in the proof of Lemma 5.4, we conclude that $\mathrm{T}_a^{\mathbb{R}} \cong S^1$, which is a contradiction as $\mathrm{T}_a^{\mathbb{R}} \cong \mathbb{R}$ by our choice of $\mathfrak{g}_{\mathbb{R}}$. q.e.d.

References

[BC1] F. Bourgeois & M. Cahen, *A variational principle for symplectic connections*, J. Geom. Phys. **30** No.3, 233–265 (1999), MR 1692232, Zbl 0963.53050.

[BC2] F. Baguis & M. Cahen, *A construction of symplectic connections through reduction*, Lett. Math. Phys. **57** No.2, 149–160 (2001), MR 1856907, Zbl 1033.53071.

[Bo] S.Bochner, *Curvature and Betti numbers, II*, Ann. Math. **50** 77–93 (1949), MR 0029252, Zbl 0039.17603.

[BR] N.R. O'Brian & J. Rawnsley, *Twistor spaces*, Ann. Glob. Anal. Geom. **3**, 29–58 (1985), MR 0812312, Zbl 0526.53057.

[Br1] R. Bryant, *Two exotic holonomies in dimension four, path geometries, and twistor theory*, Proc. Symp. in Pure Math. **53**, 33–88 (1991), MR 1141197, Zbl 0758.53017.

[Br2] R. Bryant, *Bochner-Kähler metrics*, J. AMS **14** No.3, 623–715 (2001), MR 1824987, Zbl 1006.53019.

[CGR] M. Cahen, S. Gutt & J. Rawnsley, *Symmetric symplectic spaces with Ricci-type curvature*, G.Dito, D.Sternheimer (ed.), Conférence Moshé Flato 1999, Vol.II, Math. Phys. Stud. 22, 81–91 (2000), MR 1805906, Zbl 0983.53032.

[CGHR] M. Cahen, S. Gutt, J. Horowitz & J. Rawnsley, *Homogeneous symplectic manifolds with Ricci-type curvature*, J. Geom. Phys. **38** No.2, 140–151 (2001), MR 1823665, Zbl 0999.53050.

[CGS] M. Cahen, S. Gutt & L.J. Schwachhöfer, *Construction of Ricci-type connections by reduction and induction*, in: *The breadth of symplectic and Poisson geometry*, Progr. Math., 232, Birkhäuser Boston, 41–57 (2005), MR 2103002, Zbl 1079.53117.

[CMS] Q.-S. Chi, S.A. Merkulov & L.J. Schwachhöfer, *On the Existence of Infinite Series of Exotic Holonomies*, Inv. Math. **126** No.2, 391–411 (1996), MR 1411138, Zbl 0866.53013.

[FH] W. Fulton & J. Harris, *Representation Theory*, Graduate Texts in Mathematics 129, Springer-Verlag, Berlin, New-York (1991), MR 1153249, Zbl 0744.22001.

[Hu] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics 9, Springer-Verlag, Berlin, New York (1978), MR 0499562, Zbl 0447.17001.

[K] Y. Kamishima *Uniformization of Kähler manifolds with vanishing Bochner tensor*, Acta Math. **172** No.2, 299-308 (1994), MR 1278113, Zbl 0828.53059.

[OV] A.L. Onishchik & E.B. Vinberg, *Lie groups and Lie Algebras*, Vol. 3, Encyclopaedia of Mathematical Sciences 41, Springer-Verlag, Berlin, New York (1994), MR 1349140, Zbl 0797.22001.

[MS] S.A. Merkulov & L.J. Schwachhöfer, *Classification of irreducible holonomies of torsion free affine connections*, Ann. Math. **150** No.1, 77–149 (1999);

Addendum: Classification of irreducible holonomies of torsion-free affine connections, Ann. Math. **150** No.3, 1177–1179 (1999), MR 1715321, MR 1740981, Zbl 0992.53038.

[S1] L.J. Schwachhöfer, *On the classification of holonomy representations*, Habilitationsschrift, Universität Leipzig (1998), available on the home page of the author.

[S2] L.J. Schwachhöfer, *Homogeneous connections with special symplectic holonomy*, Math. Zeit. **238** No.4, 655–688 (2001), MR 1872569, Zbl 1013.53032.

[S3] L.J. Schwachhöfer, *Connections with irreducible holonomy representations*, Adv. Math. **160** No.1, 1–80 (2001), MR 1831947, Zbl 1037.53035.

[V] I. Vaisman, *Variations on the theme of twistor spaces*, Balkan J. Geom. Appl. **3** No.2, 135–156 (1998), MR 1746886, Zbl 0926.53020.

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