SPECIAL SYMPLECTIC SIX-MANIFOLDS

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Abstract

We classify nilmanifolds with an invariant symplectic half-flat structure. We study the transverse or quotient geometry of six-manifolds with an SU(3)-structure preserved by a Killing vector field, giving characterizations in the symplectic half-flat and integrable case.

1. Introduction

A half-flat manifold is a six-dimensional manifold endowed with an SU(3)-structure with intrinsic torsion that is symmetric. An SU(3)-structure defines a non-degenerate two-form ω, an almost-complex structure J and a complex volume form Ψ; the half-flat condition is equivalent to requiring $\omega \wedge \omega$ and the real part of Ψ be closed [4]. Hypersurfaces in seven-dimensional manifolds with holonomy G_2 have a natural half-flat structure, given by the restriction of the holonomy group representation; the intrinsic torsion can then be identified with the second fundamental form. Conversely, every real analytic half-flat manifold (M, ω, Ψ), can be embedded isometrically as a hypersurface in a manifold with holonomy contained in G_2. Writing the G_2 metric explicitly is a matter of solving certain evolution equations. Hitchin [11] proved that such a solution can be viewed as an integral line for a Hamiltonian flow on an infinite-dimensional symplectic space, namely the product of cohomology classes $[\omega^2] \times [\mathfrak{Re}\,\Psi]$.

Given a six-manifold M with an SU(3)-structure, one can consider the product G_2-structure on $M \times S^1$. Half-flat manifolds satisfying special conditions on this product G_2-structure have been studied: in [3, 5], the six-dimensional nilmanifolds carrying invariant SU(3)-structures of these types have been classified. More generally, the problem of classifying nilmanifolds admitting invariant half-flat structures is open.

In this paper, we focus on the symplectic case; that is to say, we take into consideration half-flat structures for which ω is closed. Symplectic half-flat manifolds can also be viewed as symplectic manifolds (M, ω) endowed with two extra objects, namely an ω-calibrated almost-complex structure and a (3, 0)-form with closed real part which is parallel with respect to the Chern connection [7, 8].

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The evolution flow mentioned above is transverse to the space of symplectic half-flat structures, except at its critical points, that correspond to integrable (namely, Calabi–Yau) SU(3)-structures.

More generally, we consider arbitrary SU(3)-structures on six-manifolds, and we investigate the geometry determined by a Killing vector field that preserves such a structure. A similar situation has been studied in [1], in the context of seven-dimensional manifolds with holonomy G_2. A special case has been studied in [12], where it was proved that there is only one complete example of a nearly-Kähler SU(3)-structure with a Killing vector field of unit norm, namely the standard nearly-Kähler structure on $S^3 \times S^3$.

The paper is divided into two parts. In the first part, we classify nilmanifolds carrying an invariant symplectic half-flat structure; the special case $b_1 \geq 4$ was carried out by Bedulli in [2]. Since six-dimensional nilmanifolds can be realized as circle bundles over nilmanifolds of dimension five, this classification problem can be reduced to a problem in five dimensions: indeed, the symplectic half-flat structure induces an SU(2)-structure on the base of the circle bundle, satisfying certain conditions involving the curvature (see Lemma 2.2). In Lemma 2.3, we classify the five-dimensional nilmanifolds with this type of induced structure. We then use this lemma to show that every six-dimensional nilmanifold admitting a symplectic half-flat structure is modelled on one of a list of three Lie algebras (Theorem 2.4). The corresponding nilmanifolds are the torus, a torus bundle over the four-dimensional torus (see [8, 9]) and a torus bundle over a three-dimensional torus (see [2, 9]). In contrast with the first two, the third example is irreducible; moreover, the fibres of the torus fibration are special Lagrangian submanifolds.

In the second part, we generalize this situation and the construction of Lemma 2.2 to the non-invariant case. More precisely, we consider a six-manifold with an SU(3)-structure and a nowhere-vanishing Killing vector field preserving the structure. This vector field generates a rank one foliation, that we can assume regular, working locally. Then the leaf space is a five-dimensional manifold. If the leaves are compact there is a circle action on the six-manifold, and the leaf space is a quotient on which the curvature two-form is defined; in fact, a 'curvature' two-form can always be defined on the leaf space using the O'Neill tensor. The geometric structure we are interested in is an SU(2)-structure induced on the leaf space or quotient. The metric underlying this SU(2)-structure is not the same as the quotient metric, but it is obtained from the quotient metric by rescaling along certain directions, as in [1]. We compute the intrinsic torsion of the SU(2)-structure in terms of the intrinsic torsion of the SU(3)-structure, norm of the Killing vector field and curvature two-form (Proposition 3.1). As far as we know, this is the first detailed application of intrinsic torsion for SU(2)-structures on five-manifolds. We then characterize the symplectic half-flat manifolds in terms of the quotient structure (Proposition 3.2). In the case that the invariant SU(3)-structure is integrable, we obtain a stronger result: indeed, the intrinsic torsion of the quotient structure is determined by the norm of the Killing vector field, and so is the curvature form (Theorem 3.3).

2. Invariant structures on nilmanifolds

In this section, we introduce half-flat structures on six-manifolds and classify invariant symplectic half-flat structures on six-dimensional nilmanifolds.

Let M be a six-dimensional manifold. An SU(3)-structure on M is a pair (ω, Ψ), where ω is a non-degenerate two-form and Ψ is a locally decomposable complex three-form, such that the following

compatibility conditions hold:

$$\Psi \wedge \omega = 0,$$
$$\Psi \wedge \overline{\Psi} = \frac{4}{3} i\, \omega^3, \tag{1}$$

and the almost-complex structure induced by J is ω-tamed. Recall that a decomposable complex three-form $\Psi = \theta^1 \wedge \theta^2 \wedge \theta^3$ determines an almost-complex structure by the condition that θ^1, θ^2 and θ^3 span the space of forms of type $(1, 0)$. The first condition asserts that ω is of type $(1, 1)$, so J is actually calibrated by ω.

At each point x of M, the three-form Ψ_x has stabilizer conjugate to $\mathrm{SL}(3, \mathbb{C})$ for the natural action of $\mathrm{GL}^+(T_x M)$; on the other hand, the stabilizer of ω_x is conjugate to the symplectic group $\mathrm{Sp}(3, \mathbb{R})$ for the natural action of $\mathrm{GL}(T_x M)$. The compatibility conditions ensure that the intersection of the two stabilizers is conjugate to $\mathrm{SU}(3)$, so that an $\mathrm{SU}(3)$-structure is defined.

We shall denote by ψ^+ and ψ^- the real and imaginary parts of Ψ, respectively. It was shown by Hitchin [10] that having fixed the orientation, the real three-form ψ^+ is sufficient to determine the almost-complex structure J and therefore ψ^-. Thus, an $\mathrm{SU}(3)$-structure (ω, Ψ) is really determined by the pair (ω, ψ^+).

We now introduce a special class of $\mathrm{SU}(3)$-structures on six-manifolds, related to seven-dimensional Riemannian manifolds with holonomy contained in G_2 [4].

DEFINITION 2.1: An $\mathrm{SU}(3)$-structure (ω, ψ^+) on a six-manifold is *half-flat* if $\omega \wedge \omega$ and ψ^+ are closed.

The two-form ω appearing in the characterization of $\mathrm{SU}(3)$-structures is required to be non-degenerate; if it is also closed, it defines a symplectic structure. In this case, we say that the $\mathrm{SU}(3)$-structure is symplectic.

Consider a nilmanifold M, that is, a compact manifold of the form $\Gamma \backslash G$, where G is a six-dimensional nilpotent group with Lie algebra $\mathfrak{g}$ and Γ a uniform discrete subgroup of G. Recall that in six dimensions, every nilpotent Lie algebra $\mathfrak{g}$ gives rise to such a nilmanifold.

We say that a structure on the nilmanifold M is invariant if it pulls back to a left-invariant structure on G. Invariant structures can be viewed as structures on the Lie algebra $\mathfrak{g}$; using the above characterization of $\mathrm{SU}(3)$-structures in terms of differential forms, we shall mainly work with the dual $\mathfrak{g}^*$.

We start by reducing the problem to a problem in five dimensions; the idea is to realize M as a circle bundle over a five-dimensional manifold, in such a way that the $\mathrm{SU}(3)$-structure on M is invariant under the circle action. The geometry of this construction will be studied in Section 3; here, we shall use the following algebraic result.

LEMMA 2.2 *Let (ω, ψ^+) be a symplectic half-flat structure on a nilpotent Lie algebra $\mathfrak{g}$; we have an orthogonal decomposition*

$$\mathfrak{g}^* = \langle \eta \rangle \oplus V^5,$$

where η is a unit form, and

$$d(\mathfrak{g}^*) \subseteq \Lambda^2 V^5.$$

Define forms α, ω_1, ω_2 and ω_3 on $\ker \eta$ by

$$\omega = \omega_3 + \eta \wedge \alpha,$$
$$\Psi = (\omega_1 + i\omega_2) \wedge (\eta + i\alpha). \tag{2}$$

Setting $\phi = d\eta$, the following hold:

$$d\alpha = 0, \quad d\omega_1 = 0,$$
$$d\omega_3 = -\phi \wedge \alpha, \quad d(\omega_2 \wedge \alpha) = \omega_1 \wedge \phi. \tag{3}$$

Proof. By Engel's theorem, some non-zero ξ in $\mathfrak{g}$ satisfies

$$\mathrm{ad}(\xi) = 0.$$

Choosing for η a suitable multiple of $\xi^\flat$ and setting $V^5 = \eta^\perp$, the first part of the lemma is satisfied.
  By definition

$$0 = d\omega = d\omega_3 + \phi \wedge \alpha - \eta \wedge d\alpha;$$

isolating the component in $\eta \wedge \Lambda^2 V^5$, we deduce that α is closed and $d\omega_3$ satisfies the required equation.
  Similarly, the rest of (3) follows from

$$0 = d\psi^+ = d\omega_1 \wedge \eta + \omega_1 \wedge \phi - d(\omega_2 \wedge \alpha).$$

REMARK: The forms (α, ω_i) introduced in Lemma 2.2 define an $SU(2)$-structure on $\mathfrak{g}$. More generally, we recall that differential forms $(\alpha, \omega_1, \omega_2, \omega_3)$ on a five-manifold define an $SU(2)$-structure if and only if at each point (and hence locally), there exists a coframe $e^1, \ldots, e^5$ such that

$$\alpha = e^5, \quad \omega_1 = e^{12} + e^{34},$$
$$\omega_2 = e^{13} + e^{42}, \quad \omega_3 = e^{14} + e^{23}. \tag{4}$$

Here and in the sequel, e^{12} is short for $e^1 \wedge e^2$ and so on. If one fixes an orientation, the condition above is equivalent to the existence of a triplet $(\omega_1, \psi_2, \psi_3)$ with

$$\omega_1 = e^{12} + e^{34}, \quad \psi_2 = e^{135} + e^{425}, \quad \psi_3 = e^{145} + e^{235}.$$

  By construction, V^5 is itself the dual of a nilpotent Lie algebra; we shall proceed by listing the five-dimensional Lie algebras that arise this way. To describe Lie algebras, we shall use symbolic expressions such as

$$\mathfrak{g} = (0, 0, 0, 0, 0, 12),$$

meaning that $\mathfrak{g}^*$ has a basis $\eta^1, \ldots, \eta^6$ such that $d\eta^6 = \eta^1 \wedge \eta^2$, and η^i is closed for $i = 1, \ldots, 5$.

LEMMA 2.3 *In the hypotheses of Lemma* 2.2, *V^5 is one of*

$$(0, 0, 0, 0, 0), (0, 0, 0, 0, 12), (0, 0, 0, 12, 13).$$

Proof. By the same argument we used in the proof of Lemma 2.2, we can construct a filtration

$$V^0 \subset \cdots \subset V^5, \quad \text{where dim } V^i = i, \ dV^{i+1} \subset \Lambda^2 V^i.$$

We can assume that $\ker d$ coincides with some V^i. Moreover, by Lemma 2.2, we can assume that α lies in $V^1 \subset V^4$; therefore, using the fact that $SU(2)$ acts transitively on the three-dimensional sphere, we can fix a basis $e^1, \dots, e^5$ of V^5 satisfying (4), with e^4 in $(V^4)^\perp$.

We will show that the operator $d : V^5 \to \Lambda^2 V^5$ satisfies

$$\ker d \supset V^3, \quad d(V^5) \subset \Lambda^2 V^3. \tag{5}$$

Then it will follow from the classification of five-dimensional nilpotent Lie algebras that V^5 is one of the Lie algebras appearing in the statement.

By Lemma 2.2

$$0 = de^{12} + de^{34} \equiv de^3 \wedge e^4 \mod \Lambda^2 V^4,$$

implying that e^3 is closed. Thus, we can assume

$$V^2 = \langle e^3, e^5 \rangle; \quad V^4 = \langle e^1, e^2, e^3, e^5 \rangle.$$

Define a real constant h, a two-form $\gamma \in \Lambda^2 \langle e^1, e^2, e^3 \rangle$ and one-forms ϕ_4, ϕ_5 in $\langle e^1, e^2, e^3 \rangle$, such that

$$\phi = \phi_5 \wedge e^5 + \phi_4 \wedge e^4 + he^{45} + \gamma.$$

Since ϕ is closed,

$$d\phi_5 \wedge e^5 - \phi_4 \wedge de^4 + hde^4 \wedge e^5 + d\gamma = -d\phi_4 \wedge e^4; \tag{6}$$

the left-hand side lies in $\Lambda^3 V^4$, that is, it has no component containing e^4, so both sides are zero and $d\phi_4 = 0$.

By Lemma 2.2 $d\omega_3 = -\phi \wedge \alpha$, giving

$$de^{14} + de^2 \wedge e^3 = -\phi_4 \wedge e^{45} - \gamma \wedge e^5; \tag{7}$$

comparing the components in $e^4 \wedge \Lambda^2 V^4$, we obtain

$$de^1 = \phi_4 \wedge e^5.$$

Again by Lemma 2.2, $d\psi_2 = \omega_1 \wedge \phi$; on the other hand, $d\psi_2 = d\omega_2 \wedge \alpha$, so we can drop the component of ϕ not containing e^5 and write

$$(de^{13} + de^{42}) \wedge e^5 = (e^{12} + e^{34}) \wedge (\phi_5 \wedge e^5 + he^{45}). \tag{8}$$

The components containing e^4 give

$$de^2 \wedge e^5 = e^3 \wedge \phi_5 \wedge e^5 - he^{125},$$

and wedging with e^3,

$$de^2 \wedge e^{35} = -h e^{1235}.$$

Since de^2 is in $\Lambda^2 V^3$, the left-hand side is in $\Lambda^4 V^3$, and so it is zero. We conclude that $h = 0$ and

$$de^2 \wedge e^5 = e^3 \wedge \phi_5 \wedge e^5. \tag{9}$$

Now, either $V^3 = \langle e^1, e^3, e^5 \rangle$, or some linear combination $\lambda e^1 + e^2$ lies in V^3 and consequently $0 = (\lambda de^1 + de^2) \wedge e^5 = de^2 \wedge e^5$. Either way,

$$\phi_5 \in \langle e^1, e^3 \rangle.$$

By Lemma 2.2 ω_1 is closed, giving

$$\phi_4 \wedge e^{52} - e^1 \wedge de^2 = e^3 \wedge de^4. \tag{10}$$

In order to prove (5), we must distinguish three cases.

(i) Suppose that ϕ_4 is not a multiple of e^3; then

$$V^3 = \langle e^3, e^5, \phi_4 \rangle$$

and d is zero on V^3. Moreover e^1 is not closed, so $V^4 = V^3 \oplus \langle e^1 \rangle$. Since e^2 is in V^4, we have $de^2 = k\, e^5 \wedge \phi_4$ for some (possibly zero) constant k. So (10) becomes

$$\phi_4 \wedge e^{52} - k\, e^{15} \wedge \phi_4 = e^3 \wedge de^4,$$

implying that

$$de^4 \wedge e^3 \wedge \phi_4 = 0 = de^4 \wedge e^{35}.$$

Since the space of closed two-forms in $\Lambda^2 V^4$ is

$$\Lambda^2 V^3 \oplus \langle e^{15}, e^1 \wedge \phi_4 \rangle,$$

we can conclude that de^4 lies in $\Lambda^2 V^3$; we already know that $d(V^3) = 0$, so there is nothing left to prove in this case.

In general, the component of (7) in $\Lambda^3 V^4$ gives

$$-e^1 \wedge de^4 + de^2 \wedge e^3 = -\gamma \wedge e^5; \tag{11}$$

in particular, $de^4 \wedge e^{15} = 0$. Moreover, we can rewrite (6) as

$$-\phi_4 \wedge de^4 + d\gamma = 0. \tag{12}$$

(ii) Suppose now that $\phi_4 = 0$; then e^1 is closed and we can assume that $V^3 = \langle e^1, e^3, e^5 \rangle$. By (10)

$$-e^1 \wedge de^2 = e^3 \wedge de^4,$$

so clearly $de^4 \wedge e^{13} = 0$; moreover $de^4 \wedge e^{35} = 0$, since by (9) $de^2 \wedge e^{15}$ is zero. It follows that de^4 lies in $\Lambda^2 V^3$, completing the proof in this case.

(iii) The remaining case is the one where $\phi_4 = ae^3$ for some non-zero a. By (12) and (10), this condition implies

$$dy = ae^3 \wedge de^4 = a^2 e^{352} - ae^1 \wedge de^2. \tag{13}$$

Equations (9) and (10) show that

$$de^2 \wedge e^{15} = 0 = de^2 \wedge e^{13} = de^2 \wedge e^{35},$$

so de^2 lies in $\Lambda^2 \langle e^1, e^3, e^5 \rangle$. Hence the space of closed forms in $\Lambda^2 V^4$ is contained in

$$\Lambda^2 \langle e^1, e^3, e^5 \rangle \oplus e^2 \wedge \langle e^3, e^5 \rangle;$$

wedging the closed two-form $\gamma - ae^{12}$ with e^{35} we deduce

$$\gamma \wedge e^{35} = ae^{1235}.$$

Comparing with (11), we find $e^{13} \wedge de^4 = -ae^{1235}$, which together with (13) gives the contradiction

$$-a^2 e^{1235} = a^2 e^{1352}.$$

All three possibilities listed in Lemma 2.3 can occur.

- On $V^5 = (0, 0, 0, 0, 0)$, set

$$\omega_1 = \eta^{12} + \eta^{34}, \quad \psi_2 = \eta^{135} + \eta^{425}, \quad \psi_3 = \eta^{145} + \eta^{235}, \quad \phi = 0.$$

- On $V^5 = (0, 0, 0, 0, 12)$, set

$$\omega_1 = \eta^{34} + \eta^{15}, \quad \psi_2 = \eta^{312} + \eta^{542}, \quad \psi_3 = \eta^{352} + \eta^{412}, \quad \phi = -\eta^{13}.$$

- On $V^5 = (0, 0, 0, 12, 13)$, set

$$\omega_1 = \eta^{24} + \eta^{35}, \quad \psi_2 = \eta^{123} + \eta^{154}, \quad \psi_3 = \eta^{125} + \eta^{143}, \quad \phi = -2\eta^{23}; \quad \text{or}$$

$$\omega_1 = \eta^{24} - \eta^{35}, \quad \psi_2 = -\eta^{123} + \eta^{154}, \quad \psi_3 = \eta^{125} - \eta^{143}, \quad \phi = 0.$$

It is easy to verify that equations (3) are satisfied in these cases. The construction can then be inverted: define

$$\mathfrak{g}^* = \langle \eta \rangle \oplus V^5,$$

declaring that $d\eta = \phi$; clearly, $\mathfrak{g}^*$ is the dual of a nilpotent Lie algebra $\mathfrak{g}$. A straightforward calculation shows that the SU(3)-structure on $\mathfrak{g}$ defined by (2) is half-flat and symplectic.

So, there are at least three non-isomorphic nilpotent Lie algebras that admit a symplectic half-flat structure. It only remains to show that this list is complete. Indeed, we prove the following.

THEOREM 2.4 *The six-dimensional nilpotent Lie algebras whose corresponding nilmanifold carries an invariant symplectic half-flat structure are*

$$(0, 0, 0, 0, 0, 0), \quad (0, 0, 0, 0, 12, 13), \quad (0, 0, 0, 12, 13, 23).$$

Proof. We retain the notation from the proof of Lemma 2.3. We first show that ϕ_4 is zero; in other words, case (i) of Lemma 2.3 cannot occur, like case (iii), which we have already ruled out. Indeed, suppose that ϕ_4 is independent of e^3 and e^5. Hence $V^5 = (0, 0, 0, 12, 13)$; indeed, de^1 and de^4 must be independent, since otherwise a combination of e^1 and e^4 would lie in $\ker d$, which is orthogonal to e^4 by construction.

Observe that $\langle e^1, e^3, \phi^4 \rangle$ has dimension three, because ϕ_4 is closed but e^1 is not, and we are assuming that ϕ_4 is not a multiple of e^3. Therefore

$$\langle e^1, e^3, \phi^4 \rangle = \langle e^1, e^2, e^3 \rangle,$$

and we can write $\gamma = a\, e^{13} + b\, e^1 \wedge \phi_4 + c\, e^3 \wedge \phi_4$. Equation (11) then yields

$$-e^1 \wedge de^4 + ke^5 \wedge \phi_4 \wedge e^3 = -a\, e^{135} + b\, e^{15} \wedge \phi_4 + c\, e^{35} \wedge \phi_4,$$

so in particular

$$de^4 = a\, e^{35} - b\, e^5 \wedge \phi_4.$$

Substituting in (12), it follows that

$$-a\, \phi_4 \wedge e^{35} + a\, \phi_4 \wedge e^{53} = 0,$$

that is, $a = 0$; but then $de^4 = b\, e^5 \wedge \phi_4$ is a multiple of de^1, which is absurd.

We have proved that ϕ_4 is necessarily zero; now assume that e^2 is closed. Then (10) and (11) give

$$e^3 \wedge de^4 = 0, \quad e^1 \wedge de^4 = \gamma \wedge e^5.$$

It follows that $de^4 = \lambda\, e^{35} + \mu\, e^{13}$ and $\gamma = \lambda\, e^{13}$ for some constants λ and μ. The components of (8) not containing e^4 give

$$\mu\, e^{1325} = -e^{125} \wedge \phi_5.$$

Since as a consequence of (9) ϕ_5 is a multiple of e^3, it follows that $\phi_5 = -\mu\, e^3$. Summing up,

$$\phi = -\mu\, e^{35} + \lambda\, e^{13},$$

so that ϕ and de^4 are either linearly independent or both zero. The resulting six-dimensional Lie algebras are

$$(0, 0, 0, 0, 12, 13), \quad (0, 0, 0, 0, 0, 0).$$

If e^2 is not closed, V^5 is $(0, 0, 0, 12, 13)$. As both de^4 and de^2 are in $\Lambda^2 V^3$, (11) implies that γ is a multiple of e^{13}. Therefore, ϕ lies in $\Lambda^2 V^3$ as well, forcing $\mathfrak{g}$ to be either $(0, 0, 0, 12, 13, 23)$ or $(0, 0, 0, 0, 12, 13)$.

For the convenience of the reader, we write down explicitly examples of symplectic half-flat structures on the nilmanifolds associated to the Lie algebras of Theorem 2.4. Consistently with Lemma 2.2, we apply (2) to the examples of page 4. With notation as introduced

before Lemma 2.3:

(1) On $\mathfrak{g} = (0, 0, 0, 0, 12, 13)$, we set

$$\omega = \eta^{35} + \eta^{41} - \eta^{62}, \quad \Psi = (\eta^3 + i\eta^5) \wedge (\eta^4 + i\eta^1) \wedge (-\eta^6 + i\eta^2);$$

then $d\omega = 0 = d\psi^+$. Similarly,

$$\omega = \eta^{25} - \eta^{43} + \eta^{61}, \quad \Psi = (\eta^2 + i\eta^5) \wedge (\eta^4 - i\eta^3) \wedge (\eta^6 + i\eta^1)$$

is a symplectic half-flat structure on $\mathfrak{g}$.

(2) On $\mathfrak{g} = (0, 0, 0, 12, 13, 23)$, we set

$$\omega = \eta^{25} + \eta^{43} - \tfrac{1}{2}\eta^{61}, \quad \Psi = (\eta^2 + i\eta^5) \wedge (\eta^4 + i\eta^3) \wedge \left(-\tfrac{1}{2}\eta^6 + i\eta^1\right).$$

Again, $d\omega = 0 = d\psi^+$.

3. Circle bundles, quotients and intrinsic torsion

In this section, we pursue an idea introduced in Section 2, namely that of reducing a six-dimensional manifold to a five-dimensional manifold by means of a quotient, and establishing a relation between the two geometries in terms of G-structures. Here we work in a more general context, without requiring invariance under a transitive action; however, for the construction to make sense we still need invariance along one direction, that is, a Killing vector field X on the six-manifold. Then the directions orthogonal to X give a rank 5 distribution on which an SU(2)-structure is induced. Thus, if X is regular, the leaf space is a five-dimensional manifold with what we call the quotient SU(2)-structure. Otherwise, this description is valid only locally.

We are interested in the intrinsic torsion of this induced SU(2)-structure. In particular, we characterize the intrinsic torsion of the SU(2)-structures obtained by taking the quotient of a symplectic half-flat structure, generalizing Lemma 2.2. Then, in the assumption that the starting SU(3)-structure is integrable, we write down a differential equation that the norm t of the Killing vector field must satisfy and prove that the intrinsic torsion of the quotient structure depends only on t. More precisely, we give necessary and sufficient conditions on (N, α, ω_i, t) for it to arise, locally, as the quotient of a six-manifold with an integrable SU(3)-structure. Observe that in the integrable case the six-manifold cannot be compact, unless it is reducible.

We consider objects of the form (M, ω, ψ^+, X), where M is a six-dimensional manifold, (ω, ψ^+) is an SU(3)-structure on M and X is a regular vector field on M which preserves the SU(3)-structure, that is,

$$\mathcal{L}_X\omega = 0 = \mathcal{L}_X\psi^+.$$

The space of integral lines of X is a five-manifold N. Since X is a Killing vector field, the norm of X is constant along the integral lines, and so defines a smooth function t on N. Let η be the the dual form to X, rescaled so that $\eta(X) = 1$; set $\phi = d\eta$. It is easy to check that

$$\mathcal{L}_X\phi = 0, \quad X \lrcorner \phi = 0. \tag{14}$$

In general, a form ϕ on M satisfying (14) is said to be basic, and it is the pullback of a form on N. In the sequel, we shall identify basic forms on M with forms on N. Note that if M is a circle bundle

over N, its Chern class is represented by $1/2\pi$. Now define forms (α, ω_i) on M by

$$
\begin{aligned}
\alpha &= X \lrcorner \omega, \quad \omega_3 = t X \lrcorner (\omega \wedge \eta), \\
\omega_1 &= X \lrcorner \psi^+, \quad \omega_2 = X \lrcorner \psi^-.
\end{aligned}
\tag{15}
$$

By construction, all the forms appearing on the right-hand side of (15) are basic.

Now choose a local orthonormal basis of 1-forms

$$
\frac{1}{\sqrt{t}} e^1, \ldots, \frac{1}{\sqrt{t}} e^4, \frac{1}{t} e^5, e^6
$$

on M, with $\eta = t^{-1} e^6$, such that

$$
\omega = \frac{1}{t}(e^{14} + e^{23} + e^{65}), \quad \Psi = \frac{1}{t}(e^1 + i e^4) \wedge (e^2 + i e^3) \wedge \left(e^6 + \frac{1}{t} i e^5 \right).
$$

Then (4) is satisfied, so (α, ω_i) defines an SU(2)-structure on N. In particular, $e^1, \ldots, e^5$ is an orthonormal basis of 1-forms on N: so, we are not using the quotient metric on the five-manifold, but a modified metric obtained by $\mathcal{D}$-homothety.

Conversely, suppose we have

$$
(N, \alpha, \omega_1, \omega_2, \omega_3, \phi, t),
$$

where N is a five-manifold, (α, ω_i) is an SU(2)-structure on N, t is a function on N and ϕ is a closed two-form on N such that

$$
\left[\frac{1}{2\pi} \phi \right] \in H^2(N, \mathbb{Z}).
$$

Let M be a circle bundle over N with Chern class $[\phi/(2\pi)]$. Let A be the standard generator of the Lie algebra $\mathfrak{u}(1)$, and let $X = A^*$ be the associated fundamental vector field. Choose a connection form η such that $d\eta = \phi$ and define

$$
\begin{aligned}
\omega &= t^{-1} \omega_3 + \eta \wedge \alpha, \\
\Psi &= (\omega_1 + i \omega_2) \wedge (\eta + i t^{-2} \alpha).
\end{aligned}
\tag{16}
$$

It is clear that this defines an SU(3)-structure preserved by X, which has norm $t\eta(X) = t$, and equations (15) give back the original SU(2)-structure on N.

REMARK: The reduction we have chosen behaves well with respect to evolution theory. Indeed, let N be a five-manifold with an SU(2)-structure; recall that N is called hypo if the forms ω_1, $\omega_2 \wedge \alpha$, and $\omega_3 \wedge \alpha$ are closed. It is easy to verify that the SU(3)-structure induced on $N \times S^1$ by the above construction (corresponding to taking $\phi = 0$ and $t = 1$) is half-flat if and only if the structure on N is hypo. Moreover, hypo geometry also has evolution equations similar to those mentioned in the introduction, and it turns out that a one-parameter solution of the hypo evolution equations lifts to a solution of the half-flat evolution equations.

Recall from [4] that the intrinsic torsion of an SU(3)-structure takes values in a 42-dimensional space, and its components can be represented as follows:

$$
\begin{array}{|c|c|}
\hline
W_1^+ & W_1^- \\
\hline
W_2^+ & W_2^- \\
\hline
\multicolumn{2}{|c|}{W_3} \\
\hline
\multicolumn{2}{|c|}{W_4} \\
\hline
\multicolumn{2}{|c|}{W_5} \\
\hline
\end{array}
\quad \in \quad
\begin{array}{|c|c|}
\hline
\mathbb{R} & \mathbb{R} \\
\hline
[\Lambda_0^{1,1}] & [\Lambda_0^{1,1}] \\
\hline
\multicolumn{2}{|c|}{[\![\Lambda_0^{2,1}]\!]} \\
\hline
\multicolumn{2}{|c|}{[\![\Lambda^{1,0}]\!]} \\
\hline
\multicolumn{2}{|c|}{[\![\Lambda^{1,0}]\!]} \\
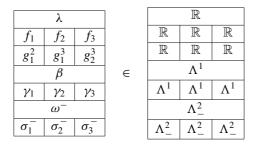\hline
\end{array}
\tag{17}
$$

meaning that the component W_1^+ takes values in $\mathbb{R}$ and so on. Explicitly, we can write

$$
\begin{aligned}
d\psi^+ &= \psi^+ \wedge W_5 + W_2^+ \wedge \omega + W_1^+ \omega^2, \\
d\psi^- &= \psi^- \wedge W_5 + W_2^- \wedge \omega + W_1^- \omega^2, \\
d\omega &= -\frac{3}{2} W_1^- \psi^+ + \frac{3}{2} W_1^+ \psi^- + W_3 + W_4 \wedge \omega.
\end{aligned}
\tag{18}
$$

We can do the same for SU(2)-structures on five-manifolds [6]; the intrinsic torsion now takes values in a 35-dimensional space, and we can arrange its components in the following table.

$$
\begin{array}{|c|c|c|}
\hline
\multicolumn{3}{|c|}{\lambda} \\
\hline
f_1 & f_2 & f_3 \\
\hline
g_1^2 & g_1^3 & g_2^3 \\
\hline
\multicolumn{3}{|c|}{\beta} \\
\hline
\gamma_1 & \gamma_2 & \gamma_3 \\
\hline
\multicolumn{3}{|c|}{\omega^-} \\
\hline
\sigma_1^- & \sigma_2^- & \sigma_3^- \\
\hline
\end{array}
\quad \in \quad
\begin{array}{|c|c|c|}
\hline
\multicolumn{3}{|c|}{\mathbb{R}} \\
\hline
\mathbb{R} & \mathbb{R} & \mathbb{R} \\
\hline
\mathbb{R} & \mathbb{R} & \mathbb{R} \\
\hline
\multicolumn{3}{|c|}{\Lambda^1} \\
\hline
\Lambda^1 & \Lambda^1 & \Lambda^1 \\
\hline
\multicolumn{3}{|c|}{\Lambda_-^2} \\
\hline
\Lambda_-^2 & \Lambda_-^2 & \Lambda_-^2 \\
\hline
\end{array}
$$

In the above table, Λ^1 is the four-dimensional representation of SU(2) such that the tangent space at a point is

$$
T = \Lambda^1 \oplus \mathbb{R},
$$

whereas Λ_-^2 is the three-dimensional representation of SU(2) consisting of anti-self-dual two-forms on Λ^1. We shall write, say, $(\omega)_{\Lambda_-^2}$ for the Λ_-^2 component of a two-form ω.

Setting $g_i^j = -g_j^i$, the components of the intrinsic torsion are given by

$$
d\alpha = \alpha \wedge \beta + \sum_{j=1}^{3} f^j \omega_j + \omega^-,
$$

$$
d\omega_i = \gamma_i \wedge \omega_i + \lambda \alpha \wedge \omega_i + \sum_{j \neq i} g_i^j \alpha \wedge \omega_j + \alpha \wedge \sigma_i^-.
\tag{19}
$$

We can now prove the following.

PROPOSITION 3.1 *Define the intrinsic torsion of M as above, and write*

$$W_i = \eta \wedge \Xi_i + \Delta_i, \quad where \quad \Xi_i = X \lrcorner W_i; \tag{20}$$

then the intrinsic torsion of the quotient is given as follows.

$-\langle \Delta_5, \alpha \rangle$		
$\frac{3}{2} W_1^-$	$-\frac{3}{2} W_1^+ - \frac{1}{2}\langle \Xi_3, \omega_2 \rangle$	$-t^{-1}\Xi_4 - \frac{1}{2}\langle \Xi_3, \omega_3 \rangle$
$-t^{-2}\Xi_5$	$-2t^{-1}W_1^+ - t^{-1}\langle \Xi_2^+, \alpha \rangle$	$-2t^{-1}W_1^- - t^{-1}\langle \Xi_2^-, \alpha \rangle$
$-\left(\Delta_4\right)_{\Lambda^1} - \alpha \lrcorner \Xi_3$		
$-\left(\Delta_5\right)_{\Lambda^1} - t^{-1}\Xi_2^+ \lrcorner \omega_2$	$-\left(\Delta_5\right)_{\Lambda^1} + t^{-1}\Xi_2^- \lrcorner \omega_1$	$\left(\Delta_4 + d\log t + \frac{1}{2}t\omega_3 \lrcorner \Delta_3\right)_{\Lambda^1}$
$-\left(\Xi_3\right)_{\Lambda_-^2}$		
$-\Delta_2^+$	$-\Delta_2^-$	$t\left(\alpha \lrcorner \Delta_3 - \phi\right)_{\Lambda_-^2}$

Proof. Taking the interior product of (18) with X, then substituting (15) in the left-hand side and (16) in the right-hand side, one easily computes

$$d\alpha = \frac{3}{2}W_1^-\omega_1 - \frac{3}{2}W_1^+\omega_2 - \Xi_3 - t^{-1}\Xi_4\,\omega_3 + \Delta_4 \wedge \alpha,$$

$$d\omega_1 = -\omega_1 \wedge \Delta_5 - t^{-2}\Xi_5\,\omega_2 \wedge \alpha - t^{-1}\Xi_2^+ \wedge \omega_3 - \Delta_2^+ \wedge \alpha - 2t^{-1}W_1^+\omega_3 \wedge \alpha,$$

$$d\omega_2 = -\omega_2 \wedge \Delta_5 + t^{-2}\Xi_5\,\omega_1 \wedge \alpha - t^{-1}\Xi_2^- \wedge \omega_3 - \Delta_2^- \wedge \alpha - 2t^{-1}W_1^-\omega_3 \wedge \alpha.$$

On the other hand, $d\omega_3 = d\log t \wedge \omega_3 - t\, X \lrcorner d(\omega \wedge \eta)$ by (15). Using (18) and then (16), we obtain

$$d(\omega \wedge \eta) = \frac{3}{2}t^{-2}W_1^-\omega_2 \wedge \alpha \wedge \eta + \frac{3}{2}t^{-2}W_1^+\omega_1 \wedge \alpha \wedge \eta + \Delta_3 \wedge \eta$$
$$+ t^{-1}\Delta_4 \wedge \omega_3 \wedge \eta + t^{-1}\omega_3 \wedge \phi + \eta \wedge \alpha \wedge \phi;$$

therefore

$$d\omega_3 = d\log t \wedge \omega_3 + \frac{3}{2}t^{-1}W_1^-\omega_2 \wedge \alpha + \frac{3}{2}t^{-1}W_1^+\omega_1 \wedge \alpha + t\Delta_3 + \Delta_4 \wedge \omega_3 - t\alpha \wedge \phi.$$

The decompositions (20) of the components W_3 and $W_2^\pm$ correspond to projections

$$[\![\Lambda_0^{2,1}]\!] \to \omega_1^\perp \oplus (\alpha \wedge \omega_1)^\perp, \quad [\Lambda_0^{1,1}] \to T \oplus \Lambda_-^2, \tag{21}$$

respectively; the statement is now a straightforward consequence of (19).

REMARK: Of the decompositions (21), the first is not surjective; in other words, the components Ξ_3 and Δ_3 are not independent.

It is clear that one can write down a converse to Proposition 3.1, because the quotient determines the intrinsic torsion of M; one can then characterize the M with special intrinsic torsion in terms of

the quotient. For example, in the symplectic half-flat case, one obtains the following generalization of Lemma 2.2.

PROPOSITION 3.2 *M is symplectic half-flat if and only if the quotient satisfies*

$$d\alpha = 0, \quad d\omega_1 = 0, \quad d\omega_2 \wedge \alpha = t^2 \omega_1 \wedge \phi + 2d \log t \wedge \omega_2 \wedge \alpha,$$
$$d\omega_3 = d \log t \wedge \omega_3 - t\alpha \wedge \phi. \tag{22}$$

Proof. This follows immediately from (16).

We now consider the case where M is integrable. In order to state our theorem, we need to introduce two differential operators on five-manifolds with an SU(2)-structure. The first one is ∂_α, which maps a function f to $\langle \alpha, df \rangle$. Secondly, consider the endomorphism J_3 of T^*N characterized by

$$J_3\alpha = 0, \quad \omega_1 \wedge \beta = \omega_2 \wedge J_3\beta \quad \text{for } \beta \in \alpha^\perp;$$

we can then define an operator d^c which maps a function f to $d^c f = J_3 df$. We can now express the intrinsic torsion in terms of the norm t of the Killing vector field.

THEOREM 3.3 *If the* SU(3)-*structure on M is integrable, t is a solution of*

$$\partial_\alpha^2 \log t - (\partial_\alpha \log t)^2 - 2t^{-1} \|(d \log t)_{\Lambda^1}\|^2 = 0. \tag{23}$$

The intrinsic torsion is determined by t as follows: α, ω_1 and ω_2 are closed, and ω_3 satisfies

$$d\omega_3 = (d \log t)_{\Lambda^1} \wedge \omega_3 + \frac{1}{\partial_\alpha t} \alpha \wedge (2d \log t \wedge d^c \log t - dd^c \log t)_{\Lambda^2_-}; \tag{24}$$

moreover the 'curvature form' is

$$\phi = t^{-1}\partial_\alpha \log t\, \omega_3 - \frac{1}{t^2 \partial_\alpha \log t}(2d \log t \wedge d^c \log t - dd^c \log t)_{\Lambda^2_-} - 2t^{-2}\alpha \wedge d^c \log t. \tag{25}$$

Conversely, let N be a five-manifold with an SU(2)-*structure* (α, ω_i) *and a function t, where α, ω_1 and ω_2 are closed, and* (23), (24) *are satisfied. Then the two-form ϕ defined by* (25) *is closed; if the cohomology class $[\phi/2\pi]$ is an element of $H^2(N, \mathbb{Z})$, there is a circle bundle over N on which an integrable* SU(3)-*structure is defined by* (16), *where η is a connection form such that $d\eta = \phi$.*

Locally, Theorem 3.3 is a characterization. Indeed, in the second part one can restrict N to a contractible open subset N', so that

$$0 = [\phi/2\pi] \in H^2(N', \mathbb{Z}).$$

Proof. It is clear from (15) that α, ω_1 and ω_2 are closed. Hence, the only non-vanishing components of the intrinsic torsion are γ_3 and σ_3^-, determined by

$$d\omega_3 = \gamma_3 \wedge \omega_3 + \alpha \wedge \sigma_3^-.$$

Using (16), we find

$$t^{-1}\omega_3 \wedge (\gamma_3 - d\log t) + \alpha \wedge (t^{-1}\sigma_3^- + \phi) = 0.$$

Hence $\gamma_3 = (d\log t)_{\Lambda^1}$, and the component of ϕ in $\Lambda^2(\alpha^\perp)$ is determined by

$$\langle\phi, \omega_1\rangle = 0 = \langle\phi, \omega_2\rangle, \quad \langle\phi, \omega_3\rangle = 2t^{-1}\partial_\alpha \log t, \quad (\phi)_{\Lambda^2_-} = -t^{-1}\sigma_3^-.$$

It also follows from (16) that

$$\omega_1 \wedge \phi + 2t^{-3}dt \wedge \omega_2 \wedge \alpha = 0,$$
$$\omega_2 \wedge \phi - 2t^{-3}dt \wedge \omega_1 \wedge \alpha = 0.$$

Therefore

$$\alpha \lrcorner \phi = -2t^{-2}d^c \log t.$$

For brevity, we set $s = \partial_\alpha \log t$. By construction ϕ is closed, so

$$0 = t^{-1}(-s\, d\log t \wedge \omega_3 + ds \wedge \omega_3 + s\,(d\log t - s\,\alpha) \wedge \omega_3 + s\,\alpha \wedge \sigma_3^-$$
$$+ d\log t \wedge \sigma_3^- d\sigma_3^-) + 2t^{-2}\alpha \wedge (-2d\log t \wedge d^c \log t + dd^c \log t). \tag{26}$$

We can split (26) into two equations by taking the wedge and the interior product with α. One of these is satisfied automatically: indeed, taking d of $d\omega_3$ we find

$$0 = \alpha \wedge (d\log t \wedge \sigma_3^- - d\sigma_3^- + ds \wedge \omega_3),$$

so the right-hand side of (26) vanishes on wedging with α. Taking the interior product gives

$$(\partial_\alpha s - s^2)\omega_3 + 2s\sigma_3^- + 2t^{-1}(-2d\log t \wedge d^c \log t + dd^c \log t)_{\Lambda^2(\alpha^\perp)}.$$

It is now clear that σ_3^- can be expressed in terms of t, giving (24). Using the general formula

$$\langle\beta \wedge J_3\beta, \omega_3\rangle = \|\beta\|^2 - \langle\beta, \alpha\rangle^2,$$

we also deduce that t satisfies (23).

Conversely, suppose (24) and (23) are satisfied, and define ϕ by (25). The above calculations show that ϕ is closed and (16) defines an integrable SU(3)-structure.

REMARK: The condition of Theorem 3.3 implies in particular that 28 out of the 35 components of the intrinsic torsion of (N, α, ω_i) vanish. A similar construction was described in [1], starting with a seven-manifold with holonomy G_2 and defining an SU(3)-structure on the quotient. In that case, the vanishing components of the intrinsic torsion of the quotient are also 28, though out of 42.

In general, (23) and (24) are not independent, because the norm on one-forms depends on ω_3. Motivated by this observation, we consider the special case

$$(d \log t)_{\Lambda^1} = 0.$$

In order to apply Theorem 3.3, we have to assume that the SU(2)-structure is integrable. Let x be a coordinate in the direction of α, so that $\alpha = dx$. Set $t = (1 - x)^{-1}$; then (23) is satisfied. Suppose that one has a circle bundle over N with Chern class $[1/2\pi \omega_3]$; then the hypotheses of Theorem 3.3 hold. Define a connection form η such that $d\eta = \omega_3$; then

$$\omega = (1 - x)\omega_3 + \eta \wedge \alpha, \quad \Psi = (\omega_1 + i\omega_2) \wedge (\eta + i(1 - x)^2\alpha)$$

defines an integrable SU(3)-structure. One can actually prove that if the original five-manifold has holonomy SU(2), then the Calabi–Yau six-manifold has holonomy SU(3).

Acknowledgements

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