

SPECIAL VALUES OF THE LERCH ZETA FUNCTION AND THE EVALUATION OF CERTAIN INTEGRALS

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ABSTRACT. The Lerch zeta function $\Phi(x, a, s)$ is defined by the series

$$\Phi(x, a, s) = \sum_{n=0}^{\infty} \frac{e^{2n\pi ix}}{(n+a)^s},$$

where x is real, $0 < a \leq 1$, and $\sigma = \operatorname{Re}(s) > 1$ if x is an integer and $\sigma > 0$ otherwise. In this paper we study the function $J(s, a) = \Phi(\frac{1}{2}, a, s)$. We use its integral representation

$$J(s, a) = \frac{a^{-s}}{2} + 2 \int_0^{\infty} (a^2 + y^2)^{-s/2} \sin\left(s \tan^{-1} \frac{y}{a}\right) \frac{e^{\pi y} dy}{e^{2\pi y} - 1}$$

to obtain the values of certain definite integrals; for example, we show that

$$\int_0^{\infty} \frac{\cosh x \log x}{\cosh 2x - \cos 2\pi a} dx = \frac{\pi}{2 \sin \pi a} \left\{ \log \frac{\Gamma((1+a)/2)}{\Gamma(a/2)} + \frac{1}{2} \log \left(2\pi \cot \frac{\pi a}{2} \right) \right\}, \quad 0 < a < 1.$$

1. INTRODUCTION

The Lerch zeta function $\Phi(x, a, s)$ is defined by the series

$$\Phi(x, a, s) = \sum_{n=0}^{\infty} \frac{e^{2n\pi ix}}{(n+a)^s},$$

where x is real, $0 < a \leq 1$, and $\sigma = \operatorname{Re}(s) > 1$ if x is an integer and $\sigma > 0$ otherwise. In this paper we consider the special case of the Lerch zeta function

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$\Phi(x, a, s)$ when $x = \frac{1}{2}$. We denote $\Phi(\frac{1}{2}, a, s)$ by $J(s, a)$, so that

$$(1.1) \quad J(s, a) = \Phi\left(\frac{1}{2}, a, s\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s}, \quad 0 < a \leq 1.$$

The function $J(s, a)$ is related to the Hurwitz zeta function $\zeta(s, a)$ by the formula

$$(1.2) \quad J(s, a) = 2^{1-s} \zeta(s, a/2) - \zeta(s, a), \quad \sigma > 1.$$

Appealing to the Hermite formula for $\zeta(s, a)$,

$$(1.3) \quad \zeta(s, a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + 2 \int_0^{\infty} (a^2 + y^2)^{-s/2} \sin\left(s \tan^{-1} \frac{y}{a}\right) \frac{dy}{e^{2\pi y} - 1},$$

we obtain in §2 the analogue of the Hermite formula for $J(s, a)$, namely,

$$(1.4) \quad J(s, a) = \frac{a^{-s}}{2} + 2 \int_0^{\infty} (a^2 + y^2)^{-s/2} \sin\left(s \tan^{-1} \frac{y}{a}\right) \frac{e^{\pi y} dy}{e^{2\pi y} - 1}.$$

This formula enables $J(s, a)$ to be continued analytically to the whole complex plane, and in §2 we obtain the values

$$(1.5) \quad J(1, a) = \frac{1}{2} \left\{ \frac{\Gamma'((1+a)/2)}{\Gamma((1+a)/2)} - \frac{\Gamma'(a/2)}{\Gamma(a/2)} \right\},$$

$$(1.6) \quad J'(0, a) = \log \frac{\Gamma(a/2)}{\Gamma((1+a)/2)} - \frac{1}{2} \log 2.$$

In §3, making use of the integral representation

$$J(s, a) = \frac{e^{-\pi i s} \Gamma(1-s)}{2\pi i} \int_C \frac{e^{(1-a)z} z^{s-1}}{e^z + 1} dz$$

of $J(s, a)$, where C is the contour consisting of the real axis from $+\infty$ to ε , the circle $|z| = \varepsilon$, and the real axis from ε to $+\infty$, we show that $J(s, a)$ can be expressed in the form

$$(1.7) \quad J(s, a) = \frac{\Gamma(1-s)}{\pi^{1-s}} \left\{ \sin \frac{\pi s}{2} C(1-s, a) + \cos \frac{\pi s}{2} S(1-s, a) \right\}, \quad \sigma < 1,$$

where

$$C(s, a) = \sum_{m=0}^{\infty} \frac{\cos(2m+1)\pi a}{(2m+1)^s}, \quad S(s, a) = \sum_{m=0}^{\infty} \frac{\sin(2m+1)\pi a}{(2m+1)^s}, \quad \sigma > 0.$$

From (1.7) and (1.6) we obtain the value of $S'(1, a)$. We also obtain integral representations of $C(s, a)$, $S(s, a)$, and $J(s, a)$. These representations are then used to evaluate some definite integrals. For example, we prove

$$(1.8) \quad \int_0^{\infty} \frac{\cosh x \log x}{\cosh 2x - \cos 2\pi a} dx = \frac{\pi}{2 \sin \pi a} \left\{ \log \frac{\Gamma((1+a)/2)}{\Gamma(a/2)} + \frac{1}{2} \log \left(2\pi \cot \frac{\pi a}{2} \right) \right\}, \quad 0 < a < 1.$$

In §4, we consider the function $S(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = 2^{-s} J(s, \frac{1}{2})$. It is shown that $S(s)$ satisfies the functional equation

$$(1.9) \quad S(s) = \left(\frac{\pi}{2}\right)^{s-1} \cos \frac{\pi s}{2} \Gamma(1-s) S(1-s).$$

Using contour integration, we derive simultaneously recurrence relations for $S(2n+1)$ and $S(2n)$.

2. THE HERMITE FORMULA FOR $J(s, a)$

The Lerch zeta function is defined by the series

$$(2.1) \quad \Phi(x, a, s) = \sum_{n=0}^{\infty} \frac{e^{2n\pi i x}}{(n+a)^s},$$

where x is real, $0 < a \leq 1$, and $\sigma = \operatorname{Re}(s) > 1$ if x is an integer and $\sigma > 0$ otherwise. It is clear that $\Phi(0, a, s) = \zeta(s, a)$ is the Hurwitz zeta function and that $\Phi(0, 1, s) = \zeta(s)$ is the Riemann zeta function. In this paper, we consider the special case of the Lerch zeta function $\Phi(x, a, s)$ when $x = \frac{1}{2}$. We denote $\Phi(\frac{1}{2}, a, s)$ by $J(s, a)$ so that

$$(2.2) \quad J(s, a) = \Phi\left(\frac{1}{2}, a, s\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s}, \quad 0 < a \leq 1.$$

We begin by obtaining the analogue for $J(s, a)$ of the Hermite formula for $\zeta(s, a)$ and use it to determine the values of $J(0, a)$, $J(1, a)$, and $J'(0, a)$. Since for $\sigma > 1$

$$\begin{aligned} J(s, a) &= \sum_{n=0}^{\infty} \frac{1}{(2n+a)^s} - \sum_{n=0}^{\infty} \frac{1}{(2n+1+a)^s} \\ &= \frac{1}{2^s} \sum_{n=0}^{\infty} \frac{1}{(n+a/2)^s} - \frac{1}{2^s} \sum_{n=0}^{\infty} \frac{1}{(n+(1+a)/2)^s}, \end{aligned}$$

we have

$$(2.3) \quad J(s, a) = \frac{1}{2^s} \zeta\left(s, \frac{a}{2}\right) - \frac{1}{2^s} \zeta\left(s, \frac{1+a}{2}\right), \quad \sigma > 1.$$

Similarly we have

$$\zeta(s, a) = \frac{1}{2^s} \zeta\left(s, \frac{a}{2}\right) + \frac{1}{2^s} \zeta\left(s, \frac{1+a}{2}\right), \quad \sigma > 1.$$

Hence we have

$$(2.4) \quad J(s, a) = \zeta(s, a) - \frac{1}{2^{s-1}} \zeta\left(s, \frac{1+a}{2}\right), \quad \sigma > 1,$$

$$(2.5) \quad J(s, a) = \frac{1}{2^{s-1}} \zeta\left(s, \frac{a}{2}\right) - \zeta(s, a), \quad \sigma > 1.$$

Since $\zeta(s, a)$ can be continued analytically to the whole complex plane except for a simple pole at $s = 1$ with residue 1, $J(s, a)$ can be continued analytically to become an entire function and (2.3)–(2.5) hold in the whole plane.

An important property of $\zeta(s, a)$ is the Hermite formula (valid for all $s \neq 1$; see, e.g., [2, p. 270])

$$(2.6) \quad \zeta(s, a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + 2 \int_0^\infty (a^2 + y^2)^{-s/2} \sin\left(s \tan^{-1} \frac{y}{a}\right) \frac{dy}{e^{2\pi y} - 1}.$$

We have

$$\zeta\left(s, \frac{a}{2}\right) = 2^{s-1} a^{-s} + \frac{2^{s-1} a^{1-s}}{s-1} + 2^s \int_0^\infty (a^2 + y^2)^{-s/2} \sin\left(s \tan^{-1} \frac{y}{a}\right) \frac{dy}{e^{\pi y} - 1},$$

and from (2.5) we obtain the following result.

Proposition 1. For all s and $0 < a \leq 1$

$$(2.7) \quad J(s, a) = \frac{a^{-s}}{2} + 2 \int_0^\infty (a^2 + y^2)^{-s/2} \sin\left(s \tan^{-1} \frac{y}{a}\right) \frac{e^{\pi y} dy}{e^{2\pi y} - 1}.$$

In particular, we have

$$(2.8) \quad J(0, a) = \frac{1}{2}$$

and

$$J(1, a) = \frac{1}{2a} + 2 \int_0^\infty \frac{\sin(\tan^{-1}(y/a)) e^{\pi y}}{\sqrt{a^2 + y^2} (e^{2\pi y} - 1)} dy.$$

Since $\sin(\tan^{-1}(y/a)) = y/\sqrt{a^2 + y^2}$, we have

$$\begin{aligned} J(1, a) &= \frac{1}{2a} + 2 \int_0^\infty \frac{y e^{\pi y} dy}{(a^2 + y^2)(e^{2\pi y} - 1)} \\ &= \frac{1}{2a} + 2 \int_0^\infty \frac{y}{a^2 + y^2} \left(\frac{1}{e^{\pi y} - 1} - \frac{1}{e^{2\pi y} - 1} \right) dy \\ &= \frac{1}{2a} + 2 \int_0^\infty \frac{y}{(a/2)^2 + y^2} \cdot \frac{dy}{e^{2\pi y} - 1} - 2 \int_0^\infty \frac{y}{a^2 + y^2} \cdot \frac{dy}{e^{2\pi y} - 1}. \end{aligned}$$

Appealing to the following formula [2, p. 251] for the gamma function $\Gamma(s)$

$$\frac{\Gamma'(a)}{\Gamma(a)} = \log a - \frac{1}{2a} - 2 \int_0^\infty \frac{y dy}{(a^2 + y^2)(e^{2\pi y} - 1)},$$

we have

$$J(1, a) = \frac{1}{2a} + \left\{ \log \frac{a}{2} - \frac{1}{a} - \frac{\Gamma'(a/2)}{\Gamma(a/2)} \right\} - \left\{ \log a - \frac{1}{2a} - \frac{\Gamma'(a)}{\Gamma(a)} \right\},$$

giving the following result.

Proposition 2. For $0 < a \leq 1$

$$(2.9) \quad J(1, a) = \frac{\Gamma'(a)}{\Gamma(a)} - \frac{\Gamma'(a/2)}{\Gamma(a/2)} - \log 2.$$

Proposition 2 enables us to determine $\Gamma'(\frac{1}{2})$. Taking $a = 1$ in Proposition 2 and recalling that

$$J(1, 1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \log 2, \quad \Gamma(1) = 1, \quad \Gamma'(1) = -\gamma, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

where γ is Euler’s constant, we obtain the well-known result

$$\Gamma'(\frac{1}{2}) = -(\gamma + 2 \log 2)\sqrt{\pi}.$$

We can also obtain (2.9) by using (2.5) as follows:

$$\zeta(s, a) = \frac{1}{s-1} - \frac{\Gamma'(a)}{\Gamma(a)} + c_1(s-1) + c_2(s-1)^2 + \dots,$$

$$\begin{aligned} J(s, a) &= \frac{1}{2^{s-1}} \zeta\left(s, \frac{a}{2}\right) - \zeta(s, a) \\ &= \{1 - (\log 2)(s-1) + \dots\} \left\{ \frac{1}{s-1} - \frac{\Gamma'(a/2)}{\Gamma(a/2)} + \dots \right\} \\ &\quad - \left\{ \frac{1}{s-1} - \frac{\Gamma'(a)}{\Gamma(a)} + \dots \right\} \\ &= \left\{ \frac{\Gamma'(a)}{\Gamma(a)} - \frac{\Gamma'(a/2)}{\Gamma(a/2)} - \log 2 \right\} + A_1(s-1) + A_2(s-1)^2 + \dots, \end{aligned}$$

which gives (2.9). From (2.3) and (2.4) we obtain two other forms of (2.9), namely,

$$(2.9)' \quad J(1, a) = \frac{1}{2} \left\{ \frac{\Gamma'((1+a)/2)}{\Gamma((1+a)/2)} - \frac{\Gamma'(a/2)}{\Gamma(a/2)} \right\},$$

$$(2.9)'' \quad J(1, a) = \frac{\Gamma'((1+a)/2)}{\Gamma((1+a)/2)} - \frac{\Gamma'(a)}{\Gamma(a)} + \log 2.$$

From (2.9)' we obtain a formula of Kummer (see, e.g., [2, p. 262]):

$$(2.10) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n+a} = \int_0^{\infty} \frac{t^{a-1}}{1+t} dt = \frac{1}{2} \left\{ \frac{\Gamma'((1+a)/2)}{\Gamma((1+a)/2)} - \frac{\Gamma'(a/2)}{\Gamma(a/2)} \right\}.$$

Next, we use Proposition 1 to evaluate $J'(0, a)$. We have

$$\begin{aligned} J'(0, a) &= -\frac{1}{2} \log a + 2 \int_0^{\infty} \frac{\tan^{-1}(y/a)e^{\pi y}}{e^{2\pi y} - 1} dy \\ &= -\frac{1}{2} \log a + 2 \int_0^{\infty} \tan^{-1}(y/a) \left(\frac{1}{e^{\pi y} - 1} - \frac{1}{e^{2\pi y} - 1} \right) dy \\ &= -\frac{1}{2} \log a + 4 \int_0^{\infty} \frac{\tan^{-1}(2y/a)}{e^{2\pi y} - 1} dy - 2 \int_0^{\infty} \frac{\tan^{-1}(y/a)}{e^{2\pi y} - 1} dy. \end{aligned}$$

In view of Binet’s formula for $\log \Gamma(a)$ [2, p. 251],

$$\log \Gamma(a) = \left(a - \frac{1}{2}\right) \log a - a + \frac{1}{2} \log(2\pi) + 2 \int_0^{\infty} \frac{\tan^{-1}(y/a)}{e^{2\pi y} - 1} dy,$$

we obtain

$$\begin{aligned} J'(0, a) &= -\frac{1}{2} \log a + 2 \left\{ \log \Gamma\left(\frac{a}{2}\right) - \left(\frac{a}{2} - \frac{1}{2}\right) \log \frac{a}{2} + \frac{a}{2} - \frac{1}{2} \log(2\pi) \right\} \\ &\quad - \left\{ \log \Gamma(a) - \left(a - \frac{1}{2}\right) \log a + a - \frac{1}{2} \log(2\pi) \right\} \\ &= 2 \log \Gamma\left(\frac{a}{2}\right) - \log \Gamma(a) - (a-1) \log 2 - \frac{1}{2} \log(2\pi). \end{aligned}$$

Finally, from the duplication formula

$$\Gamma(a) = \frac{2^{a-1}}{\sqrt{\pi}} \Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{1+a}{2}\right),$$

we obtain the following result.

Proposition 3. For $0 < a \leq 1$

$$(2.11) \quad J'(0, a) = \log \frac{\Gamma(a/2)}{\Gamma((1+a)/2)} - \frac{1}{2} \log 2.$$

We remark that Proposition 3 can also be deduced from (2.5) and the formula

$$(2.12) \quad \zeta'(0, a) = \log \frac{\Gamma(a)}{\sqrt{2\pi}}.$$

The formula (2.12) can be found in [3, Corollary 2] or can be obtained by differentiating both sides of (2.6).

3. EVALUATION OF CERTAIN DEFINITE INTEGRALS

Since

$$\frac{\Gamma(s)}{(n+a)^s} = \int_0^\infty e^{-(n+a)x} x^{s-1} dx, \quad \sigma > 0,$$

we have

$$\begin{aligned} \Gamma(s)J(s, a) &= \sum_{n=0}^{\infty} \int_0^\infty (-1)^n e^{-(n+a)x} x^{s-1} dx \\ &= \int_0^\infty \sum_{n=0}^{\infty} (-1)^n e^{-(n+a)x} x^{s-1} dx = \int_0^\infty \frac{e^{-ax} x^{s-1}}{e^{-x} + 1} dx, \quad \sigma > 0, \end{aligned}$$

that is,

$$(3.1) \quad \Gamma(s)J(s, a) = \int_0^\infty \frac{e^{(1-a)x} x^{s-1}}{e^x + 1} dx, \quad \sigma > 0.$$

By considering the integral of the function $e^{(1-a)z} z^{s-1} / (e^z + 1)$ along the contour C , which starts at infinity on the positive real axis, circles the origin once in the positive direction, and returns to positive infinity, $J(s, a)$ can be continued analytically in the whole plane. Since

$$\begin{aligned} \int_C \frac{e^{(1-a)z} z^{s-1}}{e^z + 1} dz &= (e^{2\pi is} - 1) \int_0^\infty \frac{e^{(1-a)x} x^{s-1}}{e^x + 1} dx, \\ \int_0^\infty \frac{e^{(1-a)x} x^{s-1}}{e^x + 1} dx &= \frac{1}{e^{2\pi is} - 1} \int_C \frac{e^{(1-a)z} z^{s-1}}{e^z + 1} dz \\ &= \frac{e^{-\pi is}}{2i \sin \pi s} \int_C \frac{e^{(1-a)z} z^{s-1}}{e^z + 1} dz, \end{aligned}$$

we have (as $\Gamma(s)\Gamma(1-s) = \pi / \sin \pi s$)

$$(3.2) \quad J(s, a) = \frac{e^{-\pi is} \Gamma(1-s)}{2\pi i} \int_C \frac{e^{(1-a)z} z^{s-1}}{e^z + 1} dz.$$

Next, we evaluate the residue of the function $f(z) = e^{(1-a)z} z^{s-1} / (e^z + 1)$ at $z_m = (2m + 1)\pi i$ and $z'_m = -(2m + 1)\pi i$ ($m \geq 0$):

$$\begin{aligned} \operatorname{Res}(f(z), z_m) &= e^{-az} z^{s-1} \Big|_{z=z_m} = e^{-(2m+1)a\pi i} (2m + 1)^{s-1} \pi^{s-1} e^{(s-1)\pi i/2}, \\ \operatorname{Res}(f(z), z'_m) &= e^{-az} z^{s-1} \Big|_{z=z'_m} = e^{(2m+1)a\pi i} (2m + 1)^{s-1} \pi^{s-1} e^{(3(s-1)\pi i/2)}, \\ \operatorname{Res}(f(z), z_m) + \operatorname{Res}(f(z), z'_m) &= -2(2m + 1)^{s-1} \pi^{s-1} e^{s\pi i} \sin \left\{ \frac{\pi s}{2} + (2m + 1)a\pi \right\}. \end{aligned}$$

By Cauchy’s residue theorem, we have

$$(3.3) \quad J(s, a) = 2\Gamma(1 - s)\pi^{s-1} \sum_{m=0}^{\infty} \frac{\sin(\pi s/2 + (2m + 1)a\pi)}{(2m + 1)^{1-s}}, \quad \sigma < 0,$$

•or

$$(3.4) \quad J(1 - s, a) = \frac{\Gamma(s)}{\pi^s} \left\{ \cos \frac{\pi s}{2} C(s, a) + \sin \frac{\pi s}{2} S(s, a) \right\}, \quad \sigma > 0,$$

where

$$(3.5) \quad C(s, a) = \sum_{m=0}^{\infty} \frac{\cos(2m + 1)a\pi}{(2m + 1)^s}, \quad S(s, a) = \sum_{m=0}^{\infty} \frac{\sin(2m + 1)a\pi}{(2m + 1)^s}, \quad \sigma > 0.$$

From the expansion

$$\log \frac{1 + z}{1 - z} = 2 \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n + 1}, \quad |z| < 1,$$

we have

$$(3.6) \quad S(1, a) = \frac{\pi}{4},$$

$$(3.7) \quad C(1, a) = \frac{1}{2} \log \left(\cot \frac{a\pi}{2} \right).$$

From (3.2) and the generating function of the Euler polynomials

$$\frac{2e^{az}}{e^z + 1} = \sum_{n=0}^{\infty} \frac{E_n(a)}{n!} z^n, \quad |z| < \pi,$$

where $E_n(a)$ is the Euler polynomial of degree n , we have

$$J(-n, a) = \frac{(-1)^n n!}{2 \cdot 2\pi i} \int_{|z|=\epsilon} \sum_{m=0}^{\infty} \frac{E_m(1 - a)}{m!} \cdot \frac{1}{z^{n+1}} dz,$$

that is,

$$(3.8) \quad J(-n, a) = \frac{(-1)^n}{2} E_n(1 - a) = \frac{1}{2} E_n(a) \quad (n \geq 0).$$

(In particular, when $n = 0$, we obtain (2.8) again: $J(0, a) = \frac{1}{2} E_0(a) = \frac{1}{2}$.) On the other hand, from (3.3) we have

$$(3.9) \quad J(-n, a) = \frac{2n!}{\pi^{n+1}} \sum_{m=0}^{\infty} \frac{\sin((2m + 1)a\pi - n\pi/2)}{(2m + 1)^{n+1}}, \quad n \geq 0,$$

so that

$$(3.10) \quad \sum_{m=0}^{\infty} \frac{\sin((2m+1)a\pi - n\pi/2)}{(2m+1)^{n+1}} = \frac{\pi^{n+1} E_n(a)}{4n!}, \quad n \geq 0,$$

and

$$(3.11) \quad S(2n+1, a) = \sum_{m=0}^{\infty} \frac{\sin(2m+1)a\pi}{(2m+1)^{2n+1}} = \frac{(-1)^n \pi^{2n+1} E_{2n}(a)}{4(2n)!}, \quad n \geq 0,$$

$$(3.12) \quad C(2n, a) = \sum_{m=0}^{\infty} \frac{\cos(2m+1)a\pi}{(2m+1)^{2n}} = \frac{(-1)^n \pi^{2n} E_{2n-1}(a)}{4(2n-1)!}, \quad n \geq 0.$$

The formulae (3.11) and (3.12) are the well-known Fourier expansions of the Euler polynomials.

By differentiating both sides of (3.4), we obtain

$$\begin{aligned} & \pi^s (\log \pi) J(1-s, a) - \pi^s J'(1-s, a) \\ &= 2\Gamma'(s) \cos \frac{\pi s}{2} C(s, a) - \pi \Gamma(s) \sin \frac{\pi s}{2} C(s, a) + 2\Gamma(s) \cos \frac{\pi s}{2} C'(s, a) \\ & \quad + 2\Gamma'(s) \sin \frac{\pi s}{2} S(s, a) + \pi \Gamma(s) \cos \frac{\pi s}{2} S(s, a) + 2\Gamma(s) \sin \frac{\pi s}{2} S'(s, a). \end{aligned}$$

Letting $s = 1$ and using (2.8), (3.6), and (3.7), we obtain

$$\frac{\pi}{2} \log \pi - \pi J'(0, a) = -\frac{\pi}{2} \log \left(\cot \frac{\pi a}{2} \right) - \frac{\gamma \pi}{2} + 2S'(1, a),$$

where $\gamma = -\Gamma'(1)$ is Euler's constant. Appealing to Proposition 3, we obtain

Proposition 4. For $0 < a < 1$

$$(3.13) \quad S'(1, a) = \frac{\pi}{2} \log \frac{\Gamma((1+a)/2)}{\Gamma(a/2)} + \frac{\pi}{4} \left\{ \log \left(2\pi \cot \frac{\pi a}{2} \right) + \gamma \right\}.$$

Since

$$S'(1, a) = - \sum_{m=0}^{\infty} \frac{\sin(2m+1)a\pi}{2m+1} \log(2m+1),$$

we have

Corollary. For $0 < a < 1$

$$(3.14) \quad \begin{aligned} & \sum_{m=0}^{\infty} \frac{\sin(2m+1)a\pi}{2m+1} \log(2m+1) \\ &= \frac{\pi}{2} \log \frac{\Gamma(a/2)}{\Gamma((1+a)/2)} - \frac{\pi}{4} \left\{ \log \left(2\pi \cot \frac{\pi a}{2} \right) + \gamma \right\}. \end{aligned}$$

The formula (3.14) can be obtained by using Kummer's formula (see, e.g., [3, (2.28)])

$$\begin{aligned} \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi a}{n} \log n &= \log \Gamma(a) - (\gamma + \log 2\pi) \left(\frac{1}{2} - a \right) \\ & \quad - \frac{1}{2} \log \pi + \frac{1}{2} \log(\sin \pi a), \quad 0 < a < 1, \end{aligned}$$

and the well-known result

$$\sum_{n=1}^{\infty} \frac{\sin 2n\pi a}{n} = \pi \left(\frac{1}{2} - a \right), \quad 0 < a < 1.$$

We have

$$\begin{aligned} & \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin 2(2n+1)a\pi}{2n+1} \log(2n+1) \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi a}{n} \log n - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 4n\pi a}{2n} \log(2n) \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\pi a}{n} \log n - \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin 4n\pi a}{n} \log n - \frac{\log 2}{2\pi} \sum_{n=1}^{\infty} \frac{\sin 4n\pi a}{n} \\ &= \log \Gamma(a) - \frac{1}{2} \log \Gamma(2a) - \frac{1}{4}(\gamma + \log 2\pi) \\ &\quad - \frac{1}{4} \log \pi - \frac{1}{2}(1-2a) - \frac{1}{4} \log(\cot \pi a). \end{aligned}$$

Using the duplication formula

$$\Gamma(2a) = \frac{2^{2a-1}}{\sqrt{\pi}} \Gamma(a) \Gamma\left(a + \frac{1}{2}\right),$$

we have

$$\begin{aligned} & \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin 2(2n+1)a\pi}{2n+1} \log(2n+1) \\ &= \frac{1}{2} \log \frac{\Gamma(a)}{\Gamma(a+1/2)} - \frac{1}{4}(\gamma + \log 2\pi) - \frac{1}{4} \log(\cot \pi a). \end{aligned}$$

Changing a into $a/2$ gives (3.14).

The next step is to obtain integral representations of $S(s, a)$, $C(s, a)$, and $J(s, a)$ ((3.15), (3.16), and (3.17) below). We start with the formula

$$\frac{\Gamma(s)}{(2m+1)^s} = \int_0^{\infty} e^{-(2m+1)x} x^{s-1} dx, \quad \sigma > 0.$$

Multiplying the formula by $\sin(2m+1)\pi a$ and summing over m , we have

$$\begin{aligned} \Gamma(s)S(s, a) &= \int_0^{\infty} \sum_{m=0}^{\infty} e^{-(2m+1)x} \sin(2m+1)\pi a \cdot x^{s-1} dx \\ &= \int_0^{\infty} \operatorname{Im} \left\{ \frac{e^{-x+\pi ai}}{1 - (e^{-x+\pi ai})^2} \right\} x^{s-1} dx, \end{aligned}$$

since $\sum_{m=0}^{\infty} z^{2m+1} = z/(1-z^2)$, $|z| < 1$. If we let $z = e^{-x+\pi ai}$, then

$$\begin{aligned} \Gamma(s)S(s, a) &= \int_0^{\infty} \frac{\operatorname{Im} z(1-\bar{z})}{|1-z^2|^2} x^{s-1} dx = \int_0^{\infty} \frac{\operatorname{Im}(z - |z|^2\bar{z})}{|1-z^2|^2} x^{s-1} dx \\ &= \sin \pi a \int_0^{\infty} \frac{(e^{-x} + e^{-3x})x^{s-1}}{1 - 2e^{-2x} \cos 2\pi a + e^{-4x}} dx, \end{aligned}$$

that is,

$$(3.15) \quad \Gamma(s)S(s, a) = \sin \pi a \int_0^\infty \frac{(\cosh x)x^{s-1}}{\cosh 2x - \cos 2\pi a} dx, \quad \sigma > 0, 0 < a \leq 1.$$

Similarly,

$$(3.16) \quad \Gamma(s)C(s, a) = \cos \pi a \int_0^\infty \frac{(\sinh x)x^{s-1}}{\cosh 2x - \cos 2\pi a} dx,$$

where either $\sigma > 0, 0 < a < 1$ or $\sigma > 1, a = 1$, and

$$(3.17) \quad J(s, a) = \frac{2}{\pi^s} \int_0^\infty \frac{\cos \pi a \cos(\pi s/2) \sinh x + \sin \pi a \sin(\pi s/2) \cosh x}{\cosh 2x - \cos 2\pi a} x^{s-1} dx, \\ 0 < a < 1, \sigma > 0 \text{ or } \sigma > 1, a = 1.$$

From these integral representations we will obtain integral representations of the Euler polynomials. Taking $s = 2n + 1$ in (3.15), we have

$$(2n)!S(2n + 1, a) = \sin \pi a \int_0^\infty \frac{(\cosh x)x^{2n}}{\cosh 2x - \cos 2\pi a} dx.$$

Then, from (3.11) we obtain (3.18), and from (3.12) and (3.16), we obtain (3.19).

Proposition 5.

$$(3.18) \quad E_{2n}(a) = \frac{4(-1)^n \sin \pi a}{\pi^{2n+1}} \int_0^\infty \frac{(\cosh x)x^{2n}}{\cosh 2x - \cos 2\pi a} dx, \quad 0 < a < 1,$$

$$(3.19) \quad E_{2n-1}(a) = \frac{4(-1)^n \cos \pi a}{\pi^{2n}} \int_0^\infty \frac{(\sinh x)x^{2n-1}}{\cosh 2x - \cos 2\pi a} dx, \quad 0 < a < 1.$$

Next we make use of the value of $S'(1, a)$ to evaluate a certain definite integral.

Theorem 1. For $0 < a < 1$

$$(3.20) \quad \int_0^\infty \frac{\cosh x \cdot \log x}{\cosh 2x - \cos 2\pi a} dx \\ = \frac{\pi}{2 \sin \pi a} \left\{ \log \frac{\Gamma((1+a)/2)}{\Gamma(a/2)} + \frac{1}{2} \log \left(2\pi \cot \frac{\pi a}{2} \right) \right\}.$$

Proof. We have

$$\int_0^\infty \frac{\cosh x \cdot \log x}{\cosh 2x - \cos 2\pi a} dx = \frac{1}{\sin \pi a} \{\Gamma(s)S(s, a)\}'|_{s=1} \\ = \frac{1}{\sin \pi a} \{\Gamma(1)S'(1, a) + \Gamma'(1)S(1, a)\} = \frac{1}{\sin \pi a} \{S'(1, a) - \gamma S(1, a)\}.$$

The theorem follows from (3.6) and (3.13). \square

Taking $a = \frac{1}{2}$ in (3.20), the integral on the left-hand side becomes

$$\int_0^\infty \frac{\cosh x \cdot \log x}{\cosh 2x + 1} dx = \frac{1}{2} \int_0^\infty \frac{\log x}{\cosh x} dx = \int_0^\infty \frac{e^x \log x}{e^{2x} + 1} dx \\ = \int_{\pi/4}^{\pi/2} \log \log \tan t dt,$$

where the last integral was obtained by means of the substitution $t = \tan^{-1}(e^x)$. Then Theorem 1 gives

$$\int_{\pi/4}^{\pi/2} \log \log \tan t \, dt = \frac{\pi}{2} \log \frac{\sqrt{2\pi}\Gamma(3/4)}{\Gamma(1/4)},$$

which was obtained in [3].

4. RECURRENCE RELATIONS FOR $S(2n)$ AND $S(2n + 1)$

In this section we obtain the functional equation for $S(s)$ as well as determining $S(s)$ and $S'(s)$ for certain values of s .

Taking $a = \frac{1}{2}$ in (2.2), we obtain

$$(4.1) \quad J\left(s, \frac{1}{2}\right) = 2^s \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = 2^s S(s),$$

where

$$(4.2) \quad S(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}.$$

As $C(s, \frac{1}{2}) = 0$, $S(s, \frac{1}{2}) = S(s)$, we have from (3.4)

$$2^{1-s} S(1-s) = \frac{2\Gamma(s)}{\pi^s} \sin \frac{\pi s}{2} S(s),$$

that is,

$$(4.3) \quad S(s) = \left(\frac{\pi}{2}\right)^s \cos \frac{\pi s}{2} \Gamma(1-s) S(1-s),$$

which is the well-known functional equation for $S(s)$.

From (3.1) and (4.1) we obtain

$$(4.4) \quad \Gamma(s)S(s) = \int_0^{\infty} \frac{e^x x^{s-1}}{e^{2x} + 1} dx = \frac{1}{2} \int_0^{\infty} \frac{x^{s-1}}{\cosh x} dx, \quad \sigma > 0.$$

From (4.1) and (2.11) we have

$$(4.5) \quad S'(0) = \log \frac{\Gamma(1/4)}{\Gamma(3/4)} - \log 2,$$

and from (3.13) we have

$$(4.6) \quad S'(1) = \frac{\pi}{2} \left\{ \log \frac{\Gamma(3/4)}{\Gamma(1/4)} + \frac{1}{2}(\log 2\pi + \gamma) \right\}.$$

We now return to (3.16). As $C(s, \frac{1}{2}) = \cos \frac{\pi}{2} = 0$, the value of the integral in (3.16) when $a = \frac{1}{2}$ must be considered as the limiting value:

$$\begin{aligned} \int_0^{\infty} \frac{(\sinh x)x^{s-1}}{\cosh 2x + 1} dx &= \Gamma(s) \lim_{a \rightarrow 1/2} \frac{C(s, a)}{\cos \pi a} \\ &= \frac{\Gamma(s)}{\sin \pi a} \sum_{m=0}^{\infty} \frac{\sin(2m+1)\pi a}{(2m+1)^{s-1}} \Big|_{a=1/2} = \Gamma(s)S(s-1), \quad \sigma > 0. \end{aligned}$$

Appealing to (4.15) and recalling that $S(0) = J(0, \frac{1}{2}) = \frac{1}{2}$, we obtain

$$(4.7) \quad \int_0^\infty \frac{\sinh x \cdot \log x}{\cosh 2x + 1} dx = \{\Gamma(s)S(s-1)\}'|_{s=1} = -\gamma S(0) + S'(0) \\ = \log \frac{\Gamma(1/4)}{\Gamma(3/4)} - \log 2 - \frac{\gamma}{2}.$$

The following integrals are easily deduced from (4.7):

$$\int_0^1 \frac{(1-t^2)}{(1+t^2)^2} \log \log \frac{1}{t} dt = \int_1^\infty \frac{(t^2-1)}{(1+t^2)^2} \log \log t dt = \log \frac{\Gamma(1/4)}{2\Gamma(3/4)} - \frac{\gamma}{2}$$

or

$$\int_0^{\pi/4} \cos 2x \log \log \cot x dx = - \int_{\pi/4}^{\pi/2} \cos 2x \log \log \tan x dx = \log \frac{2\Gamma(3/4)}{\Gamma(1/4)} + \frac{\gamma}{2}.$$

We remark that (4.7) can also be obtained by integrating (4.4) by parts:

$$\Gamma(s)S(s) = \frac{1}{2} \int_0^\infty \frac{x^{s-1}}{\cosh x} dx = \frac{1}{2s} \int_0^\infty \frac{dx^s}{\cosh x} \\ = \frac{x^s}{2s \cosh x} \Big|_0^\infty + \frac{1}{2s} \int_0^\infty \frac{\sinh x}{\cosh^2 x} x^s dx = \frac{1}{s} \int_0^\infty \frac{\sinh x}{\cosh 2x + 1} x^s dx,$$

that is,

$$(4.8) \quad \int_0^\infty \frac{\sinh x}{\cosh 2x + 1} x^s dx = s\Gamma(s)S(s)$$

and

$$\int_0^\infty \frac{\sinh x \cdot \log x}{\cosh 2x + 1} dx = \{s\Gamma(s)S(s)\}'|_{s=0} = -\gamma S(0) + S'(0),$$

which reproves (4.7).

Next we obtain the values of $S(-(2n+1))$ and $S(-2n)$ ($n \geq 0$). From (3.9) and (4.1) we have for $n \geq 0$

$$S(-n) = 2^n J\left(-n, \frac{1}{2}\right) = n! \left(\frac{2}{\pi}\right)^{n+1} \sum_{m=0}^\infty \frac{\sin\{(2m+1)\pi/2 - n\pi/2\}}{(2m+1)^{n+1}}.$$

Hence

$$(4.9) \quad S(1-2n) = 0, \quad n \geq 1,$$

and

$$S(-2n) = (-1)^n (2n)! \left(\frac{2}{\pi}\right)^{2n+1} \sum_{m=0}^\infty \frac{(-1)^m}{(2m+1)^{2n+1}} \\ = (-1)^n (2n)! \left(\frac{2}{\pi}\right)^{2n+1} S(2n+1), \quad n \geq 0.$$

But from (3.11) we have

$$(4.10) \quad S(2n+1) = \frac{(-1)^n \pi^{2n+1}}{2(2n)!} E_{2n} \left(\frac{1}{2}\right) = \frac{(-1)^n E_{2n}}{2(2n)!} \left(\frac{\pi}{2}\right)^{2n+1}, \quad n \geq 0,$$

giving

$$(4.11) \quad S(-2n) = \frac{1}{2}E_{2n}, \quad n \geq 0.$$

We observe that from (4.4) and (4.10)

$$(4.12) \quad \int_0^\infty \frac{x^{2n}}{\cosh x} dx = (-1)^n \left(\frac{\pi}{2}\right)^{2n+1} E_{2n}, \quad n \geq 0.$$

The values of $S(2n+1)$ have been determined in (4.10). We now turn to the evaluation of $S(2n)$. From (3.12), for $n \geq 1$,

$$E_{2n-1}(x) = \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} \sum_{m=0}^{\infty} \frac{\cos(2m+1)\pi x}{(2m+1)^{2n}}.$$

Hence

$$\int_0^1 \frac{E_{2n-1}(x)}{\cos \pi x} dx = \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n}} \int_0^1 \frac{\cos(2m+1)\pi x}{\cos \pi x} dx.$$

However, as

$$\int_0^1 \frac{\cos(2m+1)\pi x}{\cos \pi x} dx = (-1)^m, \quad m \geq 0,$$

(see [1, 332. 22b]) we have

$$\int_0^1 \frac{E_{2n-1}(x)}{\cos \pi x} dx = \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{2n}},$$

that is,

$$(4.13) \quad S(2n) = \frac{(-1)^n \pi^{2n}}{4(2n-1)!} \int_0^1 \frac{E_{2n-1}(x)}{\cos \pi x} dx, \quad n \geq 1.$$

From (4.4) we can also write (4.13) in another form, namely,

$$(4.14) \quad \int_0^\infty \frac{x^{2n-1}}{\cosh x} dx = \frac{(-1)^n \pi^{2n}}{2} \int_0^1 \frac{E_{2n-1}(x)}{\cos \pi x} dx, \quad n \geq 1.$$

From (4.13), for $k = 1, 2, \dots, n$, we have

$$\frac{(-1)^k (2k-1)!}{\pi^{2k}} S(2k) = \frac{1}{4} \int_0^1 \frac{E_{2n-1}(x)}{\cos \pi x} dx,$$

so that

$$\sum_{k=1}^n (-1)^k \binom{2n}{2k-1} \pi^{-2k} (2k-1)! S(2k) = \frac{1}{4} \int_0^1 \sum_{k=1}^n \binom{2n}{2k-1} E_{2k-1}(x) \cdot \frac{dx}{\cos \pi x}.$$

It is easy to prove that

$$(4.15) \quad \sum_{k=1}^n \binom{2n}{2k-1} E_{2k-1}(x) = x^{2n} - (1-x)^{2n}, \quad n \geq 1.$$

Hence we have the following recurrence relation for $S(2n)$:

Theorem 2. For positive integers n

$$(4.16) \quad \sum_{k=1}^n (-1)^k \binom{2n}{2k-1} \pi^{-2k} (2k-1)! S(2k) = \frac{1}{4} \int_0^1 \frac{x^{2n} - (1-x)^{2n}}{\cos \pi x} dx.$$

In [3] we gave the following recurrence relation for $S(2n+1)$:

$$(4.17) \quad \sum_{k=0}^n (-1)^k \binom{2n}{2k} \pi^{2n-2k} (2k)! S(2k+1) + (-1)^n (2n)! S(2n+1) = \left(\frac{\pi}{2}\right)^{2n+1}.$$

Finally we prove the recurrence formulas (4.16) and (4.17) simultaneously by contour integration of the function $f(z) = z^{2n}/(e^z + e^{-z})$ along the contour in the clockwise direction consisting of:

- the x -axis from $x = R$ to $x = 0$,
- the y -axis from $y = 0$ to $y = \pi/2 - \varepsilon$ ($\varepsilon > 0$),
- the semicircle $|z - \pi i/2| = \varepsilon$ in the first quadrant,
- the y -axis from $y = \pi/2 + \varepsilon$ to $y = \pi$,
- the line $y = \pi$ from $x = 0$ to $x = R$, and
- the line $x = R$ from $y = \pi$ to $y = 0$.

We have

$$\begin{aligned} \int_C f(z) dz &= \frac{1}{2} \cdot 2\pi i \operatorname{Res} \left(f(z), \frac{\pi i}{2} \right) = \pi i \frac{z^{2n}}{e^z + e^{-z}} \Big|_{z=\pi i/2} = (-1)^n \left(\frac{\pi}{2}\right)^{2n+1}, \\ &\int_0^{\pi/2-\varepsilon} f(iy) i dy = \frac{(-1)^n}{2} i \int_0^{\pi/2-\varepsilon} \frac{y^{2n}}{\cos y} dy, \\ &\int_{\pi/2+\varepsilon}^{\pi} f(iy) i dy = \frac{(-1)^{n+1}}{2} i \int_0^{\pi/2-\varepsilon} \frac{(\pi-y)^{2n}}{\cos y} dy. \end{aligned}$$

On the right vertical segment, $z = R + iy$, $0 \leq y \leq \pi$,

$$\begin{aligned} |z^{2n}| &= |z|^{2n} \leq (R^2 + \pi^2)^n, \\ |e^z + e^{-z}| &\geq \|e^z\| - \|e^{-z}\| = e^R - e^{-R} \geq \frac{1}{2} e^R \quad (R \geq \frac{1}{2} \log 2), \\ |f(z)| &\leq 2(R^2 + \pi^2)^n e^{-R}, \\ \left| \int_{x=R, 0 \leq y \leq \pi} f(z) dz \right| &\leq 2\pi(R^2 + \pi^2)^n e^{-R} \rightarrow 0 \quad (R \rightarrow +\infty). \end{aligned}$$

By Cauchy's residue theorem, we have

$$(4.18) \quad \begin{aligned} &(-1)^n \left(\frac{\pi}{2}\right)^{2n+1} + \int_0^R \frac{(x + \pi i)^{2n}}{e^{x+\pi i} + e^{-x-\pi i}} dx - \int_0^R \frac{x^{2n}}{e^x + e^{-x}} dx \\ &+ \frac{1}{2} (-1)^n i \int_0^{\pi/2-\varepsilon} \frac{y^{2n} - (\pi-y)^{2n}}{\cos y} dy + \alpha(R) = 0, \end{aligned}$$

where $\alpha(R) \rightarrow 0$ as $R \rightarrow +\infty$. Taking the real part of (4.18) and letting

$R \rightarrow +\infty$, $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned} \operatorname{Re} \int_0^\infty \frac{(x + \pi i)^{2n} + x^{2n}}{e^x + e^{-x}} dx &= (-1)^n \left(\frac{\pi}{2}\right)^{2n+1}, \\ \sum_{k=0}^n (-1)^{n-k} \binom{2n}{2k} \pi^{2n-2k} \int_0^\infty \frac{x^{2k}}{e^x + e^{-x}} dx + \int_0^\infty \frac{x^{2n}}{e^x + e^{-x}} dx \\ &= (-1)^n \left(\frac{\pi}{2}\right)^{2n+1}, \end{aligned}$$

which proves (4.17).

Similarly, taking the imaginary part of (4.18) and letting $R \rightarrow +\infty$, $\varepsilon \rightarrow 0$, we obtain

$$\sum_{k=1}^n (-1)^{k-1} \binom{2n}{2k} \pi^{2n-2k+1} \int_0^\infty \frac{x^{2k-1}}{e^x + e^{-x}} dx = \frac{\pi^{2n+1}}{2} \int_0^1 \frac{(1-x)^{2n} - x^{2n}}{\cos \pi x} dx.$$

From (4.4) and

$$\int_0^{1/2} \frac{(1-x)^{2n} - x^{2n}}{\cos \pi x} dx = \int_{1/2}^1 \frac{(1-x)^{2n} - x^{2n}}{\cos \pi x} dx$$

we deduce (4.16).

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