Specialization of Ashtekar's Formalism to Bianchi Cosmology

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Recently Ashtekar found a new canonical formalism of general relativity in which the constraints and the dynamical equations are all written formally as polynomials of canonical variables. In this paper we apply his formalism to the Bianchi cosmology and write down the constraints and the dynamical equations explicitly in terms of Ashtekar-type canonical variables which depend only on time. In particular we prove that for the vacuum Bianchi IX model the gauge freedom is decoupled from the dynamics and the canonical variables reduce to three diagonal components of the metric and their Ashtekar-type conjugate momentum, preserving the polynomial nature of the equations. Further we discuss the quantization of this model and show that the quantized hamiltonian constraint takes a quite simple and beautiful form. To illustrate the tractability of this equation, we present an explicit non-trivial solution to it.

§ 1. Introduction

Recently Ashtekar proposed a new canonical formalism of general relativity.[1],[2] In his formalism the constraint equations and the dynamical equations become polynomials when they are expressed in terms of the new complex canonical variables, the Ashtekar variables. Hence the severe nonlinearity, which has been thought to be an inevitable nature of general relativity and has bothered us so far, disappears.

In spite of this fascinating aspect, Ashtekar's formalism has various problems yet to be solved. First the role of the newly introduced gauge freedom which corresponds to rotation of triad is not clear. Second the complex nature of the canonical variables brings about new problems when one tries to quantize the theory, such as the structure of state space, interpretation and the correspondence with the Wheeler-DeWitt theory based on the conventional ADM formalism. In order to solve these problems and also to check the power of the formalism, it is desirable to see how the Ashtekar formalism works in simple systems.

From this point of view, in the present paper, we examine how the canonical dynamics of spatially homogeneous spacetime is described in terms of the Ashtekar variables. In particular, as an application we study the vacuum Bianchi IX model in detail. We show that in this model the gauge freedom is decoupled from the dynamics and the canonical variables reduce to the three diagonal components of the spatial metric and their Ashtekar-type conjugate momentum, preserving the polynomial nature of the equations. This implies that there is a direct correspondence between the Ashtekar formalism and the ADM formalism in this case. Further we show that when the theory is quantized it leads to an equation for the wavefunction which is much more tractable than the Wheeler-DeWitt equation in the conventional ADM formalism. In order to illustrate the tractability of the equation, we give an example of non-trivial exact solutions, though it is unphysical. The relation

between wavefunctions in the Ashtekar formalism and those in the ADM formalism is also discussed.

The organization of the rest of the paper is as follows. First in § 2 we briefly summarize the fundamental definitions and equations of the Ashtekar formalism partly because the notations used in the present paper are different from those adopted by Ashtekar's original paper. Then in § 3 we specialize the formalism to Bianchi models. The main job is to reduce the formalism to a canonical theory of finite degrees of freedom by separating space-coordinate dependence from the general Ashtekar variables. In § 4 the reduced formalism is applied to the vacuum Bianchi IX model and the decoupling of the gauge freedom and the off-diagonal components of the metric are proved by examining the canonical structure of the reduced theory. Finally in § 5 quantization of the vacuum Bianchi IX model is briefly discussed. Section 6 is devoted to conclusion. Throughout the paper the units $c=16\pi G=\hbar=1$ are used.

§ 2. Summary of Ashtekar's formalism

In this section we briefly summarize Ashtekar's formalism to fix the notations used in this paper some of which are slightly different from Ashtekar's.[1),2)]

There are two approaches to reach the Ashtekar formalism. One is to start from the ADM formalism and extend it.[1),2)] The other is to start directly from the action principle.[3)~5)] We adopt the former approach in this paper.

On the basis of the $(3+1)$-decomposition of the spacetime metric

$$ds^2 = -N^2 dt^2 + q_{ij}(N^i dt + dx^i)(N^j dt + dx^j), \tag{2.1}$$

the ADM formalism takes, as the fundamental variables, q_{ij} and its conjugate momentum p^{ij} defined by

$$p^{ij} := \sqrt{|q|}(q^{ij}K - K^{ij}),$$

$$K_{ij} := \frac{1}{2N}(D_i N_j + D_j N_i - \dot{q}_{ij}), \tag{2.2}$$

where $|q|$ denotes the determinant of q_{ij}, D_i the covariant derivative defined by q_{ij} and the latin indices $i, j, \cdots$ running from 1 through 3 are raised and lowered by q^{ij} and q_{ij}, respectively.

In order to reach the Ashtekar formalism starting from the ADM formalism, one must pass through two steps. The first step is to extend the fundamental variables q_{ij} and p^{ij} to the triad $e_a{}^i$ or the soldering form σ^i and its conjugate momentum M_i. The second step is the introduction of new variables which are complex combinations of σ^i and M_i.

Triad is introduced as a square root decomposition of the spatial metric q_{ij}. To be precise, letting $\{e_a\} = \{(e_a{}^i)\}$ be a triad on a three-dimensional $t = $const. hypersurface M^3 and $\{\omega^a\} = \{(\omega^a{}_i)\}$ be its dual 1-form

$$e_a{}^i \omega^a{}_j = \delta^i_j, \qquad e_a{}^i \omega^b{}_i = \delta^b_a, \tag{2.3}$$

q_{ij} and q^{ij} are expressed as

$$q_{ij} = \omega^a{}_i \omega^a{}_j , \qquad q^{ij} = e_a{}^i e_a{}^j . \tag{2.4}$$

In the present paper $a, b, c, \cdots$ represent the internal indices of the tangent vector space to a $t=$const hypersurface and $i, j, k, \cdots$ the spatial coordinate indices both of which run from 1 through 3.

In terms of the triad and a set of matrices satisfying the conditions

$$\sigma_a{}^\dagger = \sigma_a ,$$

$$[\sigma_a, \sigma_b] = \sqrt{2} i \epsilon_{abc} \sigma_c ,$$

$$\sigma_a \sigma_b + \sigma_b \sigma_a = \delta_{ab} , \tag{2.5}$$

the soldering form is defined as

$$\sigma_i = \omega^a{}_i \sigma_a , \qquad \sigma^i = e_a{}^i \sigma_a . \tag{2.6}$$

Here note that the hermitian conjugation $\dagger$ in this paper is defined to take the ordinary hermitian conjugation of matrices, unlike the definition adopted in Ashtekar's paper.[2] The metric tensor is represented in terms of the soldering form as

$$q_{ij} = \mathrm{Tr}(\sigma_i \sigma_j) , \qquad q^{ij} = \mathrm{Tr}(\sigma^i \sigma^j) . \tag{2.7}$$

Decomposing the metric q_{ij} into triad introduces three additional freedoms into the theory. For a given q_{ij} any triad e_a is always related to a fixed triad $\bar{e}_a$ by a rotation in the tangent vector space $e_a = O_{ab} \bar{e}_b$, where $O = (O_{ab})$ is a three-dimensional orthogonal matrix. On the other hand, there is a three-dimensional freedom in the choice of a set of matrices satisfying the conditions (2.5). Any set of matrices satisfying these conditions is related to the Pauli matrix σ_{Pa} with some appropriate three-dimensional orthogonal matrix O' as $\sigma_a = (1/\sqrt{2}) \sigma_{Pb} O'_{ba}$. When the triad and the sigma matrix are combined to form the soldering form, O and O' always go in the combination $O'O$. Thus the soldering form is expressed in terms of q_{ij} and a three-dimensional orthogonal matrix. In order to parametrize the latter gauge freedom, one can use either the freedom of the triad or the freedom of the sigma matrix. The latter approach will be used in the following section. Equations in the rest of this section hold for either approach.

The first step is completed by defining the momentum variable M_i conjugate to σ^i as

$$M_i := \sqrt{|q|} [\Pi_i - \sigma_i \mathrm{Tr}(\Pi_j \sigma^j)] , \tag{2.8}$$

$$\Pi_i := \frac{1}{N} (\dot{\sigma}_i - D_i N_j \sigma^j + \epsilon_{abc} m^a \sigma_b \omega^c{}_i) , \tag{2.9}$$

where $\cdot$ denotes the time derivative and m^a is an arbitrary function representing the gauge freedom parameter. The momentum p^{ij} conjugate to q_{ij} in the ADM-formalism is expressed in terms of M_i as

$$p^{ij} = \mathrm{Tr}\, M^{(i} \sigma^{j)} . \tag{2.10}$$

Now we proceed to the second step. The soldering form defines an isometric correspondence between vector fields and mixed spinor fields. Through this correspondence the metric connection induces an $SU(2)$ spinor connection. In terms of an appropriate 1-form $\Gamma_i = (\Gamma^A{}_{Bi})$ taking value in the anti-hermitian matrices of rank 2, the corresponding covariant derivative of a mixed spinor field $V = (V^A{}_B)$ is expressed in the matrix notation as

$$D_i V = \partial_i V + [\Gamma_i, V] \,. \tag{2.11}$$

Γ_i is determined by the condition

$$D_j \sigma_i = 0 \tag{2.12}$$

to be

$$\Gamma_i = \frac{1}{4} \{\omega^a{}_{bi} + \mathrm{Tr}(\partial_i \sigma_a \sigma_b)\}[\sigma_a, \sigma_b] \,, \tag{2.13}$$

where $\omega^a{}_{bi}$ is the connection form defined by

$$D_i e_a = e_b \omega^b{}_{ai} \,. \tag{2.14}$$

The connection form is expressed in terms of the triad as

$$\omega^a{}_{bi} = \omega^a{}_{bc} \omega^c{}_i \,, \qquad \omega^a{}_{bc} = f^a{}_{bc} - f_b{}^a{}_c - f_c{}^a{}_b \,, \tag{2.15}$$

where

$$d\omega^a = f^a{}_{bc} \omega^b \wedge \omega^c \,, \qquad f^a{}_{bc} = \frac{1}{2}(\partial_i \omega^a{}_j - \partial_j \omega^a{}_i) e_b{}^i e_c{}^j \,. \tag{2.16}$$

Now Ashtekar's canonical variables are defined in terms of σ^i, Γ_i and Π_i as

$$\tilde{\sigma}^i := \sqrt{|q|}\, \sigma^i = |\omega| e^i{}_a \sigma_a \,,$$

$$\mathcal{A}_i := \Gamma_i \pm \frac{1}{\sqrt{2}} \Pi_i \,, \tag{2.17}$$

where $|\omega| = \det(\omega^a{}_i)$. The Poisson brackets of $\tilde{\sigma}^i$ and $\mathcal{A}_i$ are given by

$$\{\tilde{\sigma}^i(\boldsymbol{x}), \tilde{\sigma}^j(\boldsymbol{y})\} = \{\mathcal{A}_i(\boldsymbol{x}), \mathcal{A}_j(\boldsymbol{y})\} = 0 \,,$$

$$\left\{\mathrm{Tr}\left(\mathcal{A}_i(\boldsymbol{x}) \frac{\sigma_{\mathrm{P}a}}{\sqrt{2}}\right), \mathrm{Tr}\left(\tilde{\sigma}^j(\boldsymbol{y}) \frac{\sigma_{\mathrm{P}b}}{\sqrt{2}}\right)\right\} = \mp \frac{1}{2\sqrt{2}} \delta_i{}^j \delta_{ab} \delta(\boldsymbol{x}, \boldsymbol{y}) \,. \tag{2.18}$$

$\mathcal{A}_i$ defines an $SL(2, \mathbf{C})$ spinor connection on M^3 by

$$\mathcal{D}_i := \partial_i + \mathcal{A}_i \,, \tag{2.19}$$

and the corresponding curvature form $\mathcal{F}_{ij}$,

$$\mathcal{F}_{ij} := [\mathcal{D}_i, \mathcal{D}_j] = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_j] \,. \tag{2.20}$$

In terms of the Ashtekar variables the constraint equations are expressed as

Gauge constraint

$$C_a{}^G \sigma_a := \mathcal{D}_i \tilde{\sigma}^i = \partial_i \tilde{\sigma}^i + [\mathcal{A}_i, \tilde{\sigma}^i] \approx 0 , \tag{2·21}$$

Momentum constraint

$$C_j{}^M := \mp 2\sqrt{2}\,\mathrm{Tr}\,\tilde{\sigma}^i \mathcal{F}_{ij} \approx \sqrt{|q|}\, n^\mu T_{\mu j} , \tag{2·22}$$

Hamiltonian constraint

$$\sqrt{|q|}\, C^H := 2\,\mathrm{Tr}\,\tilde{\sigma}^i \tilde{\sigma}^j \mathcal{F}_{ij} \approx |q|\, n^\mu n^\nu T_{\mu\nu} , \tag{2·23}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter and n^μ is the normal vector to the $t=$const spacelike hypersurfaces, which is expressed in terms of N and N^i as

$$(n^\mu) = \left(\frac{1}{N}, -\frac{N^i}{N}\right) . \tag{2·24}$$

The total hamiltonian is expressed in terms of these constraints as

$$\mathcal{H} = \int d^3x [N C^H + N^i C_i{}^M + \tilde{m}^a C_a{}^G] , \tag{2·25}$$

where N and N^i are the lapse function and the shift vector in the $(3+1)$-decomposition of the spactime metric $(2·1)$, and $\tilde{m}^a$ is an arbitrary function representing the gauge freedom. Comparing the expression for $\dot{\sigma}_i$ derived from the canonical equation $\dot{\tilde{\sigma}}^i = \{\mathcal{H}, \tilde{\sigma}^i\}$ and the definition of Π_i in terms of $\dot{\sigma}_i$, Eq. $(2·9)$, one finds that $\tilde{m}^a$ is expressed in terms of m^a as

$$\tilde{m}^a = \mp 2im^a - 2e_a{}^k \partial_k N - 2N^k \mathrm{Tr}(\Pi_k \sigma_a) . \tag{2·26}$$

§ 3. Specialization to spatially homogeneous spacetime

The metric of spatially homogeneous spacetime is expressed as

$$ds^2 = -N^2 dt^2 + e^{2\alpha}(e^{2\beta})_{ab}(N^a dt + \chi^a)(N^b dt + \chi^b) , \tag{3·1}$$

where N, N^a, α and the symmetric traceless matrix β_{ab} depend only on time t, and $\chi^a = \chi^a{}_i dx^i$ are the time-independent invariant 1-forms with respect to the spatial symmetry group.[7] These invariant 1-forms satisfy

$$d\chi^a = \frac{1}{2} C^a{}_{bc}\chi^b \wedge \chi^c , \tag{3·2}$$

where $C^a{}_{bc}$ is the structure constant of the spatial symmetry group. The corresponding time-independent invariant basis $X_a = X_a{}^i \partial_i$ which is dual to the invariant 1-forms satisfies

$$[X_a, X_b] = -X_c C^c{}_{ab} . \tag{3·3}$$

The structure of the metric $(3·1)$ defines the following natural triad and its dual basis,

$$e_a = e^{-\alpha}(e^{-\beta})_{ab}X_b , \tag{3·4}$$

$$\omega^a = e^\alpha (e^\beta)_{ab} \chi^b .$$ (3·5)

We do not allow the additional rotational freedom of the triad and parametrize the gauge freedom by the freedom in the choice of the sigma matrices. Then in the spatially homogeneous system the dynamical variables are N, N_a, α, β_{ab}, and the orthogonal matrix O which relates σ_a to $(1/\sqrt{2})\sigma_{Pa}$ by

$$\sigma_a = \frac{1}{\sqrt{2}} \sigma_{Pb} O_{ba} .$$ (3·6)

Among them the variables other than O depend only on time. One will expect that one can also restrict O to the one dependent only on time. In fact we will show later that this expectation is true. However, since it is not obvious at the beginning whether this restriction can be achieved without loss of generality, we will keep the space-coordinate dependence of O as general as possible for a while.

The goal of this section is to decompose the Ashtekar variables into the non-dynamical space-coordinate dependent part and the dynamical space-coordinate independent part, and to express the constraints in terms of the latter part. We only consider the vacuum case. The first job is achieved by expressing the Ashtekar variables in terms of the variables listed above and their derivatives.

For the above triad the exterior derivative of ω^a is easily calculated with the aid of Eq. (3·2) to yield

$$f^a{}_{bc} = \frac{1}{2} e^{-\alpha} (e^\beta)_{ap} C^p{}_{qr} (e^{-\beta})_{qb} (e^{-\beta})_{rc} .$$ (3·7)

Hence the connection form is expressed as

$$\omega^a{}_{bi} =: \Phi^a{}_{bc} \chi^c{}_i ,$$

$$\Phi^a{}_{bc} = \frac{1}{2} [(e^\beta)_{ap} C^p{}_{qc} (e^{-\beta})_{qb} - (e^\beta)_{bp} C^p{}_{qc} (e^{-\beta})_{qa} - (e^{2\beta})_{cp} C^p{}_{qr} (e^{-\beta})_{qa} (e^{-\beta})_{rb}] .$$

(3·8)

Inserting this expression into Eq. (2·13), Γ_i is expressed as

$$\Gamma_i = \frac{i}{\sqrt{2}} (\Phi_{ab} + W_{ab}) \chi^b{}_i \sigma_a ,$$ (3·9)

where

$$\Phi_{ab} := \frac{1}{2} \epsilon_{apq} \Phi^p{}_{qb} ,$$ (3·10)

$$W_{ab} := \frac{1}{2} \epsilon_{acd} X_b{}^j (\partial_j O_{pc}) O_{pd} .$$ (3·11)

Here W_{ab} represents the space-coordinate dependence of the gauge freedom O. We will later show that it can be put zero without loss of generality as is expected from the spatial homogeneity of the system.

Next let us calculate Π_i. From the definition of connection form (2·14) and

Eq. (3·4), the covariant derivative of the invariant basis is expressed in terms of the connection form $\omega^a{}_{bi}$ as

$$D_i X_a{}^j = X_c{}^j (e^{-\beta})_{cd} \omega^d{}_{bi} (e^{\beta})_{ba} . \tag{3·12}$$

Hence by noting that $D_i \chi^a{}_j = -\chi^b{}_j (D_i X_b{}^k) \chi^a{}_k$ and $D_i N_j = D_i (N_a \chi^a{}_j) = N_a D_i \chi^a{}_j$, Π_i is expressed as

$$\Pi_i = \frac{1}{N} [\dot{a} \delta_{ac} + B_{ac} + V_{ac} + N_{ac}] e^\alpha (e^\beta)_{cb} \chi^b{}_i \sigma_a , \tag{3·13}$$

where

$$B_{ab} := (e^\beta)_{ac} (e^{-\beta})_{cb} , \tag{3·14}$$

$$V_{ab} := O_{ca} \dot{O}_{cb} , \tag{3·15}$$

$$N_{ab} := N^p (e^\beta)_{pq} \omega^q{}_{ar} (e^{-\beta})_{rb} + m^p \epsilon_{pab} . \tag{3·16}$$

We assume that the gauge parameter m^p is independent of the space coordinates.

Substituting Eqs. (3·4), (3·5), (3·9) and (3·13) into Eqs. (2·17), one finds the following expressions for the Ashtekar variables:

$$\tilde{\sigma}^i = \Sigma_{ab} |\chi| X_b{}^i \sigma_a ,$$

$$\mathcal{A}_i = \frac{1}{\sqrt{2}} [A_{ab} + i W_{ab}] \chi^b{}_i \sigma_a , \tag{3·17}$$

where

$$\Sigma_{ab} := e^{2\alpha} (e^{-\beta})_{ab} , \tag{3·18}$$

$$A_{ab} := \pm \frac{1}{N} (\dot{a} \delta_{ac} + B_{ac} + V_{ac} + N_{ac}) e^\alpha (e^\beta)_{cb} + i \Phi_{ab} . \tag{3·19}$$

Now we can rewrite the constraint equations in terms of variables which depend only on times. First to rewrite the gauge constraint (2·21), we must calculate $\partial_i \tilde{\sigma}^i$. The derivative of the determinant of $\chi^a{}_i$ is given by

$$\partial_i |\chi| = X_a{}^j \partial_i \chi^a{}_j ,$$

and due to the duality the differential of $X_a{}^i$ is expressed in terms of the differential of $\chi^a{}_i$ as

$$dX_a{}^i = -X_a{}^j X_b{}^i d\chi^b{}_j .$$

Hence

$$\partial_i (|\chi| X_b{}^i) = |\chi| [\partial_i \chi^a{}_j - \partial_j \chi^a{}_i] X_a{}^j X_b{}^i$$

$$= |\chi| C_b , \tag{3·20}$$

where

$$C_a := C^b{}_{ab} , \tag{3·21}$$

and we have used Eq. (3·2). Thus, noting that from the definition of W_{ab}, (3·11), the spatial derivative of σ_a is expressed as

$$\partial_i \sigma_a = -\epsilon_{abc} W_{bd} \chi^d{}_i \sigma_c , \tag{3·22}$$

one obtains

$$\partial_i \tilde{\sigma}^i = |\chi| [C_a \Sigma_{ab} - \epsilon_{acb} W_{cd} \Sigma_{da}] \sigma_b . \tag{3·23}$$

On the other hand, the bracket of $\mathcal{A}_i$ and $\tilde{\sigma}^i$ is easily calculated to be

$$[\mathcal{A}_i, \tilde{\sigma}^i] = i|\chi| \epsilon_{cad} (A_{cb} + i W_{cb}) \Sigma_{ba} \sigma_d . $$

Putting together these equations, $\mathcal{D}_i \tilde{\sigma}^i$ is expressed as

$$\mathcal{D}_i \tilde{\sigma}^i = |\chi| [C_a \Sigma_{ad} + i\epsilon_{cad} A_{cb} \Sigma_{ba}] \sigma_d . \tag{3·24}$$

Hence the gauge constraint is equivalent to

$$C_a \Sigma_{ad} + i\epsilon_{cad} A_{cb} \Sigma_{ba} \approx 0 . \tag{3·25}$$

From the definitions of Φ_{ab} and Σ_{ab}, one can prove the following identity relation:

$$\epsilon_{abc} \Phi_{bd} \Sigma_{dc} = C_b \Sigma_{ba} . \tag{3·26}$$

With the aid of this relation one easily sees that the above gauge constraint is equivalent to

$$B_{[ab]} + V_{ab} + N_{[ab]} \approx 0 . \tag{3·27}$$

Since B_{ab} and N_{ab} are independent of the space coordinates, this relation implies that the gauge constraint requires that V_{ab} is also independent of the space coordinates. On the other hand, from the commutativity of the spatial derivative and the time derivative of O, $\partial_t(\partial_i O) = \partial_i(\partial_t O)$, W_{ab} and V_{ab} must satisfy the integrability condition

$$\dot{W}_{ab} + V_{ac} W_{cb} - \frac{1}{2} \epsilon_{apq} X_b{}^i \partial_i V_{pq} = 0 . \tag{3·28}$$

Hence the space-coordinate independence of V_{ab} leads to a homogeneous evolution equation for W_{ab}. Therefore, if one chooses the initial data for O to be independent of the space coordinates, W_{ab} stays zero along with time evolution. This implies that one can fix the gauge so that W_{ab} stays identically zero without conflicting with the dynamics. For this reason we assume from now on that O is independent of the space coordinates. Then the space-coordinate dependence of the Ashtekar variables is completely separated out, and the dynamics are described by variables depending only on time, Σ_{ab}, O and A_{ab}.

Since these variables are related with the Ashtekar variables nonlinearly, they are not natural variables for Ashtekar formalism. Therefore we introduce the following new variables:

$$\tilde{\Sigma}_{ab} := O_{ac} \Sigma_{cb} ,$$

$$\tilde{A}_{ab} := O_{ac} A_{cb} . \tag{3·29}$$

Then the Ashtekar variables are expressed as

$$\tilde{\sigma}^i = \tilde{\Sigma}_{ab}|\chi|X_b{}^i\frac{\sigma_{\mathrm{P}a}}{\sqrt{2}},$$

$$\mathcal{A}_i = \frac{1}{\sqrt{2}}\tilde{A}_{ab}\chi^b{}_i\frac{\sigma_{\mathrm{P}a}}{\sqrt{2}}. \tag{3.30}$$

In terms of these new variables the gauge constraint (3·25) is written as
  Gauge constraint

$$\mathcal{C}_a{}^G = |\chi|\tilde{C}_a{}^G,$$

$$\tilde{C}_a{}^G := C_b\tilde{\Sigma}_{ba} + i\epsilon_{cda}\tilde{A}_{cb}\tilde{\Sigma}_{bd} \approx 0. \tag{3.31}$$

Now we proceed to the momentum constraint and the hamiltonian constraint. With the aid of Eq. (3·2) $\mathcal{F}_{ij}$ is calculated as

$$\mathcal{F}_{ij} = \frac{1}{\sqrt{2}}[\tilde{A}_{ap}C^p{}_{bc} + i\epsilon_{apq}\tilde{A}_{pb}\tilde{A}_{qc}]\chi^b{}_i\chi^c{}_j\sigma_a. \tag{3.32}$$

From this equation the momentum constraint and the hamiltonian constraint are expressed as
  Momentum constraint

$$\mathcal{C}_j{}^M = \mp 2\tilde{C}_a{}^M|\chi|\chi^a{}_j,$$

$$\tilde{C}_a{}^M := \tilde{A}_{pb}\tilde{\Sigma}_{pc}C^b{}_{ca} + \tilde{A}_{pa}\tilde{\Sigma}_{pq}C_q \approx 0, \tag{3.33}$$

  Hamiltonian constraint

$$\mathcal{C}^H = |\chi|e^{-3a}\tilde{C}^H,$$

$$\tilde{C}^H := [i\epsilon_{abc}\tilde{A}_{ap}C^p{}_{qr} - \tilde{A}_{bq}\tilde{A}_{cr} + \tilde{A}_{br}\tilde{A}_{cq}]\tilde{\Sigma}_{bq}\tilde{\Sigma}_{cr} \approx 0. \tag{3.34}$$

Note that the quantities without tilde, Σ_{ab} and A_{ab}, satisfy the equations obtained by replacing the quantities with tilde by those without tilde from $\tilde{C}_a{}^M \approx 0$ and $\tilde{C}^H \approx 0$ as in the case of the gauge constraint $\tilde{C}_a{}^G \approx 0$ since O is an orthogonal matrix.

  To find the equations of motion, one needs the Poisson brackets among $\tilde{\Sigma}_{ab}$ and $\tilde{A}_{ab}$, and the hamiltonian. From Eqs. (3·30) $\tilde{\Sigma}_{ab}$ and $\tilde{A}_{ab}$ are related to the general spacetime-dependent Ashtekar variables as

$$\tilde{\Sigma}_{ab} = \frac{1}{\Omega}\int d^3x\, \mathrm{Tr}\,(\sigma_{\mathrm{P}a}\tilde{\sigma}^i)\chi^b{}_i,$$

$$\tilde{A}_{ab} = \frac{\sqrt{2}}{\Omega}\int d^3x|\chi|\mathrm{Tr}\,(\sigma_{\mathrm{P}a}A_i)X_b{}^i, \tag{3.35}$$

where Ω is the coordinate volume of the invariant space

$$\Omega := \int d^3x|\chi|. \tag{3.36}$$

Hence it follows from Eq. (2·18) that

$$\{\tilde{\Sigma}_{ab}, \tilde{\Sigma}_{cd}\} = \{\tilde{A}_{ab}, \tilde{A}_{cd}\} = 0 \,,$$

$$\{\tilde{A}_{ab}, \tilde{\Sigma}_{cd}\} = \mp \frac{1}{2\Omega} \delta_{ac}\delta_{bd} \,. \tag{3.37}$$

From Eq. (2·25) and the relations (3·31), (3·33) and (3·34), the hamiltonian is expressed as

$$\mathcal{H} = \Omega[Ne^{-3a}\tilde{C}^H \mp 2N^a\tilde{C}_a{}^M + \tilde{m}^a\tilde{C}_a{}^G] \,, \tag{3.38}$$

where N, N^a and $\tilde{m}^a$ are arbitrary functions of time and correspond to the lapse function, the shift vector and the gauge-freedom parameter, respectively. With the aid of these expressions the classical equations of motion are easily obtained from the canonical equations

$$\dot{\mathcal{O}} = \{\mathcal{H}, \mathcal{O}\} \,, \tag{3.39}$$

where $\mathcal{O}$ is any of $\tilde{\Sigma}_{ab}$ and $\tilde{A}_{ab}$. The right-hand sides of these equations are polynomials of third degree at most as expected. Since they will not be utilized below, we do not write them down explicitly here.

Finally we remark on the elimination of gauge freedom. As stated above, the constraint equations can be written only in terms of the variables without tilde, Σ_{ab} and A_{ab}, and do not contain the gauge variable O. Hence one might expect that the gauge freedom might decouple from the dynamics. However, this is not correct in general, since A_{ab} contains the time derivative of O. In fact, from Eq. (3·27), one sees that O can be made time-independent only if e^β can be diagonalizable under the gauge condition $N^a = 0$ (the synchronous gauge) and $\tilde{m}^a = 0$. Hence if one wants to fix the gauge so that $O = 1$, one must allow the non-vanishing shift vector. Thus one must work with the full variables with tilde in general since the diagonalization is not allowed for most of the Bianchi types. One important exception is the Bianchi type IX, which is explored in the next section.

§4. Vacuum Bianchi type IX—classical dynamics

The Bianchi type IX model is characterized by the structure constant,

$$C^a{}_{bc} = \epsilon_{abc} \,,$$

$$C_a = 0 \,. \tag{4.1}$$

As is well-known, the matrix e^β can be diagonalized without loss of generality for the Bianchi type IX model if there exists no matter.[7] In this section we will show that the gauge freedom as well as non-diagonal components of e^β decouples and the dynamics is described by three conjugate pairs of Ashtekar-type canonical variables preserving the polynomial nature of equations.

In general the symmetric matrix e^β can be diagonalized by time-dependent orthogonal matrix $R = (R_{ab})$ as

$$e^\beta = Re^{\beta_D}R^\sim \,, \tag{4.2}$$

where β_D is a diagonal matrix

$$\beta_D = [\beta_1, \beta_2, \beta_3] \tag{4·3}$$

and $R^{\sim}$ denotes the transposed matrix of R. In terms of R and the orthogonal matrix $Q = OR$ where O is the matrix representing the gauge freedom, the matrix $\tilde{\Sigma} = (\tilde{\Sigma}_{ab})$ is diagonalized as

$$\tilde{\Sigma} = Q\Sigma_D R^{\sim} , \tag{4·4}$$

where Σ_D is a diagonal matrix

$$\Sigma_D = [\Sigma_1, \Sigma_2, \Sigma_3] . \tag{4·5}$$

Corresponding to this decomposition of $\tilde{\Sigma}$, we rewrite $\tilde{A}$ in terms of α, β_D, Q and R as

$$\tilde{A} = QHR^{\sim} , \tag{4·6}$$

where the matrix $H = (H_{ab})$ is given by

$$H := \pm \frac{1}{N} e^{\alpha} \{ (\dot{\alpha} + \dot{\beta}_D) e^{\beta_D} + Q^{\sim} \dot{Q} e^{\beta_D} - e^{\beta_D} R^{\sim} \dot{R} + \tilde{N} \} + i\tilde{\Phi} \tag{4·7}$$

with

$$\tilde{N} := R^{\sim}(N_{ab})R ,$$
$$\tilde{\Phi} := R^{\sim}\Phi R .$$

The canonical structure of the system in terms of Σ_D, Q, R and H, are determined by the Poisson brackets among them. From Eq. (3·37), after a short algebra, one finds

$$\{H_{ab}, \Sigma_c\} = \mp \frac{1}{2\Omega} \delta_{ac} \delta_{bc} , \tag{4·8}$$

$$\{H_{ab}, R_{cd}\} = \mp \frac{1}{2\Omega} \frac{\Sigma_a R_{cp}}{\Sigma_d^2 - \Sigma_p^2} [\delta_{ad}\delta_{bp} + \delta_{ap}\delta_{bd} - 2\delta_{ap}\delta_{bp}\delta_{dp}] , \tag{4·9}$$

$$\{H_{ab}, Q_{cd}\} = \mp \frac{1}{2\Omega} \frac{\Sigma_b Q_{cp}}{\Sigma_d^2 - \Sigma_p^2} [\delta_{ad}\delta_{bp} + \delta_{ap}\delta_{bd} - 2\delta_{ap}\delta_{bp}\delta_{dp}] , \tag{4·10}$$

$$\begin{aligned}
\{H_{ab}, H_{cd}\} = \mp \frac{1}{2\Omega} \Big[&-\frac{\Sigma_a H_{cp}}{\Sigma_p^2 - \Sigma_d^2}(\delta_{ap}\delta_{bd} + \delta_{ad}\delta_{pb} - 2\delta_{ap}\delta_{bp}\delta_{dp}) \\
&-\frac{\Sigma_b H_{pd}}{\Sigma_p^2 - \Sigma_c^2}(\delta_{ap}\delta_{bc} + \delta_{ac}\delta_{pb} - 2\delta_{ap}\delta_{bp}\delta_{cp}) \\
&+\frac{\Sigma_c H_{ap}}{\Sigma_p^2 - \Sigma_b^2}(\delta_{cp}\delta_{bd} + \delta_{bc}\delta_{pd} - 2\delta_{bp}\delta_{cp}\delta_{dp}) \\
&+\frac{\Sigma_d H_{pb}}{\Sigma_p^2 - \Sigma_a^2}(\delta_{cp}\delta_{ad} + \delta_{ac}\delta_{pd} - 2\delta_{ap}\delta_{cp}\delta_{dp}) \Big].
\end{aligned} \tag{4·11}$$

In particular, Σ_D commutes with the off-diagonal components of H:

$$\{H_a, \Sigma_b\} = \mp \frac{1}{2\Omega} \delta_{ab} , \tag{4·12}$$

$$\{H'_{ab}, \Sigma_c\} = 0 , \tag{4·13}$$

where $H_a := H_{aa}$ and H' is the off-diagonal part of H. Further the diagonal part of H, $H_D = (H_a)$, commutes with R and Q:

$$\{H_D, R\} = \{H_D, Q\} = 0 . \tag{4·14}$$

Finally the diagonal components of H commute with each other and the Poisson brackets of H_D and H' do not contain H_D and are written only in terms of Σ_D and H':

$$\{H_a, H_b\} = 0 , \tag{4·15}$$

$$\{H_a, H'_{(cd)}\} = \mp \frac{1}{2\Omega} \frac{\Sigma_c + \Sigma_d}{\Sigma_p{}^2 - \Sigma_a{}^2} H'_{(ap)}(\delta_{ad}\delta_{pc} + \delta_{ac}\delta_{pd} - 2\delta_{ap}\delta_{cp}\delta_{dp}) , \tag{4·16}$$

$$\{H_a, H'_{[cd]}\} = \mp \frac{1}{2\Omega} \frac{\Sigma_c - \Sigma_d}{\Sigma_p{}^2 - \Sigma_a{}^2} H'_{[ap]}(\delta_{ad}\delta_{pc} + \delta_{ac}\delta_{pd} - 2\delta_{ap}\delta_{cp}\delta_{dp}) , \tag{4·17}$$

where $H'_{(ab)}$ and $H'_{[ab]}$ are the symmetric and the anti-symmetric part of the off-diagonal matrix H'_{ab}.

For the vacuum Bianchi IX model the constraint equations take quite simple forms expressed in terms of Σ_D, Q, R, H_D and H'. First since C_a vanishes, the gauge constraint and the momentum constraint are written only in terms of Σ_D and H':

Gauge constraint

$$\tilde{C}_a{}^G = Q_{ap} C_a{}^G \approx 0 ,$$

$$-i\epsilon_{abc} C_c{}^G = H_{ab}\Sigma_b - H_{ba}\Sigma_a$$

$$= (\Sigma_b - \Sigma_a)H'_{(ab)} + (\Sigma_a + \Sigma_b)H'_{[ab]} . \tag{4·18}$$

Momentum constraint

$$\tilde{C}_a{}^M = R_{ap} C_a{}^M \approx 0 ,$$

$$\epsilon_{abc} C_c{}^M = H_{ba}\Sigma_b - H_{ab}\Sigma_a$$

$$= (\Sigma_b - \Sigma_a)H'_{(ab)} - (\Sigma_a + \Sigma_b)H'_{[ab]} . \tag{4·19}$$

From these equations the off-diagonal matrix H' is expressed in terms of $C_a{}^G$ and $C_a{}^M$ as

$$H'_{(ab)} = -\frac{1}{2}\epsilon_{abp} \frac{C_p{}^M - iC_p{}^G}{\Sigma_a - \Sigma_b} ,$$

$$H'_{[ab]} = -\frac{1}{2}\epsilon_{abp} \frac{C_p{}^M + iC_p{}^G}{\Sigma_a + \Sigma_b} . \tag{4·20}$$

Hence the gauge constraint and the momentum constraint are equivalent to

$$H' \approx 0 . \tag{4·21}$$

Finally the hamiltonian constraint is written as

Hamiltonian constraint

$$\tilde{C}^H = C^H + \frac{1}{4}|\epsilon_{abp}|\Sigma_a\Sigma_b\left\{\frac{(C_p{}^M - iC_p{}^G)^2}{(\Sigma_a - \Sigma_b)^2} - \frac{(C_p{}^M + iC_p{}^G)^2}{(\Sigma_a + \Sigma_b)^2}\right\} \approx 0,$$

$$C^H = [i|\epsilon_{abc}|H_a - (1 - \delta_{bc})H_bH_c]\Sigma_b\Sigma_c$$

$$= 2[(iH_1 - H_2H_3)\Sigma_2\Sigma_3 + (iH_2 - H_3H_1)\Sigma_3\Sigma_1 + (iH_3 - H_1H_2)\Sigma_1\Sigma_2]. \qquad (4\cdot22)$$

Note that since the difference between $\tilde{C}^H$ and C^H is quadratic with respect to the constraints, we can use C^H instead of $\tilde{C}^H$ everywhere without changing the dynamics.

From the structure of the Poisson brackets among the canonical variables calculated above one easily sees that the Poisson brackets between the canonical variables and the constraints which do not vanish modulo the constraints are

$$\tilde{C}^G: \qquad \{\tilde{C}_a{}^G, Q_{bc}\} = \mp\frac{1}{2\Omega}i\epsilon_{abp}Q_{pc}, \qquad\qquad (4\cdot23)$$

$$\tilde{C}^M: \qquad \{\tilde{C}_a{}^M, R_{bc}\} = \mp\frac{1}{2\Omega}\epsilon_{abp}R_{pc}, \qquad\qquad (4\cdot24)$$

$$\tilde{C}^H: \qquad \{\tilde{C}^H, \Sigma_a\}, \{\tilde{C}^H, H_a\}.$$

Hence from Eqs. (3·38) and (3·39), the canonical equations of motion is written

$$\dot{\Sigma}_a \approx \mp Ne^{-3a}[i\Sigma_b\Sigma_c - \Sigma_a(\Sigma_bH_b + \Sigma_cH_c)],$$

$$\dot{H}_a \approx \mp Ne^{-3a}[i(\Sigma_bH_c + \Sigma_cH_b) - H_a(\Sigma_bH_b + \Sigma_cH_c)],$$

$$\dot{R}_{ab} \approx N^p\epsilon_{paq}R_{qb},$$

$$\dot{Q}_{ab} \approx \mp\frac{1}{2}i\tilde{m}^p\epsilon_{paq}Q_{qb},$$

$$\dot{H}'_{ab} \approx 0, \qquad\qquad\qquad\qquad\qquad (4\cdot25)$$

where in the first two equations the indices (a, b, c) are any of the cyclic permutation of $(1, 2, 3)$.

These equations clearly show that the dynamics of Σ_a and H_a completely decouples from other freedoms and the dynamics of R and Q becomes trivial. In particular the evolutions of R and Q decouple from each other and are determined by the shift vector N^a and the gauge parameter $\tilde{m}^a$, respectively. Hence even for the synchronous gauge $N^a = 0$, R becomes time-independent and can be set to be the unit matrix.

Since the gauge freedom is decoupled, the resultant dynamical system is completely equivalent to the ordinary ADM canonical theory. In fact, there is a simple correspondence between the Ashtekar variables and the ADM variables for the reduced system. Since the gravitational lagrangian is written in terms of Σ_a as

$$L = N\Omega(\Sigma_1\Sigma_2\Sigma_3)^{1/2}\left[\frac{1}{N^2}\left\{\frac{\dot{\Sigma}_1^2}{\Sigma_1^2} + \frac{\dot{\Sigma}_2^2}{\Sigma_2^2} + \frac{\dot{\Sigma}_3^2}{\Sigma_3^2} - \frac{1}{2}\left(\frac{\dot{\Sigma}_1}{\Sigma_1} + \frac{\dot{\Sigma}_2}{\Sigma_2} + \frac{\dot{\Sigma}_3}{\Sigma_3}\right)^2\right\}\right.$$

$$+\frac{\Sigma_1{}^2+\Sigma_2{}^2+\Sigma_3{}^2}{\Sigma_1\Sigma_2\Sigma_3}-\frac{\Sigma_2{}^4\Sigma_3{}^4+\Sigma_3{}^4\Sigma_1{}^4+\Sigma_1{}^4\Sigma_2{}^4}{2\Sigma_1{}^3\Sigma_2{}^3\Sigma_3{}^3}\bigg]\,, \tag{4.26}$$

the ADM momentum P_a conjugate to Σ_a is given by

$$P_a := \frac{\partial L}{\partial \dot{\Sigma}_a}$$

$$=\frac{\Omega}{N}\frac{(\Sigma_1\Sigma_2\Sigma_3)^{1/2}}{\Sigma_a}\bigg[2\frac{\dot{\Sigma}_a}{\Sigma_a}-\Big(\frac{\dot{\Sigma}_1}{\Sigma_1}+\frac{\dot{\Sigma}_2}{\Sigma_2}+\frac{\dot{\Sigma}_3}{\Sigma_3}\Big)\bigg]$$

$$=-\frac{2\Omega}{N}(\dot{\alpha}+\dot{\beta}_a)e^{\alpha+\beta_a}\,. \tag{4.27}$$

On the other hand, the potential Φ_{ab} in A_{ab} becomes diagonal in the gauge $R=1$ and its diagonal component Φ_a is given by

$$\Phi_a=\frac{\partial}{\partial\Sigma_a}F(\Sigma)\,,$$

$$F:=\frac{1}{2}\Big(\frac{\Sigma_2\Sigma_3}{\Sigma_1}+\frac{\Sigma_3\Sigma_1}{\Sigma_2}+\frac{\Sigma_1\Sigma_2}{\Sigma_3}\Big)\,. \tag{4.28}$$

Hence in the gauge $Q=1$ and $R=1$, the Ashtekar momentum is related with the ADM momentum by

$$H_a=\mp\frac{1}{2\Omega}P_a+i\frac{\partial}{\partial\Sigma_a}F(\Sigma)\,. \tag{4.29}$$

That is, the Ashtekar momentum is a simple combination of the ADM momentum and an anisotropy potential.

In spite of this direct correspondence the Ashtekar variables seem to be useful because the dynamical equations become much simpler. This is clearly seen by comparing the expression for C^H in terms of the Ashtekar variables, Eq. (4.22) and the corresponding expression in terms of the ADM variables

$$C^H=-\frac{1}{2\Omega^2}(\Sigma_1\Sigma_2P_1P_2+\Sigma_2\Sigma_3P_2P_3+\Sigma_3\Sigma_1P_3P_1)$$

$$-(\Sigma_1^2+\Sigma_2^2+\Sigma_3^2)+\frac{\Sigma_2{}^4\Sigma_3{}^4+\Sigma_3{}^4\Sigma_1{}^4+\Sigma_1{}^4\Sigma_2{}^4}{2\Sigma_1{}^2\Sigma_2{}^2\Sigma_3{}^2}\,. \tag{4.30}$$

§ 5. Vacuum Bianchi type IX—quantum dynamics

As explained in the last section, the gauge constraint and the momentum constraint are solved and the evolution of the gauge freedom and the off-diagonal components of the metric become trivial for the vacuum Bianchi IX model. Hence the dynamics is completely described only by the canonical variables Σ_a and H_a with the hamiltonian $\mathcal{H}=\Omega Ne^{-3\alpha}C^H$. This system is formally quantized by replacing the Poisson brackets by the corresponding commutation relations

$$[\Sigma_a, \Sigma_b] = [H_a, H_b] = 0 ,$$

$$[\Sigma_a, H_b] = \mp \frac{1}{2\Omega} i \delta_{ab} . \tag{5.1}$$

Like the ADM theory,[8] the whole dynamical information is contained in the hamiltonian constraint

$$C^H | \Psi \rangle = 0 . \tag{5.2}$$

One subtle point in this equation is the operator ordering. According to Jacobson and Smolin,[6] the constraint algebra closes in a weak sense if the operators are ordered so that the $\mathcal{A}_i$ sits left to $\tilde{\sigma}^i$. If this ordering rule is respected, by the replacement

$$\Sigma_a \longrightarrow \mp 2\Omega_i \frac{\partial}{\partial H_a} , \tag{5.3}$$

this equation takes a quite simple and beautiful form in the momentum representation:

$$\left[(H_1 H_2 - iH_3) \frac{\partial^2}{\partial H_1 \partial H_2} + (H_2 H_3 - iH_1) \frac{\partial^2}{\partial H_2 \partial H_3} \right.$$

$$\left. + (H_3 H_1 - iH_2) \frac{\partial^2}{\partial H_3 \partial H_1} \right] \Psi(H) = 0 . \tag{5.4}$$

This equation appears to be much easier to solve than the corresponding Wheeler-DeWitt equation in the ADM formalism. In fact, if the wavefunction $\Psi(H)$ is expanded in power series of H, its coefficients satisfy a simple linear recurrence relation. Though it is not easy to find the general solution of this recurrence relation, one can find some exact solutions of it. For example, by solving the recurrence relation under a certain ansatz, we found the following non-trivial solution:

$$\Psi = f_1(H_1) + f_2(H_2) + f_3(H_3)$$

$$+ c_1 \left[H_1 \ln \frac{1+H_3^2}{1+H_2^2} + H_2 \ln \frac{1+iH_3}{1-iH_3} - H_3 \ln \frac{1+iH_2}{1-iH_2} \right]$$

$$+ c_2 \left[H_2 \ln \frac{1+H_1^2}{1+H_3^2} + H_3 \ln \frac{1+iH_1}{1-iH_1} - H_1 \ln \frac{1+iH_3}{1-iH_3} \right]$$

$$+ c_3 \left[H_3 \ln \frac{1+H_2^2}{1+H_1^2} + H_1 \ln \frac{1+iH_2}{1-iH_2} - H_2 \ln \frac{1+iH_1}{1-iH_1} \right] , \tag{5.5}$$

where f_a are arbitrary functions, and c_a are arbitrary constants. Though this solution does not seem to be cosmologically interesting for the reason explained below, it is at least the first non-trivial solution of the Wheeler-DeWitt equation for the vacuum Bianchi IX model. Its existence also suggests that it might be possible to find the general solution of the equation.

It is also possible to work in the Σ-representation. For the correspondence (5.3), solutions in the two representations are related by the ordinary Fourier transformation:

$$\Psi(\Sigma) = \int dH \; e^{\mp 2i\Omega\Sigma \cdot H} \Psi(H) . \tag{5.6}$$

There are some subtle points here. First, since the Ashtekar momentum is a complex quantity, there is a large freedom in the choice of the integration path in the above transformation. Any choice formally yields a solution in the Σ representation. Even closed paths can be taken. Second the solutions to the Σ-representation of the Wheeler-DeWitt equation expressed in terms of the Ashtekar variables are different from the corresponding solutions to the Wheeler-DeWitt equation expressed in terms of the ADM variables. From the correspondence of the momentum variables, Eq. (4.29), the wavefunctions in the Ashtekar variables and those in the ADM variables are related by

$$\Psi_{\text{ADM}}(\Sigma) = e^{\pm 2\Omega F(\Sigma)} \Psi_{\text{Ashtekar}}(\Sigma) . \tag{5.7}$$

When this transformation procedure is applied to the solution (5.5), there appear singular functions in $\Psi_{\text{Ashtekar}}(\Sigma)$ such as $\delta(H_1)$ and $\delta'(H_2)$. Due to this singular behavior, we obtain a physically meaningless $\Psi_{\text{ADM}}(\Sigma)$ because the prefactor in Eq. (5.7) vanishes or diverges when any of Σ_a becomes zero. This example shows that one must look for solutions of Eq. (5.4) which, for each of H_a, have poles or fall off at large $|H_a|$, in order to obtain physically meaningful wavefunctions in the ADM representation.

§ 6. Conclusion

In this paper we have studied how the canonical dynamics of spatially homogeneous spacetime is described in terms of the Ashtekar variables. We have shown that it is not possible in general to eliminate the gauge freedom introduced by Ashtekar preserving the polynomial nature of the constraint equations and the canonical equations, which is the sales point of the Ashtekar formalism, except for the vacuum Bianchi IX model. We have examined the structure of this exceptional but important case of the vacuum Bianchi IX model in detail to prove that the dynamics of the system is described by polynomial equations with respect to a reduced set of Ashtekar-type variables which are in one-to-one correspondence to the ADM variables.

One of the most important results of the paper is the simple and beautiful equation for the wavefunction of the quantized vacuum Bianchi IX universe. It seems possible to find a large class of exact solutions to this equation.

Of course, there exist some new problems provoked by adopting the Ashtekar variables. First if one starts from the momentum representation as proposed by Jacobson and Smolin,[6] one must find solutions which satisfy appropriate asymptotic structure or pole structure in order to obtain a physically meaningful solution when transformed into the metric representation of the ADM formalism. This restriction makes it difficult to find a solution in the momentum representation with the aid of power series expansion. One might expect that the formal procedure found by Jacobson and Smolin[6] to construct the general solutions to the quantum hamiltonian

constraint might work since the momentum and gauge constraints become trivial for the vacuum Bianchi IX model. Unfortunately, however, it seems not to be the case, though not exactly proved. This is because in Jacobson and Smolin's proof it plays an essential role for fields to be varied locally, which is not allowed when the fields are restricted to be spatially homogeneous.

Second even if this condition is satisfied, solutions do not necessarily yield cosmologically acceptable universes due to the prefactor in Eq. (5·7). This prefactor is a real-valued function which is rapidly changing but not oscillatory. Hence unless this prefactor is canceled by a factor coming from the Ashtekar wavefunction itself, there exists no region where semi-classical approximation holds. Note that this requirement of the cancellation of the prefactor is the quantum expression of the reality condition on the ADM variables in the classical dynamics.

Further detailed investigation of these problems in this simple case will give us insight to construct a meaningful quantum theory in the framework of the Ashtekar formalism.

Acknowledgements

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