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SPECIALIZATION OF MACPHERSON'S CHERN CLASSES

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In [4] MacPherson created a theory of Chern classes for singular algebraic varieties over the field of complex numbers. He defined a natural transformation between two covariant functors:

(a) constructible functions
(b) an appropriate homology theory; let us say rational equivalence theory [2].

In [3] it is observed that, via results of Sabbah [5], this theory can be extended to varieties over an arbitrary algebraically closed field of characteristic zero. Verdier [6], using derived categories, has shown that MacPherson's Chern classes enjoy a nice specialization property: there are specializations of constructible functions and of homology classes compatible with the Chern class. In the present paper it is shown that this specialization property can likewise be extended to varieties over an arbitrary algebraically closed field of characteristic zero.

As in [3] we rely on Sabbah's conormal space constructions, which show in his words that "la théorie des classes de Chern se ramène à une théorie de Chow ..., qui ne fait intervenir que des classes fondamentales". Our sole contribution is to combine Sabbah's constructions with the specialization apparatus of [2].

I am indebted to Bernard Teissier for his series of lectures on Lagrangian varieties at the University of North Carolina in 1986, and have borrowed freely from a preliminary version of a joint paper of Lê and Teissier.

1. Lagrangian varieties; MacPherson's Chern classes.

Let k be an algebraically closed field of characteristic zero. Consider the category of divisorial schemes (separated and of finite type) over k, and of proper morphisms. From this category to the category of abelian groups there are two covariant functors:

– The functor A_* assigns to each scheme X the group A_*X of cycles modulo rational equivalence. See [2, Chap. 1] for details.

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– The constructible function functor $\mathscr{C}$ assigns to each scheme X the group $\mathscr{C}X$ of constructible functions. The definition of the pushforward homomorphisms is given in [3, Sec. 3] and summarized below. In case $k = \mathbf{C}$ the definition is essentially "calculate the topological Euler characteristic of the fiber".

In [3, Sec. 4] it is shown that there is a natural transformation c_* from $\mathscr{C}$ to A_*. If X is nonsingular then $c_*: \mathscr{C}X \to A_*X$ takes the characteristic function to the Poincaré dual of the total Chern class of the tangent bundle. Of particular interest is the case $k = \mathbf{C}$, which was studied by MacPherson [4].

The natural transformation c_* is defined by introducing a third functor $\mathscr{L}$, which we call the *Lagrangian functor*. To avoid an annoying technicality in the definition of $\mathscr{L}X$, let us assume that X is a closed subscheme of a nonsingular variety M. (We deal with this technicality in the final paragraph). Let m denote the dimension of M. Let $\pi: T^\vee M \to M$ denote the cotangent bundle of M. The total space $T^\vee M$ is equipped with a canonical differential form α: if λ is a point of $T^\vee M$, i.e., a linear form on $T_{\pi(\lambda)}M$, and v is a tangent vector to $T^\vee M$ at λ, then by definition

$$\alpha(v) = \lambda(d\pi(v)).$$

A closed subvariety Λ of $T^\vee M$ is called *conical Lagrangrian* (or *homogeneous Lagrangian*) if it is of dimension m, and if α annihilates each tangent vector at each nonsingular point of Λ.

Let $T^\vee M|_X$ denote the restriction of $T^\vee M$ to X. A conical Lagrangian subvariety of $T^\vee M|_X$ will be called a *conical Lagrangian variety over X*. We define $\mathscr{L}X$ to be the free abelian group on the set of conical Lagrangian varieties over X. Note that $\mathscr{L}X$ is a subgroup of the group of cycles on $T^\vee M|_X$; we call an element of $\mathscr{L}X$ a *conical Lagrangian cycle over X*.

An example (in fact the only example) of such a subvariety is the *conormal space* $C(W)$ of a closed subvariety $W \subset X$. The fiber of $C(W)$ over a nonsingular point $x \in W$ consists of all forms annihilating the tangent space T_xW; $C(W)$ is obtained by taking the closure in $T^\vee M$.

LEMMA. *Every conical Lagrangian subvariety of $T^\vee M$ is the conormal space of a closed subvariety of M. Hence $\mathscr{L}X$ is isomorphic to the free abelian group on the set of conormal spaces of subvarieties of X.*

PROOF. Suppose that Λ is a conical Lagrangian subvariety. Let $W = \pi(\Lambda)$. For a generic point $\lambda \in \Lambda$ the derivative $T_\lambda\Lambda \to T_{\pi(\lambda)}W$ is surjective. Hence λ annihilates $T_{\pi(\lambda)}W$. Therefore $\Lambda \subset C(W)$.

Now suppose that f is a proper morphism from X to Y. We assume that Y is a subscheme of a nonsingular variety N, and that X is a subscheme of $M \times N$, with M nonsingular and f the restriction of the projection p of $M \times N$ onto N.

The total space $p^* T^\vee N$ is naturally contained in $T^\vee(M \times N)$, and there is a natural morphism from $T^\vee(M \times N)$ to $T^\vee N$ covering p. If Λ is a conical Lagrangian cycle over X, we can obtain a conical Lagrangian cycle over Y by intersecting Λ and $p^* T^\vee N$, then pushing forward to $T^\vee N$ by the natural morphism. In this fashion we obtain a homomorphism from $\mathscr{L}X$ to $\mathscr{L}Y$; in [3, Sec. 2] it is shown that such homomorphisms make $\mathscr{L}$ into a covariant functor.

Again suppose that X is embedded in M. The space $T^\vee M$ carries a tautological bundle of rank $m - 1$ whose fiber over a point $x \in M$ is a hyperplane in $T_x M$; we denote this bundle "taut". If W is a closed subvariety of X, its *Chern-Mather class* is the element of $A_* X$ defined by

$$\check{c}(W) = (-1)^{m - \dim W - 1} \kappa_*(c(\text{taut}) \cap [C(W)]),$$

where $\kappa\colon C(W) \to W$ is the projection. We define a homomorphism from $\mathscr{L}X$ to $A_* X$ by sending $C(W)$ to $(-1)^{\dim W} \check{c}(W)$. It is shown in [3, Sec. 4] that these definitions are independent of the choice of M, and that the homomorphisms give a natural transformation from $\mathscr{L}$ to A_*.

The *multiplicity* of $\check{c}(W)$ at $x \in X$ is the constructible function μ_W given by

$$\mu_W(x) = (-1)^{m - \dim W - 1} \deg(c(\text{taut}) \cap s(\kappa^{-1} x, C(W))),$$

where s denotes the Segre class [2, Chap. 4]. We define a homomorphism from $\mathscr{L}X$ to $\mathscr{C}X$ by sending $C(W)$ to $(-1)^{\dim W} \mu_W$. Observe that $\mu_W(x) = 0$ if $x \notin W$ and that $\mu_W(x) = 1$ if x is a nonsingular point of W; therefore $\mathscr{L}X \to \mathscr{C}X$ is an isomorphism. Now $\mathscr{L}$ is a functor; hence by transport of structure we can define an isomorphic functor $\mathscr{C}$, and automatically will have a natural transformation c_* from $\mathscr{C}$ to A_*. We call c_* the *Chern-MacPherson natural transformation.*

2. Specialization.

Suppose that $f\colon X \to S$ is a morphism to a nonsingular curve, and that s is a nonsingular point of S. Let X_s denote the fiber over s. We wish to define a homomorphism from $\mathscr{L}X$ to $\mathscr{L}X_S$.

We assume that X is a closed subscheme of $M \times S$, where M is nonsingular and f is the restriction of the projection of $M \times S$ onto S. Let us identify M with the fiber of $M \times S$ over s. Then there is a specialization homomorphism from the cycles on $T^\vee(M \times S)$ to those on $T^\vee(M \times S)|_M$. (The hypothesis $\dim S = 1$ is essential here; otherwise one obtains a specialization homomorphism only on the level of rational equivalence. See [2, p. 76].)

The *specialization* of an element Λ of $\mathscr{L}X$ is obtained by applying this specialization homomorphism, pushing the resulting cycle forward via the natural projection from $T^\vee(M \times S)|_M$ to $T^\vee M$, and changing the sign. The cycle thus obtained is a conical Lagrangian cycle over X_S.

The specialization apparatus of intersection theory shows that specialization is compatible with pushforward, i.e., that for a morphism $f\colon X \to Y$ over S we have a commutative diagram

$$\begin{array}{ccc} \mathscr{L}X & \longrightarrow & \mathscr{L}X_s \\ \downarrow & & \downarrow \\ \mathscr{L}Y & \longrightarrow & \mathscr{L}Y_s \end{array}$$

It likewise shows that specialization in $\mathscr{L}$ is compatible with specialization in A_* via the natural transformation, i.e., that there are commutative diagrams

$$\begin{array}{ccc} \mathscr{L}X & \longrightarrow & \mathscr{L}X_s \\ \downarrow & & \downarrow \\ A_*X & \longrightarrow & A_*X_s \end{array}$$

(The relevant facts from [2] are summarized in Proposition 10.1(a).)

Specialization of constructible functions is defined by transport of structure from $\mathscr{L}$. Explicitly, if W is a subvariety of X and $C(W)$ specializes to $\sum m_V C(V)$, then μ_W specializes to $\sum m_V \mu_V$. In [5] Sabbah shows that in case $k = \mathbf{C}$ this specialization agrees with the specialization defined by Verdier [6].

THEOREM. 1) *Specialization of constructible functions is compatible with pushforward.*

2) *Specialization of constructible functions is compatible with specialization in A_* via the Chern-MacPherson natural transformation.*

PROOF. These statements follow automatically from the corresponding statements for $\mathscr{L}$.

NOTE. We have assumed throughout that each scheme X is embedded as a closed subscheme of a nonsingular variety. Such an embedding serves merely to present the sheaf of differentials on X as the quotient of a locally free sheaf. If X is divisorial then by [1, Theorem 3.3] every coherent sheaf on X is a quotient of a locally free sheaf. Using this fact one can systematically extend the theory of MacPherson's Chern classes to divisorial schemes.

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