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Spectra of graphs resulting from various graph operations and products: a survey

Abstract: Let G be a graph on n vertices and $A(G)$, $L(G)$, and $|L|(G)$ be the adjacency matrix, Laplacian matrix and signless Laplacian matrix of G, respectively. The paper is essentially a survey of known results about the spectra of the adjacency, Laplacian and signless Laplacian matrix of graphs resulting from various graph operations with special emphasis on corona and graph products. In most cases, we have described the eigenvalues of the resulting graphs along with an explicit description of the structure of the corresponding eigenvectors.

1 Introduction

The study of spectral graph theory is concerned with the relationships between the spectra of certain matrices associated with a graph and the structural properties of that graph. In literature, there are a wide variety of matrices associated with graphs from which the spectrum can be extracted. Among these, frequently studied matrices are the adjacency matrix, the Laplacian matrix and the signless Laplacian matrix, see [1, 14–16, 18, 30, 31, 45, 47]. Several researchers have studied various spectral properties of these matrices. The study of the spectra has found its applications in several subjects like biology, geography, economics, social sciences, computer science, information and communication technologies, see for example [23, 51] and references therein. The study of graph spectra is a vast area. This article aims to survey some specific topics of this area that are described in the next few paragraphs.

All graphs considered in this article are assumed to be simple and connected, unless otherwise mentioned. Let G be a graph on vertices $1, 2, \ldots, n$. At times, we use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G, respectively. We use the notation $i \sim j$ to mean the existence of an edge between the vertices i and j of G. The *adjacency matrix* of G, denoted by $A(G)$, is an $n \times n$ matrix with entries $a_{ij} = 1$ or 0, depending on whether $i \sim j$ or otherwise, respectively. The *Laplacian matrix* of G, denoted by $L(G)$, is defined as $D(G) - A(G)$, where $D(G)$ is the diagonal matrix with degree of the vertex i as the i-th diagonal entry. It is well known that $L(G)$ is a positive semidefinite matrix with the smallest eigenvalue 0. There is an extensive literature available on the adjacency and Laplacian matrices of graphs. We refer the reader to a classical book by Cvetković, Doob, and Sachs [15] and two survey articles by Merris [45] and Mohar [47], for more background on these two matrices. Fiedler [25] proved that 0 is a simple eigenvalue of $L(G)$ if and only

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if G is connected, which led Fiedler to coin the term *algebraic connectivity* of a graph to mean the second smallest eigenvalue of the Laplacian matrix of that graph. Since its introduction to the literature, the algebraic connectivity of a graph has received a good deal of attention (see [3, 19, 24, 25, 29, 45, 48]). The matrix $|L|(G)$ defined as $|L|(G) = D(G) + A(G)$, was first termed as the *signless Laplacian matrix* of G by Haemers and Spence in [32]. Like the Laplacian matrix of a graph the signless Laplacian matrix is positive semidefinite. Cvetković [18] proved that the least eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite and the multiplicity of the eigenvalue 0 is equal to the number of bipartite components. Recently, the signless Laplacian matrix of a graph and its spectrum have attracted the attention of researchers (see [10, 13, 18, 20–22, 53, 54]).

One of the interesting question in spectral graph theory is: *looking at the structure of a graph, is it possible to predict the spectrum of that graph?* One way to deal with this problem is to use various graph operations. Several researchers have introduced many graph operations such as complement, disjoint union, join, graph products (namely the Cartesian product, the direct product, the strong product and the lexicographic product), the corona, the edge corona, the neighbourhood corona, the subdivision-vertex join, the subdivision-edge join, the subdivision-vertex corona, the subdivision-edge corona, the subdivision-vertex neighbourhood corona, the subdivision-edge neighbourhood corona, the R-vertex corona, the R-edge corona, the R-vertex neighbourhood corona, the R-edge neighbourhood corona, see for example [4, 6–9, 15, 26–28, 33, 34, 36, 39, 40, 42, 44] and the references therein.

It is well known that G^c, the *complement* of G is the graph whose vertex set is same as that of G and two vertices are adjacent in G^c if and only if they are not adjacent in G. The *union* of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$ is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and the edge set is $E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$ is the graph obtained from $G_1 \cup G_2$ by adding all possible edges from the vertices of G_1 to those in G_2. Join operation is also known as complete product ([15]).

The graph products are useful in constructing many important classes of graphs. Let G_1 and G_2 be two graphs with disjoint vertex sets $\{u_1, \ldots, u_m\}$ and $\{v_1 \ldots, v_n\}$, respectively. A *graph product* of G_1 and G_2 is a new graph whose vertex set is $V(G_1) \times V(G_2)$, the Cartesian product of $V(G_1)$ and $V(G_2)$. The adjacency of two distinct vertices (u_i, v_j) and (u_r, v_s) in the product graph is determined entirely by the *adjacency/equality/non adjacency* of u_i and u_r in G_1 and that of v_j and v_s in G_2. Thus, one can define 256 different types of graph products. The graphs obtained by taking the products of two graphs are called the *product graphs*, and the two graphs are called the *factors*. The four graph products, namely the Cartesian product, the direct product, the strong product and the lexicographic product are known as the standard graph products and have been studied by many researchers. We refer the reader to the book by Imrich and Klavžar [35] for a study of graph products and their structural properties. The *Cartesian product* of G_1 and G_2, denoted by $G_1 \square G_2$ is the graph, where $(u_i, v_j) \sim (u_r, v_s)$ if either ($u_i = u_r$ and $v_j \sim v_s$ in G_2) or ($u_i \sim u_r$ in G_1 and $v_j = v_s$). The *Kronecker product* or *direct product* of G_1 and G_2, denoted by $G_1 \otimes G_2$, is the graph where $(u_i, v_j) \sim (u_r, v_s)$ if $u_i \sim u_r$ in G_1 and $v_j \sim v_s$ in G_2. The *strong product* of G_1 and G_2, denoted by $G_1 \boxtimes G_2$, is the graph where $(u_i, v_j) \sim (u_r, v_s)$ if either ($u_i = u_r$ and $v_j \sim v_s$ in G_2) or ($u_i \sim u_r$ in G_1 and $v_j = v_s$) or ($u_i \sim u_r$ in G_1 and $v_j \sim v_s$ in G_2). The *lexicographic product* of G_1 and G_2, denoted by $G_1[G_2]$, is the graph where $(u_i, v_j) \sim (u_r, v_s)$ if either ($u_i \sim u_r$ in G_1) or ($u_i = u_r$ and $v_j \sim v_s$ in G_2). Investigation of the various spectra of product graphs is one interesting topic for researchers. Some results describing the adjacency and the Laplacian spectra of product graphs can be found in [2, 6, 15, 27, 45, 50] and the references therein.

Like the above mentioned graph operations, the corona is another operation which is used in constructing many important classes of graphs. The corona of two graphs was first introduced by Frucht and Harary [26]. Let G_1 and G_2 be two graphs on disjoint sets of n and m vertices, respectively. The *corona* of G_1 and G_2, denoted by $G_1 \circ G_2$, is defined as the graph obtained by taking one copy of G_1 and n copies of G_2, and then joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2. Recently, two variants of the corona operation, namely the edge corona and the neighbourhood corona were introduced by Hou and Shiu [34] and Gopalapillai [28], respectively. Let G_1 and G_2 be two graphs on disjoint sets of n_1 and n_2 vertices, m_1 and m_2 edges, respectively. The *edge corona* [34] of G_1 and G_2, denoted by $G_1 \diamondsuit G_2$ is the graph obtained by taking one copy of G_1 and m_1 copies of G_2, and then joining two end vertices of the i-th edge of G_1 to every vertex in the i-th copy of G_2. The *neighbourhood corona* [28] of G_1 and G_2, denoted by $G_1 \bullet G_2$ is the graph obtained

by taking one copy of G_1 and n_1 copies of G_2, and joining every neighbour of the i-th vertex of G_1 to every vertex of the i-th copy of G_2 by a new edge. Results on the adjacency, the Laplacian and the signless Laplacian spectra of different coronae of two graphs can be found in [4, 28, 34, 44, 52].

Let G be a connected graph on n vertices and m edges. The *subdivision graph* $S(G)$ of G is the graph obtained by inserting a new vertex into every edge of G. The *Q-graph* of G, denoted by $Q(G)$ is the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G. The *total graph* of G, denoted by $T(G)$, is the graph whose set of vertices is the union of the set of vertices and set of edges of G, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident (see [15]). Note that $S(G)$, $Q(G)$ and $T(G)$ have $n+m$ vertices each. In the above three graphs, let us call the n vertices taken from G as the *old-vertices* and the new inserted m vertices as the *new-vertices*. In [49], Shinodab has described the characteristic polynomial of the adjacency matrix of a subdivision graph.

Two graph operations based on subdivision graphs, namely the subdivision-vertex join and the subdivision-edge join were introduced by Indulal [37]. The *subdivision-vertex join* of two graphs G_1 and G_2, denoted by $G_1 \dot\vee G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining each old-vertex of G_1 with every vertex of G_2. The *subdivision-edge join* of G_1 and G_2, denoted by $G_1 \veebar G_2$, is the graph obtained from $S(G_1)$ and G_2 by joining each new-vertex of G_1 with every vertex of G_2. In [37], the author described the adjacency spectra of $G_1 \dot\vee G_2$ and $G_1 \veebar G_2$ in terms of the adjacency spectra of G_1 and G_2, when both G_1 and G_2 are regular. In [41], Liu and Zhang described the adjacency, Laplacian, and signless Laplacian spectra of $G_1 \dot\vee G_2$ and $G_1 \veebar G_2$, when G_1 is regular and G_2 is an arbitrary graph using the adjacency, Laplacian and signless Laplacian spectra of G_1 and G_2, respectively.

In [43], Lu and Miao defined two new operations on subdivision graphs, namely, subdivision-vertex corona and subdivision-edge corona. Let G_1 and G_2 be two vertex disjoint graphs. The *subdivision-vertex corona* of G_1 and G_2, denoted by $G_1^{(S)} \odot G_2$, is the graph obtained from $S(G_1)$ and $|V(G_1)|$ copies of G_2, all vertex disjoint, and joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2. The *subdivision-edge corona* of G_1 and G_2, denoted by $G_1^{(S)} \ominus G_2$, is the graph obtained from $S(G_1)$ and $|I(G_1)|$ copies of G_2, all vertex disjoint, and joining the i-th vertex of $I(G_1)$ to every vertex in the i-th copy of G_2, where $I(G_1)$ is the set of inserted new-vertices of $S(G_1)$. Liu and Lu [40] introduced two new graph operations based on subdivision graphs, namely, the subdivision vertex-neighbourhood corona and the subdivision edge-neighbourhood corona and discussed their adjacency, Laplacian and signless Laplacian spectra . Let G_1 and G_2 be two vertex disjoint graphs. The *subdivision-vertex neighbourhood corona* of G_1 and G_2, denoted by $G_1^{(S)} \boxdot G_2$, is the graph obtained from $S(G_1)$ and $|V(G_1)|$ copies of G_2, all vertex disjoint, and joining the neighbours of the i-th vertex of G_1 to every vertex in the i-th copy of G_2. The *subdivision-edge neighbourhood corona* of G_1 and G_2, denoted by $G_1^{(S)} \boxminus G_2$, is the graph obtained from $S(G_1)$ and $|I(G_1)|$ copies of G_2, all vertex disjoint, and joining the neighbours of the i-th vertex of $I(G_1)$ to every vertex in the i-th copy of G_2.

The *R-graph* of G, denoted by $R(G)$ is the graph obtained from G by adding a vertex u_e and joining u_e to the end vertices of e, for each $e \in E(G)$ [15]. Observe that $R(G)$ is just the edge corona of G and K_1. Lan and Zhou [39] defined four new graph operations based on R-graphs, namely the *R*-vertex corona, *R*-edge corona, *R*-vertex neighbourhood corona and *R*-edge neighbourhood corona. Let G_1 and G_2 be two vertex disjoint graphs. The *R-vertex corona* of G_1 and G_2, denoted by $G_1^{(R)} \odot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the i-th vertex of G_1 to every vertex in the i-th copy of G_2. The *R-edge corona* of G_1 and G_2, denoted by $G_1^{(R)} \ominus G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the i-th vertex of $I(G_1)$ to every vertex in the i-th copy of G_2, where $I(G_1) = V(R(G_1)) - V(G_1)$. The *R-vertex neighbourhood corona* of G_1 and G_2, denoted by $G_1^{(R)} \boxdot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the neighbours of the i-th vertex of G_1 in $R(G_1)$ to every vertex in the i-th copy of G_2. The *R-edge neighbourhood corona* of G_1 and G_2, denoted by $G_1^{(R)} \boxminus G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the neighbours of the i-th vertex of $I(G_1)$ in R_{G_1} to every vertex in the i-th copy of G_2. In [39], the authors discussed the adjacency, Laplacian and signless Laplacian spectra of *R*-vertex corona, *R*-edge corona, *R*-vertex neighbourhood corona and *R*-edge neighbourhood corona, when G_1 is regular.

Barik and Sahoo in [5] introduced some more variants of corona graphs such as subdivision double corona, Q-graph double corona, R-graph double corona, total double corona, subdivision double neighbourhood corona, Q-graph double neighbourhood corona, R-graph double neighbourhood corona and total double neighbourhood corona. Let G be a connected graph on n vertices and m edges. Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. The *subdivision double corona* of G, G_1 and G_2, denoted by $G^{(S)} \circ (G_1, G_2)$, is the graph obtained by taking one copy of $S(G)$, n copies of G_1 and m copies of G_2 and then by joining the i-th old-vertex of $S(G)$ to every vertex of the i-th copy of G_1 and the j-th new-vertex of $S(G)$ to every vertex of the j-th copy of G_2. In place of $S(G)$, if we take $Q(G)$ ($R(G)$, $T(G)$), then the resulting graph is called as *Q-graph (R-graph, total) double corona* and denoted by $G^{(Q)} \circ (G_1, G_2)$ ($G^{(R)} \circ (G_1, G_2)$, $G^{(T)} \circ (G_1, G_2)$). The *subdivision double neighbourhood corona* of G, G_1 and G_2, denoted by $G^{(S)} \bullet (G_1, G_2)$, is the graph obtained by taking one copy of $S(G)$, n copies of G_1 and m copies of G_2 and then by joining the neighbourhood new vertices of the i-th old-vertex of $S(G)$ to every vertex of the i-th copy of G_1 and the neighbourhood old vertices of the j-th new-vertex of $S(G)$ to every vertex of the j-th copy of G_2. In place of $S(G)$, if we take $Q(G)$ (resp. $R(G)$, $T(G)$), then the resulting graph is called as *Q-graph (resp. R-graph, total graph) double neighbourhood corona* and denoted by $G^{(Q)} \bullet (G_1, G_2)$ (resp. $G^{(R)} \bullet (G_1, G_2)$, $G^{(T)} \bullet (G_1, G_2)$). In [5], the authors have described the spectra of these graphs.

In this article, we have listed together all the results on the adjacency, Laplacian and signless Laplacian spectra of graphs obtained from the above mentioned graph operations and graph products in tables. In most cases, we have provided the structure of eigenvectors. The constants k, k_1, k_2 used in the tables can be obtained easily from the eigen-equations.

Following notations are being used in the rest of the paper. By the A-eigenvalues (resp. L-eigenvalues, $|L|$-eigenvalues) of G we mean the eigenvalues of $A(G)$ (resp. $L(G)$, $|L|(G)$). The $n \times 1$ vector with each entry 1 is denoted by $\mathbf{1}_n$. The zero matrix of appropriate order is denoted by $\mathbf{0}$. By I_n, we denote the identity matrix of size n. The *Kronecker product* of matrices $R = [r_{ij}]$ and S is defined to be the partitioned matrix $[r_{ij}S]$ and is denoted by $R \otimes S$. The vector with i-th entry equal to one and all other entries zero is denoted by e_i.

2 Some unary operations on a graph and their eigenvalues

In this section, we consider operations on a single graph and describe the adjacency, Laplacian and signless Laplacian eigenvalues, and the corresponding eigenvectors of the new graphs.

Let G be a connected graph on n vertices and m edges. If the graph is regular, we denote its regularity by r with $r \geq 2$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, $0 = \lambda_1^L \leq \lambda_2^L \leq \ldots \leq \lambda_n^L$ and $\lambda_1^{|L|} \geq \lambda_2^{|L|} \geq \ldots \geq \lambda_n^{|L|}$ be the eigenvalues of $A(G)$, $L(G)$ and $|L|(G)$, respectively. Let x_i, x_i^L and $x^{|L|}$ be the eigenvectors of $A(G)$, $L(G)$ and $|L|(G)$ corresponding to the eigenvalues λ_i, λ_i^L and $\lambda_i^{|L|}$, respectively, for $i = 1, 2, \ldots, n$.

Among all the graph operations, the complement of a graph is the simplest one. When G is regular, all the adjacency eigenvalues of G^c are obtained by Sachs in 1962. As $L(G^c) = nI - J - L(G)$ and the eigenvectors of $L(G)$ are also eigenvectors of J, the eigenvalues and eigenvectors of $L(G^c)$ can be easily obtained from that of $L(G)$. For a r-regular graph G, the well known relation between $A(G)$, $L(G)$ and $|L|(G)$ is given by $A(G) = rI - L(G) = |L|(G) - rI$. So finding the eigenvalues of $|L|(G^c)$ in terms of eigenvalues of $|L|(G)$ for regular graphs is immediate.

The *line graph* G_ℓ of a graph G is the graph whose vertex set is in one-to-one correspondence with the set of edges of the graph and two vertices of G_ℓ are adjacent if and only if the corresponding edges in G have a vertex in common. Line graphs have one special property that their least (adjacency) eigenvalue is always greater than or equal to -2. This fact is evident from the relations $M(G)M^T(G) = A(G) + D(G)$ and $M^T(G)M(G) = A(G_\ell) + 2I$. Besides, the adjacency matrix of a line graph comes into role while expressing matrices related to other graphs like the total graph, Q-graph, etc. For more spectral properties of line graphs the reader is referred to the book by Cvetković, Rowlinson and Simić [17]. The characterization of the characteristic polynomial of $A(G_\ell)$ can be found in the book by Cvetković, Doob, and Sachs [15]. The fact that if 'G is r-regular, then G_ℓ is $(2r-2)$-regular' helps in determining the Laplacian as well as signless Laplacian eigenvalues of these graphs.

The subdivision graph $S(G)$ of a graph G is always a bipartite graph whose adjacency matrix can be written as

$$\begin{bmatrix} \mathbf{0}_n & M(G) \\ M^T(G) & \mathbf{0}_m \end{bmatrix},$$

where $M(G)$ is the 0-1 incidence matrix of G. Now consider the rectangular matrix $M(G)$ of order $n \times m$. Let ξ_i and ζ_i are singular vector pairs of $M(G)$ corresponding to the singular value s_i, for $i = 1, 2, \ldots, n$. That is, $M(G)\xi_i = s_i\zeta_i$ and $M^T(G)\zeta_i = s_i\xi_i$, for $i = 1, \ldots, n$. For an r-regular graph, the following relation holds:

$$s_i^2 = \lambda_i + r = 2r - \lambda_i^L = \lambda_i^{|L|}, \text{ for } i = 1, 2, \ldots, |m - n|.$$

Further, let η_j be orthogonal vectors such that $M(G)\eta_j = \mathbf{0}_n$, for $j = 1, \ldots, m - n$. Now it becomes easy to observe that

$$\begin{pmatrix} \pm\zeta_i \\ \xi_i \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{0}_n \\ \eta_j \end{pmatrix}$$

are the eigenvectors of $A(S(G))$ corresponding to eigenvalues $\pm s_i$ and 0, respectively. Similar observations can be made for the Q-graph and the R-graph of G whose adjacency matrices are

$$\begin{bmatrix} \mathbf{0}_n & M(G) \\ M^T(G) & A(G_\ell) \end{bmatrix} \text{ and } \begin{bmatrix} A(G) & M(G) \\ M^T(G) & \mathbf{0}_m \end{bmatrix},$$

respectively. Cvetković [12] in 1975 first obtained the characteristic polynomials of adjacency matrices of $S(G)$, $Q(G)$ and $R(G)$. In [11], a relationship between the spectra of a regular graph G and its total graph, $T(G)$ has been obtained.

Table 1: Spectral properties under unary operations

Graphs	A-eigenvalues	L-eigenvalues	$|L|$-eigenvalues	Multiplicity	Eigenvectors
G	λ_i	λ_i^L	$\lambda_i^{|L|}$	1	X_i
G^c	$-\lambda_t - 1$	$n - \lambda_t^L$	$n - 2 - \lambda_t^{|L|}$	1	X_t
	$n - r - 1$	0	$2(n - r - 1)$	1	$\mathbf{1}_n$
G_ℓ	$\lambda_i + r - 2$	$2 - \lambda_i^L$	$\lambda_i^{|L|} + 2r - 4$	1	ζ_i
	-2	$r - 2$	$2r - 4$	$m - n$	η_j
$S(G)$	$\pm\sqrt{\lambda_i + r}$	$\frac{r+2\pm\sqrt{(r+2)^2-4\lambda_i^L}}{2}$	$\frac{r+2\pm\sqrt{(r-2)^2+4\lambda_i^{|L|}}}{2}$	1	$\begin{pmatrix} \pm k\zeta_i \\ \xi_i \end{pmatrix}$
	0	2	2	$m - n$	$\begin{pmatrix} \eta_j \\ \mathbf{0} \end{pmatrix}$
$Q(G)$	$\frac{\lambda_i+r-2\pm\sqrt{(\lambda_i+r)^2+4}}{2}$	$\frac{\lambda_i^L+r+2\pm\sqrt{(\lambda_i^L-r)^2+4(r+1)}}{2}$	$\frac{\lambda_i^{|L|}+3r-2\pm\sqrt{(\lambda_i^{|L|}+r-2)^2+4\lambda_i^{|L|}}}{2}$	1	$\begin{pmatrix} \pm k\zeta_i \\ \xi_i \end{pmatrix}$
	-2	$2r + 2$	$2r - 2$	$m - n$	$\begin{pmatrix} \eta_j \\ \mathbf{0} \end{pmatrix}$
$R(G)$	$\frac{\lambda_i\pm\sqrt{\lambda_i^2+4(\lambda_i+r)}}{2}$	$\frac{\lambda_i^L+r+2\pm\sqrt{(\lambda_i^L+r+2)^2-12\lambda_i^L}}{2}$	$\frac{\lambda_i^{|L|}+r+2\pm\sqrt{(\lambda_i^{|L|}+r-2)^2+4\lambda_i^{|L|}}}{2}$	1	$\begin{pmatrix} \pm k\zeta_i \\ \xi_i \end{pmatrix}$
	0	2	2	$m - n$	$\begin{pmatrix} \eta_j \\ \mathbf{0} \end{pmatrix}$
$T(G)$	$\frac{2\lambda_i+r-2\pm\sqrt{4\lambda_i+r^2+4}}{2}$	$\frac{2\lambda_i^L+r+2\pm\sqrt{(r+2)^2-4\lambda_i^L}}{2}$	$\frac{2\lambda_i^{|L|}+3r-2\pm\sqrt{(r-2)^2+4\lambda_i^{|L|}}}{2}$	1	$\begin{pmatrix} \pm k\zeta_i \\ \xi_i \end{pmatrix}$
	-2	$2r + 2$	$2r - 2$	$m - n$	$\begin{pmatrix} \eta_j \\ \mathbf{0} \end{pmatrix}$

$$t = 2, \ldots, n; \ i = 1, 2, \ldots, n; \ j = 1, 2, \ldots, m - n$$

In the above table, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the eigenvalues of $A(G)$, $\lambda_1^L \leq \lambda_2^L \leq \ldots \leq \lambda_n^L$ are the eigenvalues of $L(G)$, $\lambda_1^{|L|} \geq \lambda_2^{|L|} \geq \ldots \geq \lambda_n^{|L|}$ are the eigenvalues of $|L|(G)$. Note that $\lambda_i, \lambda_i^L, \lambda_i^{|L|}$ correspond to the same eigenvector X_i.

In Table 1, G is considered to be a connected graph on n vertices and with m edges. All the different eigenvalues (adjacency, Laplacian and signless Laplacian) of the graph produced from G by the above stated unary operations are listed in it. Further, the possible form of corresponding eigenvectors are given. The 'k' appearing in the last column is an arbitrary constant. For a graph operation on G, if 'r' is used in the list of its A-eigenvalues (L-eigenvalues, $|L|$-eigenvalues), then for that operation to find its A-eigenvalues (L-eigenvalues, $|L|$-eigenvalues), G is assumed r-regular ($r \geq 2$). In the last column of Table 1, $X_t = x_t$ (resp. x_t^L, $x_t^{|L|}$) for adjacency (resp. Laplacian, signless Laplacian) eigenvector of G while ξ, ζ_i and η_i are singular vectors of $M(G)$.

3 Some binary operations on graphs and their eigenvalues

In this section, we consider two different graphs and some binary operations on them. Let G_1, G_2 be two graphs on n_1, n_2 vertices and m_1, m_2 edges. We assume $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$. If regularity of G_1 is required (or G_2 is required), then it is assumed r_1 regular (or r_2 regular). Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n_1}$, $0 = \lambda_1^L \leq \lambda_2^L \leq \ldots \leq \lambda_{n_1}^L$ and $\lambda_1^{|L|} \geq \lambda_2^{|L|} \geq \ldots \geq \lambda_{n_1}^{|L|}$ be the eigenvalues of $A(G_1)$, $L(G_1)$ and $|L|(G_1)$, respectively and let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n_2}$, $0 = \mu_1^L \leq \mu_2^L \leq \ldots \leq \mu_{n_2}^L$ and $\mu_1^{|L|} \geq \mu_2^{|L|} \geq \ldots \geq \mu_{n_2}^{|L|}$ be the eigenvalues of $A(G_2)$, $L(G_2)$ and $|L|(G_2)$, respectively. Let x_i (resp. x_i^L, $x_i^{|L|}$) and y_j (resp. y_j^L, $y_j^{|L|}$) be the eigenvectors of $A(G_1)$ (resp. $L(G_1)$, $|L|(G_1)$) and $A(G_2)$ (resp. $L(G_2)$, $|L|(G_2)$) corresponding to eigenvalues λ_i (resp. λ_i^L, $\lambda_i^{|L|}$) and μ_j (resp. μ_j^L, $\mu_j^{|L|}$), respectively, for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$.

Table 2: Spectral properties of the graphs obtained by disjoint union and join

Graphs	A-eigenvalues	L-eigenvalues	$|L|$-eigenvalues	Eigenvectors
G_1	λ_i	λ_i^L	$\lambda_i^{|L|}$	X_i
G_2	μ_j	μ_j^L	$\mu_j^{|L|}$	Y_j
$G_1 \cup G_2$	λ_i	λ_i^L	$\lambda_i^{|L|}$	$\begin{pmatrix} X_i \\ \mathbf{0} \end{pmatrix}$
	μ_j	μ_j^L	$\mu_j^{|L|}$	$\begin{pmatrix} \mathbf{0} \\ Y_j \end{pmatrix}$
$G_1 \vee G_2$	λ_i	$\lambda_i^L + n_2$	$\lambda_i^{|L|} + n_2$	$\begin{pmatrix} X_i \\ \mathbf{0} \end{pmatrix}$
	μ_j	$\mu_j^L + n_1$	$\mu_j^{|L|} + n_1$	$\begin{pmatrix} \mathbf{0} \\ Y_j \end{pmatrix}$
	$\frac{r_1 + r_2 \pm \sqrt{(r_1 - r_2)^2 + 4n_1 n_2}}{2}$	$0, n_1 + n_2$	$\frac{n_1 + n_2 + 2(r_1 + r_2) \pm \sqrt{\Delta}}{2}$ where $\Delta = (n_1 + n_2)^2$ $-4(r_1 - r_2)(n_1 - r_1 - (n_2 - r_2))$	$\begin{pmatrix} k\mathbf{1}_{n_1} \\ \mathbf{1}_{n_2} \end{pmatrix}$

$$i = 2, \ldots, n_1; \; j = 2, \ldots, n_2$$

As the adjacency matrix of $G_1 \cup G_2$ is the direct sum of $A(G_1)$ and $A(G_2)$, its adjacency eigenvalues are all the adjacency eigenvalues of G_1 and G_2. Similar type of relationship holds for Laplacian and signless Laplacian eigenvalues of $G_1 \cup G_2$. The join of G_1 and G_2 can be expressed as $G_1 \vee G_2 = (G_1^c \cup G_2^c)^c$. Thus, when both G_1 and G_2 are regular, the adjacency and signless Laplacian eigenvalues of $G_1 \cup G_2$ can be obtained using the operations complement and union. Notice that even if G_1 and G_2 are regular, $G_1 \vee G_2$ may not be regular. The complete description of Laplacian eigenvalues of $G_1 \vee G_2$ in terms of the Laplacian eigenvalues of G_1 and G_2 is given by Merris in 1998 [46]. Table 2 lists the adjacency, Laplacian and signless Laplacian eigenvalues of $G_1 \cup G_2$ and $G_1 \vee G_2$. In the last column of the table when the adjacency (Laplacian, signless Laplacian) case is considered, then $X_i = x_i$ (x_i^L, $x_i^{|L|}$) and $Y_j = y_j$ (y_j^L, $y_j^{|L|}$).

By finding suitable eigenvectors, Indulal [37] in 2012 obtained the spectrum of $G_1 \dot{\vee} G_2$ and $G_1 \underline{\vee} G_2$ in terms of the spectra of G_1 and G_2, when both G_1 and G_2 are regular. Table 3 lists all eigenvalues and eigenvectors of

Table 3: Spectral properties of subdivision-vertex and subdivision-edge join graphs

Graphs	A-eigenvalues	L-eigenvalues	$\lvert L\rvert$-eigenvalues	Multiplicity	Eigenvectors
G_1	λ_i	λ_i^L	$\lambda_i^{\lvert L\rvert}$	1	X_i
G_2	μ_j	μ_j^L	$\mu_j^{\lvert L\rvert}$	1	Y_j
$G_1 \dot{\vee} G_2$	$\pm\sqrt{\lambda_i + r_1}$	zeros of $x^2 - x(n_2 - r_1 + 2) + 2n_2 - 4r_1 + \lambda_i^L$	zeros of $(x-2)(x - n_2 - r_1) - \lambda_i^{\lvert L\rvert}$	1	$\begin{pmatrix} k\zeta_i \\ \xi_i \\ \mathbf{0}_{n_2} \end{pmatrix}$
	μ_j	$\mu_j^L + n_1$	$\mu_j^{\lvert L\rvert} + n_1$	1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ Y_j \end{pmatrix}$
	0	2	2	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \eta_l \\ \mathbf{0}_{n_2} \end{pmatrix}$
	zeros of $x^3 - r_2 x^2 - (n_1 n_2 + 2r_1)x + 2r_1 r_2$	zeros of $x^3 - x^2(n_1 + n_2 + r_1 + 2) + x\big(2(n_1 + n_2) + n_1 r_1\big)$	zeros of $x^3 - x^2(n_1 + 2r_2 + n_2 + 2 + r_1) + x\big(2(n_1 + n_2) + r_1(n_1 + 2r_2) + 2r_2(n_2 + 2)\big) - 4n_2 r_2$	1	$\begin{pmatrix} k_1 \mathbf{1}_{n_1} \\ k_2 \mathbf{1}_{m_1} \\ \mathbf{1}_{n_2} \end{pmatrix}$
$G_1 \veebar G_2$	$\pm\sqrt{\lambda_i + r_1}$	zeros of $(x-r_1)(x - n_2 - 2) - 2r_1 + \lambda_i^L$	zeros of $(x-r_1)(x - n_2 - 2) - \lambda_i^L$	1	$\begin{pmatrix} k\zeta_i \\ \xi_i \\ \mathbf{0}_{n_2} \end{pmatrix}$
	μ_j	$\mu_j^L + m_1$	$\mu_j^{\lvert L\rvert} + m_1$	1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ Y_j \end{pmatrix}$
	0	$n_2 + 2$	$n_2 + 2$	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \eta_l \\ \mathbf{0}_{n_2} \end{pmatrix}$
	zeros of $x^3 - r_2 x^2 - (m_1 n_2 + 2r_1)x + 2r_1 r_2$	zeros of $x^3 - x^2(m_1 + n_2 + r_1 + 2) + x\big(m_1(r_1 + 2) + n_2 r_1\big)$	zeros of $x^3 - x^2(m_1 + n_2 + r_1 + 2r_2 + 2) + x\big(2m_1(n_2+1) + r_1(m_1 + n_2) + 2r_2(n_2 + r_1 + 2)\big) - 2n_2 r_1(m_1 + r_2)$	1	$\begin{pmatrix} k_1 \mathbf{1}_{n_1} \\ k_2 \mathbf{1}_{m_1} \\ \mathbf{1}_{n_2} \end{pmatrix}$

$$i = 2, \ldots, n_1; \; j = 2, \ldots, n_2; \; l = 1, \ldots, m_1 - n_1$$

$G_1 \dot{\vee} G_2$ and $G_1 \veebar G_2$. The notation Y_j is used in a similar way as that used in Table 2. ξ_i, ζ_i and η_l are singular vectors of $M(G_1)$ for $i = 1, \ldots, n_1$ and $l = 1, \ldots, m_1 - n_1$.

In the last few decades, graph products have been studied extensively and applied to many problems in structural mechanics, see for example Kaveh and Alinejad [38] and the references therein. Among many graph products, the four standard products are the Cartesian, the direct, the strong and the lexicographic product of graphs. The Kronecker product of matrices play a crucial role while expressing the adjacency (resp. Laplacian, signless Laplacian) matrices of these graph products. Results describing the adjacency matrix and its spectra of the product graphs can be found in Brouwer and Haemers [6] and Cvetković, Doob and Sachs [15]. If $G_1 \square G_2$, $G_1 \otimes G_2$, $G_1 \boxtimes G_2$ and $G_1[G_2]$ represent the Cartesian product, the direct product, the strong product and the lexicographic product of two graphs G_1 and G_2, then we have

$$A(G_1 \square G_2) = A(G_1) \otimes I_{n_2} + I_{n_1} \otimes A(G_2),$$
$$A(G_1 \otimes G_2) = A(G_1) \otimes A(G_2),$$
$$A(G_1 \boxtimes G_2) = A(G_1 \square G_2) + A(G_1 \times G_2) \text{ and}$$
$$A(G_1[G_2]) = I_{n_1} \otimes A(G_2) + A(G_1) \otimes J_{n_2}.$$

Now if $\{x_i, \; i = 1, \ldots, n_1\}$ and $\{y_j, \; j = 1, \ldots, n_2\}$ are orthogonal sets of eigenvectors of $A(G_1)$ and $A(G_2)$, then $\{x_i \otimes y_j, \text{ for } i = 1, \ldots, n_1, j = 1, \ldots, n_2\}$, forms a set of orthogonal eigenvectors for $A(G_1 \square G_2)$, $A(G_1 \otimes G_2)$ and $A(G_1 \boxtimes G_2)$. But, as in the case of lexicographic product, the second term of $A(G_1[G_2])$ involves J_{n_2}, thus, to

find a set of orthogonal eigenvectors $A(G_1[G_2])$, G_2 is chosen regular. The Laplacian spectra of the Cartesian and the lexicographic product of graphs have been described completely using the Laplacian spectra of the factor graphs in Merris[45] and Barik, Bapat and Pati [2]. However, the Laplacian spectra of the direct product and the strong product of graphs are expressed in terms of the Laplacian spectra of its factor graphs only when the factor graphs are regular, see [2]. Table 4 lists all the eigenvalues of the four products along with their corresponding eigenvectors. Notations used in Table 4 are similar to that of Table 2.

Table 4: Spectral properties of product graphs

Graphs	A-eigenvalues	L-eigenvalues	$|L|$-eigenvalues	Eigenvectors
G_1	λ_i	λ_i^L	$\lambda_i^{|L|}$	X_i
G_2	μ_j	μ_j^L	$\mu_j^{|L|}$	Y_j
$G_1 \square G_2$	$\lambda_i + \mu_j$	$\lambda_i^L + \mu_j^L$	$\lambda_i^{|L|} + \mu_j^{|L|}$	$X_i \otimes Y_j$
$G_1 \otimes G_2$	$\lambda_i \mu_j$	$r_1 \mu_j^L + r_2 \lambda_i^L - \lambda_i^L \mu_j^L$	$(\lambda_i^{|L|} - r_1)(\mu_j^{|L|} - r_2) + r_1 r_2$	$X_i \otimes Y_j$
$G_1 \boxtimes G_2$	$\lambda_i + \mu_j + \lambda_i \mu_j$	$(1 + r_1)\mu_j^L + (1 + r_2)\lambda_i^L$ $- \lambda_i^L \mu_j^L$	$(\lambda_i^{|L|} - r_1)(\mu_j^{|L|} - r_2) + r_1 r_2 + \lambda_i^{|L|} +$ $\mu_j^{|L|}$	$X_i \otimes Y_j$
$G_1[G_2]$	$\lambda_i n_2 + r_2$	$\lambda_i^L n_2$	$\lambda_i^{|L|} n_2 + 2r_2$	$X_i \otimes \mathbf{1}_{n_2}$
	μ_j	$\mu_j^L + n_2 \deg(v_i)$	$\mu_j^{|L|} + n_2 \deg(v_i)$	$e_i \otimes Y_j$

$$i = 1, 2, \ldots, n_1; \; j = 1, 2, \ldots, n_2.$$

Like the above mentioned graph operations, the corona of two graphs fascinates many researchers because of its almost symmetrical structure and many more important spectral properties. Subsequently, many variants of corona (like the edge corona, the neighbourhood corona, etc.) are defined and their spectral properties are observed. In 2007, Barik, Pati and Sarma [4] provided complete information about the spectrum of $G_1 \circ G_2$ in terms of the spectrum of G_1 and G_2, when G_2 is regular. In the same paper, all trees with the property SR (A graph G is said to have property SR if $\frac{1}{\lambda}$ is an eigenvalue $A(G)$ if and only if λ is an eigenvalue of $A(G)$ and λ, $\frac{1}{\lambda}$ have the same multiplicity) are characterized and it is shown that such a tree is the corona product of some tree and an isolated vertex. Further, the authors obtained complete information about the Laplacian spectrum of $G_1 \circ G_2$. The adjacency matrix of $G_1 \circ G_2$ can be expressed as

$$A(G_1 \circ G_2) = \begin{bmatrix} A(G_1) & \mathbf{1}_{n_2}^T \otimes I_{n_1} \\ \mathbf{1}_{n_2} \otimes I_{n_1} & A(G_2) \otimes I_{n_1} \end{bmatrix}.$$

McLeman and McNicholas [44] in 2011, observed the presence of the term $\mathbf{1}_{n_2}^T A^{-1}(G_2)\mathbf{1}_{n_2}$ in the characteristic polynomial of $A(G_1 \circ G_2)$ and named it as *coronal* of the graph. They have described the characteristic polynomial of $A(G_1 \circ G_2)$ using that of $A(G_1)$, $A(G_2)$, and the coronal of G_2. But they have obtained simple expressions (of the spectrum) only for the graphs which are regular or complete bipartite.

In 2010, Hou and Shiu [34] defined the edge corona operation on two graphs and described the spectrum (resp. the Laplacian spectrum) of $G_1 \Diamond G_2$ in terms of the spectra (resp. Laplacian spectra) of G_1 and G_2, when both G_1 and G_2 are regular (resp. when G_1 is regular). In [28], Gopalapillai described the spectrum (resp. Laplacian spectrum) and eigenvectors of $G_1 \bullet G_2$, when G_2 is regular (resp. G_1 is regular). Table 5 describes all the eigenvalues of corona, edge corona and neighbourhood corona. The notations used in the table are similar to those used in the previous tables.

Lu and Miao [43] in 2013 defined subdivision-vertex and subdivision-edge corona of graphs. Using the coronal of the second graph G_2, the authors [43] described the characteristic polynomial of the adjacency, the Laplacian and the signless Laplacian matrices of subdivision-vertex and subdivision-edge corona graphs, when G_1, G_2 are regular. In Table 6, all the eigenvalues of $G_1^{(S)} \odot G_2$ and $G_1^{(S)} \ominus G_2$ are listed.

In 2013, Liu and Lu [40] used different coronals of G_2 to express the characteristic polynomials of the adjacency (Laplacian, signless Laplacian) matrices of $G_1^{(S)} \boxdot G_2$ (subdivision-vertex neighbourhood corona)

Table 5: Spectral properties of corona, edge corona and neighbourhood corona graphs

Graphs	A-eigenvalues	L-eigenvalues	$|L|$-eigenvalues	Multi-plicity	Eigenvectors
G_1	λ_i	λ_i^L	$\lambda_i^{|L|}$	1	X_i
G_2	μ_j	μ_j^L	$\mu_j^{|L|}$	1	Y_j
$G_1 \circ G_2$	$\frac{\lambda_i + r_2 \pm \sqrt{(\lambda_i - r_2)^2 + 4n_2}}{2}$	$\frac{\Delta \pm \sqrt{\Delta^2 - 4\lambda_i^L}}{2}$, where $\Delta = \lambda_i^L + n_2 + 1$	$\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_2}}{2}$, where $\Delta_1 = \lambda_i^{|L|} + n_2 + 2r_2 + 1,$ $\Delta_2 = \lambda_i^{|L|}(2r_2 + 1) + 2n_2 r_2$	1	$\begin{pmatrix} kX_i \\ \mathbf{1}_{n_2} \otimes X_i \end{pmatrix}$
	μ_j	$\mu_j^L + 1$	$\mu_j^{|L|} + 1$	n_1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ Y_j \otimes e_i \end{pmatrix}$
$G_1 \diamond G_2$	$\frac{\lambda_i + r_2 \pm \sqrt{(\lambda_i - r_2)^2 + 4\Delta}}{2}$, where $\Delta = n_2(\lambda_i + r_1)$	$\frac{\Delta \pm \sqrt{\Delta^2 - 4\lambda_i^L(n_2 + 2)}}{2}$, where $\Delta = \lambda_i^L + n_2 r_1 + 2$	$\frac{\Delta_1 \pm \sqrt{\Delta_1^2 + 4n_2\lambda_i^{|L|}}}{2}$, where $\Delta_1 = n_2 r_1 + \lambda_i^{|L|} + 2(r_2 + 1),$ $\Delta_2 = n_2 r_1 + \lambda_i^{|L|} - 2(r_2 + 1)$	1	$\begin{pmatrix} kX_i \\ \mathbf{1}_{n_2} \otimes X_i \end{pmatrix}$
	μ_j	$\mu_j^L + 2$	$\mu_j^{|L|} + 2$	m_1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ Y_j \otimes e_i \end{pmatrix}$
	r_2	2	$2r_2 + 2$	$m_1 - n_1$	$\begin{pmatrix} \eta_j \\ \mathbf{0} \end{pmatrix}$
$G_1 \bullet G_2$	$\frac{\Delta \pm \sqrt{\Delta^2 + 4\lambda_i(\lambda_i n_2 - r_2)}}{2}$, where $\Delta = \lambda_i + r_2$	$\frac{\Delta_1 \pm \sqrt{\Delta_1^2 + 4\lambda_i^L \Delta_2}}{2}$, where $\Delta_1 = \lambda_i^L + (n_2 + 1)r_1,$ $\Delta_2 = n_2\lambda_i^L - r_1(2n_2 + 1)$	$\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_2}}{2}$, where $\Delta_1 = r_1(n_2 + 1) + 2r_2 + \lambda_i^{|L|},$ $\Delta_2 = 2n_2 r_1 r_2 + \lambda_i^{|L|}(2n_2 r_1 + 2r_2 + r_1 - n_2\lambda_i^{|L|})$	1	$\begin{pmatrix} kX_i \\ \mathbf{1}_{n_2} \otimes X_i \end{pmatrix}$
	μ_j	$\mu_j^L + r_1$	$\mu_j^{|L|} + r_1$	n_1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ Y_j \otimes e_i \end{pmatrix}$

$$i = 1, 2, \ldots, n_1; \; j = 2, \ldots, n_2$$

and $G_1^{(S)} \boxminus G_2$ (subdivision-edge neighbourhood corona). In Table 7, all the eigenvalues of $G_1^{(S)} \boxdot G_2$ and $G_1^{(S)} \boxminus G_2$ are listed, respectively.

Lan and Zhou [39] introduced four new graph operations based on R-graphs, namely the R-vertex corona, R-edge corona, R-vertex neighbourhood corona and R-edge neighbourhood corona. In [39], the authors have provided characterstic polynomials of adjacency, Laplacian and signless Laplacian matrices of R-vertex corona, R-edge corona and R-edge neighbourhood corona, when G_1 is regular. In Table 8, we have listed different eigenvalues of $G_1^{(R)} \odot G_2$ and $G_1^{(R)} \ominus G_2$ along with their multiplicities and Table 9 contains all the eigenvalues of $G_1^{(R)} \boxdot G_2$ and $G_1^{(R)} \boxminus G_2$.

4 Some ternary operations on graphs and their eigenvalues

In this section, G is considered to be a connected graph on n vertices and with m edges. Furthermore, G_1 and G_2 are two graphs on n_1 and n_2 vertices, respectively. If regularity of G (resp. G_1, G_2) is required, then it is assumed to be r (resp. r_1, r_2) regular. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, $0 = \lambda_1^L \leq \lambda_2^L \leq \ldots \leq \lambda_n^L$ and $\lambda_1^{|L|} \geq \lambda_2^{|L|} \geq \ldots \geq \lambda_n^{|L|}$ be the eigenvalues of $A(G)$, $L(G)$ and $|L|(G)$, respectively. Similarly, let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n_1}$, $0 = \mu_1^L \leq \mu_2^L \leq \ldots \leq \mu_{n_1}^L$ and $\mu_1^{|L|} \geq \mu_2^{|L|} \geq \ldots \geq \mu_{n_1}^{|L|}$ be the eigenvalues of $A(G_1)$, $L(G_1)$ and $|L|(G_1)$, respectively and let $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_{n_2}$, $0 = \nu_1^L \leq \nu_2^L \leq \ldots \leq \nu_{n_2}^L$ and $\nu_1^{|L|} \geq \nu_2^{|L|} \geq \ldots \geq \nu_{n_2}^{|L|}$ be the eigenvalues of $A(G_2)$, $L(G_2)$ and $|L|(G_2)$, respectively. Let y_i (y_i^L, $y_i^{|L|}$) and z_j (z_j^L, $z_j^{|L|}$) be the eigenvectors of $A(G_1)$ ($L(G_1)$, $|L|(G_1)$) and $A(G_2)$ ($L(G_2)$, $|L|(G_2)$) corresponding to eigenvalues μ_i (μ_i^L, $\mu_i^{|L|}$) and ν_j (ν_j^L, $\nu_j^{|L|}$) for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$, respectively. Let ξ_i and ζ_i are singular vector pairs of $M(G)$ corresponding to the singular value s_i, for $i = 1, 2, \ldots, n$. Further, let η_j, for $j = 1, \ldots, m - n$, be orthogonal vectors such that $M(G)\eta_j = \mathbf{0}_n$, for $j = 1, \ldots, m - n$.

Table 6: Spectral properties of subdivision-vertex and subdivision-edge corona graphs

Graphs	A-eigenvalues	L-eigenvalues	$\|L\|$-eigenvalues	Multi-plicity	Eigenvector
$G_1^{(S)} \odot G_2$	zeros of $x^3 - r_2 x^2 - (\lambda_i + r_1 + n_2)x + r_2(\lambda_i + r_1)$	zeros of $x^3 - (n_2 + r_1 + 3)x^2 + (2n_2 + r_1 + \lambda_i^L + 2)x - \lambda_i^L$	zeros of $x^3 - (n_2 + r_1 + 2r_2 + 3)x^2 + ((2r_2 + 3)(n_2 + r_1) + 2(2r_2 + 1) - n_2 - \lambda_i^{\|L\|})x + (2r_2 + 1)\lambda_i^{\|L\|} - 2r_1 - 4r_2(n_2 + r_1)$	1	$\begin{pmatrix} k_1 \zeta_i \\ k_2 \xi_i \\ \mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}$
	μ_j	$\mu_j^L + 1$	$\mu_j^{\|L\|} + 1$	n_1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ Y_j \otimes e_i \end{pmatrix}$
	0	2	2	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \eta_p \\ \mathbf{0}_{n_2 n_1} \end{pmatrix}$
$G_1^{(S)} \ominus G_2$	zeros of $x^3 - r_2 x^2 - (\lambda_i + r_1 + n_2)x + r_2(\lambda_i + r_1)$	zeros of $x^3 - (n_2 + r_1 + 3)x^2 + (r_1(n_2 + 1) + 2 + \lambda_i^L)x - \lambda_i^L$	zeros of $x^3 - (n_2 + r_1 + 2r_2 + 3)x^2 + (3r_1 + 4r_2 + n_2 r_1 + 2n_2 r_2 + 2r_1 r_2 + 2 - \lambda_i^{\|L\|})x - 2n_2 r_1 r_2 - 4r_1 r_2 - 2r_1 + (2r_2 + 1)\lambda_i^{\|L\|}$	1	$\begin{pmatrix} k_1 \zeta_i \\ k_2 \xi_i \\ \mathbf{1}_{n_2} \otimes \xi_i \end{pmatrix}$
	μ_j	$\mu_j^L + 1$	$\mu_j^{\|L\|} + 1$	m_1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ Y_j \otimes e_q \end{pmatrix}$
	$\frac{r_2 \pm \sqrt{r_2^2 + 4n_2}}{2}$	$\frac{(n_2+3) \pm \sqrt{(n_2+3)^2 - 8}}{2}$	zeros of $x^2 - (n_2 + 2r_2 + 3)x + 2(r_2 n_2 + 2r_2 + 1)$	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ k\eta_p \\ \mathbf{1}_{n_2} \otimes \eta_p \end{pmatrix}$

$i = 1, 2, \ldots, n_1;\ j = 2, \ldots, n_2;\ p = 1, 2, \ldots, m_1 - n_1;\ q = 1, 2, \ldots, m_1$

Recently, Barik and Sahoo [5], defined some ternary operations which are generalizations of operations like subdivision-vertex neighbourhood corona, subdivision-edge neighbourhood corona etc. These includes *subdivision double corona* denoted by $G^{(S)} \circ (G_1, G_2)$, *Q-graph double corona* denoted by $G^{(Q)} \circ (G_1, G_2)$, *R-graph double corona* denoted by $G^{(R)} \circ (G_1, G_2)$ and *total graph double corona* denoted by $G^{(T)} \circ (G_1, G_2)$ of G, G_1 and G_2, respectively. Observe that the adjacency matrix of $G^{(S)} \circ (G_1, G_2)$ can be expressed as

$$A(G^{(S)} \circ (G_1, G_2)) = \begin{pmatrix} \mathbf{0}_n & M(G) & \mathbf{1}_{n_1}^T \otimes I_n & \mathbf{0} \\ M^T(G) & \mathbf{0}_m & \mathbf{0} & \mathbf{1}_{n_2}^T \otimes I_m \\ \mathbf{1} \otimes I_n & \mathbf{0} & A(G_1) \otimes I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \otimes I_m & \mathbf{0} & A(G_2) \otimes I_m \end{pmatrix}.$$

The adjacency matrices of $G^{(Q)} \circ (G_1, G_2)$, $G^{(R)} \circ (G_1, G_2)$ and $G^{(T)} \circ (G_1, G_2)$ are given by

$$A(G^{(Q)} \circ (G_1, G_2)) = A(G^{(S)} \circ (G_1, G_2)) + \text{diag}(\mathbf{0}_n, A(G_\ell), \mathbf{0}_{nn_1}, \mathbf{0}_{mn_2}),$$

$$A(G^{(R)} \circ (G_1, G_2)) = A(G^{(S)} \circ (G_1, G_2)) + \text{diag}(A(G), \mathbf{0}_m, \mathbf{0}_{nn_1}, \mathbf{0}_{mn_2})\ \text{and}$$

$$A(G^{(T)} \circ (G_1, G_2)) = A(G^{(S)} \circ (G_1, G_2)) + \text{diag}(A(G), A(G_\ell), \mathbf{0}_{nn_1}, \mathbf{0}_{mn_2}).$$

Furthermore, the degree diagonal matrix of these graphs are given by

$$D(G^{(S)} \circ (G_1, G_2)) = \text{diag}\big((n_1 + r)I_n, (n_2 + 2)I_m, I_{nn_1}, I_{mn_2}\big),$$

$$D(G^{(Q)} \circ (G_1, G_2)) = \text{diag}\big((n_1 + r)I_n, D(G_\ell) + (n_2 + 2)I_m, I_{nn_1}, I_{mn_2}\big),$$

$$D(G^{(R)} \circ (G_1, G_2)) = \text{diag}\big(D(G) + (n_1 + r)I_n, (n_2 + 2)I_m, I_{nn_1}, I_{mn_2}\big)\ \text{and}$$

$$D(G^{(T)} \circ (G_1, G_2)) = \text{diag}\big(D(G) + (n_1 + r)I_n, D(G_\ell) + (n_2 + 2)I_m, I_{nn_1}, I_{mn_2}\big).$$

Now the expressions for the Laplacian and signless Laplacian matrices can be easily obtained.

Table 7: Spectral properties of subdivision-vertex neighbourhood and subdivision-edge neighbourhood corona graphs

Graphs	A-eigenvalues	L-eigenvalues	$\|L\|$-eigenvalues	Multi-plicity	Eigenvector
$G_1^{(S)} \square G_2$	zeros of $x^3 - r_2 x^2 - (n_2+1)(\lambda_i + r_1)x + r_2(\lambda_i + r_1)$	zeros of $(x - r_1)\big(x^2 - (2n_2 + r_1 + 2)x + (n_2+1)\lambda_i^L\big)$	zeros of $(x - r_1 - 2r_2)(x - r_1)(x - 2 - 2n_2) - \lambda_i^{\|L\|}\big((n_2+1)x - 2r_2 - r_1 n_2 - r_1\big)$	1	$\begin{pmatrix} k_1 \zeta_i \\ k_2 \xi_i \\ \mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}$
	μ_j	$\mu_j^L + r_1$	$\mu_j^{\|L\|} + r_1$	n_1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ Y_j \otimes e_i \end{pmatrix}$
	0	$2n_2 + 2$	$2n_2 + 2$	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \eta_p \\ \mathbf{0}_{n_2 n_1} \end{pmatrix}$
$G_1^{(S)} \boxminus G_2$	zeros of $x^3 - r_2 x^2 - (n_2+1)(\lambda_i + r_1)x + r_2(\lambda_i + r_1)$	zeros of $(x - 2)\big(x^2 - (r_1 n_2 + r_1 + 2)x + (n_2+1)\lambda_i^L\big)$	zeros of $(x - r_1 - r_1 n_2)(x - 2)(x - 2 - 2r_2) - \lambda_i^{\|L\|}\big((n_2+1)x - 2r_2 - 2n_2 - 2\big)$	1	$\begin{pmatrix} k_1 \zeta_i \\ k_2 \xi_i \\ \mathbf{1}_{n_2} \otimes \xi_i \end{pmatrix}$
	μ_j	$\mu_j^L + 2$	$\mu_j^{\|L\|} + 2$	m_1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ Y_j \otimes e_q \end{pmatrix}$
	0	2	2	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \eta_p \\ \mathbf{0}_{n_2 m_1} \end{pmatrix}$
	r_2	2	$2r_2 + 2$	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ \mathbf{1}_{n_2} \otimes \eta_p \end{pmatrix}$

$$i = 1, 2, \ldots, n_1;\ j = 2, \ldots, n_2;\ p = 1, 2, \ldots, m_1 - n_1;\ q = 1, 2, \ldots, m_1$$

Table 10, lists the A, L and $\|L\|$-eigenvalues of all the above described double corona graphs. The '$*$' appearing in the first column of Table 10 stands for S, Q, R or T. Further in the table, ξ_i, ζ_i and η_p are the singular vectors of $M(G)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m - n$ and $Y_q = y_q\ (y_q^L, y_q^{\|L\|})$ for $q = 2, \ldots, n_1$ and $Z_l = z_l\ (z_l^L, z_l^{\|L\|})$ for $l = 2, \ldots, n_2$ if we are determining the adjacency (Laplacian, signless Laplacian) eigenvalues. The coefficients of the polynomials appearing in Table 10 are given below (outside the table).

Table 8: Spectral properties of R-vertex and R-edge corona graphs

Graphs	A-eigenvalues	L-eigenvalues	$\|L\|$-eigenvalues	Multiplicity	Eigenvector
$G_1^{(R)} \odot G_2$	zeros of $x^3 - (r_2 + \lambda_i)x^2 - (n_2 + r_1 - \lambda_i(r_2 - 1))x + r_2(r_1 + \lambda_i)$	zeros of $x^3 - (n_2 + r_1 + \lambda_i^L + 3)x^2 + (2n_2 + r_1 + 4\lambda_i^L + 2)x - 3\lambda_i^L$	zeros of $x^3 - (n_2 + r_1 + 2r_2 + \lambda_i^{\|L\|} + 3)x^2 + (2(r_2 + 1)(n_2 + \lambda_i^{\|L\|}) + 2(r_1 r_2 + 1) + 3r_1 + 4r_2)x - (2r_1 + \lambda_i^{\|L\|})(2r_2 + 1) - 4n_2 r_2$	1	$\begin{pmatrix} k_1 \zeta_i \\ k_2 \xi_i \\ \mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}$
	μ_j	$\mu_j^L + 1$	$\mu_j^{\|L\|} + 1$	n_1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ Y_j \otimes e_i \end{pmatrix}$
	0	2	2	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \eta_p \\ \mathbf{0}_{n_2 n_1} \end{pmatrix}$
$G_1^{(R)} \ominus G_2$	zeros of $x^3 - (r_2 + \lambda_i)x^2 - (n_2 + r_1 - \lambda_i(r_2 - 1))x + r_1 r_2 + \lambda_i(n_2 + r_2)$	zeros of $x^3 - (n_2 + r_1 + \lambda_i^L + 3)x^2 + (r_1(n_2 + 1) + \lambda_i^L(n_2 + 4) + 2)x - 3\lambda_i^L$	zeros of $x^3 - (n_2 + r_1 + 2r_2 + \lambda_i^{\|L\|} + 3)x^2 + ((n_2 + 2)(2r_2 + \lambda_i^{\|L\|}) + 2r_2(r_1 + \lambda_i^{\|L\|}) + r(n_2 + 3) + 2)x - 2r_1(r_2(n_2 + 2) + 1) - \lambda_i^{\|L\|}(2r_2(n_2 + 1) + 1)$	1	$\begin{pmatrix} k_1 \zeta_i \\ k_2 \xi_i \\ \mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}$
	μ_j	$\mu_j^L + 1$	$\mu_j^{\|L\|} + 1$	m_1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ Y_j \otimes e_i \end{pmatrix}$
	$\dfrac{r_2 \pm \sqrt{r_2^2 + 4n_2}}{2}$	$\dfrac{n_2 + 3 \pm \sqrt{(n_2 + 3)^2 - 8}}{2}$	zeros of $x^2 - (n_2 + 2r_2 + 3)x + 2(2r_2 + n_2 r_2 + 1)$	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ k\eta_p \\ \mathbf{1}_{n_2} \otimes \eta_p \end{pmatrix}$

$$i = 1, 2, \ldots, n_1; \; j = 2, \ldots, n_2; \; p = 1, 2, \ldots, m_1 - n_1; \; q = 1, 2, \ldots, m_1$$

$$\mathring{A}_1 = r_1 + r_2 + \mathring{a}_1 + \mathring{b}_1,$$
$$\mathring{B}_1 = (r_1 + r_2)(\mathring{a}_1 + \mathring{b}_1) + r_1 r_2 + \mathring{a}_1 \mathring{b}_1 - (n_1 + n_2 + r + \lambda_i),$$
$$\mathring{C}_1 = (r_1 + r_2)(\mathring{a}_1 \mathring{b}_1 - \lambda_i - r) + r_1 r_2(\mathring{a}_1 + \mathring{b}_1) - n_1(r_2 + \mathring{b}_1) - n_2(r_1 + \mathring{a}_1),$$
$$\mathring{D}_1 = n_1 n_2 - \mathring{a}_1 n_2 r_1 - \mathring{b}_1 n_1 r_2 - r_1 r_2(\lambda_i + r - \mathring{a}_1 \mathring{b}_1);$$

$$\mathring{A}_2 = n_1 + n_2 + r + \mathring{a}_2 + \mathring{b}_2 + 4,$$
$$\mathring{B}_2 = (n_1 + 1)(n_2 + 3) + 2(r + 1) + n_2 r + \mathring{a}_2(n_2 + 4) + \mathring{b}_2(n_1 + r + 2) + \mathring{a}_2 \mathring{b}_2 + \lambda_i^L,$$
$$\mathring{C}_2 = \mathring{a}_2(n_2 + 5) + \mathring{b}_2(n_1 + 2r + 1) + \mathring{a}_2 \mathring{b}_2 + 2(n_1 + \lambda_i^L + 1) + r(n_2 + 1),$$
$$\mathring{D}_2 = 2\mathring{a}_2 + \mathring{b}_2 r + \mathring{a}_2 \mathring{b}_2 + \lambda_i^L;$$

$$\mathring{A}_3 = n_1 + n_2 + r + 2(r_1 + r_2) + \mathring{a}_3 + \mathring{b}_3 + 4,$$
$$\mathring{B}_3 = 2(r_1 + r_2)(n_1 + n_2 + r + \mathring{a}_3 + \mathring{b}_3 + 3) + 4r_1 r_2 + n_1 n_2 + \mathring{a}_3 \mathring{b}_3 + r(n_2 + \mathring{b}_3 + 4)$$
$$\quad + n_1(\mathring{b}_3 + 3) + n_2(\mathring{a}_3 + 1) + 4\mathring{a}_3 + 2\mathring{b}_3 - \lambda_i^{\|L\|} + 5,$$
$$\mathring{C}_3 = 2(r_1 + r_2)\left(n_1 n_2 + (r + \mathring{a}_3)(n_2 + 3) + \mathring{b}_3(n_1 + r + \mathring{a}_3 + 1) + 2 - \lambda_i^{\|L\|}\right)$$
$$\quad + 4r_1 r_2(n_1 + n_2 + r + \mathring{a}_3 + \mathring{b}_3 + 2) + n_1(2(3r_1 + 2r_2 + 1) + \mathring{b}_3) + n_2(r + 2r_2 + \mathring{a}_3)$$
$$\quad + (r + \mathring{a}_3)(2\mathring{b}_3 + 5) + 2(1 - \lambda_i^{\|L\|}) + \mathring{b}_3,$$
$$\mathring{D}_3 = 4r_1 r_2\left(n_1 n_2 + n_2(r + \mathring{a}_3) + (n_1 + r + \mathring{a}_3)(\mathring{b}_3 + 2) - \lambda_i^{\|L\|}\right)$$
$$\quad + 2(r_1 + r_2)\left((r + \mathring{a}_3)(\mathring{b}_3 + 2) - \lambda_i^{\|L\|}\right) + (2n_1 r_1 + r + \mathring{a}_3)(\mathring{b}_3 + 2) + 2n_2 r_2(r + \mathring{a}_3) - \lambda_i^{\|L\|},$$

Table 9: Spectral properties of R-vertex neighbourhood and R-edge neighbourhood corona graphs

Graphs	A-eigenvalues	L-eigenvalues	$\|L\|$-eigenvalues	Multi-plicity	Eigenvector
$G_1^{(R)} \boxdot G_2$	zeros of $x^3 - (r_2 + \lambda_i)x^2 - ((n_2 + 1)(\lambda_i + r_1) + \lambda_i(n_2\lambda_i - r_2))x + (\lambda_i + r_1)(r_2 - n_2\lambda_i)$	zeros of $x^3 - (2n_2 + n_2r_1 + 3r_1 + \lambda_i^L + 2)x^2 + (2n_2r_1(n_2 + \lambda_i^L) + r_1^2(n_2 + 2) + \lambda_i^L n_2(3 - \lambda_i^L) + \lambda_i^L(2r_1 + 3) + 2r_1(3n_2 + 2))x - 6\lambda_i^L r_1 - \lambda_i^L n_2(5n_2r_1 + 13r_1 - 3\lambda_i^L - 2\lambda_i^L n_2)$	zeros of $(x + n_2(\lambda_i^{\|L\|} - r_1) - 2(r_1 + r_2))((\lambda_i^{\|L\|} - r_1)(x - 2n_2 - 2) + \lambda_i^{\|L\|}) + (x - r_1(n_2 + 2))(n_2\lambda_i^{\|L\|} - (x - 2r_1 - 2r_2)(x - 2n_2 - 2))$	1	$\begin{pmatrix} k_1\zeta_i \\ k_2\xi_i \\ \mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}$
	μ_j	$\mu_j^L + 2r_1$	$\mu_j^{\|L\|} + 2r_1$	n_1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ Y_j \otimes e_i \end{pmatrix}$
	0	$2(n_2 + 1)$	$2(n_2 + 1)$	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \eta_p \\ \mathbf{0}_{n_2 n_1} \end{pmatrix}$
$G_1^{(R)} \boxminus G_2$	zeros of $x^3 - (r_2 + \lambda_i)x^2 - (\lambda_i(n_2 - r_2 + 1) + r_1(n_2 + 1))x + r_2(\lambda_i + r_1)$	zeros of $(x-2)(x^2 - (r_1(n_2+1)+\lambda_i^L)x + (n_2 + 3)\lambda_i^L)$	zeros of $x^3 - (r_1(n_2 + 1) + 2(r_2 + 2) + \lambda_i^{\|L\|})x^2 + (2r_1(n_2+1)(r_2+2) + 4(r_2+1)+\lambda_i^{\|L\|}(2r_2-n_2+3))x - 2(r_2 + 1)(2r_1(n_2 + 1) + \lambda_i^{\|L\|}) + 2n_2\lambda_i^{\|L\|}$	1	$\begin{pmatrix} k_1\zeta_i \\ k_2\xi_i \\ \mathbf{1}_{n_2} \otimes \zeta_i \end{pmatrix}$
	μ_j	$\mu_j^L + 2$	$\mu_j^{\|L\|} + 2$	m_1	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ Y_j \otimes e_q \end{pmatrix}$
	0	2	2	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \eta_p \\ \mathbf{0}_{n_2 m_1} \end{pmatrix}$
	r_2	2	$2r_2 + 2$	$m_1 - n_1$	$\begin{pmatrix} \mathbf{0}_{n_1} \\ \mathbf{0}_{m_1} \\ \mathbf{1}_{n_2} \otimes \eta_p \end{pmatrix}$

$$i = 1, 2, \ldots, n_1;\ j = 2, \ldots, n_2;\ p = 1, 2, \ldots, m_1 - n_1;\ q = 1, 2, \ldots, m_1$$

Table 10: Spectral properties of double corona graphs

Graphs	A-eigenvalues	L-eigenvalues	$\|L\|$-eigenvalues	Multi-plicity	Eigenvectors
$G^{(*)} \circ (G_1, G_2)$	zeros of $x^4 - \mathring{A}_1 x^3 + \mathring{B}_1 x^2 - \mathring{C}_1 x + \mathring{D}_1$	zeros of $x^4 - \mathring{A}_2 x^3 + \mathring{B}_2 x^2 - \mathring{C}_2 x + \mathring{D}_2$	zeros of $x^4 - \mathring{A}_3 x^3 + \mathring{B}_3 x^2 - \mathring{C}_3 x + \mathring{D}_3$	1	$\begin{pmatrix} k_1\zeta_i \\ k_2\xi_i \\ k_3\mathbf{1}_{n_1} \otimes \zeta_i \\ \mathbf{1}_{n_2} \otimes \xi_i \end{pmatrix}$
	$\frac{r_2 \pm \sqrt{r_2^2 + 4(n_2 + \mathring{c}_1)}}{2}$	$\frac{\Delta \pm \sqrt{\Delta^2 - 4(\mathring{c}_2 + 2)}}{2},$ where $\Delta = n_2 + \mathring{c}_2 + 3$	$\frac{\Delta \pm \sqrt{\Delta^2 + 4(2r_2 - \mathring{c}_3 - 2)}}{2},$ where $\Delta = n_2 - 2r_2 + \mathring{c}_3 + 3$	$m - n$	$\begin{pmatrix} \mathbf{0}_n \\ k\eta_p \\ \mathbf{0}_{n_1 n} \\ \mathbf{1}_{n_2} \otimes \eta_p \end{pmatrix}$
	μ_j	$\mu_j^L + 1$	$\mu_j^{\|L\|} + 1$	n	$\begin{pmatrix} \mathbf{0}_{n+m} \\ Y_q \otimes e_i \\ \mathbf{0}_{n_2 m} \end{pmatrix}$
	ν_l	$\nu_l^L + 1$	$\nu_l^{\|L\|}$	m	$\begin{pmatrix} \mathbf{0}_{n+m+n_1 n} \\ Z_l \otimes e_j \end{pmatrix}$

$$i = 1, 2, \ldots, n;\ j = 1, \ldots, m;\ p = 1, \ldots, m - n;\ q = 2, \ldots, n_1;\ l = 2, \ldots, n_2$$

where for

$$S_G : \mathring{a}_1 = \mathring{b}_1 = \mathring{c}_1 = 0; \quad \mathring{a}_2 = \mathring{b}_2 = \mathring{c}_2 = 0; \quad \mathring{a}_3 = \mathring{b}_3 = \mathring{c}_3 = 0;$$

$$Q_G : \mathring{a}_1 = 0, \mathring{b}_1 = \lambda_i + r - 2, \mathring{c}_1 = -2; \quad \mathring{a}_2 = 0, \mathring{b}_2 = \lambda_i^L, \mathring{c}_2 = 2r; \quad \mathring{a}_3 = 0, \mathring{b}_3 = \lambda_i^{\|L\|} + 2r - 4, \mathring{c}_3 = 2r - 4;$$

$$R_G : \mathring{a}_1 = \lambda_i, \mathring{b}_1 = \mathring{c}_1 = 0; \quad \mathring{a}_2 = \lambda_i^L, \mathring{b}_2 = \mathring{c}_2 = 0; \quad \mathring{a}_3 = \lambda^{\|L\|}, \mathring{b}_3 = \mathring{c}_3 = 0;$$

$$T_G : \mathring{a}_1 = \lambda_i, \mathring{b}_1 = \lambda_i + r - 2, \mathring{c}_1 = -2; \mathring{a}_2 = \mathring{b}_2 = \lambda_i^L, \mathring{c}_2 = 2r; \mathring{a}_3 = \lambda_i^{\|L\|}, \mathring{b}_3 = \lambda_i^{\|L\|} + 2r - 4, \mathring{c}_3 = 2r - 4.$$

Likewise, Barik and Sahoo [5] defined another four variants of subdivision double corona, namely the *subdivision double neighbourhood corona* of G, G_1 and G_2, denoted by $G^{(S)} \bullet (G_1, G_2)$, *Q-graph double neighbourhood corona* of G, G_1 and G_2, denoted by $G^{(Q)} \bullet (G_1, G_2)$, the *R-graph double neighbourhood corona* of G, G_1 and G_2, denoted by $G^{(R)} \bullet (G_1, G_2)$, and the *total double neighbourhood corona* of G, G_1 and G_2, denoted by $G^{(T)} \bullet (G_1, G_2)$.

Observe that, the adjacency matrix of $G^{(S)} \bullet (G_1, G_2)$ can be expressed as

$$A(G^{(S)} \bullet (G_1, G_2)) = \begin{pmatrix} \mathbf{0}_n & M(G) & \mathbf{0} & \mathbf{1}_{n_2}^T \otimes M(G) \\ M^T(G) & \mathbf{0}_m & \mathbf{1}_{n_1}^T \otimes M^T(G) & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \otimes M(G) & A(G_1) \otimes I_n & \mathbf{0} \\ \mathbf{1} \otimes M^T(G) & \mathbf{0} & \mathbf{0} & A(G_2) \otimes I_m \end{pmatrix}.$$

Then the adjacency matrices of $G^{(Q)} \bullet (G_1, G_2)$, $G^{(R)} \bullet (G_1, G_2)$ and $G^{(T)} \bullet (G_1, G_2)$ are given by

$$A(G^{(Q)} \bullet (G_1, G_2)) = A(G^{(S)} \bullet (G_1, G_2)) + \mathrm{diag}(\mathbf{0}_n, A(G_\ell), \mathbf{0}_{nn_1}, \mathbf{0}_{mn_2}),$$

$$A(G^{(R)} \bullet (G_1, G_2)) = A(G^{(S)} \bullet (G_1, G_2)) + \mathrm{diag}(A(G), \mathbf{0}_m, \mathbf{0}_{nn_1}, \mathbf{0}_{mn_2}) \text{ and}$$

$$A(G^{(T)} \bullet (G_1, G_2)) = A(G^{(S)} \bullet (G_1, G_2)) + \mathrm{diag}(A(G), A(G_\ell), \mathbf{0}_{nn_1}, \mathbf{0}_{mn_2}).$$

The degree diagonal matrix of these graphs are given by

$$D(G^{(S)} \bullet (G_1, G_2)) = \mathrm{diag}\big(r(n_2 + 1)I_n, 2(n_1 + 1)I_m, rI_{nn_1}, 2I_{mn_2}\big),$$

$$D(G^{(Q)} \bullet (G_1, G_2)) = \mathrm{diag}\big(r(n_2 + 1)I_n, 2(n_1 + 1)I_m, D(G_\ell) + rI_{nn_1}, 2I_{mn_2}\big),$$

$$D(G^{(R)} \bullet (G_1, G_2)) = \mathrm{diag}\big(D(G) + r(n_2 + 1)I_n, 2(n_1 + 1)I_m, rI_{nn_1}, 2I_{mn_2}\big) \text{ and}$$

$$D(G^{(T)} \bullet (G_1, G_2)) = \mathrm{diag}\big(D(G) + r(n_2 + 1)I_n, D(G_\ell) + 2(n_1 + 1)I_m, rI_{nn_1}, 2I_{mn_2}\big).$$

Similar expressions for the Laplacian and signless Laplacian matrices can be easily obtained.

The A, L and $|L|$-eigenvalues eigenvalues of the above described double neighbourhood corona operations are listed in Table 11. The notations used in this table are similar to that in Table 10.

Table 11: Spectral properties of double neighbourhood corona graphs

Graphs	A-eigenvalues	L-eigenvalues	$\|L\|$-eigenvalues	Multiplicity	Eigenvectors
$G^{(*)} \bullet (G_1, G_2)$	zeros of $x^4 - \dot{A}_1 x^3 + \dot{B}_1 x^2 - \dot{C}_1 x + \dot{D}_1$	zeros of $x^4 - \dot{A}_2 x^3 + \dot{B}_2 x^2 - \dot{C}_2 x + \dot{D}_2$	zeros of $x^4 - \dot{A}_3 x^3 + \dot{B}_3 x^2 - \dot{C}_3 x + \dot{D}_3$	1	$\begin{pmatrix} k_1 \zeta_i \\ k_2 \xi_i \\ k_3 \mathbf{1}_{n_1} \otimes \zeta_i \\ \mathbf{1}_{n_2} \otimes \xi_i \end{pmatrix}$
	-2	$2(n_1 + 1) + \dot{c}_2$	$2(n_1 + 1) + \dot{c}_3$	$m - n$	$\begin{pmatrix} \mathbf{0}_n \\ \eta_p \\ \mathbf{0}_{n_1 n + n_2 m} \end{pmatrix}$
	r_2	2	$2r_2 + 2$	$m - n$	$\begin{pmatrix} \mathbf{0}_{n+m+n_1 n} \\ \mathbf{1}_{n_2} \otimes \eta_p \end{pmatrix}$
	μ_q	$\mu_q^L + r$	$\mu_q^{|L|} + r$	n	$\begin{pmatrix} \mathbf{0}_{n+m} \\ Y_q \otimes e_i \\ \mathbf{0}_{n_2 m} \end{pmatrix}$
	ν_l	$\nu_l^L + 2$	$\nu_l^{|L|} + 2$	m	$\begin{pmatrix} \mathbf{0}_{n+m+n_1 n} \\ Z_l \otimes e_j \end{pmatrix}$

$$i = 1, 2, \ldots, n; j = 1, \ldots, m; p = 1, \ldots, m - n; q = 2, \ldots, n_1; l = 2, \ldots, n_2$$

In Table 11,

$\dot{A}_1 = r_1 + r_2 + \dot{a}_1 + \dot{b}_1,$

$\dot{B}_1 = (r_1 + r_2)(\dot{a}_1 + \dot{b}_1) + r_1 r_2 + \dot{a}_1 \dot{b}_1 - n_1(r + \lambda_i) - n_2 - 1,$

$\dot{C}_1 = (r_1 + r_2)(\dot{a}_1 \dot{b}_1 + 1) - r_1 r_2(\dot{a}_1 + \dot{b}_1) + n_1(r + \lambda_i)(r_2 + \dot{a}_1) + n_2(r_1 + \dot{b}_1),$

$\dot{D}_1 = n_1(r + \lambda_i)(n_2 - r_2 \dot{a}_1) + r_1 r_2(\dot{a}_1 \dot{b}_1 - 1) - n_2 r_1 \dot{a}_1;$

$\dot{A}_2 = 2(n_1 + 2) + r(n_2 + 2) + \dot{a}_2 + \dot{b}_2,$

$\dot{B}_2 = (n_1 + n_2)(2r + \lambda_i^L) + \dot{a}_2(2(n_1 + 2) + r) + \dot{b}_2(r(n_2 + 2) + 2) + (2n_1 + r)(rn_2 + 2) + (r + 2)^2 + \dot{a}_2 \dot{b}_2 + \lambda_i^L,$

$\dot{C}_2 = \lambda_i^L(r + 2)(n_1 + n_1 n_2 + n_2) + (r + 2)(\lambda_i^L + \dot{a}_2 \dot{b}_2) + 4r(\dot{a}_2 + \dot{b}_2) + \dot{b}_2 n_2(\lambda_i^L + r^2) + \dot{a}_2 n_1(\lambda_i^L + 4)$
$\qquad + 2r(2(n_1 + 1) + r(n_2 + 1)) + 4\dot{a}_2 + \dot{b}_2 r^2,$

$\dot{D}_2 = \lambda_i^L(\lambda_i^L n_1 n_2 + 2r + 2\dot{a}_2 n_1 + r\dot{b}_2 n_2) + 2r(\lambda_i^L(n_1 + n_2) + \dot{a}_2 \dot{b}_2 + 2\dot{a}_2 + r\dot{b}_2);$

$\dot{A}_3 = 2(n_1 + r_2) + r_1(n_2 + 3) + r + 4 + \dot{a}_3 + \dot{b}_3,$

$\dot{B}_3 = r(\dot{a}_3 + \dot{b}_3) + \dot{a}_3 \dot{b}_3 + 2\dot{a}_3(n_1 + r_1 + r_2 + 2) + \dot{b}_3(r_1(n_2 + 3) + 2(r_2 + 1)) + 2(n_1 + r_2)(r + r_1(n_2 + 3))$
$\qquad + r_1(2r_1 + r)(n_2 + 1) + 4(n_1(r_2 + 1) + r_1(n_2 + 3) + r + r_2 + 1) - \lambda_i^{|L|}(n_1 + n_2 + 1),$

$\dot{C} = \left(2(n_1 + r_1 + 1) + r + \dot{b}_3\right)\left(2(r_2 + 1)(r_1(n_2 + 1) + \dot{a}_3) - n_2 \lambda_i^{|L|}\right)$
$\qquad + \left(2(r_2 + 1) + r_1(n_2 + 1) + \dot{a}_3\right)\left((2r_1 + r)(2(n_1 + 1) + \dot{b}_3) - n_1 \lambda_i^{|L|}\right) - \lambda_i^{|L|}(2(r_1 + r_2 + 1) + r),$

$\dot{D}_3 = \left((2r_1 + r)(2(n_1 + 1) + \dot{b}_3) - n_1 \lambda_i^{|L|}\right)\left(2(r_2 + 1)(r_1(n_2 + 1) + \dot{a}_3) - n_2 \lambda_i^{|L|}\right) - 2(2r_1 + r)(r_2 + 1)\lambda_i^{|L|}$

where for

$S_G : \quad \dot{a}_1 = \dot{b}_1 = \dot{c}_1 = 0; \quad \dot{a}_2 = \dot{b}_2 = \dot{c}_2 = 0; \quad \dot{a}_3 = \dot{b}_3 = \dot{c}_3 = 0;$

$Q_G : \quad \dot{a}_1 = 0, \dot{b}_1 = \lambda_i + r - 2, \dot{c}_1 = -2; \quad \dot{a}_2 = 0, \dot{b}_2 = \lambda_i^L, \dot{c}_2 = 2r; \quad \dot{a}_3 = 0, \dot{b}_3 = \lambda_i^{|L|} + 2r - 4, \dot{c}_3 = 2r - 4;$

$R_G : \quad \dot{a}_1 = \lambda_i, \dot{b}_1 = \dot{c}_1 = 0; \quad \dot{a}_2 = \lambda_i^L, \dot{b}_2 = \dot{c}_2 = 0; \quad \dot{a}_3 = \lambda^{|L|}, \dot{b}_3 = \dot{c}_3 = 0;$

$T_G : \quad \dot{a}_1 = \lambda_i, \dot{b}_1 = \lambda_i + r - 2, \dot{c}_1 = -2; \dot{a}_2 = \dot{b}_2 = \lambda_i^L, \dot{c}_2 = 2r; \dot{a}_3 = \lambda_i^{|L|}, \dot{b}_3 = \lambda_i^{|L|} + 2r - 4, \dot{c}_3 = 2r - 4.$

Remark 4.1. *Observe that when $n_2 = 0$, then $A(G^{(S)} \circ (G_1, G_2))$ reduces to $A(G^{(S)} \odot G_1)$. Similarly, when $n_1 = 0$, then $A(G^{(S)} \circ (G_1, G_2))$ reduces to $A(G^{(S)} \ominus G_2)$. Therefore, the subdivision double corona operation is a more general operation than the subdivision-vertex corona and the subdivision-edge corona operations. In a similar manner it can be observed that $G^{(R)} \odot G_1$ and $G^{(R)} \ominus G_2$ are subcases of $G^{(R)} \circ (G_1, G_2)$.*

Remark 4.2. *When $n_2 = 0$, then $A(G^{(S)} \bullet (G_1, G_2))$ reduces to $A(G^{(S)} \boxdot G_1)$. Similarly, when $n_1 = 0$, then $A(G^{(S)} \bullet (G_1, G_2))$ get reduced to $A(G^{(S)} \boxminus G_2)$. Therefore, the subdivision double corona operation is the more general operation than the subdivision-vertex neighbourhood corona and the subdivision-edge neighbourhood corona operations. Also, $G^{(R)} \boxminus G_2$ is a special case of $G^{(R)} \circ (G_1, G_2)$, when we choose $n_1 = 0$. But $G^{(R)} \boxdot G_2$ is not a special case of $G^{(R)} \circ (G_1, G_2)$. The reason for this lies in the way both the operations are defined. In the definition for R-vertex neighbourhood corona (defined by Lan and Zhou in [39]) of G_1 and G_2 (denoted by $G_1^{(R)} \boxdot G_2$), each neighbour of the i-th old-vertex of $R(G_1)$ are joined to every vertex in the i-th copy of G_2. But in case of R-graph double neighbourhood corona (as defined in [5]) of G, G_1 and G_2 (denoted by $G^{(R)} \bullet (G_1, G_2)$), only the new-vertex neighbours of the i-th old-vertex of $R(G)$ are joined to every vertex in the i-th copy of G_1.*

5 Notations

$u \sim v$	:	vertex u is adjacent to vertex v
$\deg(v)$	:	the degree of a vertex v in a graph
e_i	:	the vector whose i-th component is one and all the other components are zeros
$\mathbf{0}_n$	:	a vector of length n and whose each component is zero
$\mathbf{1}_n$	:	a vector of length n and whose each component is one
I_n	:	the identity matrix of order $n \times n$
J_n	:	the matrix of order $n \times n$ whose each entry is one
$A \otimes B$	:	the Kronecker product of a matrix A with another matrix B
$A(G)$	:	the adjacency matrix of a graph G
$M(G)$	:	the vertex-edge incidence matrix of a graph G
$L(G)$	:	the Laplacian matrix of a graph G
$|L|(G)$	:	the signless Laplacian matrix of a graph G
G^c	:	complement of a graph G
G_ℓ	:	line graph of a graph G
$S(G)$	:	subdivision of a graph G
$Q(G)$	:	Q-graph of a graph G
$R(G)$	:	R-graph of a graph G
$T(G)$	:	total graph of a graph G
$G_1 \cup G_2$	:	disjoint union of two graphs G_1 and G_2
$G_1 \vee G_2$	:	join of two graphs G_1 and G_2
$G_1 \dot{\vee} G_2$	:	subdivision-vertex join of two graphs G_1 and G_2
$G_1 \underline{\vee} G_2$	:	subdivision-edge join of two graphs G_1 and G_2
$G_1 \square G_2$	:	Cartesian product of two graphs G_1 and G_2
$G_1 \otimes G_2$	:	direct product of two graphs G_1 and G_2
$G_1 \boxtimes G_2$	:	strong product of two graphs G_1 and G_2
$G_1[G_2]$	:	lexicographic product of two graphs G_1 and G_2
$G_1 \circ G_2$	:	corona of two graphs G_1 and G_2
$G_1 \diamond G_2$	:	edge corona of two graphs G_1 and G_2
$G_1 \bullet G_2$	:	neighbourhood corona of two graphs G_1 and G_2
$G_1^{(S)} \boxdot G_2$	:	subdivision-vertex neighbourhood corona of two graphs G_1 and G_2
$G_1^{(S)} \boxminus G_2$	:	subdivision-edge neighbourhood corona of two graphs G_1 and G_2
$G_1^{(R)} \odot G_2$	:	R-vertex corona of two graphs G_1 and G_2
$G_1^{(R)} \ominus G_2$	:	R-edge corona of two graphs G_1 and G_2
$G_1^{(R)} \boxminus G_2$	:	R-edge neighbourhood corona of two graphs G_1 and G_2
$G^{(*)} \circ (G_1, G_2)$	:	double corona of three graphs G, G_1 and G_2 where $^{(*)}$ stands for $S/Q/R/T$
$G^{(*)} \bullet (G_1, G_2)$	:	double neighbourhood corona of three graphs G, G_1 and G_2 where $^{(*)}$ stands for $S/Q/R/T$

Appendix

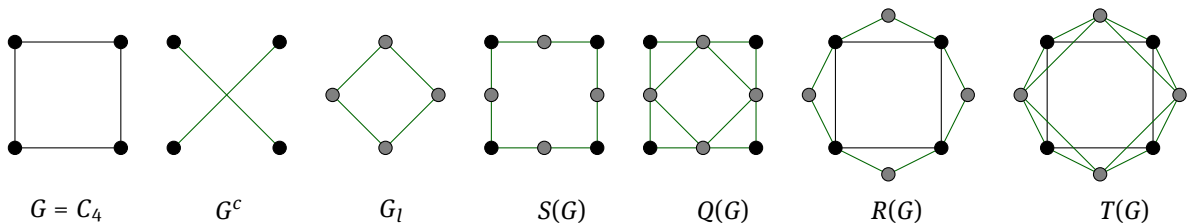| $G = C_4$	G^c	G_l	$S(G)$	$Q(G)$	$R(G)$	$T(G)$

Figure 1: Graphs resulting from unary operations on $G = C_4$
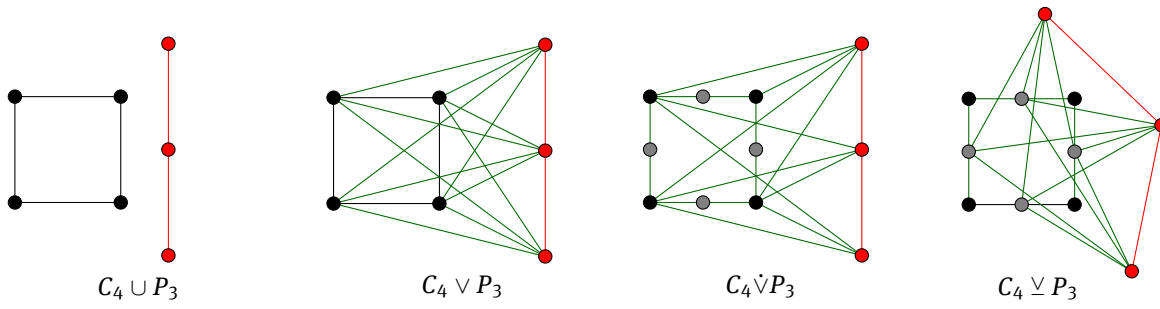
Figure 2: Union, Join, subdivision-vertex join and subdivision-edge join of graphs C_4 and P_3
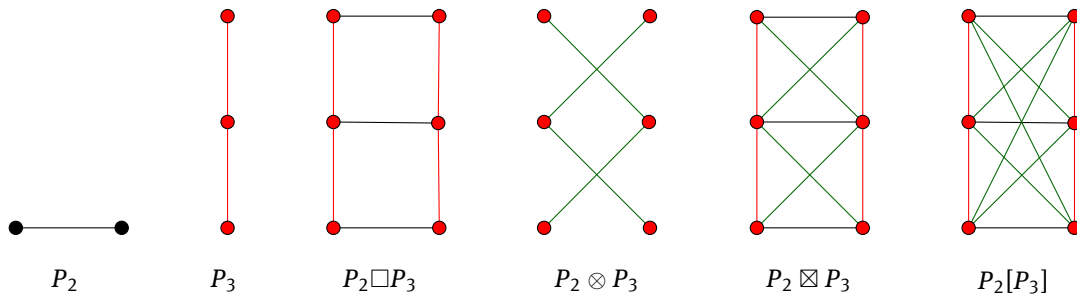

Figure 3: Cartesian, direct, strong and lexicographic product of graphs P_2 and P_3
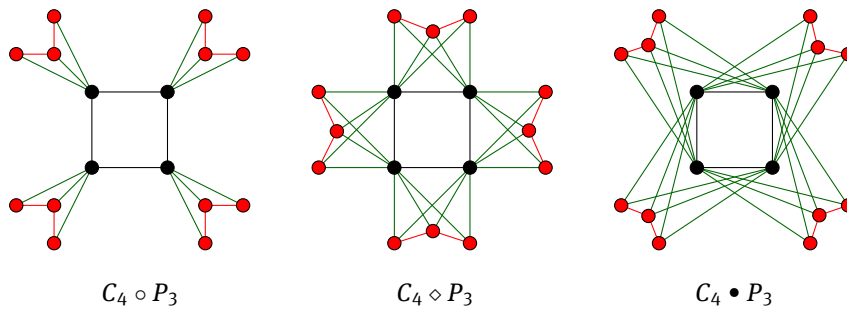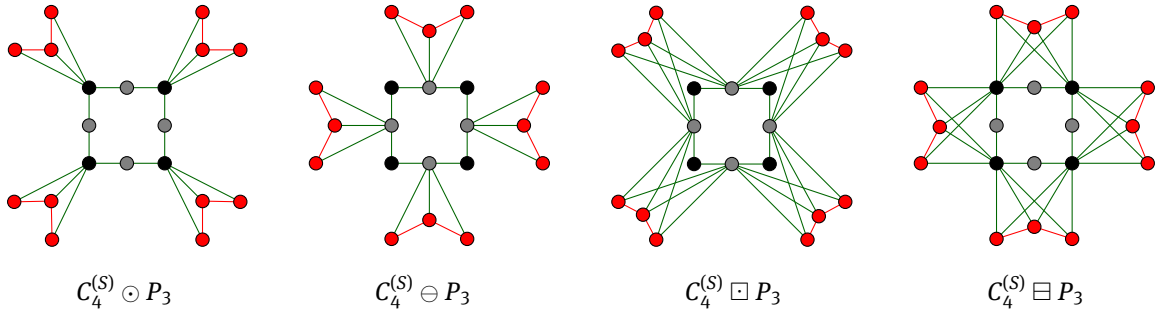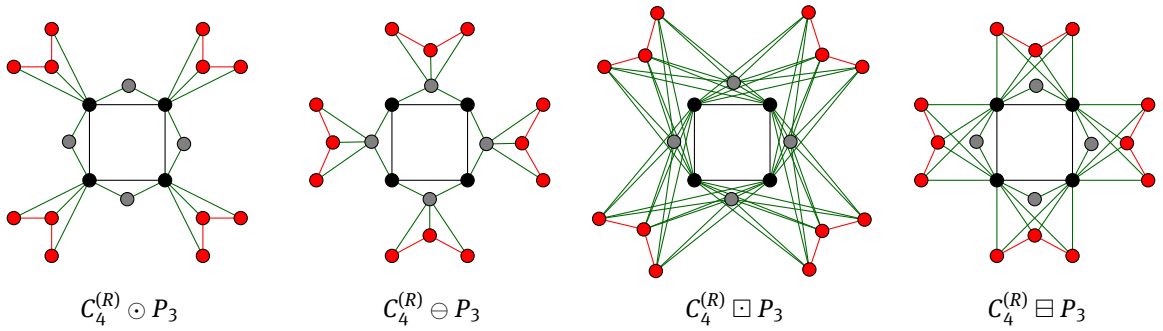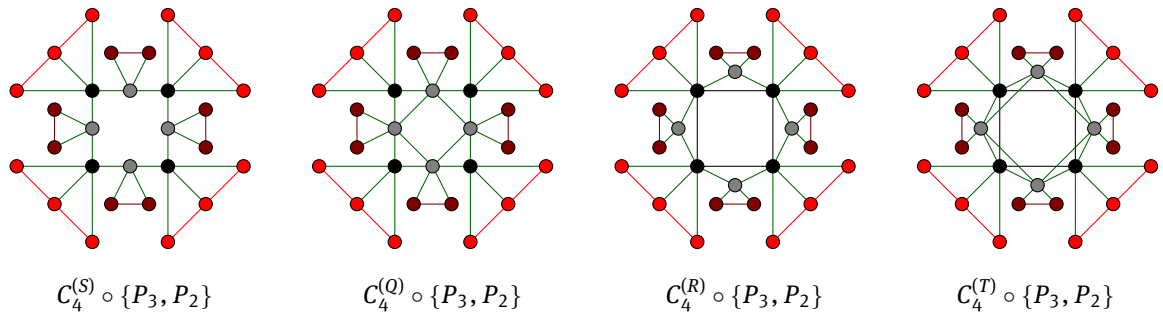

Figure 4: Corona, edge corona and neighbourhood corona of C_4 and P_3
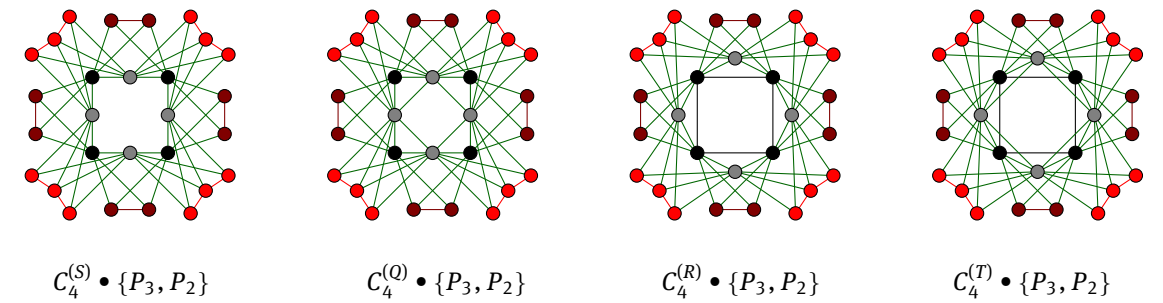
Figure 5: Subdivision-vertex corona, Subdivision-edge corona, subdivision-vertex neighbourhood corona and subdivision-edge neighbourhood corona of graphs C_4 and P_3

Figure 6: R-vertex corona, R-edge corona, R-vertex neighbourhood corona, R-edge neighbourhood corona of graphs C_4 and P_3

Figure 7: Double coronas of C_4 with P_3 and P_2

Figure 8: Double neighbourhood coronas of C_4 with P_3 and P_2

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