# Spectra of unit tangent bundles of compact hyperbolic Riemann surfaces

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**Abstract**

Recent results of C. Gordon -Y. Mao and H. Pesce imply that isospectral compact hyperbolic Riemann surfaces have Laplace and length isospectral unit tangent bundles. In this note we give explicit formulae relating the spectra of such surfaces and those of their unit tangent bundles, and use them to prove the converses.

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## 1 Introduction

The problem whether within a family of compact Riemannian manifolds the Laplace spectrum and the length spectrum determine each other has been extensively studied. For example, by means of Poisson summation and of the Selberg trace formula, one has a solution to this problem for the family of flat tori of a given dimension, and the family of compact hyperbolic Riemann surfaces, respectively (by a hyperbolic Riemann surface we understand an oriented surface of constant curvature $-1$). Here we obtain the following result, which implies in particular that within the family $\mathcal{U}$ of unit tangent bundles of compact hyperbolic Riemann surfaces, the Laplace spectrum and the length spectrum determine each other.

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**Theorem 1** *Let $S_1$ and $S_2$ be compact hyperbolic Riemann surfaces and let $T^1S_1$, $T^1S_2$ be their unit tangent bundles, endowed with the canonical (Sasaki) Riemannian metric. Then the following assertions are equivalent.*

*(a) $S_1$ and $S_2$ are isospectral.*

*(b) $T^1S_1$ and $T^1S_2$ are length isospectral.*

*(c) $T^1S_1$ and $T^1S_2$ are Laplace isospectral.*

This theorem is a consequence of theorems 2 and 4 below: the former implies that (a) and (b) are equivalent, and the latter asserts the equivalence between (a) and (c). On the other hand, the proposition in IV B of [P] yields that (a) implies (c). Besides, (a) implies (b) essentially as a consequence of this proposition and Proposition 1.3 and Theorem A in [G M]. We do not comment on this since both implications also follow from explicit formulae in the first items of Theorems 2 and 4 in this note.

At least with regard to the techniques, this is related to some extent to the work of C. Gordon and E. Wilson ([G],[G W]) on the spectra of compact Heisenberg manifolds. As for the unit tangent bundles considered here, they are geometrically non trivial $S^1$-bundles over manifolds about them much spectral information is available (flat tori and compact hyperbolic Riemann surfaces).

Next we recall some definitions and state the theorems referred to above.

Let $M$ be a Riemannian manifold. The *length spectrum* of $M$ is the function $m_M : \mathbf{R} \to \mathbf{N} \cup \{0, \infty\}$ defined as follows: $m_M(\ell)$ is the number of free homotopy classes which contain a closed geodesic of length $\ell$. If $M$ is a Riemannian covering of a compact manifold, then the support of $m_M$ (called the *weak length spectrum* of $M$) consists of a discrete sequence $0 < \ell_1 < \ell_2 < \dots$ (the lengths) and $m_M(\ell)$ is the multiplicity of $\ell$. The *primitive length spectrum* $\mathcal{P}m_M$ of $M$ is defined analogously, replacing closed geodesic with primitive closed geodesic. If $M$ is compact these functions are finite and every free homotopy class contains a closed geodesic.

Let $H$ be the hyperbolic plane and let $T^1H$ be its unit tangent bundle endowed with the Sasaki metric. For $\ell \in \mathbf{R}$ let $\mathcal{X}_\ell : \mathbf{R} \to \mathbf{N} \cup \{0\}$ be defined by $\mathcal{X}_\ell(t) = \# \left\{ (p,q) \in \mathbf{N} \times \mathbf{Z} \mid \sqrt{2\pi^2 q^2 + \ell^2 p^2} = t \right\}$.

**Theorem 2** *Let $S$ be a compact hyperbolic Riemann surface and let $T^1S$ be its unit tangent bundle endowed with the Sasaki metric.*

*(a) The following formula gives the length spectrum of $T^1S$ in terms of that of $S$.*

$$m_{T^1S} = \sum_{\ell>0} \mathcal{P}m_S(\ell)\, \mathcal{X}_\ell + m_{T^1H} \tag{1}$$

*(b) Conversely, the length spectrum of $T^1S$ determines that of $S$.*

**Theorem 3** *The weak length spectrum of $T^1S$ can be expressed in terms of that of $S$ as $\mathcal{L}_{hor} \cup \mathcal{L}_{ver}$, where*

$$\mathcal{L}_{hor} = \left\{ \sqrt{2\pi^2 q^2 + \ell^2 p^2} \mid (p,q) \in \mathbf{N} \times \mathbf{Z} \text{ and } \ell \in supp\,(\mathcal{P}m_S) \right\}$$

$$\text{and} \quad \mathcal{L}_{ver} = \left\{ \pi\sqrt{2q^2 - 4p^2} \mid q,p \in \mathbf{N} \text{ and } q \geq 2p \right\}$$

*(here $supp\,(\mathcal{P}m_S)$ is the primitive weak length spectrum of $S$). If we require $p$ and $q$ to be coprime, we obtain the primitive weak length spectrum of $T^1S$.*

The Laplace spectrum of a compact Riemannian manifold $M$ is the function $\mu_M : \mathbf{R} \to \mathbf{N} \cup \{0\}$ defined by $\mu_M(\lambda) = \dim(V_\lambda)$, where $V_\lambda = \{f \in L^2(M) \mid \Delta f = \lambda f\}$. The support of $\mu_M$ consists of a discrete sequence $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty$ (the eigenvalues) and $\mu_M(\lambda)$ is the multiplicity of $\lambda$.

For $\lambda \in \mathbf{R}$, let $\delta_\lambda$ and $\zeta_\lambda$ be respectively the characteristic functions of $\{\lambda\}$ and $\{\lambda + k^2 \mid k \in \mathbf{N}\}$. Let $\widetilde{\zeta}_m$ be the characteristic function of $\{m - m^2 + k^2 \mid k \in \mathbf{N}, k \geq m\}$ if $m \in \mathbf{N}$.

**Theorem 4** *Let $S$ be a compact hyperbolic Riemann surface and let $T^1S$ be its unit tangent bundle endowed with the Sasaki metric. Let $g = g(S)$ denote the genus of $S$, which is a spectral invariant.*

*(a) The following formula gives the Laplace spectrum of $T^1S$ in terms of that of $S$.*

$$\mu_{T^1S} = \delta_0 + \sum_{\lambda>0} \mu_S(\lambda)\,(\delta_\lambda + 2\zeta_\lambda) + 2\,g\,\widetilde{\zeta}_1 + 2\,(g-1) \sum_{2\leq m \in \mathbf{N}} (2m-1)\,\widetilde{\zeta}_m$$

*(b) Conversely, the Laplace spectrum of $T^1S$ determines that of $S$.*

Y. Colin de Verdière proved in [C de V] that for a generic compact Riemannian manifold (all critical submanifolds of the energy function on the loop space are non degenerate) the Laplace spectrum determines the length spectrum. In the last section we show that no manifold in the family $\mathcal{U}$ is generic in this sense. We also see that $\mathcal{U}$ contains non isometric isospectral manifolds.

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# 2 Preliminaries

Let $M$ be a smooth manifold. A smoothly closed (or briefly a closed) curve $\gamma$ in $M$ is a non constant smooth function $\gamma : [0, a] \to M$ such that $\dot{\gamma}(a) = \dot{\gamma}(0)$. Such a curve extends uniquely to a periodic curve in $M$ defined on the whole real line, with period $t_0$ satisfying that $a$ is an integral multiple of $t_0$. $\gamma$ is said to be primitive if $a = t_0$.

Two closed curves $\gamma_i : [0, a_i] \to M$ $(i = 0, 1)$ are said to be *free homotopic* $(\gamma_1 \sim \gamma_2)$ if there is a continuous map $h : [0, 1] \times [0, 1] \to M$ such that $h(t, i)$ is an increasing reparametrization of $\gamma_i$ for $i = 0, 1$, and $h(0, s) = h(1, s)$ for all $s$. Free homotopy is an equivalence relation.

Let $\tilde{M}$ denote the universal covering of $M$, let $\Gamma$ be the fundamental group of $M$, and let conj denote conjugation in $\Gamma$. The map

$$i : \{\text{closed curves in } M\} \, /\!\sim \; \longrightarrow \Gamma/\text{conj}$$

given by $i[\gamma] = [g]$ if $\tilde{\gamma}(a) = g\tilde{\gamma}(0)$ with $g \in \Gamma$, where $\tilde{\gamma}$ is a lift of $\gamma$ to $\tilde{M}$ of the closed curve $\gamma$ defined on the interval $[0, a]$, is a well defined bijection. If $M$ is a Lie group and $\tilde{M}$ carries the induced multiplication, then $i[\gamma] = \left[\tilde{\gamma}(a)\, \tilde{\gamma}(0)^{-1}\right]$.

Suppose now that $M$ is Riemannian. If $\gamma : \mathbf{R} \to M$ is periodic and $t_0$ is its period, then we define $\text{length}(\gamma) = \text{length}\left(\gamma|_{[0,t_0]}\right)$. Thus the length of a closed curve is an integral multiple of the length of its periodic extension. We denote by $\mathcal{G}(M)$ the set of all closed geodesics and by $\mathcal{P}(M)$ the set of all primitive closed geodesics in $M$.

Let $G = PSl(2, \mathbf{R}) = \{g \in M_2(\mathbf{R}) \,|\, \det g = 1\}/\{\pm I\}$ and let $\mathfrak{g} = \{X \in M_2(\mathbf{R}) \,|\, \operatorname{tr} X = 0\}$ be its Lie algebra. Consider on $G$ the left invariant Riemannian metric $\langle , \rangle$ such that $\langle X, Y \rangle = 2\, tr\, (XY^t)$ for all $X, Y \in \mathfrak{g}$. Consider the Cartan decomposition $\mathfrak{g} = \mathbf{R}Z \oplus \mathfrak{m}$, where $Z = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathfrak{m} = \{X \in \mathfrak{g} \mid X = X^t\}$. Let $K$ be the subgroup of $G$ with Lie algebra

$\mathfrak{k} = \mathbf{R}Z$ and let $H = G/K$ be endowed with the Riemannian metric such that the canonical projection $\pi : G \to H$ is a Riemannian submersion. As usual we shall identify $T_{eK}H$ with $\mathfrak{m}$. $H$ is the hyperbolic plane with constant curvature $-1$.

$G$ can be identified via left multiplication with the set of orientation preserving isometries of $H$, and hence it acts canonically on $T^1H$. We consider on $T^1H$ the canonical (Sasaki) metric, defined by $\|\xi\|^2 = \|\pi_{*v}\xi\|^2 + \|\mathcal{K}(\xi)\|^2$ for $\xi \in T_vT^1H$, $v \in T^1H$, where $\mathcal{K}$ is the connection operator.

$H$ carries a canonical complex structure, which comes from the $G$-invariant quasi complex structure $i$ induced by $\mathrm{ad}_Z : \mathfrak{m} \to \mathfrak{m}$. Let $X_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $X_2 = iX_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By the usual identification of $\mathfrak{m}$ with $T_{eK}H$ we can write $X_i \in T^1_{eK}H$ since $\|X_i\| = 1$.

The map $\Phi : G \to T^1H$ defined by $\Phi(g) = g_{*eK}(X_1)$ is an isometry (notice that for $Y \in \mathfrak{m}$, $(\exp tY)_{*eK}X_1$ is the parallel transport of $X_1$ along the curve $(\exp tY)K$, since $(G, K)$ is a symmetric pair).

Let $S$ be an oriented surface of constant curvature $-1$ and positive injectivity radius, and let $\Gamma$ be its fundamental group. $S$ is naturally isometric to $\Gamma\backslash H$ with the induced metric. $\Gamma$ is a discrete subgroup of $G$ which acts freely and properly discontinuously on $H$ and contains no parabolic isometries. Hence every $g \in \Gamma$, $g \neq e$, translates a unique (up to an increasing reparametrization) unit speed geodesic in $H$.

An element $g \in \Gamma$, $g \neq e$, is called primitive if whenever $g = g_0^p$ for some $g_0 \in \Gamma$ and $p \in \mathbf{N}$, then $p = 1$. Let $\mathcal{P}\Gamma$ denote the set of all primitive elements of $\Gamma$. For each $g \in \Gamma$ there exist unique $g_0 \in \mathcal{P}\Gamma$ and $p \in \mathbf{N}$ such that $g = g_0^p$. If two elements of $\Gamma$ are conjugate in $\Gamma$, then the primitive elements associated to them in this way are conjugate and the powers coincide.

If $g \in \mathcal{P}\Gamma$ translates a unit speed geodesic $\sigma$ in $H$ in $a > 0$ (that is $g\sigma(t) = \sigma(t + a)$ for all $t$), then $\Gamma\sigma|_{[0,a]}$ is a closed geodesic in $S$. This induces a well defined map

$$\mathcal{P}i : \mathcal{P}\Gamma/\mathrm{conj} \to \mathcal{P}(S)/\!\sim, \qquad \mathcal{P}i\,[g] = \left[\Gamma\sigma|_{[0,a]}\right]$$

Let $\tilde{\pi} : \tilde{G} \to G$ be the universal covering of $G$ and let $\widetilde{\exp} : \mathfrak{g} \to \tilde{G}$ be the exponential map. $\tilde{G}$ is diffeomorphic to $\mathbf{R}^3$ and its center is $\mathcal{Z} :=$

$\tilde{\pi}^{-1}(e) = \{\widetilde{\exp}(2k\pi Z) \mid k \in \mathbf{Z}\} \approx \mathbf{Z}$. $\mathcal{Z}$ is also the center of the group $\tilde{\Gamma} := \tilde{\pi}^{-1}(\Gamma)$, which is the fundamental group of $T^1 S \approx \Gamma\backslash G \approx \tilde{\Gamma}\backslash\tilde{G}$.

**Lemma 5** *Let $f : \tilde{\Gamma} - \mathcal{Z} \to \mathcal{P}\Gamma \times \mathbf{N} \times \mathbf{Z}$ be given by $f(g) = (g_0, p, q)$, where $\tilde{\pi}(g) = g_0^p$ and $q$ is the unique integer satisfying $g\tilde{\sigma}(0) = \tilde{\sigma}(pa)\widetilde{\exp}(2\pi q Z)$, where $\tilde{\sigma}$ is any lift to $\tilde{G}$ of a geodesic $\sigma$ in $H$ translated by $g_0$ in $a$. Then $f$ is well defined and can be pushed down to the quotients through $\tilde{f} : \left(\tilde{\Gamma} - \mathcal{Z}\right)/conj \to (\mathcal{P}\Gamma/conj) \times \mathbf{N} \times \mathbf{Z}$.*

**Proof.** If $\sigma_1$ is another geodesic translated by $g_0$ (in $a > 0$), then $\sigma_1 = g_1\sigma$ with $g_1$ commuting with $g_0$. Hence any lift $\tilde{\sigma}_1$ of $\dot{\sigma}_1$ to $\tilde{G}$ satisfies $\tilde{\sigma}_1 = \tilde{g}\tilde{\sigma}$ with $\tilde{g}_1$ commuting with $g$. Therefore $\tilde{\sigma}(pa)^{-1} g\tilde{\sigma}(0)$ depends only on $g$. On the other hand, $\tilde{\pi}\left(\tilde{\sigma}(pa)^{-1} g\tilde{\sigma}(0)\right) = e$ since $\tilde{\pi}$ is a homomorphism and $\sigma(pa) = g_0^p\sigma(0)$. Hence $q$ is well defined. The validity of the last assertion follows from similar considerations. □

# 3 The length spectra of $S$ and $T^1 S$

We begin the section describing the geodesics in the unit tangent bundle of the hyperbolic plane. If $V$ is a smooth curve in $T^1 H$, then $V'$ will denote the covariant derivative along the projection of $V$ to $H$.

**Proposition 6** *(a) Let $V$ be a geodesic in $T^1 H$ and let $c = \pi \circ V$. Then $\|V'\| = const$, $\|\dot{c}\| = const =: \lambda$ and one of the following possibilities holds:*

*(i) If $\lambda = 0$, then $V$ is a constant speed curve in the circle $T^1_{c(0)}H$.*

*(ii) If $\lambda \neq 0$, then the geodesic curvature $\kappa$ of $c$ with respect to the normal $i\dot{c}/\lambda$ is also constant and for $t \in \mathbf{R}$*

$$V(t) = e^{-2\lambda\kappa t i} z\dot{c}(t), \tag{2}$$

*where $z \in \mathbf{C}$ is such that $V(0) = z\dot{c}(0)$.*

*(b) Conversely, each curve $V$ in $T^1 H$ which satisfies (i) or (ii) is a constant speed geodesic. Moreover, given a constant speed curve $c$ in $H$ with constant geodesic curvature, and $V_0 \in T^1_{c(0)}H$, there is a unique geodesic $V$ in $T^1 H$ which projects to $c$ and such that $V(0) = V_0$.*

*(c) The norm of the vertical component of $\dot{V}$ with respect to the submersion $T^1 H \to H$ is $\|V'\| = \lambda|\kappa|$. Furthermore, $V$ has unit speed if and only if $\lambda^2(1 + \kappa^2) = 1$.*

**Proof.** (cf. Corollary 5.3 of [B B B]). We recall Sasaki equations for geodesics in the unit tangent bundle of a Riemannian manifold ([Sa]):

$$V'' = -\left\|V'\right\|^2 V \qquad \text{and} \qquad \dot{c}' = R\left(V', V\right)\dot{c},$$

where $R$ is the curvature tensor. Since $\|V\| \equiv 1$ and $\dim H = 2$, the first equation may be replaced with $\|V'\| = \text{const}$. Suppose $\lambda$ and $\kappa$ are constant. Clearly $V$ is a geodesic if $V$ satisfies either (i) or (ii) with $\lambda \neq 0$ and $\kappa = 0$. Let $V$ be as in (2) with $\kappa \neq 0$. By definition of geodesic curvature we have $\dot{c}'(t) = \lambda\kappa i\dot{c}(t)$, hence

$$V'(t) = ze^{-2\lambda\kappa ti}\left(\dot{c}'(t) - 2\lambda\kappa i\dot{c}(t)\right) = -\lambda\kappa iV(t)$$

Then $\|V'\| = \lambda|\kappa|$ is constant, and the second Sasaki equation is also satisfied, since constant sectional curvature $-1$ implies $R(V, iV)W = iW$ for all $W$. All geodesics are as in (i) or (ii) since one can show that any tangent vector at $T^1H$ is the initial velocity of a geodesic of either of the stated forms. □

From now on we consider the upper half plane model of the hyperbolic plane (with the metric $ds^2 = \left(dx^2 + dy^2\right)/y^2$) and the canonical action of $G$ on it by Möbius transformations. The geodesic curvature of a regular curve $c$ in $H$ or in some oriented quotient of $H$ will be always computed with respect to the normal $i\dot{c}/\|\dot{c}\|$ and denoted by $\kappa$.

**Lemma 7** *Let $c$ be a complete curve in $H$ of constant geodesic curvature $\kappa$. Given $\theta \in (0, \pi)$, let $c_\theta$ be the curve in $H$ defined by $c_\theta(t) = e^t e^{i\theta}$.*

*(a) If $|\kappa| > 1$, the image of $c$ is a geodesic circle of radius $|r|$ and length $|2\pi \sinh r|$, where $\coth r = \kappa$ (this implies that the length is $2\pi/\sqrt{\kappa^2 - 1}$).*

*(b) If $|\kappa| = 1$, the image of $c$ is a horocycle.*

*(c) If $\kappa = \cos\theta$, the image of $c$ is congruent to that of $c_\theta$.*

*(d) $c$ is injective $\Leftrightarrow$ $c$ is not bounded $\Leftrightarrow$ $|\kappa| \leq 1$.*

**Proof.** The lemma follows from straightforward computations, using polar coordinates for (a) and standard coordinates in the upper half plane for (b) and (c). (d) is an immediate consequence of the preceding items. □

Now let $S$ be as in the preliminaries an oriented surface of constant curvature $-1$ and positive injectivity radius, and let $\Gamma$ be its fundamental group. A unit speed geodesic $v$ in $T^1S$ is said to be *horizontal-like* (*vertical-like*) if $|\kappa| < 1$ ($|\kappa| > 1$ or $\pi v$ is constant). By Proposition 6 (c) this is equivalent to the fact that the (constant) norm of the vertical component of $\dot{v}$ is smaller (greater) than the norm of the horizontal component. We denote by $\mathcal{G}_{\text{hor}}(T^1S)$ the set of all closed horizontal-like geodesics in $T^1S$. The definition of $\mathcal{G}_{\text{ver}}(T^1S)$ is analogous. Every periodic geodesic in $T^1S$ is either horizontal-like or vertical-like, since curves with $|\kappa| \equiv 1$ lift to horocycles in $H$ (Lemma 7 (b)) and $\Gamma$ contains no parabolic isometries.

**Lemma 8** *Let $S$ be an oriented surface of constant curvature $-1$ and consider on $T^1S$ the canonical metric. Let $v$ be a geodesic in $T^1S$ such that $c = \pi v$ is not constant and has geodesic curvature $\kappa$. Then $v$ is closed if and only if $c$ is periodic and $p\kappa \operatorname{length}(c) = -\pi q$ for some coprime integers $p, q$, with $p > 0$. In this case,* $\operatorname{length}(v) = p(\operatorname{length} c)\sqrt{1+\kappa^2}$.

**Proof.** By Proposition 6, $v = \Gamma V$ with $V$ as in (2). If $v$ is closed, then clearly so is $c$, and the period $t_0$ of $v$ is the least positive number $t$ satisfying simultaneously $\dot{c}(t) = \dot{c}(0)$ and $-2\lambda\kappa t \in 2\pi\mathbf{Z}$ (notice that $\dot{v}(0) = \dot{v}(s)$ if $\dot{c}(s) = \dot{c}(0)$ and $v(0) = v(s)$). Equivalently, $\lambda t_0 = n \operatorname{length} c$, where $n$ is the least positive integer such that $-n\kappa \operatorname{length} c \in \pi\mathbf{Z}$. This proves the first assertion. We continue the argument in order to compute the length of $v$. Since $p$ and $q$ are coprime, then $n = p$ and

$$\operatorname{length} v = t_0 \|\dot{v}\| = \tfrac{p}{\lambda}(\operatorname{length} c)\left\|\dot{V}\right\|.$$

Now by definition of the Sasaki metric, $\left\|\dot{V}\right\|^2 = \|\dot{c}\|^2 + \|V'\|^2 = \lambda^2(1+\kappa^2)$ by Lemma 6 (c) and the last assertion follows. □

Let $\phi_t$ be the isometry of $H$ defined by $\phi_t(z) = e^t z$. Given $z \in H$, let $c_z$ be the curve in $H$ defined by $c_z(t) = \phi_t(z)$. If $z = re^{i\theta}$ with $0 < \theta < \pi$, then by Lemma 7 the curve $c_z$ has constant geodesic curvature $\kappa = \cos\theta$, since it is a reparametrization of $c_\theta$ defined in that lemma. Let $V_z$ the unique unit speed geodesic in $T^1H$ which projects to an orientation preserving reparametrization of $c_z$ and such that $V_z(0)$ is a positive multiple of $\dot{c}_z(0)$.

Now we describe the periodic horizontal-like geodesics in the unit tangent bundle of a hyperbolic cylinder of positive injectivity radius. Given $\ell > 0$, let

$\Gamma_\ell$ be the subgroup $\{\phi_\ell^m \mid m \in \mathbf{Z}\}$ and let $S_\ell$ the cylinder $\Gamma_\ell \backslash H$ with the induced Riemannian metric. Objects associated to $S_\ell$ are indicated with the subscript $\ell$.

**Lemma 9** *(a) $\Gamma_\ell c_z$ is a periodic curve in $S_\ell$ of length $\ell/\sqrt{1-\kappa^2}$.*

*(b) $\Gamma_\ell V_z$ is periodic in $T^1 S_\ell$ if and only if $\ell\kappa/\pi\sqrt{1-\kappa^2} \in \mathbf{Q}$, say*

$$-q\pi\sqrt{1-\kappa^2} = \ell\kappa p \tag{3}$$

*for some coprime integers $p$ and $q$, with $p > 0$. In this case we have that*

*(i) length $(\Gamma_\ell V_z) = \sqrt{2\pi^2 q^2 + \ell^2 p^2} =: L$*

*(ii) $\tilde{f}_\ell \circ i_\ell \left[\Gamma_\ell V_z|_{[0,L]}\right]_\ell = \left([\phi_\ell]_\ell, p, q\right)$.*

*(c) If $v$ is a unit speed horizontal-like periodic geodesic in $T^1 S_\ell$, then $v = u\Gamma_\ell V_z$ for some $z \in H$ and $u \in S^1$.*

**Proof.** Clearly $\Gamma_\ell c_z$ is a periodic curve of period $\ell$. Hence length $(\Gamma_\ell c_z) = \ell \left\|\dot{c}_z(0)\right\|$, since $\left\|\dot{c}_z\right\|$ is constant. Thus (a) follows from the following computation.

$$\left\|\dot{c}_z(0)\right\| = \left|\dot{c}_z(0)\right| / \operatorname{Im} c_z(0) = \left|re^{i\theta}\right| / r\sin\theta = 1/\sin\theta = 1/\sqrt{1-\kappa^2} \tag{4}$$

The first assertion of (b) is an immediate consequence of (a) and Lemma 8. Solving in (3) for $\kappa$ in terms of $q/p$ and substituting in (a), we obtain (i) by the expression for length $(v)$ in Lemma 8.

Now we prove (ii). By Proposition 6, $V_z(t) = \frac{1}{\lambda} e^{-2\lambda\kappa t i} \dot{C}_z(t)$, where $C_z(t) = c_z(at)$, $\left\|\dot{C}_z\right\| \equiv \lambda$ and $\lambda^2(1+\kappa^2) = 1$, for some $a \in \mathbf{R}$. Clearly $\dot{C}_z(t) = (\phi_{at})_* \dot{C}_z(0)$. Suppose that $\frac{1}{\lambda}\dot{C}_z(0) = \Phi(g)$ with $g \in G$ (we recall that $\Phi$ identifies $G$ with $T^1 H$). Since $\phi_s$ is the Möbius transformation associated to $\exp(sX_1)$, and multiplication by $e^{si}$ in $T^1 H$ corresponds to right multiplication by $\exp(sZ)$ in $G$, we have that

$$V_z(t) = \Phi\left(\exp(atX_1)\, g \exp(-2\lambda\kappa t Z)\right)$$

Hence $V_z(t) = \tilde{\pi}(\gamma(t))$, where $\gamma(t) = \widetilde{\exp}(atX_1)\, \tilde{g}\, \widetilde{\exp}(-2\lambda\kappa t Z)$, with $\tilde{\pi}\tilde{g} = g$. We obtain from Lemma 8 that $-2\lambda\kappa L = 2\pi q$, since by its proof $p\,\text{length}(c) = \lambda L$. Furthermore, replacing length $(c)$ in the expression for length $(v)$ in Lemma 8, with its value given in (a), we have that $aL = \ell p$, since a straightforward computation using (4) yields $a = \sqrt{1-\kappa^2}/\sqrt{1+\kappa^2}$. Therefore

$$i_\ell \left[ \Gamma_\ell \, V_z|_{[0,L]} \right] = \left[ \gamma(L) \, \gamma(0)^{-1} \right] = \left[ \widetilde{\exp}(p\ell X_1) \, \widetilde{\exp}(2\pi q Z) \right]$$

(notice that $\widetilde{\exp}(2\pi q Z)$ lies in the center of $\tilde{G}$). Thus (ii) follows from the fact that $\exp(p\ell X_1) = \phi_\ell^p$ and $\phi_\ell$ is primitive in $\Gamma_\ell$.

Now we prove (c). Suppose that $v$ is a unit speed periodic geodesic in $T^1 S_\ell$ of period $L$, that projects to a periodic curve $c$ of constant geodesic curvature $\kappa$, with $|\kappa| < 1$. Let $V$ be a lift of $v$ to $T^1 H$ and let $C$ be its projection to $H$. Then $C$ is a lift of $c$ and there exists $m \in \mathbf{Z}$ such that $(d\phi_\ell^m) \, \dot{C}(0) = \dot{C}(L)$. But by Lemma 7 there exists a unique constant speed curve $C_1$ in $H$ with constat geodesic curvature and $\dot{C}_1(0) = \dot{C}(0)$ and $C_1(L) = \phi_\ell^m C(0)$ and it must be a reparametrization of $t \mapsto \phi_t C(0) = c_{C(0)}(t)$. Hence $V = u V_{C(0)}$ for some $u \in S^1$ and the assertion follows.

**Lemma 10** *All horizontal-like closed geodesics in a free homotopy class have the same length. Thus length is well defined on $\mathcal{G}_{hor}(T^1 S)/\sim$.*

**Proof.** Let $v_j$ $(j = 1, 2)$ be free homotopic horizontal-like unit speed closed geodesics in $T^1 S$, defined on the interval $[0, L_j]$, and suppose that $i[v_j] = [\tilde{g}]$ with $\tilde{g} \in \tilde{G}$ and $\tilde{f}[\tilde{g}] = ([g_0], p, q)$. By conjugating $\Gamma$ in $G$ we may suppose that $g_0 = \phi_\ell$ for some $\ell > 0$. By lifting the free homotopy, we can get lifts $\gamma_j$ of $v_j$ to $\tilde{G}$ such that $g_0^p \dot{\gamma}_j(0) = \dot{\gamma}_j(L_j)$. Denoting by $V_j$ the projection of $\gamma_j$ to $T^1 H$, we have then that $g_0^p \dot{V}_j(0) = \dot{V}_j(L_j)$. Hence $\Gamma_\ell V_j$ are free homotopic (horizontal-like) closed geodesics in $T^1 S_\ell$, which project to $v_j$ on $T^1(\Gamma \backslash H)$ and whose (common) image under $\tilde{f}_\ell \circ i_\ell$ is $([g_0], p, q)$. Now it is easy to see that the formula in Lemma 9 for the length of a closed geodesic in $T^1 S_\ell$ in terms of $p$ and $q$ holds even if this numbers are not coprime. Hence $L_1 = L_2$.

We observe that in fact $v_1(t) = u v_2(t + t_0)$ for some $u \in S^1$, $t_0 \in \mathbf{R}$ and all $t \in \mathbf{R}$, but we do not need this in the following. $\square$

**Theorem 11** *(a) No horizontal-like closed geodesic in $T^1 S$ is free homotopic to a vertical-like one.*

*(b) There is a bijection $F_{hor}$ such that the following diagram commutes.*

$$\begin{array}{ccc} (\mathcal{P}(S)/\sim) \times \mathbf{N} \times \mathbf{Z} & \xrightarrow{F_{hor}} & \mathcal{G}_{hor}(T^1S)/\sim \\ \mathit{length}_S \times id \downarrow & & \downarrow \mathit{length}_{T^1S} \\ \mathbf{R} \times \mathbf{N} \times \mathbf{Z} & \xrightarrow{L} & \mathbf{R} \end{array}$$

*where* $L(\ell, p, q) = \sqrt{2\pi^2 q^2 + \ell^2 p^2}$.

*(c) The map* $F_{ver} : \mathcal{G}(T^1H)/\sim \longrightarrow \mathcal{G}_{ver}(T^1S)/\sim$, $F_{ver}[V] = [\Gamma V]$, *is a well defined bijection which preserves the collection of lengths of closed geodesics in free homotopy classes.*

**Proof.** (a) Let $v : [0, a] \to T^1S$ be a closed geodesic and let $\gamma$ be a lift of $v$ to $\tilde{G}$. Suppose that $\pi v$ is not constant and denote by $c$ the periodic extension of the projection of $\gamma$ to $H$. There exists $\tilde{g} \in \tilde{\Gamma}$ satisfying $\tilde{g}\dot{\gamma}(0) = \dot{\gamma}(a)$. Its projection $g = \tilde{\pi}(\tilde{g}) \in \Gamma$ translates $c$ in $a$. If $v$ is vertical-like, by Lemma 7, the image of $c$ is a geodesic circle in $H$ whose center is fixed by $g$. Hence $g = e$ (since $\Gamma$ contains no elliptic isometries) and thus $i[v] \in \mathcal{Z}/\text{conj}$. The same holds clearly if $\pi v$ is constant. If $v$ is horizontal-like, then $c$ is injective by Lemma 7. Thus $g \neq e$ and $i([v]) \notin \mathcal{Z}/\text{conj}$.

(b) Let $\sigma$ be a primitive closed geodesic in $S$ of length $\ell$. Let $\tilde{\sigma}$ be a lift of $\sigma$ to $H$ and let $g$ be the unique element of $\mathcal{P}\Gamma$ which translates $\tilde{\sigma}$ in $\ell$. There exists $h \in G$ such that $g = h\phi_\ell h^{-1}$. The group $\Gamma_h = h^{-1}\Gamma h$ contains $\Gamma_\ell$. Let $p, q$ integers with $p > 0$ and let $\kappa$ be the unique number ($|\kappa| < 1$) satisfying (3). Setting $L = L(\ell, p, q)$ and $z = e^{i\theta}$, with $\cos\theta = \kappa$, we have that $\Gamma h V_z|_{[0,L]}$ is a closed geodesic in $T^1S$ since it corresponds via the obvious isometry $\Gamma\backslash H \to \Gamma_h\backslash H$ to a geodesic in $T^1(\Gamma_h\backslash H)$ that lifts to $\Gamma_\ell V_z$, which is a closed geodesic in $T^1S_\ell$ by Lemma 9. Now

$$F_{\text{hor}}([\sigma], p, q) = \left[\Gamma h V_z|_{[0,L]}\right] \tag{5}$$

is well defined, since if $\tilde{\sigma}_1$ is another lift of $\sigma$ to $H$, then the unique element $g_1 \in \mathcal{P}\Gamma$ which translates $\tilde{\sigma}_1$ in $\ell$ is conjugate to $g$ in $\Gamma$ by $h_1 \in \Gamma h$.

First we prove that $F_{\text{hor}}$ is onto. Let $v$ be a horizontal-like closed geodesic in $T^1S$ defined on the interval $[0, L]$. Since $L(\ell, mp, mq) = mL(\ell, p, q)$ for all $m \in \mathbf{N}$, we may suppose that $L$ is the period of $v$. Let $V$ be a lift of $v$ to $G \approx T^1H$ and let $c$ its projection to $H$. $c$ has constant geodesic curvature, say $\kappa$, with $|\kappa| < 1$ and $\Gamma c$ is closed in $S$ of period $L/p$ for some $p \in \mathbf{N}$. Hence there exists $g \in \mathcal{P}\Gamma$ such that $gc\,(t) = c\,(t + L/p)$ for all $t$. Now $g$ translates a geodesic $\sigma$ in $H$ in some positive number, say $\ell$. On the other hand, by Lemma 7 and Proposition 6 (b) there exist $h \in G$ such that $c = hc_\theta$ and $u \in S^1$ such that $V = uhV_z$. Hence $g = h\phi_\ell h^{-1}$ and $\Gamma_\ell V_z$ is periodic in $T^1 S_\ell$ with period $L$. Thus we have by the proof of Lemma 8 and Lemma 9 that (3) holds for some integer $q$ coprime with $p$. By the way in which $\sigma, h, p$ and $q$ were introduced and the definition of $F_{\text{hor}}$, we have that $L = \sqrt{2\pi^2 q^2 + \ell^2 p^2}$, that (5) holds, and moreover the free homotopy class of $\Gamma h V_z|_{[0,L]}$ contains $v$.

Now we prove that the composition of the following maps yields the identity in $(\mathcal{P}\,(S)\ /\!\sim) \times \mathbf{N} \times \mathbf{Z}$, hence $F_{\text{hor}}$ is injective.

$$(\mathcal{P}\,(S)\ /\!\sim) \times \mathbf{N} \times \mathbf{Z} \xrightarrow{F_{hor}} \mathcal{G}_{\text{hor}}\,(T^1 S)\ /\!\sim\ \xrightarrow{i_{hor}} \left(\tilde{\Gamma} - \mathcal{Z}\right) /\text{conj}$$

$$\xrightarrow{(\mathcal{P}i \times \text{id})\circ \tilde{f}} (\mathcal{P}\,(S)\ /\!\sim) \times \mathbf{N} \times \mathbf{Z}$$

Let $\sigma, g, \ell$, etc. be as in the definition of $F_{\text{hor}}$ and let $\gamma$ be a lift of $hV_z|_{[0,L]}$ to $\tilde{G}$. We have to show that $\tilde{f}\left[\gamma\,(L)\,\gamma\,(0)^{-1}\right] = ([g]\,, p, q)$. By conjugating $\Gamma$ in $G$ we may suppose that $g = \phi_\ell$. In this case we can take $h = e$ and the identity follows from Lemma 9 (b) (ii).

(c) Let $V : [0, a] \to T^1 H$ be a closed geodesic. Its projection $c$ to $H$ is either constant or closed. If it is not constant, then by Lemma 7 it has constant geodesic curvature $\kappa$ with $|\kappa| > 1$. Hence $V$ and $v = \Gamma V$ are vertical-like. Next we show that $F_{\text{ver}}$ is onto. Let $v : [0, a] \to T^1 S$ be a vertical-like closed geodesic. With the notation of (a) we have that $\tilde{g} \in \mathcal{Z}$, hence $V = \mathcal{Z}\gamma$ is a vertical-like closed geodesic in $T^1 H \approx \mathcal{Z}\backslash\tilde{G}$ such that $F_{\text{ver}}\,[V] = [v]$. Now $F_{\text{ver}}$ is injective since the conjugacy class in $\Gamma$ of $\tilde{g} \in \mathcal{Z}$ consists only of $\tilde{g}$ itself. The last assertion follows from the facts that $V$ and $\Gamma V$ have clearly the same length, and the same happens for $v$ and $V = \mathcal{Z}\gamma$ as above. □

**Proof of Theorem 2.** (a) is just a reformulation of Theorem 11. Now we prove (b). Let $m_1 = m_{T^1 S} - m_{T^1 H}$ and define inductively $\ell_k = \min \operatorname{supp}(m_k)$ and $m_{k+1} = m_k - m_k\,(\ell_k)\,\mathcal{X}_{\ell_k}$. Since $\mathcal{X}_{\ell_k}\,(\ell_k) = 1$ and $\mathcal{X}_\ell\,(\ell_k) = 0$ if $\ell > \ell_k$, we have that $\mathcal{P}m_S\,(\ell_k) = m_k\,(\ell_k)$ and $\mathcal{P}m_S\,(\ell) = 0$ if $\ell \neq \ell_k$, for all $k$. Thus we obtain the primitive length spectrum of $S$, from which the length spectrum can be easily computed. □

**Proof of Theorem 3.** By Theorem 11 $\mathcal{L}_{\text{hor}}$ is the collection of lengths of horizontal-like closed geodesics in $T^1S$. By (a) and (c) of the same theorem it remains only to show that $\mathcal{L}_{\text{ver}}$ is the weak length spectrum of $T^1H$. Let $V$ be a vertical-like periodic geodesic in $T^1H$ and let $c = \pi V$ be its projection to $H$. If $c$ is constant, then length $(V) = 2\pi$, which belongs to $\mathcal{L}_{\text{ver}}$ (taking $q = 2p = 2$). Otherwise $c$ is periodic and by Lemmas 6 and 7 it has constant geodesic curvature, say $\kappa$, with $|\kappa| > 1$, and length $2\pi/\sqrt{\kappa^2 - 1}$. Replacing length $(c)$ with this value in Lemma 8, we obtain that

$$2p\kappa/\sqrt{\kappa^2 - 1} = -q \tag{6}$$

for some coprime integers $p, q$ with $p > 0$, and that

$$\text{length } (V) = 2\pi p\sqrt{\kappa^2 + 1}/\sqrt{\kappa^2 - 1} \tag{7}$$

Now, solving in (6) for $\kappa$ in terms of $q/p$ and substituting in (7), we have that length $(V) = \pi\sqrt{2q^2 - 4p^2}$. If we take a closed geodesic with periodic extension $V$, then its length is an integral multiple of length $(V)$, which is also of the required form for some $p, q$ (not necessarily coprime). To show the remaining inclusion we may suppose that $p, q$ are coprime integers satisfying $q \geq 2p > 0$. If $q = 2p$, then $p = 1$, $q = 2$ and so $\mathcal{L}_{\text{ver}}$ contains the number $2\pi$, which is the length of a geodesic $V$ in $T^1H$ with $c =$ constant. If $q > 2p$, by a similar argument we get that any geodesic in $T^1H$ whose projection to $H$ runs $p$ times along a geodesic circle in $H$ of constant geodesic curvature $\kappa$ satisfying (6), is closed and its length is $\pi\sqrt{2q^2 - 4p^2}$. $\square$

# 4 The Laplace spectra of $\Gamma\backslash H$ and $\Gamma\backslash G$

Let $M$ be a compact connected Riemannian manifold of dimension $n$. For $f \in \mathcal{C}^\infty(M)$ and $p \in M$, the Laplacian of $f$ at $p$ is defined by

$$(\Delta f)(p) = -\sum_{i=1}^{n} \left.\frac{d^2}{dt^2}\right|_0 f(c_i(t)),$$

where $c_i$ is the geodesic in $M$ with initial velocity $X_i$, and $\{X_1, \ldots, X_n\}$ is an arbitrary orthonormal basis of $T_pM$. As usual we denote also by $\Delta$ the unique self-adjoint extension to $L^2(M)$.

In order to prove Theorem 4, we recall from [L] some well known facts about unitary representations of $G = PSl\,(2, \mathbf{R})$. Let $\hat{G}$ be the set of equivalence classes of nontrivial irreducible unitary representations of $G$. $\hat{G}$ consists of the principal and complementary series $H^\nu$ with $\nu \in i\mathbf{R}^+ \cup [0,1)$, and the discrete series $H^\nu$, with $\nu$ varying among the odd integers. The Casimir operator $\mathcal{C} = X_1^2 + X_2^2 - Z^2$ acts on $H^\nu$ by multiplication by $\lambda\,(\nu) = (\nu^2 - 1)\,/4$. Furthermore, $H^\nu$ decomposes under the action of $SO\,(2)$ into irreducibles as follows:

$$H^\nu = \hat{\oplus}_{k\in\mathbf{Z}} H^\nu_{2k} \qquad \text{if} \qquad \nu \in i\mathbf{R}^+ \cup [0,1)$$

$$H^\nu = \hat{\oplus}_{k\geq m} H^\nu_{2k} \quad \text{and} \quad H^{-\nu} = \hat{\oplus}_{k\leq -m} H^{-\nu}_{2k} \qquad \text{if} \qquad \nu = 2m-1, \;\; m \in \mathbf{N}$$

One has that $H^\nu_{2k} \approx \mathbf{C}$ and that $Z^2$ acts on it by multiplication by $-k^2$. Let

$$L^2\,(\Gamma\backslash G) = \mathbf{C}\,1 \;\oplus\; \hat{\oplus}_{\nu\in\hat{G}}\, n_\Gamma\,(\nu)\, H^\nu$$

be the decomposition of $L^2\,(\Gamma\backslash G)$ associated to the quasi-regular representation of $G$. Now

$$L^2\,(\Gamma\backslash H) = L^2\,(\Gamma\backslash G)^K = \mathbf{C}\,1 \oplus \hat{\oplus}_{\nu\in\hat{G}}\, n_\Gamma\,(\nu)\,\left(H^\nu\right)^K = \mathbf{C}\,1 \oplus \hat{\oplus}_{\nu\in i\mathbf{R}^+\cup[0,1)}\, n_\Gamma\,(\nu)\, H^\nu_0$$

This corresponds to identifying functions $f$ on $\Gamma\backslash H$ with functions $\tilde{f}$ on $\Gamma\backslash G$ which are constant on the fibers of $\Gamma\backslash G \to \Gamma\backslash H$. Under this identification,

$$\Delta_{\Gamma\backslash H}\,(f) = -\,\mathcal{C}\left(\tilde{f}\right) \tag{8}$$

Given $X \in \mathfrak{m}$, the curve $V\,(t) = \Phi\,(\exp tX)$ is the parallel transport of $X_1$ along the geodesic $c\,(t) = \exp\,(tX)\,K$ in $H$. On the other hand, $\Phi\,(\exp tZ)$ is a constant speed curve in the circle $T^1_{eK}H$. Hence by Proposition 6, $t \mapsto \exp\,(tX)$ is a geodesic in $G$ for $X \in \mathfrak{m} \cup \mathfrak{k}$, $\|X\| = 1$. Therefore by the $G$-invariance of the metric, we have for $f \in \mathcal{C}^\infty\,(\Gamma\backslash G)$ that

$$\begin{aligned}\left(\Delta_{\Gamma/G} f\right)(\Gamma g) &= -\textstyle\sum_{i=1}^2 \frac{d^2}{dt^2}\Big|_0 f\,(\Gamma g \exp tX_i) - \frac{d^2}{dt^2}\Big|_0 f\,(\Gamma g \exp tZ) \\ &= -\,\mathcal{C}\,(f)\,(\Gamma g) - Z^2_{\Gamma g}\,(f)\end{aligned} \tag{9}$$

**Proof of Theorem 4.** (a) Let us consider first the Laplacian restricted to the subspace of $L^2\,(\Gamma\backslash G)$ associated to the principal and complementary series. Suppose $\nu \in i\mathbf{R}^{+} \cup [0,1)$, then $(1-\nu^2)/4$ is an eigenvalue of the Laplacian on $\Gamma\backslash H$ whit multiplicity $n_\Gamma(\nu)$. By (8), (9) and the description of the action of $Z^2$, the numbers $\lambda(\nu)+k^2$, with $k \in \mathbf{N} \cup \{0\}$, are Laplace eigenvalues with multiplicities $n_\Gamma(\nu) = \mu_{\Gamma\backslash H}(\lambda(\nu))$ if $k=0$, and $2n_\Gamma(\nu) = 2\mu_{\Gamma\backslash H}(\lambda(\nu))$ if $k \neq 0$. Now, arguing similarly for the Laplacian restricted to the subspace of $L^2(\Gamma\backslash G)$ associated to the discrete series, we obtain that if $m \in \mathbf{N}$ and $\nu = 2m-1$, then the numbers $\lambda(\nu)+k^2$ with $k \geq m$ are eigenvalues with multiplicities $2n_\Gamma(\nu)$. The formula follows now from the facts that $\lambda : i\mathbf{R}^{+} \cup [0,1) \to \mathbf{R}^{+}$ is a bijection and that if $\nu$ is an odd integer, $n_\Gamma(\nu)$ equals $g(\Gamma\backslash H)$ if $|\nu| = 1$ and $|\nu|(\mathrm{g}(\Gamma\backslash H)-1)$ if $|\nu| \neq 1$ (see for example [H P]).

(b) If $\mu_{\Gamma\backslash G}$ is given, we can compute from it the volume of $\Gamma\backslash G$ and hence we know area$(\Gamma\backslash H) = \mathrm{vol}(\Gamma\backslash G)/2\pi$ and from this the genus of $\Gamma\backslash H$. Thus we must obtain $\mu_{\Gamma\backslash H}$ in terms of the second term $\sum_{\lambda>0}\mu_S(\lambda)(\delta_\lambda + 2\zeta_\lambda)$ of the expression for $\mu_{\Gamma\backslash G}$ given in (a). Let $\mu_1$ denote this second term and let $\lambda_1 = \min \operatorname{supp}(\mu_1)$, which is a positive number. Clearly $\mu_{\Gamma\backslash H}(\lambda) = 0$ if $0 < \lambda < \lambda_1$. Hence we have that $\mu_{\Gamma\backslash H}(\lambda_1) = \mu_1(\lambda_1)$, since $\zeta_{\lambda_1}(\lambda_1) = 1$ and $\zeta_\lambda(\lambda_1) = 0$ if $\lambda > \lambda_1$. We define recursively $\lambda_n = \min \operatorname{supp}(\mu_n)$ and $\mu_{n+1} = \mu_n - \mu_n(\lambda_n)(\delta_{\lambda_n} + 2\zeta_{\lambda_n})$ and obtain by the same argument that $\mu_{\Gamma\backslash H}(\lambda_n) = \mu_n(\lambda_n)$ and $\mu_{\Gamma\backslash H}(\lambda) = 0$ if $0 \neq \lambda \neq \lambda_n$, for all $n$.

# 5 Isometry classes and genericity in $\mathcal{U}$

**Remark 12** *(a) No manifold in the family $\mathcal{U}$ of unit tangent bundles of compact hyperbolic Riemann surfaces is generic in the sense of Y. Colin de Verdière.*

*(b) Let $S_1$ and $S_2$ be two compact hyperbolic Riemann surfaces and suppose that $T^1S_1$ and $T^1S_2$ are isometric. Then $S_1$ and $S_2$ are isometric. In particular the family $\mathcal{U}$ contains non isometric isospectral manifolds.*

**Proof.** (a) Consider in $\Gamma\backslash G \approx T^1(\Gamma\backslash H)$ the closed geodesic $v(t) = \Gamma\exp(tZ)$, $t \in [0,2\pi]$, whose image is the circle $T^1_{eK}H$. From the proof of Theorem 3 we see that the only closed geodesics in $T^1(\Gamma\backslash H)$ free homotopic to $v$ with length $2\pi$ are $\Gamma g\exp(tZ)$, with $t \in [0,2\pi]$ and $g \in G$. All geodesics of this type in a neighborhood of $v$ can be obtained as a one-parameter group of local isometries of $T^1(\Gamma\backslash H)$ applied to $v$. Hence the connected critical submanifold in the loop space of energy $4\pi^2$ containing $v$ has dimension less than or equal to 4, the dimension of $\mathrm{Isom}_0\left(\tilde{G}\right) \approx \tilde{G} \times \mathbf{R}$ (see [Sc]).

On the other hand, using the method in [Z], we obtain that the dimension of periodic Jacobi vector fields along the periodic extension of $v$ is greater or equal to 5. Hence there is a periodic Jacobi field that is not the restriction of a Killing field. Therefore the critical submanifold of energy $4\pi^2$ in the loop space is degenerate and thus $T^1(\Gamma \backslash H)$ is not generic in the sense of [C de V] (see also [B]). Now we follow [Z] to show that the linearized Poincaré map of the periodic geodesic $\gamma : \mathbf{R} \to G$, $\gamma(t) = \exp(tZ)$, is the identity. The product $G \times K$ acts on the left on $G$ by isometries as follows: $(g,k).g_1 = gg_1k^{-1}$, with isotropy group the diagonal $\Delta(K) = \{(k,k) \mid k \in K\}$. Let $\mathfrak{h} = \{(Y,Y) \mid Y \in \mathfrak{k}\}$ be the Lie algebra of $\Delta(K)$ and let $\mathfrak{p} = \{(X+Y, 2Y) \mid X \in \mathfrak{m} \text{ and } Y \in \mathfrak{k}\}$. One verifies easily that $\mathfrak{h} \oplus \mathfrak{p}$ is a naturally reductive decomposition for $G$ expressed as the quotient $(G \times K)/\Delta(K)$. Identifying $\mathfrak{p} \approx T_eG$ in the usual way, we have in the notation of [Z] that $v = \dot{\gamma}(0) = (Z,Z)$ and $E = v^{\perp} = \{(X,0) \mid X \in \mathfrak{m}\}$. Given $U = (X,0) \in E$ we compute $[v,U] = (iX,0) = [v,U]_{\mathfrak{p}}$ and hence $[v,U]_{\mathfrak{h}} = 0$. Therefore in the notation of [Z] we have $E_1 = E_2 = 0$ and hence $E \oplus E = V_1 \oplus V_4$. Since $\exp(2\pi Z) = e$ we have that the linear isometry $A$ is the identity on $\mathfrak{p}$. Consequently the linearized Poincaré map for $\gamma$ is the identity. Moreover, since the geodesic spray of $\gamma$ is a periodic Jacobi field, we obtain finally that the dimension of periodic Jacobi fields along $\gamma$ is at least $1+\dim(E \oplus E) = 5$.

(b) For $j = 1, 2$ and for each $v \in T^1S_j$ there are exactly two unit vectors $\pm\xi \in T_v(T^1S_j)$ where $\mathrm{Ricci}_v$ attains its maximum (see [M]). The maximal connected leaves of the one-dimensional distribution $\mathcal{D}_j(v) = \mathbf{R}\xi_v$ defined on $T^1S_j$ are the fibers of the bundle $T^1S_j \to S_j$. If $F : T^1S_1 \to T^1S_2$ is an isometry, then $F_*$ preserves the distributions and hence the leaves. Moreover the function $f : S_1 \to S_2$ which lifts to $F$ is an isometry since the projection $T^1S \to S$ is a Riemannian submersion. The last assertion follows from the fact that there exist non isometric isospectral compact hyperbolic Riemann surfaces (see [V]).

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