SPECTRAL ANALYSIS OF HYPERBOLIC SYSTEMS WITH SINGULARITIES

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Abstract. We study the statistical properties of a general class of two-dimensional hyperbolic systems with singularities by constructing Banach spaces on which the associated transfer operators are quasi-compact. When the map is mixing, the transfer operator has a spectral gap and many related statistical properties follow, such as exponential decay of correlations, the central limit theorem, the identification of Ruelle resonances, large deviation estimates and an almost-sure invariance principle. To demonstrate the utility of this approach, we give two applications to specific systems: dispersing billiards with corner points and the reduced maps for certain billiards with focusing boundaries.

1. Introduction

The study of the statistical properties of hyperbolic systems with singularities is motivated in large part by mathematical billiards, introduced in [Si] and since studied extensively by many authors. A general class of such systems was introduced in the fundamental work by Katok and Strelcyn [KS] in which the following assumptions were made on the singularity set $\mathcal{S}$: the derivatives of the map T can only grow mildly near $\mathcal{S}$ (bounded by a negative power of the distance to $\mathcal{S}$) and T preserves an invariant measure μ with the property that every ε-neighborhood $N_\varepsilon(\mathcal{S})$ of $\mathcal{S}$ satisfies

$$\mu(N_\varepsilon(\mathcal{S})) = \mathcal{O}(\varepsilon^a) \tag{1.1}$$

for some constant $a > 0$. These together with several other mild assumptions are sufficient for the construction of stable and unstable manifolds, their absolute continuity, and certain formulas for the entropy of T [KS].

In later studies of finer statistical properties of billiards and related models, the fact that $a = 1$ in (1.1) played a vital role in the work on dispersing billiards [BSC1, Y, Ch1], Bunimovich's stadium [Ma], higher-dimensional Lorentz gases [BT], systems of two hard balls of different masses [CD1], certain abstract multidimensional models [Ch2], and others. Only recently, Chernov and Zhang [CZ4] extended these studies to cover systems with more general singularities, i.e. with $a < 1$ in (1.1), and obtained exponential decay of correlations under certain assumptions, using coupling methods. The coupling scheme is simple and intuitive as it captures the geometrical properties of the dynamical systems, but it can only be used under the assumption of the existence of an SRB measure.

In this paper we present a functional analytic framework in which to study certain general classes of hyperbolic systems with singularities in two dimensions. We drop two assumptions used in [CZ4]: the *a priori* existence of an SRB measure, and the absolute continuity of the holonomy map between unstable manifolds.[1] For our class of maps, we construct Banach spaces on which the associated transfer operators are quasi-compact. When the map is mixing, the transfer operator has a spectral gap and many results follow immediately: exponential decay of correlations, the

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[1]We drop such conditions as part of our formal assumptions since they are not needed for the present approach; however, these properties follow from our other assumptions on the hyperbolicity of the map T (see for example [KS, P, S]).

identification of Ruelle resonances, local large deviation estimates and an almost-sure invariance principle. We remark that our large deviation estimate has a uniform rate function with respect to any probability measure in our Banach space; this includes both Lebesgue measure and the SRB measure for the system, even though such measures may be singular with respect to one another (see Corollary 2.5).

The functional analytic approach we adopt in this paper traces back to classical results of Doeblin and Fortet regarding Markov chains [DF, IM, N]. This approach was adapted to overcome the problem of discontinuities for expanding maps by using the smoothing effect of the transfer operator on functions of bounded variation [LY, K, Sa, Bu, T1, T2, BK]. Its extension to hyperbolic maps followed, although the required Banach spaces were no longer spaces of functions, but of distributions: first to Anosov diffeomorphisms [R1, R2, R3, BKL, B2, BaT, GL] and then to piecewise hyperbolic maps [DL, BG1, BG2], and recently to the billiard map associated with a periodic Lorentz gas [DZ1].

In addition to the many limit theorems that follow from the existence of a spectral gap mentioned above, the quasi-compactness of the transfer operator has several important applications which serve to highlight the strengths of this approach. It can be used to determine the stability of statistical properties under perturbations, for example using perturbation theory [Ka] or the looser perturbative framework of [KL], as done recently for perturbations of the Lorentz gas in [DZ2]. It can be used to study the mixing rates of flows following the approach of [L1, BL]; indeed, a version of the norms presented here is expected to resolve the long-standing open conjecture of exponential decay of correlations for finite horizon billiard flows. As a final example, we mention the application to slowly mixing systems via the renewal theory developed by Sarig [Sr].

Our purpose in this paper is to formulate the approach used for the Lorentz gas in [DZ1] in as broad a framework as possible, which we present as abstract assumptions **(H1)**-**(H5)** in Section 2.1. To this end, we allow tangencies between the singularity curves and stable and unstable cones (see **(H3)**), and weaken the one-step expansion condition used in [DZ1] to admit more general singularities of the form (1.1) with $a < 1$ (see **(H5)**). We also formulate condition **(H1)** on the Jacobian of the map to allow perturbations of classical billiards, such as billiards under external forces or those subject to twists or kicks at collisions (see [DZ2]). In order to accommodate this more general setting, we have adapted and generalized the Banach space norms used in [DZ1] and prove new growth lemmas to derive the necessary Lasota-Yorke inequalities. To demonstrate the broad applicability of these abstract results, we then apply this framework to dispersing billiards with corner points as well as to the reduced maps for two types of billiard systems with focusing boundaries that were studied in [CZ4]: nonsmooth stadia and Bunimovich tables. We also recover all the results from [DZ1] for both the finite and infinite horizon Lorentz gas in this general framework.

The paper is organized as follows. In Section 2, we state our abstract conditions **(H1)**-**(H5)**, define the Banach spaces on which we will study the transfer operator and state our main results. In Section 3, we prove the necessary estimates to control the cutting generated by singularities in the presence of the weakened one-step expansion condition **(H5)** and prove preliminary properties of our Banach spaces including embeddings and compactness. Section 4 contains the required Lasota-Yorke inequalities and in Section 5 we characterize the peripheral spectrum and prove some related statistical properties, including limit theorems. Section 6 contains the application to billiards with corner points; Section 7 applies the present framework to the reduced maps corresponding to the two types of billiards with focusing boundaries mentioned above.

2. Setting and Statement of Results

2.1. **Assumptions on the hyperbolic map T.** We begin by defining the class of hyperbolic maps to which our results apply. Let M be a smooth two-dimensional Riemannian manifold (possibly with boundary and not necessarily connected). We consider maps T defined on an open subset of M which are piecewise hyperbolic in the sense described precisely below. Let d denote the

Riemannian metric on M and for a curve W, let d_W denote the metric induced by restricting d to W. The corresponding unnormalized arclength measure on W is denoted by m_W.

(H1) Smoothness of the map. Let $1-\eta_0 \le \kappa \le 1$, where η_0 will be specified after the statement of Proposition 2.3 in Section 2.4. Assume $f \ge 0$ is a $\mathcal{C}^1$ smooth function on M and $f_0 \ge \kappa$ is a piecewise $\mathcal{C}^1$ function such that

$$|D_xT| := |\det D_xT| = \frac{f(x)}{f(Tx)} \cdot f_0(x), \tag{2.1}$$

wherever D_xT exists. We assume that T is nondegenerate in the sense that the level sets of f are finite unions of smooth compact curves.

When $f_0 \equiv 1$, assumption (2.1) implies that the map T preserves the measure $f dm$ on the phase space M, where m denotes the Riemannian volume on M. This is the case for classical billiards. The inclusion of f_0 allows us to apply this framework to perturbations of billiards; for example, to dispersing billiards subject to external forces and twists or kicks at reflections as in [DZ2].

(H2) Hyperbolicity. Let $\mathcal{S}_0$ be a finite union of compact $\mathcal{C}^2$ smooth curves in M such that $\partial M \cup f^{-1}(0) \subset \mathcal{S}_0$. Denote[2] $\mathcal{S}_{\pm 1} = \mathcal{S}_0 \cup T^{\mp 1}\mathcal{S}_0$. We require that $T : M \setminus \mathcal{S}_1 \to M \setminus \mathcal{S}_{-1}$ be a $\mathcal{C}^2$ diffeomorphism. Note that while $\mathcal{S}_0$ is assumed to be a finite union of compact smooth curves, $\mathcal{S}_{\pm 1}$ may have countably many such curves.

We assume there exist two families of cones $C^u(x)$ (unstable) and $C^s(x)$ (stable) in the tangent spaces $\mathcal{T}_xM$, continuous on the closure of each component of $M \setminus \mathcal{S}_0$, such that for all $x \in M$ the angle between $C^u(x)$ and $C^s(x)$ is uniformly bounded away from zero. In addition there exists $\Lambda > 1$ with the following properties:

(1) $D_xT(C^u(x)) \subset C^u(Tx)$ and $D_xT^{-1}(C^s(x)) \subset C^s(T^{-1}x)$ whenever D_xT and D_xT^{-1} exist.
(2) $\|D_xTv\|_* \ge \Lambda\|v\|_*, \forall v \in C^u(x)$ and $\|D_xT^{-1}v\|_* \ge \Lambda\|v\|_*, \forall v \in C^s(x)$, where $\|\cdot\|_*$ is an adapted norm, uniformly equivalent to the Euclidean norm, $\|\cdot\|$.

In order to control distortion when $\|D_xT^{-1}v\|$, $v \in C^s(x)$, becomes unbounded, we introduce the concept of homogeneity regions, inspired by the study of billiards. We fix an exponent $r_h > 1$ which will determine the spacing of the boundaries of the homogeneity regions.[3] First, we define these regions in a neighborhood of $f^{-1}(0)$. Let $S_0^H = f^{-1}(0)$ and $S_k^H = f^{-1}(k^{-r_h+1})$ for $k > k_0$, where k_0 is a fixed integer with value chosen from **(H5)**. Due to **(H1)**, S_0^H and S_k^H are finite unions of smooth curves. The region between S_k^H and S_{k+1}^H is called a homogeneity region with index k, and denoted as $\mathbb{H}_k$. It is not essential here that S_k^H be precisely $f^{-1}(k^{-r_h+1})$; in applications, it may be convenient to allow some flexibility, $S_k^H \approx f^{-1}(k^{-r_h+1})$; see for example our application to nonsmooth stadia in Section 7.

It may be that $\|D_xT^{-1}\|$ becomes unbounded even when $f(T^{-1}x) \neq 0$. This may happen, for example, in the area-preserving case $|D_xT| = 1$. (See [W, BBN] for an example of such a map derived from a system of bouncing balls.[4]) In this case, we may define homogeneity regions analogous to $\mathbb{H}_k$ above with the same spacing exponent. Thus we may define homogeneity regions in the image of wherever the expansion becomes unbounded, and in particular always near $f^{-1}(0)$. In all cases, however, the $\mathbb{H}_k$ must accumulate on a finite number of smooth, compact curves in $\mathcal{S}_0$ as defined in **(H2)**. We call this set of curves S_0^H and allow the $\mathbb{H}_k$ to accumulate at single points. In applications, the decision whether to introduce these extra cuts will depend on whether

[2] If for some set A, there exists $x \in A$, such that $T^{-1}x$ is not well-defined, we extend our notation by $T^{-1}A := \{x \in M : Tx \in A\}$.

[3] The standard choice for dispersing billiards is $r_h = 3$, following [BSC1, BSC2].

[4] We are not claiming this system as an application of our method at the present time, but rather that we expect axioms **(H1)**-**(H5)** will apply, possibly with minor modifications.

the singularities of the map already provide the required bounded distortion (see Section 7 for an example of a map where additional cuts are not required even though the derivative becomes unbounded). The required properties of the homogeneity regions $\mathbb{H}_k$ are listed in **(H3)-(H5)** below. We denote by $\mathbb{H}_{k_0}$ the region in M that comprises the complement of the closures, $M \setminus (\cup_{k>k_0} \overline{\mathbb{H}}_k)$.

We say that a smooth curve $W \subset M$ is a stable (unstable) curve if at every point $x \in W$ the tangent line $\mathcal{T}_x W$ belongs in the stable (unstable) cone $C^s(x)$ ($C^u(x)$). We call a stable (unstable) curve homogeneous if it lies entirely in one homogeneity region $\mathbb{H}_k$. We will work with families of homogeneous stable and unstable curves, $\mathcal{W}^s$ and $\mathcal{W}^u$, defined below in (**H4**).

(H3) Structure of Singularities.

(1) There exist constants $C_0 > 0$ and $\xi \leq 1$ such that if $W \in W^s$ and T^{-1} is smooth on W such that $T^{-1}W \in \mathcal{W}^s$, then $|T^{-1}W| \leq C_0 |W|^\xi$.
(2) If D is a connected component of $M \setminus \mathcal{S}_{-1}$, then ∂D consists of finitely many smooth compact curves. Moreover, for each $\varepsilon > 0$, there are at most finitely many connected components of $M \setminus \mathcal{S}_{-1}$ containing stable curves of length greater than ε.
(3) There exist constants $C_1 > 0$ and $0 < t_0 \leq 1$ such that for any stable curve W and any smooth curve $S \subset \mathcal{S}_{-n}$, we have $m_W(N_\varepsilon(S) \cap W) \leq C_1 \varepsilon^{t_0}$ for all $\varepsilon > 0$ sufficiently small, where $N_\varepsilon(\cdot)$ denotes the ε-neighborhood of a set in M.
(4) The homogeneity curves S_k^H, $k \geq k_0$, satisfy the same weak transversality condition as in (3) above. In addition, there exists $C_2 > 0$ such that for all $k > k_0$, if $W \in \mathcal{W}^s$ with $W \subset \mathbb{H}_k$, then $|W| \leq C_2 k^{-r_h}$.
(5) On each connected component of $\mathbb{H}_k \cap (M \setminus \mathcal{S}_0)$, $k > k_0$, we choose a smooth foliation $\{W_\xi\}_{\xi \in E_k} \subset \mathcal{W}^s$ whose elements completely cross that component of $\mathbb{H}_k \cap (M \setminus \mathcal{S}_0)$. This is possible by **(H3)**(4) above. We decompose the Riemannian volume m on this component into $dm = \lambda(d\xi)\rho_\xi dm_W$ where m_W is arclength on W_ξ, ρ_ξ is a smooth function depending on the choice of foliation, and λ is the transverse measure on E_k. We assume that
$$\sum_{k>k_0} \int_{E_k} f(W_\xi)|W_\xi|^\varepsilon d\lambda(\xi) < \infty \quad \text{for all } \varepsilon > 0,$$
where $f(W_\xi)$ is the average value of f on W_ξ (taken with respect to arclength).[5]

Since the items in **(H3)** are quite technical, we briefly explain the significance of each and where it is used in our proofs. **(H3)**(1) is used in the Lasota-Yorke estimate in Section 4.3. The assumption on the finiteness of ∂D in **(H3)**(2) is used in Lemma 3.6. The shortness of stable curves from **(H3)**(2) is used in the graph transform argument of Lemma 3.2.

The weak transversality assumptions in **(H3)**(3) and **(H3)**(4) are standard assumptions to control the interaction between hyperbolicity and singularities; they are essential throughout this paper. The introduction of the exponent t_0 allows for the types of singularity sets with 'non-degenerate tangencies' found in billiards with corner points. Of course, if $C^s(x)$ is uniformly transverse to $\mathcal{S}_{-n}$, then one can take $t_0 = 1$; however, without loss of generality in the arguments that follow we will take $t_0 \leq 1/2$ as it simplifies the proof of Lemma 3.5 which otherwise would have to be split into two cases.

The second part of **(H3)**(4) is a spacing requirement for the homogeneity strips so that stable curves in strips of high index are short. This is necessary for the graph transform argument (Lemma 3.2), approximation by smooth functions in our Banach space (Lemma 3.5) and compactness (Lemma 3.9). Finally, **(H3)**(5) is a kind of summability condition over homogeneity strips

[5] The assumption **(H3)**(5) is automatically satisfied when $r_h \geq 2$ for the homogeneity strips $\mathbb{H}_k$ defined by $f^{-1}(k^{-r_h+1})$ as long as $\lambda(E_k)$ remains uniformly bounded. For then using **(H3)**(4), the series is majorized by $\sum_{k\geq k_0} k^{-r_h+1-\varepsilon r_h} < \infty$. This is the case for the billiards we consider in Sections 6 and 7.

used only in the proof of Lemma 3.7 in order to control the distributional norm of elements of our Banach spaces. It is easily satisfied for the billiards considered in Sections 6 and 7.

Remark 2.1. *One can replace* **(H3)***(1) with the following assumption on the blowup of the derivative: There exist constants $C_0 > 0$ and $0 < a < 1$ such that,*

$$\|D_x T^{-1} v\| \le C_0 \|v\| d(x, \mathcal{S}_{-1})^{-a} \quad \text{for all } v \in C^s(x), \tag{2.2}$$

wherever $D_x T^{-1}$ is defined.

This, together with **(H3)***(3), yields the bound $|T^{-1}W| \le C|W|^{t_0 - a}$, which is useful if $a < t_0$.*

(H4) Invariant families of stable and unstable curves. Let $\mathcal{W}^s$ denote the set of homogeneous $\mathcal{C}^2$ stable curves with length less than some positive constant δ_0 (to be chosen in (2.6)) and with curvature bounded above by some uniform constant $B > 0$. We assume there exists a choice of B such that $\mathcal{W}^s$ is invariant under T^{-1} in the following sense: The connected components of $T^{-1}W$ belong to $\mathcal{W}^s$ whenever $W \in \mathcal{W}^s$ (up to subdivision of the connected components to guarantee length at most δ_0).

We require the following distortion bounds: There exist $p_0 \in (0, 1]$ and $C_3 \ge 1$ such that if $W \in \mathcal{W}^s$ is such that $T^{-1}W \in \mathcal{W}^s$ or $W \in \mathcal{W}^u$ is such that $T^{-1}W \in \mathcal{W}^u$, then for any $x, y \in W$,

$$\|D_x T^{-1} - D_y T^{-1}\| \le C_3 \|D_x T^{-1}\| \max\{d(x, y)^{p_0}, d(T^{-1}x, T^{-1}y)^{p_0}\}. \tag{2.3}$$

We also require the analogous distortion bound for the full Jacobian of the map. If $W \in \mathcal{W}^s$ is is such that $TW \in \mathcal{W}^s$ or if $W \in \mathcal{W}^u$ is such that $TW \in \mathcal{W}^u$, then for any $x, y \in W$,

$$\left| \frac{|D_x T|}{|D_y T|} - 1 \right| \le C_3 \max\{d_W(x, y)^{p_0}, d_W(Tx, Ty)^{p_0}\}. \tag{2.4}$$

Our final assumption is on the complexity of the singularities of T^{-1}. It says that the expansion due to hyperbolicity dominates the cutting due to singularities, which is a standard assumption for hyperbolic maps with singularities. The version we use here is the weakened form introduced in [CZ4] as described in the introduction.[6]

(H5) One-step expansion. Let $W \in \mathcal{W}^s$ and partition the connected components of $T^{-1}W$ into maximal pieces V_i such that each V_i is a homogeneous stable curve (not necessarily of length at most δ_0). Let $|J_{V_i} T|_*$ denote the minimum contraction on V_i under T in the metric induced by the adapted norm $\| \cdot \|_*$, and let $|W|_*$ denote the length of $W \in \mathcal{W}^s$ in this metric. We assume there exists a constant $\gamma_0 \in [0, 1/r_h)$ and a choice of k_0 for the homogeneity strips such that

$$\limsup_{\delta \to 0} \sup_{\substack{W \in \mathcal{W}^s \\ |W| < \delta}} \sum_i \left(\frac{|V_i|_*}{|W|_*} \right)^{\gamma_0} |J_{V_i} T|_* < 1. \tag{2.5}$$

In light of **(H5)**, we fix $\delta_0 > 0$ in the definition of $\mathcal{W}^s$ sufficiently small that,

$$\sup_{W \in \mathcal{W}^s} \sum_i \left(\frac{|V_i|_*}{|W|_*} \right)^{\gamma_0} |J_{V_i} T|_* =: \theta_* < 1. \tag{2.6}$$

In the proof of Lemma 3.2 (which is essentially a graph transform argument), the index k_0 from **(H5)** may be increased and the maximum length scale δ_0 may be decreased, but this will not affect θ_* fixed above.

[6]Note that our parameter γ_0 is $1 - q$ in the notation of [CZ4].

2.2. **Transfer Operator.** Notice that if ψ is a smooth test function, then $\psi \circ T$ is only piecewise smooth due to the singularities of T. For this reason, we introduce scales of spaces, defined using the invariant family of curves $\mathcal{W}^s$ from **(H4)**, on which to describe the action of the *transfer operator* $\mathcal{L}$ associated with T.

Define $T^{-n}\mathcal{W}^s$ to be the set of homogeneous stable curves W such that T^n is smooth on W and $T^i W \in \mathcal{W}^s$ for $0 \le i \le n$. Then $T^{-n}\mathcal{W}^s \subset \mathcal{W}^s$ and it follows from **(H4)** that the connected components of $T^{-n}W$ belong to $\mathcal{W}^s$ whenever $W \in \mathcal{W}^s$ (up to subdividing long pieces).

We denote (normalized) Lebesgue measure on M by m. For $W \in T^{-n}\mathcal{W}^s$, a complex-valued test function $\psi : M \to \mathbb{C}$ and $0 < p \le 1$, define $H^p_W(\psi)$ to be the Hölder constant of ψ on W with exponent p measured in the metric d_W. Define $H^p_n(\psi) = \sup_{W \in T^{-n}\mathcal{W}^s} H^p_W(\psi)$ and let $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s) = \{\psi : M \to \mathbb{C} : |\psi|_\infty + H^p_n(\psi) < \infty\}$, denote the set of bounded complex-valued functions which are Hölder continuous on elements of $T^{-n}\mathcal{W}^s$. The set $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s)$ equipped with the norm $|\psi|_{\mathcal{C}^p(T^{-n}\mathcal{W}^s)} = |\psi|_\infty + H^p_n(\psi)$ is a Banach space. We define $\mathcal{C}^p(T^{-n}\mathcal{W}^s)$ to be the closure of $\tilde{\mathcal{C}}^1(T^{-n}\mathcal{W}^s)$ in $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s)$.[7]

It follows from (3.26) that if $\psi \in \tilde{\mathcal{C}}^p(T^{-(n-1)}\mathcal{W}^s)$, then $\psi \circ T \in \tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s)$. Similarly, if $\zeta \in \tilde{\mathcal{C}}^1(T^{-(n-1)}\mathcal{W}^s)$, then $\zeta \circ T \in \tilde{\mathcal{C}}^1(T^{-n}\mathcal{W}^s)$. These two facts together imply that for $p < 1$, if $\psi \in \mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s)$, then $\psi \circ T \in \mathcal{C}^p(T^{-n}\mathcal{W}^s)$.

If $h \in (\mathcal{C}^p(T^{-n}\mathcal{W}^s))'$, is an element of the dual of $\mathcal{C}^p(T^{-n}\mathcal{W}^s)$, then $\mathcal{L} : (\mathcal{C}^p(T^{-n}\mathcal{W}^s))' \to (\mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s))'$ acts on h by

$$\mathcal{L}h(\psi) = h(\psi \circ T), \quad \forall \psi \in \mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s).$$

If $h \in L^1(M, m)$, then h is canonically identified with a signed measure absolutely continuous with respect to Lebesgue, which we shall also call h, i.e.,

$$h(\psi) = \int_M \psi h \, dm.$$

With the above identification, we write $L^1(M, m) \subset (\mathcal{C}^p(T^{-n}\mathcal{W}^s))'$ for each $n \in \mathbb{N}$. Then restricted to $L^1(M, m)$, $\mathcal{L}$ acts according to the familiar expression

$$\mathcal{L}^n h = h \circ T^{-n} \, |DT^n(T^{-n})|^{-1} \quad \text{for any } n \ge 0 \text{ and any } h \in L^1(M, m).$$

2.3. **Definition of the Norms.** The following norms are defined via integration on the set of admissible stable curves $\mathcal{W}^s$ given by **(H4)**. In Section 3.1 we define precisely the notion of a distance $d_{\mathcal{W}^s}(\cdot, \cdot)$ between such curves as well as a distance $d_q(\cdot, \cdot)$ defined among functions supported on these curves.

Given a curve $W \in \mathcal{W}^s$, set $|W| = m_W(W)$, where as before m_W denotes the (unnormalized) arclength measure on W. With a slight abuse of notation, we define $f(W)$ to be the average value of $f(x)$ on $W \in \mathcal{W}^s$, i.e. $f(W) = |W|^{-1} \int_W f(x) \, dm_W$, where f is defined by (2.1).

For $0 \le p \le 1$, as in Section 2.2 we denote by $\tilde{\mathcal{C}}^p(W)$ the set of continuous complex-valued functions on W with Hölder exponent p and by $\mathcal{C}^p(W)$ the closure of $\tilde{\mathcal{C}}^1(W)$ in the $\tilde{\mathcal{C}}^p$-norm[8]: $|\psi|_{\mathcal{C}^p(W)} = |\psi|_{\mathcal{C}^0(W)} + H^p_W(\psi)$, where $H^p_W(\psi)$ is the Hölder constant of ψ along W. Notice that with this definition, $|\psi_1\psi_2|_{\mathcal{C}^p(W)} \le |\psi_1|_{\mathcal{C}^p(W)} |\psi_2|_{\mathcal{C}^p(W)}$. We define $\tilde{\mathcal{C}}^p(M)$ and $\mathcal{C}^p(M)$ similarly.

For $\alpha, p \ge 0$, define the following norms for test functions,

$$|\psi|_{W,\alpha,p} := |W|^\alpha \cdot f(W) \cdot |\psi|_{\mathcal{C}^p(W)}.$$

We now fix the following choices of parameters for our norms, based on **(H1)**-**(H5)**: First choose $\alpha, \gamma > 0$ such that $\gamma_0 < \gamma < \alpha < 1/r_h$, where γ_0 is from **(H5)** and r_h determines the spacing of $\mathbb{H}_k$;

[7] Here by $\tilde{\mathcal{C}}^1(\mathcal{W}^s)$ we mean to indicate $\tilde{\mathcal{C}}^p(\mathcal{W}^s)$ with $p = 1$, i.e. functions which are Lipschitz on elements of $\mathcal{W}^s$.

[8] Note that while $\mathcal{C}^p(W)$ may not contain all of $\tilde{\mathcal{C}}^p(W)$, it does contain $\mathcal{C}^{p'}(W)$ for all $p' > p$.

next choose $0 < q < p \le p_0$ such that $p < \gamma$, where p_0 is the Holder exponent from **(H4)**; finally, choose

$$0 < \beta < \min\left\{ \ p - q, \ \xi t_0(\alpha - \gamma), \ \tfrac{t_0(\alpha-\gamma)}{2(1-t_0)}, \ \tfrac{1}{r_h} - \alpha \ \right\}, \tag{2.7}$$

where ξ is from **(H3)**(1) and $t_0 \le 1/2$ is from **(H3)**(2).

Given a function $h \in \mathcal{C}^1(M)$, we define the *weak norm* of h by

$$|h|_w := \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^p(W) \\ |\psi|_{W,\gamma,p} \le 1}} \int_W h\psi \, dm_W. \tag{2.8}$$

We define the *strong stable norm* of h as

$$\|h\|_s := \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^q(W) \\ |\psi|_{W,\alpha,q} \le 1}} \int_W h\psi \, dm_W \tag{2.9}$$

and the *strong unstable norm* of h as

$$\|h\|_u := \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d_{\mathcal{W}^s}(W_1,W_2) \le \varepsilon}} \sup_{\substack{\psi_i \in \mathcal{C}^p(W_i) \\ |\psi_i|_{W_i,\gamma,p} \le 1 \\ d_q(\psi_1,\psi_2) \le \varepsilon}} \frac{1}{\varepsilon^\beta} \left| \int_{W_1} h\psi_1 \, dm_W - \int_{W_2} h\psi_2 \, dm_W \right| \tag{2.10}$$

where $\varepsilon_0 > 0$ is chosen less than δ_0, the maximum length of $W \in \mathcal{W}^s$ which is determined by (2.6). Here $d_{\mathcal{W}^s}(W_1, W_2)$ and $d_q(\psi_1, \psi_2)$ are defined in Section 3.1. We then define the *strong norm* of h by

$$\|h\|_{\mathcal{B}} = \|h\|_s + c_u \|h\|_u$$

where c_u is a small constant chosen in (2.14).

We define $\mathcal{B}$ to be the completion of $\mathcal{C}^1(M)$ in the strong norm and $\mathcal{B}_w$ to be the completion of $\mathcal{C}^1(M)$ in the weak norm.

2.4. **Statement of Results.** We assume throughout that T satisfies assumptions **(H1)-(H5)** as described in Section 2.1. The first result gives a more concrete description of the abstract spaces $\mathcal{B}$ and $\mathcal{B}_w$ introduced above.

Lemma 2.2. *For $\lambda > \beta/(1-\beta)$ and each $n \ge 0$, $\mathcal{C}^\lambda(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (\mathcal{C}^p(T^{-n}\mathcal{W}^s))'$, and each of the embeddings is continuous and injective. Moreover, $\mathcal{L}$ is well defined as a continuous operator on both $\mathcal{B}$ and $\mathcal{B}_w$.*

Proof. The continuity of the embeddings follows from the following three estimates, respectively: $\|h\|_B \le C|h|_{\mathcal{C}^\lambda(M)}$ by (3.23) in the proof of Lemma 3.5, $|\cdot|_w \le \|\cdot\|_{\mathcal{B}}$ by definition of the norms, and $|h(\psi)| \le C|h|_w |\psi|_{\mathcal{C}^p(T^{-n}\mathcal{W}^s)}$ from Lemma 3.7.

The injectivity of the first embedding is immediate while that of the second follows from the fact that our test functions for $\|\cdot\|_s$ are in $\mathcal{C}^q(M)$ rather than $\tilde{\mathcal{C}}^q(M)$. The injectivity of the third embedding follows from Lemma 3.8 since $(\mathcal{C}^p(T^{-n}\mathcal{W}^s)' \subset (\mathcal{C}^p(M))'$ for each $n \ge 0$.

By Lemma 3.6, if $h \in \mathcal{C}^1(M)$, then $\mathcal{L}h \in \mathcal{B}$. Indeed, the estimates of Section 4 prove that $\|\mathcal{L}h\|_{\mathcal{B}} \le C\|h\|_{\mathcal{B}}$ for $h \in \mathcal{C}^1(M)$. Now identify $g \in \mathcal{B}$ with a Cauchy sequence $\{h_n\}_{n \ge 0} \subset \mathcal{C}^1(M)$. Since $\mathcal{L}$ is bounded when applied to functions in $\mathcal{C}^1(M)$, it follows that $\{\mathcal{L}h_n\}_{n \ge 0}$ is a Cauchy sequence in $\mathcal{B}$. By the injectivity of the inclusion $\mathcal{B} \hookrightarrow (\mathcal{C}^p(T^{-n}\mathcal{W}^s))'$, $n \ge 0$, we identify its limit with $\mathcal{L}g$ and so $\|\mathcal{L}g\|_{\mathcal{B}} = \lim_n \|\mathcal{L}h_n\|_{\mathcal{B}} \le \lim_n C\|h_n\|_{\mathcal{B}} = C\|g\|_{\mathcal{B}}$. Thus $\mathcal{L}$ is bounded and therefore continuous on $\mathcal{B}$. A similar argument holds for $\mathcal{B}_w$. □

The following proposition is proved in Section 4.

Proposition 2.3. *Let $\Lambda > 1$ be the minimum expansion from* **(H2)***(2), let $\kappa \le 1$ be from* **(H1)***, and let $\delta_0 > 0$, $\theta_* < 1$ be constants defined by* (2.6)*. There exists $C > 0$ such that for all $h \in \mathcal{B}$ and $n \ge 0$,*

$$|\mathcal{L}^n h|_w \le C\kappa^{-n}|h|_w \,, \tag{2.11}$$

$$\|\mathcal{L}^n h\|_s \le C(\theta_*^{n/s_0} + \Lambda^{-qn})\kappa^{-n}\|h\|_s + C\delta_0^{\gamma-\alpha}\kappa^{-n}|h|_w \,, \tag{2.12}$$

$$\|\mathcal{L}^n h\|_u \le C\Lambda^{-\beta n}\kappa^{-n}\|h\|_u + Cn\kappa^{-n}\|h\|_s \,, \tag{2.13}$$

where $s_0 = \frac{1-\gamma_0}{1-\alpha}$.

We now state the restriction on η_0 referred to in **(H1)**. We take $\eta_0 > 0$ to be sufficiently small that $1 - \eta_0 > \max\{\Lambda^{-\beta}, \theta_*^{1/s_0}, \Lambda^{-q}\}$.

Then since $\kappa \ge 1 - \eta_0$, we may choose $1 > \sigma_0 > \kappa^{-1}\max\{\Lambda^{-\beta}, \theta_*^{1/s_0}, \Lambda^{-q}\}$ and there exists $N \ge 0$ such that

$$\begin{aligned}\|\mathcal{L}^N h\|_{\mathcal{B}} = \|\mathcal{L}^N h\|_s + c_u\|\mathcal{L}^N h\|_u &\le \frac{\sigma_0^N}{2}\|h\|_s + C\delta_0^{\gamma-\alpha}\kappa^{-N}|h|_w + c_u\sigma_0^N\|h\|_u + c_uCN\kappa^{-N}\|h\|_s \\ &\le \sigma_0^N\|h\|_{\mathcal{B}} + C\delta_0^{\gamma-\alpha}\kappa^{-N}|h|_w\end{aligned} \tag{2.14}$$

provided c_u is chosen small enough with respect to N. The above represents the traditional Lasota-Yorke inequality.

The final ingredient in the strategy to prove the quasi-compactness of the operator $\mathcal{L}$ is the relative compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$. This is proven in Lemma 3.9. It thus follows by standard arguments (see [B1, HH]) that the essential spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is bounded by σ_0, while the spectral radius is at most κ^{-1}.

Despite this, we prove in Section 5 that the spectral radius is in fact 1, along with the following theorem which characterizes the spectral properties of $\mathcal{L}$ and their consequences for the statistical properties of T. Let Π_θ denote the eigenprojector onto $\mathbb{V}_\theta$, the eigenspace of $\mathcal{L}$ in $\mathcal{B}$ corresponding to the eigenvalue $e^{2\pi i\theta}$.

Theorem 2.4. *The spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is 1 while its essential spectral radius is bounded by $\sigma_0 < 1$. The peripheral spectrum of $\mathcal{L}$ on $\mathcal{B}$ consists of finitely many cyclic groups with no Jordan blocks. The maps $\{T^n\}_{n\in\mathbb{N}}$ admit only finitely many physical measures*[9]*, they form a basis for $\mathbb{V} := \oplus_\theta \mathbb{V}_\theta$ and the cycles correspond to the cyclic groups. Moreover,*

1. *Each element of $\mathbb{V}$ is a signed measure absolutely continuous with respect to the probability measure $\overline{\mu} := \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathcal{L}^i m$. In particular, all the physical measures are absolutely continuous with respect to $\overline{\mu}$.*
2. *Let $\mathcal{S}_0^{\mathbb{H}} = \mathcal{S}_0 \cup (\cup_{k\ge k_0} S_k^H)$ and $\mathcal{S}_{\pm n}^{\mathbb{H}} = \cup_{i=0}^n T^{\mp i}(\mathcal{S}_0^{\mathbb{H}})$. If $\mu \in \mathbb{V}$ and $\mathcal{S}_{-1,\varepsilon}^{\mathbb{H}}$ is an ε-neighborhood of $\mathcal{S}_{-1}^{\mathbb{H}}$, then for each $\varepsilon > 0$, $\mu(\mathcal{S}_{-1,\varepsilon}^{\mathbb{H}}) \le C\varepsilon^{\xi t_0(\alpha-\gamma)}$ for some uniform constant C. In particular, $\mu(\cup_{n\in\mathbb{Z}}T^n(\mathcal{S}_{-1,\varepsilon n^{-2/\xi t_0(\alpha-\gamma)}}^{\mathbb{H}})) \le C\varepsilon^{\xi t_0(\alpha-\gamma)}$ and $\mu(\mathcal{S}_{\pm n}) = 0$.*
3. *The supports of the physical measures correspond to the ergodic decomposition with respect to Lebesgue.*
4. *If $(T, \overline{\mu})$ is ergodic, then 1 is a simple eigenvalue.*

The next three items all assume that $(T^n, \overline{\mu})$ is ergodic for all $n \ge 1$.

5. *If $(T^n, \overline{\mu})$ is ergodic for all n, then 1 is the only eigenvalue of modulus one and $\mathcal{L}$ enjoys a spectral gap. For any probability measure $\nu \in \mathcal{B}$, we have $\lim_{n\to\infty}\|\mathcal{L}^n\nu - \overline{\mu}\|_{\mathcal{B}} = 0$, and the convergence is at an exponential rate.*

[9] An ergodic, invariant probability measure μ is called a physical measure if there exists a positive Lebesgue measure set B_μ, with $\mu(B_\mu) = 1$, such that $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) = \mu(f)$ for all $x \in B_\mu$ and all continuous functions f.

(6) *Let $\lambda > \max\{p, \beta/(1-\beta)\}$, where p and β are from Section 2.3, and suppose $\mathcal{P}$ is a partition of M satisfying the assumptions of Lemma 5.3. If ϕ is a function satisfying $\sup_{P\in\mathcal{P}} |\phi|_{\mathcal{C}^\lambda(P)} < \infty$ and $\psi \in \mathcal{C}^p(T^{-k}\mathcal{W}^s)$ for some $k \geq 0$, then*

$$\Big| \int_M \phi\psi \circ T^n d\overline{\mu} - \int_M \phi d\overline{\mu} \int_M \psi d\overline{\mu} \Big| \leq C\sigma_1^n \big(\sup_{P\in\mathcal{P}} |\phi|_{\mathcal{C}^\lambda(P)} \big) (|\psi|_\infty + H_k^p(\psi))$$

for some $\sigma_1 < 1$ and all $n \geq 0$.

(7) *The Fourier transform of the correlation function (sometimes called the power spectrum) admits a meromorphic extension in the annulus $\{z \in \mathbb{C} \; ; \; \sigma_0 < |z| < \sigma_0^{-1}\}$ and the poles (Ruelle resonances) correspond exactly to the eigenvalues of $\mathcal{L}$, where $\sigma_0 < 1$ is from (2.14).*

When T has a spectral gap, the following limit theorems (among others) follow by standard methods. For a function g on M, define $S_n g = \sum_{k=0}^{n-1} g \circ T^k$.

Corollary 2.5. *Assume T has a spectral gap. As in Theorem 2.4, let $\lambda > \max\{p, \beta/(1-\beta)\}$ and suppose $\mathcal{P}$ is a partition of M satisfying the assumptions of Lemma 5.3.*

(a) *(Local large deviation estimate.) Let g satisfy $\sup_{P\in\mathcal{P}} |g|_{\mathcal{C}^\lambda(P)} < \infty$. For any (not necessarily invariant) probability measure $\nu \in \mathcal{B}$,*

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log \nu\Big(x \in M : \frac{1}{n} S_n g(x) \in [t-\varepsilon, t+\varepsilon] \Big) = -I(t)$$

where the rate function $I(t)$ is independent of $\nu \in \mathcal{B}$, and t is in a neighborhood of the mean $\overline{\mu}(g)$.

(b) *(Vector-valued almost-sure invariance principle.) Suppose $g : M \to \mathbb{R}^d$ is an $\mathbb{R}^d$-valued observable with $\overline{\mu}(g) = 0$ and such that $\sup_{P\in\mathcal{P}} |g_i|_{\mathcal{C}^\lambda(P)} < \infty$ for each of its component functions g_i, $i = 1, \dots, d$. Distribute $(g \circ T^j)_{j\in\mathbb{N}}$ according to a probability measure $\nu \in \mathcal{B}$. Then there exists a probability space Ω with random variables $\{X_n\}$ satisfying $S_n g \overset{dist.}{=} X_n$, and a Brownian motion W with mean 0 and covariance matrix Σ^2 such that*

$$X_n = W(n) + o(n^r) \qquad \textit{for any } r > 1/4 \textit{ almost-surely in } \Omega.$$

Theorem 2.4 and Corollary 2.5 are proved in Section 5.

2.4.1. *Application to Dispersing Billiards with Corner Points.* We apply our abstract framework to dispersing billiards with corner points. Let $Q \subset \mathbb{R}^2$ be a compact region whose boundary consists of finitely many $\mathcal{C}^3$ curves positioned so that they are convex inward to Q with strictly positive curvature. We assume the interior of Q is connected, but not necessarily simply connected. Thus the boundary of Q comprises a finite number b_0 of connected components, Γ_i and each Γ_i consists of a finite number of smooth curves as described above. The intersections of the smooth curves comprising ∂Q are called corner points and we assume that all such intersections are transverse, i.e. the angle at each corner point is positive.[10]

We consider the billiard flow on the table Q induced by a particle traveling at unit speed and undergoing elastic collisions at the boundaries. The phase space for the billiard flow is $\mathcal{M} = Q\times\mathbb{S}^1/\sim$ with the conventional identifications at the boundaries. Define $M = \cup_{i=1}^{b_0}(\Gamma_i\times[-\pi/2, \pi/2])$ to be a union of cylinders. The billiard map $\mathcal{F} : M \to M$ is the Poincaré map corresponding to collisions with the scatterers. We will denote coordinates on M by (r, φ), where $r \in \Gamma_i$ is parametrized by arclength (oriented according to convention so that Q is always on the left when traversing ∂Q in the positive direction) and φ is the angle that the velocity vector at r makes with the normal pointing into the domain Q just after the moment of collision. $\mathcal{F}$ preserves a measure μ_{SRB} defined by $d\mu_{SRB} = c \cos\varphi \, dr \, d\varphi$ on M, where c is the normalizing constant.

[10] In the presence of cusps (corner points whose angle is zero), it was proved in [CM2, CZ3] that such billiards have only polynomial decay of correlations.

Chernov [Ch1] proved exponential decay of correlations for billiards with corner points under an additional complexity assumption, which now can be removed due to the recent advance contained in [DT]. Here we apply our approach based on the spectral analysis of the transfer operator by establishing **(H1)**-**(H5)** for an iterate of the map $\mathcal{F}$. In addition to exponential decay of correlations, the present method also implies a wide variety of other limit theorems, including those given by Corollary 2.5. Such limit theorems with respect to the smooth invariant measure were already proved using Young towers ([MN1, MN2, RY]) but the extension to non-invariant (and even singular) measures is new for this class of maps. The following theorem is proved in Section 6.

Theorem 2.6. *Under the assumptions above, there exists $n_1 \in \mathbb{N}$ such that $T := \mathcal{F}^{n_1}$ satisfies properties* **(H1)**-**(H5)***. In terms of the quantities introduced there: in* **(H1)***, $f = \cos\varphi$, $f_0 \equiv \kappa = 1$; in* **(H3)***, $\xi = (3/5)^{n_1}$, $t_0 = 1/2$, and $r_h = 3$; in* **(H4)***, $p_0 = 1/3$; in* **(H5)***, $\gamma_0 = 0$.*

Fixing the choice of constants in the norms according to Section 2.3 defines a Banach space $\mathcal{B}$ on which $\mathcal{L}_{\mathcal{F}}$ is quasi-compact and enjoys a spectral gap. Thus all the items of Theorem 2.4 and Corollary 2.5 apply to the billiard map $\mathcal{F}$.

2.4.2. *Application to Certain Billiards with Focusing Boundaries.* Next we consider two specific classes of billiards that were studied in [CZ4]. The first is a non-smooth stadium, which is a convex domain Q bounded by two parallel straight segments and two minor circular arcs (i.e., arcs smaller than a semicircle) with radii $\mathbf{r}_1 \le \mathbf{r}_2$. We assume that Q satisfies the standard Bunimovich assumptions [Bu], i.e. the complement of each arc in ∂Q to a full circle crosses both straight segments in ∂Q, but does not cross the other arc. We will also need a complexity assumption, which is easily satisfied for for certain choices of the geometric parameters for this type of stadium; this is formulated precisely in (7.8) of Section 7.1.

We present the application to non-smooth stadia rather than the standard smooth stadium in order to demonstrate the wider applicability of our weakened one-step expansion condition **(H5)**. It is known that **(H5)** is satisfied for the smooth stadium with $\gamma_0 = 0$ (the traditional one-step expansion), while for non-smooth stadia, it fails. Thus we need to choose $\gamma_0 > 0$ in **(H5)** [CZ4].

Our second class of billiards corresponds to Bunimovich tables [Bu, CM1] whose focusing boundaries contain major arcs (i.e. arcs greater than a semicircle). Such arcs add a new type of 'bad spot' where the hyperbolicity is weak due to nearly diametrical reflections, see [CZ1]. For simplicity, we assume that the major arcs are less than 240°, to prevent even further technical complications. Also we assume that the boundary components are either focusing or dispersing, and that they intersect each other transversally (do not make cusps). Finally, we assume that every focusing component Γ_i is an arc of a circle such that there are no points of ∂Q on that circle or inside it, other than the arc Γ_i itself; this is known as Bunimovich's Defocusing Condition. Finally, we formulate the required complexity assumption on the billiard table as (7.9) and (7.10) of Section 7.2.

For both types of billiards, we set $M = \cup_i \Gamma_i \times [-\pi/2, \pi/2]$, where Γ_i denote the smooth components of ∂Q, and let $\mathcal{F} : M \to M$ denote the collision map as in Section 2.4.1. We adopt the same canonical coordinates (r, φ) as in Section 2.4.1 and $\mathcal{F}$ preserves the same smooth measure μ_{SRB}. Under our assumptions, in each case the billiard dynamics is hyperbolic, ergodic, and mixing.

For billiards with focusing boundary components, the hyperbolicity may be weak during long series of successive reflections along certain trajectories. To study the mixing rates, one needs to find and remove the spots in the phase space where expansion (contraction) slows down. Such spots come in several types and are easy to identify, for example, see [CZ1] and [CM1, Chapter 8]. Traditionally, the collision space can be naturally divided into focusing, dispersing and neutral parts:

$$\mathcal{M}_0 = \{(r, \varphi) \in M \,:\, r \in \partial^0 Q\}, \qquad \mathcal{M}_\pm = \{(r, \varphi) \in M \,:\, r \in \partial^\pm Q\},$$

where $\partial^0 Q$ is the union of flat boundaries, $\partial^- Q$ contains focusing boundaries and $\partial^+ Q$ corresponds to dispersing boundaries. Let

$$\bar{M} = \{x \in \mathcal{M}_- :\; \pi(x) \in \Gamma_i,\, \pi(\mathcal{F}x) \in \Gamma_j,\, j \neq i\} \cup \mathcal{M}_+, \tag{2.15}$$

where $\pi(x)$ denotes the projection onto the position coordinate. Note that $\bar{M}$ contains only the *last* collisions with each focusing arc. The reduced map $T : \bar{M} \to \bar{M}$ is the first return map to $\bar{M}$ and preserves the measure μ_{SRB} conditioned on $\bar{M}$, which we denote by $\mu_0 = [\mu_{SRB}(\bar{M})]^{-1}\mu_{SRB}$. Furthermore, T has uniform expansion and contraction, since we omit all collisions too close to the 'bad spots' in the collision space; however, T has a larger singularity set than the original map.

We remark that in [CM1, CZ1], $\bar{M}$ is defined to contain only the first collision with each focusing arc rather the last collision that we have chosen here. We make this choice for $\bar{M}$ since we are interested in the propagation of stable curves under T^{-1}. Thus by symmetry, the properties for unstable curves mapped forward in the first entry space defined in [CM1, CZ1] will hold for stable curves mapped backward in the last exit space we define here. Indeed, our definition of $\bar{M}$ coincides with that used in [BSC2, Ma].

To characterize the mixing rates for the original billiard maps, it is essential to prove the reduced system $(T, \bar{M}, \mu_0)$ enjoys exponential decay of correlations. Chernov and Zhang [CZ4] proved exponential decay of correlations for these reduced systems under the same assumptions as above. Here we use our approach based on the spectral analysis of the transfer operator permitted by establishing **(H1)**-**(H5)** for the map T. As before, this method also allows us to apply the limit theorems of Corollary 2.5 to the reduced map T. Note that because the original map $\mathcal{F}$ has polynomial decay of correlations, the spectral gap for $\mathcal{L}_T$ does not imply a spectral gap for $\mathcal{L}_{\mathcal{F}}$. This is in contrast to the case of dispersing billiards with corner points described in Theorem 2.6 in which we are able to obtain a spectral gap for $\mathcal{L}_{\mathcal{F}}$. The following theorem is proved in Section 7.

Theorem 2.7. *For the two types of reduced systems $(T, \bar{M}, \mu_0)$ described above, there exists $n_1 \in \mathbb{N}$ such that the map $T_1 = T^{n_1}$ satisfies properties* **(H1)**-**(H5)**. *In terms of the quantities introduced there, in* **(H1)**, *$f = \cos\varphi$, $f_0 = \kappa = 1$; in* **(H3)**, *$\xi = (\frac{1}{2})^{n_1}$, $t_0 = 1$, and $r_h = 3$; in* **(H4)**, *$p_0 = 1/3$; in* **(H5)**, *γ_0 can be taken to be any number in $(0, 1/3)$, but for definiteness we choose $\gamma_0 = 1/4$.*

Fixing the choice of constants in the norms according to Section 2.3 defines a Banach space $\mathcal{B}$ on which $\mathcal{L}_T$ is quasi-compact and enjoys a spectral gap. Thus all the items of Theorem 2.4 and Corollary 2.5 apply to T.

3. Preliminary Estimates and Properties of the Banach Spaces

3.1. **Representation of Admissible Stable Curves via Charts.** Recall that $\mathcal{S}_0^{\mathbb{H}} = \mathcal{S}_0 \cup (\cup_{k\geq k_0} S_k^H)$. On each connected component of $M \setminus \mathcal{S}_0^{\mathbb{H}}$, by **(H2)** we may choose a finite number of coordinate charts $\{\chi_j\}_{j=1}^K$, whose domains R_j depend on whether they contain part of a curve in $\mathcal{S}_0^{\mathbb{H}}$.

If χ_j maps only to the interior of $M \setminus \mathcal{S}_0^{\mathbb{H}}$, then $R_j = (-r_j, r_j)^2$. If χ_j maps to a part of the boundary of $\mathcal{S}_0^{\mathbb{H}}$, then we take R_j to be $(-r_j, r_j)^2$ restricted to one side of a $\mathcal{C}^1$ curve (the preimage of the boundary curve or singularity) which we position so that it passes through the origin in R_j. On the other hand, if the image of χ_j contains a point of intersection of two boundary curves, we place this intersection point at the origin and consider R_j to be $(-r_j, r_j)^2$ intersected with one of the sectors created by the intersection (we use a separate chart for each sector). Finally, in homogeneity strips of high index, charts will have two nonintersecting smooth curves which map to part of the boundary of $\mathbb{H}_k$. In these cases, the domain of the chart will be the usual square intersected with the region between these two curves.

Let $E^s(x)$ and $E^u(x)$ denote the stable and unstable subspaces at x respectively. We denote by y_j the centroid of R_j and construct each χ_j to satisfy,

(a) $D\chi_j(y_j)$ is an isometry and the $\mathcal{C}^2$ norms of χ_j and χ_j^{-1} are bounded by a constant $C_c > 0$ on R_j.

(b) $D\chi_j(y_j) \cdot (\mathbb{R} \times \{0\}) = E^s(\chi_j(y_j))$ and $D\chi_j(y_j) \cdot (\{0\} \times \mathbb{R}) = E^u(\chi_j(y_j))$.

(c) There exists $a_j < 1$ such that the cone $C^s_j = \{u + v \in \mathbb{R}^2 : u \in \mathbb{R} \times \{0\}, v \in \{0\} \times \mathbb{R}, \|v\| \le a_j \|u\|\}$ has the following property: For $x \in R_j$ such that $\chi_j(x) \notin \mathcal{S}_0^{\mathbb{H}}$, $D\chi_j(x) C^s_j \supseteq C^s(\chi_j(x))$. Similarly, there exists an unstable cone in the chart, containing the vertical direction, and enjoying the analogous property with respect to $C^u(x)$.

(d) $M \setminus \mathcal{S}_0^{\mathbb{H}}$ is covered by the sets $\{\chi_j(R_j \cap (-\frac{r_j}{2}, \frac{r_j}{2})^2)\}_j$.

Note that although this collection of charts is finite on each component of $M \setminus \mathcal{S}_0^{\mathbb{H}}$, it forms a countable cover of $M \setminus \mathcal{S}_0^{\mathbb{H}}$. Also, these charts do not take into account cuts necessitated by $\mathcal{S}_{-1}$ since we use them only to represent curves in $\mathcal{W}^s$ and $\mathcal{W}^u$ and not to iterate the dynamics. When we do iterate the dynamics, we must use smaller charts and it is a consequence of Lemma 3.2 (graph transform argument) that for k large enough, on each component of $\mathbb{H}_k$ these smaller charts can be chosen large enough to cross $\mathbb{H}_k$ completely in the direction of the stable cone.

Let $r_0 = \min_{1 \le j \le K} r_j > 0$ and $a_0 = \max_{1 \le j \le K} a_j < 1$. Fix $B < \infty$ and consider the set of functions

$$\Xi := \{F \in \mathcal{C}^2([-r_0, r_0], \mathbb{R}) : F(0) = 0, |F|_{\mathcal{C}^1} \le a_0, |F|_{\mathcal{C}^2} \le B\}.$$

Assumption **(H4)** implies that we may realize elements of $\mathcal{W}^s$ as graphs of functions in Ξ as follows. Let $I_r = (-r, r)$, $r \le r_0$. For $x \in R_j \cap (-\frac{r_j}{2}, \frac{r_j}{2})^2$ such that $x + (t, F(t)) \in R_j$ for $t \in I_r$, we define $G(x, r, F)(t) = \chi_j(x + (t, F(t)))$, $t \in I_r$, to be a lift of the graph of F to M. For brevity, we often write G_F for $G(x, r, F)$. Note that $\mathrm{Lip}(G_F) \le C_c(1 + a_j)$ and $\mathrm{Lip}(G_F^{-1}) \le C_c$, where $\mathrm{Lip}(\cdot)$ denotes the Lipschitz constant of a function on I_r. Then each $W \in \mathcal{W}^s$ can be written as $W = G(x, r, F)(I_r)$ for an appropriate choice of x, r and F. If necessary, we shrink r_0 further so that $\sup_{W \in \mathcal{W}^s} |W| \le \delta_0$, where δ_0 is chosen in (2.6). Note that although r_0 is fixed on each component of $M \setminus \mathcal{S}_0^{\mathbb{H}}$, it is not uniform on M.

Let $W_j = W_j(\chi_{i_j}, x_j, r_j, F_j) \in \mathcal{W}^s$, $j = 1, 2$, be two stable curves and let $\mathbb{H}_{k_j}$ be the homogeneity strip containing W_j. We define the distance between W_1 and W_2 to be,

$$d_{\mathcal{W}^s}(W_1, W_2) = \eta(k_1, k_2) + \eta(i_1, i_2) + |r_2 - r_1| + |x_1 - x_2| + |F_1 - F_2|_{\mathcal{C}^1(I_{r_1} \cap I_{r_2})},$$

where $\eta(A, B) = 0$ if $A = B$ and $\eta(A, B) = \infty$ otherwise, i.e., we only compare curves which lie in the same homogeneity region and are mapped under the same chart.

Given two functions $\psi_i \in \mathcal{C}^q(W_i, \mathbb{C})$, we define the distance between ψ_1, ψ_2 as

$$d_q(\psi_1, \psi_2) = |\psi_1 \circ G_{F_1} - \psi_2 \circ G_{F_2}|_{\mathcal{C}^q(I_1 \cap I_2)}.$$

3.2. **Distortion Bounds.**

In this section, we derive several distortion bounds which we shall use throughout the paper. The statements are quite standard for hyperbolic maps and follow from assumptions **(H2) - (H4)**.

Lemma 3.1. *There exists $C_d > 0$ such that for any stable curve $W \in \mathcal{W}^s$, with $T^i W \in \mathcal{W}^s$ for $i = 0, 1, \ldots, n$, and any $x, y \in W$,*

$$\left| \frac{J_W T^n(x)}{J_W T^n(y)} - 1 \right| \le C_d d_W(x, y)^{p_0}, \tag{3.1}$$

where $J_W T(x) = |\det(D_x T | \mathcal{T}_x W)|$ denotes the Jacobian of T along W and $d_W(\cdot, \cdot)$ is the arclength distance on W.

If $T^i W$ is a homogeneous stable curve for $0 \le i \le n$, or if $T^i W$ is a homogeneous unstable curve for $0 \le i \le n$, then for any $x, y \in W$,

$$\left| \frac{|D_x T^n|}{|D_y T^n|} - 1 \right| \le C_d \max\{d_W(x, y)^{p_0}, d_W(T^n x, T^n y)^{p_0}\}. \tag{3.2}$$

Proof. First we prove (3.1). Suppose T^iW is a stable curve for $i = 0, \dots n$. It is equivalent to estimate,

$$\log \frac{J_{T^nW}T^{-n}(T^nx)}{J_{T^nW}T^n(T^ny)} \leq \sum_{i=1}^{n} \frac{1}{A_i} |J_{T^iW}T^{-1}(T^ix) - J_{T^iW}T^{-1}(T^iy)|, \tag{3.3}$$

where $A_i = \min\{J_{T^iW}T^{-1}(T^ix), J_{T^iW}T^{-1}(T^iy)\}$.

We estimate the differences one term at a time and assume without loss of generality that the minimum for A_i is attained at T^ix. Set $x_i = T^ix$, $y_i = T^iy$. Let $\vec{u}_1(x_i)$ denote the unit tangent vector to T^iW at x_i and notice that $J_{T^iW}T^{-1}(x_i) = \|D_{x_i}T^{-1}\vec{u}_1\|$. Define $\vec{u}_2(y_i)$ similarly. Then using (2.3) of **(H4)**,

$$\begin{aligned} &|\,\|D_{x_i}T^{-1}\vec{u}_1\| - \|D_{x_i}T^{-1}\vec{u}_2\|\,| \\ &\qquad \leq |\,\|D_{x_i}T^{-1}\vec{u}_1\| - \|D_{x_i}T^{-1}\vec{u}_2\|| + ||\|D_{x_i}T^{-1}\vec{u}_2\| - \|D_{y_i}T^{-1}\vec{u}_2\|\,| \\ &\qquad \leq \|D_{x_i}T^{-1}\| \left(\|\vec{u}_1 - \vec{u}_2\| + C_d d_W(x_{i-1}, y_{i-1})^{p_0}\right). \end{aligned}$$

Now since T^iW has bounded curvature, we have $\|\vec{u}_1 - \vec{u}_2\| \leq C d_W(x_i, y_i) \leq C d_W(x_{i-1}, y_{i-1})$, where in the last inequality we have used the fact that T^iW is expanded under T^{-1} from **(H2)**. Finally, note that $\|D_{x_i}T^{-1}\|/\|D_{x_i}T^{-1}\vec{u}\| \leq C$ where C is some uniform constant for all unit vectors $\vec{u} \in C^s(x_i)$. Putting these estimates together with (3.3), we obtain the required distortion bound,

$$\log \frac{J_{T^nW}T^{-n}(T^nx)}{J_{T^nW}T^n(T^ny)} \leq \sum_{i=1}^{n} C d_W(x_{i-1}, y_{i-1})^{p_0} \leq \sum_{i=1}^{n} C\Lambda^{-p_0(i-1)} d_W(x, y)^{p_0}.$$

The proof of (3.2) follows similarly from (2.4) and is omitted. □

Next we prove a distortion bound for the stable Jacobian of T along different stable curves as well as the exponential contraction of those curves in the following context. Let $W^1, W^2 \in \mathcal{W}^s$ and suppose there exist $U^j \subset T^{-n}W^j$, $j = 1, 2$, such that for $0 \leq i \leq n$,

(i) $T^iU^j \in \mathcal{W}^s$ and the curves T^iU^1 and T^iU^2 lie in the same homogeneity strip;
(ii) U^1 and U^2 can be put into a 1-1 correspondence by a smooth foliation $\{\gamma_x\}_{x \in U^1}$ of curves $\gamma_x \in \mathcal{W}^u$ such that $\{T^n\gamma_x\} \subset \mathcal{W}^u$ creates a 1-1 correspondence between T^nU^1 and T^nU^2;
(iii) $|T^i\gamma_x| \leq 2\max\{|T^iU^1|, |T^iU^2|\}$, for all $x \in U^1$.

Let $J_{U^k}T^n$ denote the stable Jacobian of T^n along the curve U^k with respect to arclength.

Lemma 3.2. *Assume (i)-(iii) above, and for $x \in U^1$, define $x^* = \gamma_x \cap U^2$. There exists $C_* > 0$, independent of $W^1, W^2 \in \mathcal{W}^s$, such that for all $n \geq 0$,*

(a) $d_{\mathcal{W}^s}(U^1, U^2) \leq C_*\Lambda^{-n} d_{\mathcal{W}^s}(W^1, W^2)$;
(b) $\left|\dfrac{J_{U^1}T^n(x)}{J_{U^2}T^n(x^*)} - 1\right| \leq C_*[d(T^nx, T^nx^*)^{p_0} + \theta(T^nx, T^nx^*)]$,

where $\theta(T^nx, T^nx^*)$ is the angle formed by the tangent lines of T^nU^1 and T^nU_2 at T^nx and T^nx^*, respectively.

Proof. (a) This is essentially a graph transform argument adapted for this class of maps satisfying **(H2) - (H4)**. What we need to show here is that we do not need to cut curves lying in homogeneity strips any further in order to get the required contraction and control on distortion. The assumptions of the lemma imply that T^iU^1 and T^iU^2 can be viewed as lying in a single chart for each iterate, $0 \leq i \leq n$. The purpose of this lemma is to show that locally DT is comparable along T^iU^1 and T^iU^2. Note that by assumption (i) before the statement of the lemma, the curves we work with always lie in the stable cones of the relevant charts.

Due to the uniform expansion of γ_x under T^n given by **(H2)**, we have

$$|\gamma_x| \leq C_e C_t \Lambda^{-n} d_{\mathcal{W}^s}(W^1, W^2),$$

where where C_e is a uniform constant relating the Euclidean norm to the adapted norm $\|\cdot\|_*$, and C_t is a constant depending only on the maximum angular diameter of $C^u(x)$ (which must be less than $\pi/2$ by definition of the charts).

In the notation of Section 3.1, we write $U^j = G_j(x_j, r_j, F_j)(I_{r_j})$, $j = 1, 2$. By the uniform transversality of $C^s(x)$ with $C^u(x)$ as well as the smoothness of the charts χ_i, there exists a constant C such that $|r_1 - r_2| + |x_1 - x_2| \le C|\gamma_x| \le C'\Lambda^{-n} d_{\mathcal{W}^s}(W^1, W^2)$, where for the last inequality, we have used the previous paragraph.

Letting $I = I_{r_1} \cap I_{r_2}$ and recalling the definition of $d_{\mathcal{W}^s}(\cdot,\cdot)$ from Section 3.1, it remains to estimate $|F_1 - F_2|_{\mathcal{C}^1(I)}$. Using again the estimate on $|\gamma_x|$ together with the maximum angular diameter of the unstable cone, we have $|F_1 - F_2|_{\mathcal{C}^0(I)} \le C\Lambda^{-n} d_{W^s}(W^1, W^2)$. In order to show that the slopes of these curves also contract exponentially, we make the usual graph transform argument using charts in the adapted norm $\|\cdot\|_*$ from **(H2)**.

Fix $x \in U^1$ and define charts along the orbit of x so that $x_i := T^i x$, $0 \le i \le n$, corresponds to the origin in each chart with the stable direction at x_i given by the horizontal axis and the unstable direction by the vertical axis in the charts. Let $\vartheta < 1$ denote the maximum absolute value of slopes of stable curves in the chart. Due to property (iii) before the statement of the lemma, we may choose the size of the charts to have stable and unstable diameters $\le C|T^i U_1|$ for each i, for some uniform constant C. The dynamics induced by T^{-1} on these charts is defined by

$$\tilde{T}_{x_i}^{-1} = \chi_{i-1}^{-1} \circ T^{-1} \circ \chi_i,$$

where the domain of the charts χ_i are possibly much smaller than those defined in Section 3.1 since these charts must avoid singularity curves $\mathcal{S}_{-1}$. Nevertheless, it holds that the charts can be chosen such that $|\chi_{x_i}|_{\mathcal{C}^2}, |\chi_{x_i}^{-1}|_{\mathcal{C}^2} \le C$ for some uniform constant C.

Note that $D\tilde{T}_{x_i}^{-1}$ satisfies **(H4)** with possibly larger constant $C_3 > 0$. In the chart coordinates, since $\tilde{T}_{x_i}^{-1}(0) = 0$, we have

$$\tilde{T}_{x_i}^{-1}(s,t) = (A_i s + \alpha_i(s,t), B_i t + \beta_i(s,t)),$$

where A_i is the expansion at x_i in the stable direction and B_i is the contraction at x_i in the unstable direction given by $D\tilde{T}_{x_i}^{-1}(0)$. The nonlinear functions α_i, β_i satisfy $\alpha_i(0,0) = \beta_i(0,0) = 0$ and their Lipschitz constants $\mathrm{Lip}(\cdot)$ are bounded by $\mathrm{Lip}(\tilde{T}_{x_i}^{-1} - D\tilde{T}_{x_i}^{-1}(0))$, which we estimate using (2.3) of **(H4)** as the maximum of

$$\|D\tilde{T}_{x_i}^{-1}(u) - D\tilde{T}_{x_i}^{-1}(0)\| \le C\|D\tilde{T}_{x_i}^{-1}(0)\| \max\{\|u\|^{p_0}, \|\tilde{T}_{x_i}^{-1}(u)\|^{p_0}, \tag{3.4}$$

where u ranges over the chart at x_i.

We fix i and let g_1, g_2 denote two Lipschitz functions whose graphs lie in the stable cone of the chart at x_i and satisfy $g_j(0) = 0$, $j = 1, 2$. Define $L(g_1, g_2) = \sup_{s \neq 0} \frac{|g_1(s) - g_2(s)|}{|s|}$. Let $\tilde{g}_1 = \tilde{T}_*^{-1} g_1$ and $\tilde{g}_2 = \tilde{T}_*^{-1} g_2$ denote the graphs of the images of these two curves in the chart at x_{i-1} and suppose that $\tilde{g}_1$, $\tilde{g}_2$ lie in the stable cone at x_{i-1}. We wish to estimate $L(\tilde{g}_1, \tilde{g}_2)$. For s on the horizontal axis in the chart at x_i, we write,

$$\begin{aligned}
|\tilde{g}_1(A_i s + \alpha_i(s, g_1(s))) - \tilde{g}_2(A_i s + \alpha_i(s, g_1(s)))| &\le |\tilde{g}_1(A_i s + \alpha_i(s, g_1(s))) - \tilde{g}_2(A_i s + \alpha_i(s, g_2(s)))| \\
&\quad + |\tilde{g}_2(A_i s + \alpha_i(s, g_2(s))) - \tilde{g}_2(A_i s + \alpha_i(s, g_1(s)))| \\
&\le |B_i||g_1(s) - g_2(s)| + |\beta_i(s, g_1(s)) - \beta_i(s, g_2(s))| + \vartheta|\alpha_i(s, g_1(s)) - \alpha_i(s, g_2(s))| \\
&\le (|B_i| + \mathrm{Lip}(\beta_i) + \vartheta \mathrm{Lip}(\alpha_i))|g_1(s) - g_2(s)|.
\end{aligned}$$

On the other hand,

$$|A_i s + \alpha_i(s, g_1(s))| \ge \big(|A_i| - \mathrm{Lip}(\alpha_i)(1 + \vartheta)\big)|s|.$$

Putting these together, we see that,

$$\begin{aligned} L(\tilde{g}_1, \tilde{g}_2) &\leq \sup_{s \neq 0} \frac{(|B_i| + \mathrm{Lip}(\beta_i) + \vartheta \mathrm{Lip}(\alpha_i))|g_1(s) - g_2(s)|}{(|A_i| - \mathrm{Lip}(\alpha_i)(1+\vartheta))|s|} \\ &\leq \frac{|B_i| + \mathrm{Lip}(\beta_i) + \vartheta \mathrm{Lip}(\alpha_i)}{|A_i| - \mathrm{Lip}(\alpha_i)(1+\vartheta)} L(g_1, g_2). \end{aligned} \tag{3.5}$$

Fix $\varepsilon_1 > 0$. Using **(H3)**(2), there are at most finitely many connected components D of $M \setminus \mathcal{S}_{-1}$ such that the stable diameter (the maximum length of a stable curve) in D is greater than ε_1. Suppose the chart at x_i lies in one of the countably many components with stable diameter less than ε_1. Since the image of the chart under T^{-1} lies in one homogeneity region by assumption, using **(H3)**(1) the length of the images of each of these curves is at most $C_0 \varepsilon_1^{\xi}$. By assumption (iii) before the statement of the lemma, the unstable diameter in both charts is at most of the same order and so we may bound both $\|u\|$ and $\|\tilde{T}_{x_i}^{-1}(u)\|$ in (3.4) by $C_0 \varepsilon_1^{\xi}$. Putting these estimates together with (3.5) yields,

$$L(\tilde{g}_1, \tilde{g}_2) \leq \frac{\Lambda^{-1} + C\|D_{x_i}T^{-1}\| C_0 \varepsilon_1^{\xi p_0}}{\|D_{x_i}T^{-1}\|(1 - CC_0 \varepsilon_1^{\xi p_0})} L(g_1, g_2) \leq \left(\Lambda^{-2} + \mathcal{O}(\varepsilon_1^{\xi p_0})\right) L(g_1, g_2),$$

and the contracting factor can be made smaller than Λ^{-1} for ε_1 small enough. In particular, the contraction is smaller than Λ^{-1} on all curves landing in a homogeneity region $\mathbb{H}_k$ with k sufficiently large by **(H3)**(4).

Thus we may choose $\varepsilon_1 > 0$ such that the contraction is less than Λ^{-1} on all curves lying in components of $M \setminus \mathcal{S}_{-1}$ with stable diameter less than ε_1. On the remainder of M, by **(H4)** the norm and distortion constant of $D_x T^{-1}$ are uniformly bounded by constants depending on ε_1. For curves in this part of M, we choose δ_0, the maximum length of stable curves in $\mathcal{W}^s$, sufficiently small that the distortion given by (3.4) is less than $\frac{1}{2}(\Lambda^{-1/2} - \Lambda^{-1})$. Then by (3.5), since $\vartheta < 1$, the contraction on these pieces is less than Λ^{-1} as well.

Applying these estimates successively along the orbit of x completes the proof of item (a).

(b) It is equivalent to estimate $\log \frac{J_{T^n U_1} T^{-n}(T^n x)}{J_{T^n U_2} T^{-n}(T^n x^*)}$, for $x \in U^1$. Recalling that $x^* = \gamma_x \cap U^2$, we write

$$\log \frac{J_{T^n U_1} T^{-n}(T^n x)}{J_{T^n_U 2} T^{-n}(T^n x^*)} \leq \sum_{i=1}^{n} \frac{1}{A_i} |J_{T^i U_1} T^{-1}(T^i x) - J_{T^i U_2} T^{-1}(T^i x^*)|$$

where $A_i = \min\{J_{T^i U_1} T^{-1}(T^i x), J_{T^i U_2} T^{-1}(T^i x^*)\}$. Following the proof of Lemma 3.1 after (3.3) and using again (2.3), we arrive at the estimate,

$$\log \frac{J_{T^n U_1} T^{-n}(T^n x)}{J_{T^n_U 2} T^{-n}(T^n x^*)} \leq C \sum_{i=1}^{n} \|\vec{u}_1(x_i) - \vec{u}_2(x_i^*)\| + d(x_i, x_i^*)^{p_0},$$

where as before, $x_i = T^i x$ and $x_i^* = T^i x^*$. Now $\|\vec{u}_1(x_i) - \vec{u}_2(x_i^*)\| \leq \theta(x_i, x_i^*) \leq C_0 \Lambda^{i-n} \theta(T^n x, T^n x^*)$ by part (a) of the lemma together with the fact that curves in $\mathcal{W}^s$ have $\mathcal{C}^2$ norm uniformly bounded above. Also, $d(x_i, x_i^*) \leq C_e \Lambda^{i-n} d(T^n x, T^n x^*)$ by **(H2)**(2), which completes the proof of the lemma. □

3.3. **Growth Lemma.** In order to prove the characterization of our Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$ given by Lemma 2.2 as well as the estimates of Proposition 2.3, we need some understanding of the properties of $T^{-n}W$ for $W \in \mathcal{W}^s$. To ensure that each connected component V_i of $T^{-1}W$ is again in $\mathcal{W}^s$, we subdivide any of the long pieces V_i whose length is $> \delta_0$, where δ_0 is from (2.6). This process is then iterated so that given $W \in \mathcal{W}^s$, we construct the components of $T^{-n}W$, which we call the n^{th} generation $\mathcal{G}_n(W)$, inductively as follows.

Let $\mathcal{G}_0(W) = \{W\}$ and suppose we have defined $\mathcal{G}_{n-1}(W) \subset \mathcal{W}^s$. First, for any $W' \in \mathcal{G}_{n-1}(W)$, we partition $T^{-1}W'$ into maximal components W'_i so that T is smooth on each W'_i and each W'_i is a homogeneous stable curve. If any W'_i have length greater than δ_0, we subdivide those pieces into pieces of length between $\delta_0/2$ and δ_0. We define $\mathcal{G}_n(W)$ to be the collection of all pieces $W^n_i \subset T^{-n}W$ obtained in this way. Note that each W^n_i is in $\mathcal{W}^s$ by construction and **(H4)**.

For $W \in \mathcal{W}^s$, $n \geq 0$, and $0 \leq k \leq n$, let $\mathcal{G}_k(W) = \{W^k_i\}$ denote the k^{th} generation pieces in $T^{-k}W$. Let $B_k(W) = \{i : |W^k_i| < \delta_0/3\}$ and $L_k(W) = \{i : |W^k_i| \geq \delta_0/3\}$ denote the index of the short and long elements of $\mathcal{G}_k(W)$, respectively. We consider $\{\mathcal{G}_k(W)\}_{k=0}^n$ as a tree with W as its root and $\mathcal{G}_k(W)$ as the k^{th} level.

We group the pieces in $\mathcal{G}_n(W)$ as follows. Let $W^n_{i_0} \in \mathcal{G}_n(W)$ and let $W^k_j \in L_k(W)$ denote the most recent long "ancestor" of $W^n_{i_0}$, i.e. $k = \max\{0 \leq \ell \leq n : T^{n-\ell}(W^n_{i_0}) \subset W^\ell_j \text{ and } j \in L_\ell(W)\}$. If no such ancestor exists, set $k = 0$ and $W^k_j = W$. Note that if $W^n_{i_0}$ itself is long, then $W^k_j = W^n_{i_0}$. Let

$$\mathcal{I}_n(W^k_j) = \{i : W^k_j \text{ is the most recent long ancestor of } W^n_i\}.$$

The set $\mathcal{I}_n(W)$ represents those curves W^n_i that belong to short pieces in $\mathcal{G}_k(W)$ at each time step $1 \leq k \leq n$, i.e. such W^n_i are never part of a piece that has grown to length $\geq \delta_0/3$.

We prove here a growth lemma essential for controlling the iterates of $\mathcal{L}$.

Lemma 3.3. *Let $W \in \mathcal{W}^s$ and for $n \geq 0$, let $\mathcal{I}_n(W)$ and $\mathcal{G}_n(W)$ be defined as above. For $\gamma_0 \leq \varsigma < 1$, set $s = (1-\gamma_0)/(1-\varsigma)$. There exist constants $C_4, C_5 \geq 1$, independent of W, such that for any $n \geq 0$,*

(a) $\displaystyle\sum_{i \in \mathcal{I}_n(W)} \frac{|W^n_i|^\varsigma}{|W|^\varsigma} |J_{W^n_i} T^n|_{\mathcal{C}^0(W^n_i)} \leq C_4 \theta_*^{n/s}$;

(b) $\displaystyle\sum_{W^n_i \in \mathcal{G}_n(W)} \frac{|W^n_i|^\varsigma}{|W|^\varsigma} \, |J_{W^n_i} T^n|_{\mathcal{C}^0(W^n_i)} \leq C_5 = C_5(\varsigma)$.

Proof. (a) We first prove this by induction on n for $\varsigma = \gamma_0$ in the adapted metric with $C_4 = 1$. The case $n = 1$ follows from assumption **(H5)** since short pieces do not require extra subdivision in the creation of $\mathcal{G}_1(W)$. Now assume (a) holds with $C_4 = 1$ for all times up to $n-1$. Fix $W \in \mathcal{W}^s$ and for $W^{n-1}_j \in \mathcal{I}_{n-1}(W)$, let $A(W^{n-1}_j) = \{i : W^n_i \in \mathcal{I}_n(W), TW^n_i \subseteq W^{n-1}_j\}$.

Note that at each iterate between 1 and n, every $W^n_i \in \mathcal{I}_n(W)$ is created by cuts due to singularities or the boundaries of homogeneity regions and not by any artificial subdivisions since these only occur when a piece has grown to length greater than δ_0. Thus the indices in $A(W^{n-1}_j)$ form a subset of the pieces V_i of $T^{-1}W^{n-1}_j$ referred to in **(H5)**. So we may estimate,

$$\begin{aligned}
\sum_{i \in \mathcal{I}_n(W)} \frac{|W^n_i|_*^{\gamma_0}}{|W|_*^{\gamma_0}} |J_{W^n_i} T^n|_* &\leq \sum_{j \in \mathcal{I}_{n-1}(W)} \sum_{i \in A(W^{n-1}_j)} \frac{|W^n_i|_*^{\gamma_0}}{|W|_*^{\gamma_0}} \frac{|W^{n-1}_j|_*^{\gamma_0}}{|W^{n-1}_j|_*^{\gamma_0}} |J_{W^{n-1}_j} T^{n-1}|_* |J_{W^n_i} T|_* \\
&\leq \theta_* \sum_{j \in \mathcal{I}_{n-1}(W)} \frac{|W^{n-1}_j|_*^{\gamma_0}}{|W|_*^{\gamma_0}} |J_{W^{n-1}_j} T^{n-1}|_* \leq \theta_*^n.
\end{aligned} \tag{3.6}$$

The analogous estimate in the Euclidean norm then follows up to a constant C'_4 depending on the uniform constant relating $\|\cdot\|$ to $\|\cdot\|_*$.

Next we extend (a) to $\gamma_0 < \varsigma < 1$ via a Hölder inequality. Fix $\varsigma > \gamma_0$ and define $s = (1-\gamma_0)/(1-\varsigma) > 1$. We will use repeatedly that by (3.1),

$$|T^n W^n_i|/|W^n_i| \leq |J_{W^n_i} T^n|_{\mathcal{C}^0(W^n_i)} \leq (1 + C_d)|T^n W^n_i|/|W^n_i|. \tag{3.7}$$

Now multiplying by $|W|/|W|$, we have

$$\begin{aligned}\sum_{i\in\mathcal{I}_n(W)} \frac{|W_i^n|^\varsigma}{|W|^\varsigma}|J_{W_i^n}T^n|_{\mathcal{C}^0(W_i^n)} &\le (1+C_d)\sum_{i\in\mathcal{I}_n(W)}\frac{|W|^{1-\varsigma}}{|W_i^n|^{1-\varsigma}}\frac{|T^nW_i^n|}{|W|} \\ &\le (1+C_d)\left(\sum_{i\in\mathcal{I}_n(W)}\frac{|W|^{(1-\varsigma)s}}{|W_i^n|^{(1-\varsigma)s}}\frac{|T^nW_i^n|}{|W|}\right)^{1/s}\left(\sum_{i\in\mathcal{I}_n(W)}\frac{|T^nW_i^n|}{|W|}\right)^{1-1/s} \\ &\le (1+C_d)\left(\sum_{i\in\mathcal{I}_n(W)}\frac{|W_i^n|^{\gamma_0}}{|W|^{\gamma_0}}|J_{W_i^n}T^n|_{\mathcal{C}^0(W_i^n)}\right)^{1/s} \le (1+C_d)C_4'\theta_*^{n/s}\end{aligned}$$

by (3.6) since $\sum_{W_i^n\in\mathcal{G}_n(W)}\frac{|T^nW_i^n|}{|W|}=1$. Part (a) follows with $C_4=(1+C_d)C_4'$.

(b) Fix $\varsigma\ge\gamma_0$, $W\in\mathcal{W}^s$ and $n>0$. We group $W_i^n\in\mathcal{G}_n(W)$ by most recent long ancestor $W_j^k\in L_k(W)$ as described before the statement of the lemma. Then using the fact that $|J_{W_i^n}T^n|_{\mathcal{C}^0(W_i^n)}\le |J_{W_j^k}T^k|_{\mathcal{C}^0(W_j^k)}|J_{W_i^n}T^{n-k}|_{\mathcal{C}^0(W_i^n)}$, we estimate

$$\begin{aligned}&\sum_{W_i^n\in\mathcal{G}_n(W)}\frac{|W_i^n|^\varsigma}{|W|^\varsigma}|J_{W_i^n}T^n|_{\mathcal{C}^0(W_i^n)} \\ &\qquad\le \sum_{k=0}^n\left(\sum_{W_j^k\in L_k(W)}\frac{|W_j^k|^\varsigma}{|W|^\varsigma}|J_{W_j^k}T^k|_{\mathcal{C}^0(W_j^k)}\sum_{i\in\mathcal{I}_{n-k}(W_j^k)}\frac{|W_i^n|^\varsigma}{|W_j^k|^\varsigma}|J_{W_i^n}T^{n-k}|_{\mathcal{C}^0(W_i^n)}\right).\end{aligned}$$

Note that $\mathcal{I}_n(W_j^k)$ (with W as root) and $\mathcal{I}_{n-k}(W_j^k)$ (with W_j^k as root) correspond to the same set of short pieces in the $(n-k)^{\text{th}}$ generation of W_j^k, so we can apply part (a) of the lemma to each of these sums separately with $s=(1-\gamma_0)/(1-\varsigma)$ as before. Since $|W_j^k|\ge\delta_0/3$, we split off the term for $k=0$ and use (3.7) to estimate

$$\begin{aligned}\sum_{W_i^n\in\mathcal{G}_n(W)}\frac{|W_i^n|^\varsigma}{|W|^\varsigma}|J_{W_i^n}T^n|_{\mathcal{C}^0(W_i^n)} &\le \sum_{k=1}^{n-1}\sum_{W_j^k\in L_k(W)}3\delta_0^{\varsigma-1}C_4\theta_*^{(n-k)/s}(1+C_d)|W|^{-\varsigma}|T^kW_j^k| \;+\; C_4\theta_*^{n/s} \\ &\le C\delta_0^{\varsigma-1}\sum_{k=1}^{n-1}|W|^{1-\varsigma}\theta_*^{(n-k)/s}+C_4\theta_*^{n/s},\end{aligned}$$

which is uniformly bounded in n, where we have used $\sum_{W_j^k\in L_k(W)}|T^kW_j^k|\le|W|$. □

3.4. **Properties of the Banach spaces.** We begin by verifying that our Banach spaces contain an interesting class of measures. We first record the following simple observations.

Lemma 3.4. *(a) There exists a constant $C_f>0$ such that for any homogeneous stable curve W and any $x\in W$,*

$$C_f^{-1}\le\frac{f(x)}{f(W)}\le C_f$$

where $f(W)$ is as defined in Section 2.3. In addition if two curves $W,W'\in\mathcal{W}^s$ lie in the homogeneity region, then $f(W)/f(W')$ satisfies the same bounds as above.

(b) There exists $C_w>0$ such that if $W_1,W_2\in\mathcal{W}^s$ with $d_{\mathcal{W}^s}(W_1,W_2)\le\varepsilon$ and $|W_2|\ge\varepsilon$, then $|W_1|/|W_2|\le C_w$.

(c) There exists $C>0$ such that for any $W\in\mathcal{W}^s$,

$$|W|^{1-\alpha}f(W)^{-1}\le Ck^{-r_h\delta_1},\tag{3.8}$$

where $\delta_1 := \frac{1}{r_h} - \alpha > 0$ *by choice of* α *in Section 2.3.*

Proof. (a) Since $f(x)$ is continuous on W, there exists $y \in W$ such that $f(y) = f(W)$. The bound is trivial if f is not close to 0. On those $\mathbb{H}_k$ with f close to 0, the first bound follows from the definition of $\mathbb{H}_k$: There exists $C > 0$ such that if $x \in \mathbb{H}_k$, then $Ck^{-r_h+1} \le f(x) \le C^{-1}k^{-r_h+1}$. If W and W' lie in the same homogeneity strip, $f(W)/f(W')$ satisfies the same bounds by an identical argument.

(b) Recalling the definition of $d_{\mathcal{W}^s}(\cdot,\cdot)$ from Section 3.1, there exists $\vartheta > 0$, depending on the maximum slope of functions $F \in \Xi$, such that $||W_1| - |W_2|| \le \varepsilon(1+\vartheta) + \varepsilon|I_{W_1} \cap I_{W_2}|$, where I_{W_j} are the intervals where G_{W_j} is defined, $j = 1, 2$. If $|W_2| \ge \varepsilon$, we may divide by $|W_2|$ to obtain,

$$|W_1|/|W_2| \le 1 + (1+\vartheta) + |I_{W_2}|/|W_2|,$$

which is uniformly bounded.

(c) Consider the expression $|W|^{1-\alpha} f(W)^{-1}$. Since W is a homogeneous curve, it lies either in $\mathbb{H}_{k_0}$ or in a homogeneity strip indexed by $k > k_0$. In the former case, $f(W) \ge k_0^{-r_h+1}$ so that the above expression is bounded. In the latter case, $f(W) \ge Ck^{-r_h+1}$ while by **(H3)**(4), $|W| \le C_2 k^{-r_h}$. Thus

$$|W|^{1-\alpha} f(W)^{-1} \le Ck^{r_h\alpha - 1} \le Ck^{-r_h\delta_1},$$

where $\delta_1 = \frac{1}{r_h} - \alpha > 0$ as defined in the statement of the lemma. □

The first main lemma of this section, Lemma 3.5, shows that $\mathcal{B}$ contains functions with certain types of discontinuities. The argument uses the fact that the contribution to the norm of the function we must approximate from homogeneity strips of high index is small. The proof is similar to [DZ1, Lemma 3.7], but is modified to (a) allow tangencies between the discontinuities of the given function and the stable cone, and (b) respect the additional constants and restrictions introduced into the norms to exploit the weak form of the one-step expansion given by (2.5).

The subsequent lemmas 3.6, 3.7 and 3.9 are similar to lemmas appearing in [DZ1], but we have adapted their proofs to this more general setting. In particular, the proof of Lemma 3.7 is significantly changed to accommodate **(H3)** and requires the summability condition **(H3)**(5) since we allow additional homogeneity strips where f is not close to 0. Lemma 3.8 is new and does not appear in [DZ1].

Lemma 3.5. *Let* $\mathcal{P}$ *be a (mod 0) countable partition of* M *into open, simply connected sets such that (1) for each* $k \in \mathbb{N}$, *there is an* $N_k < \infty$ *such that at most* N_k *elements* $P \in \mathcal{P}$ *intersect* $\mathbb{H}_k$; *(2) there are constants* $K, C_5 > 0$ *such that for each* $P \in \mathcal{P}$ *and* $W \in \mathcal{W}^s$, $P \cap W$ *comprises at most* K *connected components and for any* $\varepsilon > 0$, $m_W(N_\varepsilon(\partial P) \cap W) \le C_5 \varepsilon^{t_0}$.

Let $\lambda > \beta/(1-\beta)$. *If* $h \in \mathcal{C}^\lambda(P)$ *for each* $P \in \mathcal{P}$ *and* $\sup_{P\in\mathcal{P}} |h|_{\mathcal{C}^\lambda(P)} < \infty$, *then* $h \in \mathcal{B}$. *In particular,* $\mathcal{C}^\lambda(M) \subset \mathcal{B}$ *for each* $\lambda > \beta/(1-\beta)$ *and Lebesgue measure is in* $\mathcal{B}$.

Proof. Since $\mathcal{B}$ is defined as the completion of $\mathcal{C}^1(M)$, we must show that h as above can be approximated by functions in $\mathcal{C}^1(M)$ in the $\|\cdot\|_{\mathcal{B}}$ norm.

For $P \in \mathcal{P}$ we define P_k to be a simply connected component of $P \cap \mathbb{H}_k$. The label P_k may not be unique, but there are only finitely many such elements for each $k \ge k_0$ by assumption (1) of the lemma. Let h be as in the statement of the lemma. Since $\|h\|_{\mathcal{B}} = \sup_k \|h|_{\mathbb{H}_k}\|_{\mathcal{B}}$ by definition of $\mathcal{W}^s$, we may fix k and approximate h on one $\mathbb{H}_k$ at a time. We fix P_k and for simplicity first consider $h \equiv 0$ off of P_k.

Choose $\eta > 0$ and define $\hat{P}_k = P_k \backslash (B_{\eta/k^{r_h}}(\partial P_k))$, the part of P_k which is at least η/k^{r_h} away from the boundary of P_k. Let $\rho_\eta(x,y)$ be a nonnegative $\mathcal{C}^\infty$ bump function such (1) $\int_{\hat{P}_k} \rho_\eta(x,y) dm(y) = 1$ for each $x \in \hat{P}_k$, and (2) $\rho_\eta(x,y) = 0$ whenever $d(x,y) > \eta/(2k^{r_h})$. Define

$$f_\eta(x) = \int_{\hat{P}_k} h(y)\rho_\eta(x,y)\, dm(y), \quad \text{for } x \in M.$$

Note that $f_\eta \in \mathcal{C}^\infty(M)$ and that $f_\eta(x) \equiv 0$ for $x \notin P_k$. It is also straightforward to check that $|f_\eta|_{\mathcal{C}^\lambda(\hat{P}_k)} \le |h|_{\mathcal{C}^\lambda(P_k)}$ and $|f_\eta|_\infty \le |h|_\infty$.

Now let $W \in \mathcal{W}^s$, $W \subset \mathbb{H}_k$, and take $\psi \in \mathcal{C}^q(W)$, $|\psi|_{W,\alpha,q} \le 1$. Notice that $|\psi|_\infty \le |W|^{-\alpha} f(W)^{-1}$. Thus,

$$\begin{aligned}\int_W (h - f_\eta)\psi \, dm_W &= \int_{W\cap \hat{P}_k} (h - f_\eta)\psi \, dm_W + \int_{W\setminus \hat{P}_k} (h - f_\eta)\psi \, dm_W \\ &\le |h - f_\eta|_{\mathcal{C}^0(W\cap\hat{P}_k)} |W|^{1-\alpha} f(W)^{-1} + 2|h|_\infty |W \cap (P_k \setminus \hat{P}_k)||W|^{-\alpha} f(W)^{-1},\end{aligned} \tag{3.9}$$

since the supports of h and f_η lie entirely in the closure of P_k.

For the first term above, we estimate the difference in functions for $x \in W \cap \hat{P}_k$ by,

$$|h(x) - f_\eta(x)| \le \int_{\hat{P}_k} |h(x) - h(y)| \rho_\eta(x, y) \, dm(y).$$

The integrand is 0 whenever, $d(x, y) > \eta/(2k^{r_h})$, thus

$$|h(x) - f_\eta(x)| \le C|h|_{\mathcal{C}^\lambda(P)} \eta^\lambda k^{-r_h\lambda}.$$

Thus by Lemma 3.4(c), we obtain for the first term of (3.9),

$$|h - f_\eta|_{\mathcal{C}^0(W\cap P_k)} |W|^{1-\alpha} f(W)^{-1} \le C|h|_{\mathcal{C}^\lambda(P)} \eta^\lambda k^{-r_h(\lambda+\delta_1)}. \tag{3.10}$$

For the second term of (3.9), note that $|W \cap (P_k \setminus \hat{P}_k)||W|^{-\alpha} \le |W \cap (P_k \setminus \hat{P}_k)|^{1-\alpha}$. By assumption (2) of the lemma, $W \cap (P_k \setminus \hat{P}_k)$ comprises at most K connected components, each of length at most $\min\{C_4 k^{-r_h}, C(\eta/k^{r_h})^{t_0}\}$ due to weak transversality and **(H3)**(4). Recalling our convention that $t_0 \le 1/2$, this minimum is largest when the two quantities are equal,[11] i.e., when $\eta = k^{-r_h(1-t_0)/t_0}$. Thus

$$|h|_\infty |W \cap (P_k \setminus \hat{P}_k)||W|^{-\alpha} f(W)^{-1} \le C|h|_\infty k^{r_h\alpha - 1} \le C|h|_\infty \eta^{\delta_1 t_0/(1-t_0)}, \tag{3.11}$$

where $\delta_1 = \frac{1}{r_h} - \alpha > 0$ as before. Putting together these estimates and taking the suprema over $W \subset \mathbb{H}_k$ and $\psi \in \mathcal{C}^q(W)$, we have by (3.9),

$$\|(h - f_\eta^{P_k})|_{\mathbb{H}_k}\|_s \le C|h|_{\mathcal{C}^\lambda(P)} (\eta^\lambda + \eta^{\delta_1 t_0/(1-t_0)}).$$

Notice that if we were not concerned with approximating h by f_η, but only estimating $\|h\|_s$, then (3.8) and (3.9) would imply,

$$\|h|_{\mathbb{H}_k}\|_s \le C|h|_\infty k^{-r_h\delta_1} \quad \text{for all bounded functions } h. \tag{3.12}$$

To estimate $\|(h - f_\eta)|_{\mathbb{H}_k}\|_u$, fix $0 < \varepsilon \le \varepsilon_0$, where ε_0 is from (2.10), and let $W_1, W_2 \subset \mathbb{H}_k$ be two admissible stable curves such that $d_{\mathcal{W}^s}(W_1, W_2) \le \varepsilon$. In the notation of Section 3.1, we identify W_i with $G_{W_i}(t)$, $t \in I_i$. Let ψ_1, ψ_2 be two test functions satisfying $|\psi_i|_{W_i,\gamma,p} \le 1$, $i = 1, 2$, and $|\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{\mathcal{C}^q(I_1\cap I_2)} \le \varepsilon$. Without loss of generality, assume $\lambda = \beta/(1-\beta) + \delta_2 \le 1/2$, for some $\delta_2 > 0$. This is always possible since by (2.7) in the definition of the norms, $\beta < 1/3$.

First assume that $\varepsilon \ge \eta^{(1+\delta_2)/(1-\beta)} k^{-(r_h - 1 + r_h\gamma)/(1-\beta)}$. Following the analogous estimate on the stable norm given by (3.9) with γ in place of α (this is possible since $\gamma < \alpha$), (3.11) becomes,

$$|h|_\infty |W \cap (P_k \setminus \hat{P}_k)||W|^{-\gamma} f(W)^{-1} \le C|h|_\infty k^{r_h\gamma - 1} = C|h|_\infty k^{-r_h\delta_1 - r_h(\alpha-\gamma)} \le C|h|_\infty \eta^{z/(1-t_0)} k^{-r_h\delta_1},$$

where $z := t_0(\alpha - \gamma)$, remembering that $\eta = k^{-r_h(1-t_0)/t_0}$ in (3.11). Putting this together with the analogue of (3.10), we have

$$\varepsilon^{-\beta} \left| \int_{W_1} (h - f_\eta)\psi_1 \, dm_W - \int_{W_2} (h - f_\eta)\psi_2 \, dm_W \right| \le C\varepsilon^{-\beta} |h|_{\mathcal{C}^\lambda(P)} (\eta^\lambda + \eta^{z/(1-t_0)}) k^{-r_h\delta_1}. \tag{3.13}$$

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[11]If $t_0 = 1$, the minimum is $C\eta k^{-r_h}$ and the estimate in (3.11) becomes $\le C|h|_\infty \eta^{1-\alpha} k^{\alpha-1}$. One can carry this change through to get improved estimates on the exponents in this case.

Since $\varepsilon \geq \eta^{(1+\delta_2)/(1-\beta)} k^{-(r_h-1+r_h\gamma)/(1-\beta)}$, the exponent of k in the above expression is given by

$$\beta \frac{r_h - 1 + r_h\gamma}{1-\beta} - r_h\delta_1 < 0$$

as $\beta < \delta_1$ from the definition of the norms so that $\frac{r_h-1+r_h\gamma}{1-\beta} < \frac{r_h(1-\delta_1)}{1-\beta} < r_h$. Again using the fact that $\beta < 1/3$ and the definition of δ_2, we have $\frac{1+\delta_2}{1-\beta} \leq 2$, so that the exponent of η is given by

$$\eta^{-\beta\frac{1+\delta_2}{1-\beta}} \left(\eta^{\frac{\beta}{1-\beta}+\delta_2} + \eta^{z/(1-t_0)} \right) \leq \eta^{\delta_2(1-\frac{\beta}{1-\beta})} + \eta^{\frac{z}{1-t_0}-2\beta},$$

and both terms have positive exponents since $\beta < z/2(1-t_0)$ by (2.7).

It remains to estimate the case $\varepsilon < \eta^{(1+\delta_2)/(1-\beta)} k^{-(r_h-1+r_h\gamma)/(1-\beta)}$. For this estimate, we split up the terms involving h and f_η,

$$\begin{aligned} &\int_{W_1} (h - f_\eta)\psi_1 \, dm_W - \int_{W_2} (h - f_\eta)\psi_2 \, dm_W \\ &\qquad = \int_{W_1} h\psi_1 \, dm_W - \int_{W_2} h\psi_2 \, dm_W \; + \int_{W_2} f_\eta\psi_2 \, dm_W - \int_{W_1} f_\eta\psi_1 \, dm_W. \end{aligned} \tag{3.14}$$

We first estimate the difference involving h.

We match W_1 and W_2 using a foliation of homogeneous unstable curves which are vertical line segments of length at most ε in the chart on which G_{W_i} is defined, $i = 1, 2$. This partitions W_1 in the following way: curves $U_1^i \subset W_1$ for which the unstable curve connecting U_1^i to W_2 lies entirely in P_k; curves $V_1^j \subset W_1$ which either are not matched to W_2 (near the endpoints of W_1) or for which the vertical segment connecting V_1^i to W_2 does not lie entirely in P_k. In particular if $|W_2| < \varepsilon$, we set $V_\ell = W_\ell$, $\ell = 1, 2$, and declare W_ℓ to be unmatched. This induces a corresponding partition on W_2 into curves U_2^i and V_2^j. We call $U_\ell^i \subset W_\ell$ the matched pieces and $V_\ell^j \subset W_\ell$ the unmatched pieces and note that by assumption on $\mathcal{P}$, there can be no more than K matched pieces and $K + 2$ unmatched pieces.

We split up the integrals on W_1 and W_2 on matched and unmatched pieces,

$$\int_{W_1} h\psi_1 \, dm_W - \int_{W_2} h\psi_2 \, dm_W = \sum_i \int_{U_1^i} h\psi_1 \, dm_W - \int_{U_2^i} h\psi_2 \, dm_W + \sum_{j,\ell} \int_{V_\ell^j} h\psi_\ell \, dm_W. \tag{3.15}$$

We estimate the integrals on the unmatched pieces first. Since $h \equiv 0$ off of P_k, and ∂P_k and the unstable curves are either uniformly transverse to the stable cone or have the type of tangency allowed by assumption (2) in the statement of the lemma, we have $|\mathrm{supp}(h) \cap V_\ell^j| \leq C\varepsilon^{t_0}$ for each V_j^ℓ. Then using (3.12), we estimate

$$\left| \int_{V_\ell^j} h\psi_i \, dm_W \right| \leq \|h|_{\mathbb{H}_k}\|_s |\mathrm{supp}(h) \cap V_\ell^j|^\alpha f(V_\ell^j) |\psi_\ell|_{\mathcal{C}^q(W_\ell)} \leq C|h|_\infty k^{-r_h\delta_1} |\mathrm{supp}(h) \cap V_j^\ell|^\alpha |W_\ell|^{-\gamma},$$

where in the last inequality, $|\psi_\ell|_{\mathcal{C}^q(W_\ell)} \leq f(W_\ell)^{-1}|W_\ell|^{-\gamma}$ and we have used Lemma 3.4 to bound $f(V_\ell^j)/f(W_\ell)$. Since $(\mathrm{supp}(h) \cap V_j^\ell) \subset W_\ell$, remembering that $z = t_0(\alpha - \gamma)$ we have

$$|\mathrm{supp}(h) \cap V_j^\ell|^\alpha |W_\ell|^{-\gamma} \leq |\mathrm{supp}(h) \cap V_j^\ell|^{\alpha-\gamma} \leq C\varepsilon^{t_0(\alpha-\gamma)} = C\varepsilon^z.$$

Putting these estimates together, we obtain our bound on unmatched pieces,

$$\left| \int_{V_k^j} h\psi_i \, dm_W \right| \leq C|h|_\infty \varepsilon^z k^{-r_h\delta_1}. \tag{3.16}$$

Next we estimate the difference on matched pieces in (3.15). To do this, we change variables to the intervals I_i common to U_1^i and U_2^i.

$$\left| \int_{I_i} (h\psi_1) \circ G_{U_1^i} \, JG_{U_1^i} - (h\psi_2) \circ G_{U_2^i} \, JG_{U_2^i} \, dt \right| \leq \ell(I_i) |(h\psi_1) \circ G_{U_1^i} \, JG_{U_1^i} - (h\psi_2) \circ G_{U_2^i} \, JG_{U_2^i}|_{\mathcal{C}^0(I_i)},$$

where $JG_{U_k^i}$ denotes the Jacobian of $G_{U_k^i}$. Due to the uniform upper bound on the slopes of curves in the stable cone, there exists $C_g > 0$ such that

$$|JG_{U_k^i}|_{\mathcal{C}^0(I_i)} \le C_g. \tag{3.17}$$

We split the difference on matched pieces into the sum of three terms. The first term is,

$$\begin{aligned} A &:= |h \circ G_{U_1^i} - h \circ G_{U_2^i}|_{\mathcal{C}^0(I_i)} |\psi_1 \circ G_{U_1^i} JG_{U_1^i}|_{\mathcal{C}^0(I_i)} \\ &\le \frac{C_g H^\lambda(h)}{f(W_1)|W_1|^\gamma} \sup_{t \in I_i} \left(d(G_{U_1^i}(t), G_{U_2^i}(t))^\lambda \right), \end{aligned}$$

where $H^\lambda(h)$ denotes the Hölder constant of h with exponent λ. Now $d(G_{U_1^i}(t), G_{U_2^i}(t)) = |F_{U_1^i}(t) - F_{U_2^i}(t)| \le \varepsilon$ by definition of $d_{\mathcal{W}^s}(\cdot,\cdot)$. Thus,

$$A \le C_g H^\lambda(h) \frac{\varepsilon^\lambda}{f(W_1)|W_1|^\gamma} \tag{3.18}$$

The second term of the difference is,

$$B := |\psi_1 \circ G_{U_1^i} - \psi_2 \circ G_{U_2^i}|_{\mathcal{C}^0(I_i)} |h \circ G_{U_2^i} JG_{U_1^i}|_{\mathcal{C}^0(I_i)} \le \varepsilon |h|_\infty C_g, \tag{3.19}$$

by assumption on ψ_1 and ψ_2. Finally, the last difference we must estimate is,

$$E := |h \circ G_{U_2^i} \psi_2 \circ G_{U_2^i}|_{\mathcal{C}^0(I_i)} |JG_{U_1^i} - JG_{U_2^i}|_{\mathcal{C}^0(I_i)} \le |h|_\infty |\psi_2|_\infty |F'_{U_1^i} - F'_{U_2^i}|_{\mathcal{C}^0(I_i)} \le \frac{|h|_\infty \varepsilon}{f(W_2)|W_2|^\gamma}, \tag{3.20}$$

again by definition of $d_{\mathcal{W}^s}(\cdot,\cdot)$, where $F'_{U_k^i} = dF_{U_k^i}/dt$.

Putting together the estimates for A, B and E, as well as (3.16), into (3.15), we have

$$\begin{aligned} \varepsilon^{-\beta} \left| \int_{U_1^j} h\psi_1 \, dm_W - \int_{U_2^j} h\psi_2 \, dm_W \right| &\le C|W_1| \Big(\frac{H^\lambda(h)\varepsilon^{\lambda-\beta}}{f(W_1)|W_1|^\gamma} + \frac{|h|_\infty \varepsilon^{1-\beta}}{f(W_2)|W_2|^\gamma} \Big) + C|h|_\infty \varepsilon^{z-\beta} k^{-r_h\delta_1} \\ &\le C\frac{|W_1|^{1-\gamma}}{f(W_1)} |h|_{\mathcal{C}^\lambda(P_k)} \varepsilon^{\lambda-\beta} + C|h|_\infty \varepsilon^{z-\beta} k^{-r_h\delta_1} \le C|h|_{\mathcal{C}^\lambda(P_k)} \varepsilon^{z-\beta} k^{-r_h\delta_1}, \end{aligned} \tag{3.21}$$

where we have used Lemma 3.4(b) to bound $|W_1|/|W_2|$ and $f(W_1)/f(W_2)$, and (3.8) for the last step. Also, $z - \beta > 0$ since β is chosen $< t_0(\alpha - \gamma)$ in the definition of the norms. Notice that (3.21) holds without the assumption $\varepsilon < \eta^{(1+\delta_2)/(1-\beta)} k^{-(r_h - 1 + r_h\gamma)/(1-\beta)}$ which is what makes (3.23) possible.

A similar estimate holds for f_η. Indeed the estimate is simpler since f_η is Lipschitz continuous on all of M with $H^1(f_\eta) \le C|h|_\infty k^{r_h}/\eta$. Thus we may partition W_1 and W_2 into one matched piece and at most two unmatched pieces near their endpoints. The unmatched pieces have length at most $C\varepsilon^{t_0}$ so that an estimate similar to (3.16) holds for f_η. Then since f_η is Lipschitz continuous everywhere, estimates A, B and E hold on the single matched piece with $\lambda = 1$ and so,

$$\varepsilon^{-\beta} \left| \int_{U_1} f_\eta \psi_1 \, dm_W - \int_{U_2} f_\eta \psi_2 \, dm_W \right| \le C|W_1| \Big(\frac{H^1(f_\eta)\varepsilon^{1-\beta}}{f(W_1)|W_1|^\gamma} + \frac{|h|_\infty \varepsilon^{1-\beta}}{f(W_2)|W_2|^\gamma} \Big) + C|h|_\infty \varepsilon^{z-\beta}. \tag{3.22}$$

Following the same estimate as in (3.21), it is clear that the only term that can cause a problem is the first one in (3.22) due to the size of $H^1(f_\eta)$. We estimate using the analogue of (3.8) with γ in place of α,

$$\frac{|W_1|^{1-\gamma}}{f(W_1)} \frac{\varepsilon^{1-\beta} k^{r_h}}{\eta} \le C \frac{1}{k^{1-r_h\gamma}} \frac{\eta^{1+\delta_2} k^{r_h}}{\eta \, k^{r_h - 1 + r_h\gamma}} \le C\eta^{\delta_2}.$$

Putting together the estimates in (3.13), (3.21) and (3.22), we have shown that $\|(h - f_\eta)|_{\mathbb{H}_k}\|_u \le CK(H^\lambda(h) + |h|_\infty)\eta^{\delta_3}$, where $\delta_3 := \min\{\delta_2(1 - \frac{\beta}{1-\beta}), z - \beta, \frac{z}{1-t_0} - 2\beta\} > 0$. This together with the estimate on the strong stable norm implies that $\|(h - f_\eta)|_{\mathbb{H}_k}\|_{\mathcal{B}} \le C|h|_{\mathcal{C}^\lambda(P)}\eta^\delta$, where $\delta = \min\{\delta_1, \delta_3\}$. Notice that if we are not concerned with approximating h by f_η, then (3.12) and (3.21) together imply that

$$\|h|_{\mathbb{H}_k}\|_{\mathcal{B}} \le C \sup_{P \in \mathcal{P}} |h|_{\mathcal{C}^\lambda(P)} k^{-r_h \delta_1}. \tag{3.23}$$

In making this approximation argument, we have assumed that $h \equiv 0$ outside P_k. More general h can be expressed as $h = \sum_k \sum_{P_k} h\mathbf{1}_P$ where $h\mathbf{1}_{P_k} \equiv 0$ outside of P_k and so can be approximated by a $\mathcal{C}^1$ function $f_\eta^{P_k}$ as above. Due to (3.23), given $\epsilon > 0$, we first choose K'_ϵ so that $\|h|_{\mathbb{H}_k}\|_{\mathcal{B}} < \epsilon$ for all $k > K'_\epsilon$. By property (1) of $\mathcal{P}$, there exists $N_\epsilon > 0$ such that for each $k_0 \le k \le K'_\epsilon$, $\mathbb{H}_k$ intersects at most N_ϵ elements of $\mathcal{P}$. We thus form the finite sum $\sum_{k_0 \le k \le K'_\epsilon} \sum_{P_k} f_\eta^{P_k}$ and approximate h by 0 on $\cup_{k > K'_\epsilon} \mathbb{H}_k$. Since there are at most N_ϵ elements P_k for each $k \le K'_\epsilon$,

$$\Big\| \big(h - \sum_{k \le K'_\epsilon} \sum_{P_k} f_\eta^{P_k}\big) \Big\|_{\mathcal{B}} \le \epsilon + \sup_{k \le K'_\epsilon} \Big\| \sum_{P_k} \big(h\mathbf{1}_{P_k} - f_\eta^{P_k}\big)\Big|_{\mathbb{H}_k} \Big\|_{\mathcal{B}} \le \epsilon + CN_\epsilon \eta^\delta \sup_{P \in \mathcal{P}} |h|_{\mathcal{C}^\lambda(P)},$$

and finally we choose η sufficiently small that $\eta^\delta N_\epsilon < \epsilon$. □

Next we prove that $\mathcal{L}$ is well-defined as an operator on $\mathcal{B}$. Its proof uses the fact that $\|\mathcal{L}h\|_{\mathcal{B}} < \infty$ for $h \in \mathcal{C}^1(M)$ from Section 4.

Lemma 3.6. *If $h \in \mathcal{C}^1(M)$, then $\mathcal{L}h \in \mathcal{B}$.*

Proof. Let $h \in \mathcal{C}^1(M)$. As in the proof of Lemma 3.5, we must approximate $\mathcal{L}h$ by $\mathcal{C}^1$ functions in the norm $\|\cdot\|_{\mathcal{B}}$. Note that $\mathcal{L}h$ has a countable number of smooth discontinuity curves given by $\mathcal{S}_{-1}^{\mathbb{H}} = \mathcal{S}_{-1} \cup T(\cup_{k \ge k_0} S_k^H)$ (we include the images of boundaries of the homogeneity regions). These curves define a countable partition $\mathcal{P}$ of M into open simply connected sets which does not satisfy assumption (1) of Lemma 3.5 since each $\mathbb{H}_k$ can intersect countably many $P \in \mathcal{P}$. In addition, the $\mathcal{C}^1$ norm of $\mathcal{L}h$ blows up near the curves $T(S_0^H)$.

Let $\{P_j\}_{j \in \mathbb{N}}$ be an enumeration of the elements of $\mathcal{P}$. For $J > k_0$, let $P^J = \cup_{j > J} P_j$. Given $\epsilon > 0$, we claim that $\|\mathcal{L}h|_{P^J}\|_{\mathcal{B}} < \epsilon$ for J sufficiently large.

Indeed, the claim is trivial using the estimates of Section 4. For example, we must estimate $\|\mathcal{L}h|_{P^J}\|_s = \|1_{P^J}\mathcal{L}h\|_s$. Taking $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^q(W)$ with $|\psi|_{W,\alpha,q} \le 1$, we write

$$\int_W 1_{P^J} \mathcal{L}h \, \psi \, dm_W = \int_{T^{-1}(W \cap P^J)} h |DT|^{-1} J_{T^{-1}W} T \, \psi \circ T \, dm_W,$$

and the homogeneous stable components of $T^{-1}(W \cap P^J)$ correspond precisely to the tail of the series considered in (4.2) and following and so can be made arbitrarily small by choosing J large (notice that we do not need contraction here so that we may use the simpler estimate similar to Section 4.1 applied to the strong stable norm rather than the estimate of Section 4.2).

Similarly, in estimating $\|\mathcal{L}h\|_u$, one can see that the contribution from P^J corresponds to the tail of the series from the estimates of Section 4.3, and so this too can be made arbitrarily small by choosing J large.

Now fix $\epsilon > 0$ and choose J such that $\|\mathcal{L}h|_{P^J}\|_{\mathcal{B}} < \epsilon$. On the finite set of P_j with $j \le J$, the $\mathcal{C}^1$ norm of $\mathcal{L}h$ is bounded by a constant $C_J < \infty$ and can be approximated using Lemma 3.5 as follows. Since the partition $\mathcal{P}^* = \{P_j\}_{j \le J} \cup \{P^J\}$ is finite, it satisfies assumption (1) of Lemma 3.5. To verify assumption (2) of Lemma 3.5, note that by **(H3)**(2) we have only finitely many curves in $\mathcal{S}_{-1}^{\mathbb{H}}$ comprising ∂P_j for $j \le J$. Thus there is a uniform upper bound $K_J < \infty$ on the number of connected components of $P_j \cap W$ for all $W \in \mathcal{W}^s$, $W \subset M \setminus P^J$. Finally, the weak transversality assumption (2) of Lemma 3.5 is guaranteed by **(H3)**(3),(4).

Now we approximate $\mathcal{L}h$ as in Lemma 3.5 on the finitely many elements P_j, $j \le J$, choosing η in the approximating function f_η small compared to C_J and K_J, and approximate $\mathcal{L}h$ by 0 on P^J. □

The next lemma allows us to establish a connection between our Banach spaces and the space of distributions introduced in Section 2.2. Recall that $H^p_n(\psi) = \sup_{W \in T^{-n}\mathcal{W}^s} H^p_W(\psi)$.

Lemma 3.7. *For each $h \in \mathcal{C}^1(M)$, $n \geq 0$, and $\psi \in \mathcal{C}^p(T^{-n}\mathcal{W}^s)$ we have*

$$\left| \int_M h\psi \, dm \right| \leq C|h|_w(|\psi|_\infty + H^p_n(\psi)).$$

Proof. We partition each component of $\mathbb{H}_{k_0} \cap (M \setminus \mathcal{S}_0)$ into finitely many boxes B_j whose boundary curves are elements of $\mathcal{W}^s$ and $\mathcal{W}^u$ as well as the boundary of $\mathbb{H}_{k_0} \cap (M \setminus \mathcal{S}_0)$. We construct the boxes so that each B_j has diameter $\leq \delta_0$ and is foliated by curves $W \in \mathcal{W}^s$. On each B_j, we choose a smooth foliation $\{W_\xi\}_{\xi \in E_j} \subset \mathcal{W}^s$, each of whose elements completely crosses B_j in the approximate stable direction. This is possible since by **(H2)**, $M \setminus \mathcal{S}_0$ has finitely many connected components and the cones are continuous up to the closure of each component.

We decompose Lebesgue measure on B_j into $dm = \lambda(d\xi)dm_\xi$, where m_ξ is the conditional measure of m on W_ξ and λ is the transverse measure on E_j. We normalize the measures so that $m_\xi(W_\xi) = |W_\xi|$. Since the foliation is smooth, $dm_\xi = \rho_\xi dm_W$ where $C^{-1} \leq |\rho_\xi|_{\mathcal{C}^1(W_\xi)} \leq C$ for some constant C independent of ξ. Note that $\lambda(E_j) \leq C\delta_0$ due to the transversality of curves in $\mathcal{W}^s$ and $\mathcal{W}^u$.

Next in each homogeneity region, on each connected component of $\mathbb{H}_k \cap (M \setminus \mathcal{S}_0)$, $k > k_0$, we choose a smooth foliation $\{W_\xi\}_{\xi \in E_k} \subset \mathcal{W}^s$ whose elements all cross the component of $\mathbb{H}_k \cap (M \setminus \mathcal{S}_0)$ in which they lie. This is possible due to **(H3)**(4). We again decompose m on each component of $\mathbb{H}_k \cap (M \setminus \mathcal{S}_0)$ into $dm = \lambda(d\xi)dm_\xi$, $\xi \in E_k$, and $dm_\xi = \rho_\xi dm_W$ is normalized as above.

Now let $h \in \mathcal{C}^1(M)$ and $\psi \in \mathcal{C}^p(T^{-n}\mathcal{W}^s)$. Notice that since $M = T^{-n}M$ (mod 0), we have $\int_M h\psi \, dm = \int_M \mathcal{L}^n h \, \psi \circ T^{-n} \, dm$. We estimate the second integral on each connected component M_ℓ of $M \setminus \mathcal{S}_0$, $\ell \leq L$, where L is finite due to **(H2)**.

$$\begin{aligned} \int_{M_\ell} \mathcal{L}^n h \, \psi \circ T^{-n} \, dm &= \sum_j \int_{B_j} \mathcal{L}^n h \, \psi \circ T^{-n} \, dm + \sum_{k > k_0} \int_{\mathbb{H}_k} \mathcal{L}^n h \, \psi \circ T^{-n} \, dm \\ &= \sum_j \int_{E_j} \int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, dm_W d\lambda(\xi) + \sum_{k > k_0} \int_{E_k} \int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, dm_W d\lambda(\xi). \end{aligned} \tag{3.24}$$

We change variables and estimate the integrals on one W_ξ at a time. Letting $W^n_{\xi,i}$ denote the components of $\mathcal{G}_n(W_\xi)$ defined in Section 3.3, we define $J_{W^n_{\xi,i}} T^n$ to be the stable Jacobian of T^n along the curve $W^n_{\xi,i}$, and write

$$\begin{aligned} \left| \int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, dm_W \right| &= \sum_i \int_{W^n_{\xi,i}} h\psi |DT^n|^{-1} J_{W^n_{\xi,i}} T^n \, \rho_\xi \circ T^n \, dm_W \\ &\leq \sum_i |h|_w f(W^n_{\xi,i}) |W^n_{\xi,i}|^\gamma |\psi|_{\mathcal{C}^p(W^n_{\xi,i})} |\rho_\xi \circ T^n|_{\mathcal{C}^p(W^n_{\xi,i})} ||DT^n|^{-1} J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^p(W^n_{\xi,i})}. \end{aligned}$$

The distortion bounds given by Lemma 3.1 imply that

$$||DT^n|^{-1} J_{W^n_i} T^n|_{\mathcal{C}^p(W^n_i)} \leq (1 + 2C_d) ||DT^n|^{-1} J_{W^n_i} T^n|_{\mathcal{C}^0(W^n_i)}. \tag{3.25}$$

Moreover, for $x, y \in W^n_{\xi,i}$, it follows from **(H2)**(2) that

$$\frac{|\rho_\xi(T^n x) - \rho_\xi(T^n y)|}{d_W(T^n x, T^n y)^p} \cdot \frac{d_W(T^n x, T^n y)^p}{d_W(x,y)^p} \leq C|\rho_\xi|_{\mathcal{C}^p(W_\xi)} |J_{W^n_{\xi,i}} T^n|^p_{\mathcal{C}^0(W^n_{\xi,i})} \leq C\Lambda^{-pn} |\rho_\xi|_{\mathcal{C}^p(W_\xi)}, \tag{3.26}$$

and so $|\rho_\xi \circ T^n|_{\mathcal{C}^p(W^n_{\xi,i})} \le C|\rho_\xi|_{\mathcal{C}^p(W_\xi)} \le C$ for some uniform constant C. Putting these estimates together yields,

$$|\int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, dm_W| \le C|h|_w(|\psi|_\infty + H^p_n(\psi)) \sum_i f(W^n_{\xi i})|W^n_{\xi,i}|^\gamma ||DT^n|^{-1} J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^0(W^n_{\xi,i})}. \tag{3.27}$$

We group pieces $W^n_{\xi,i} \in \mathcal{G}_n(W_\xi)$ according to most recent long ancestor $W^k_{\xi,j} \in L_k(W_\xi)$ as described in Section 3.3. Since by **(H1)**,

$$|DT^n|^{-1}(x) = \frac{f(T^n x)}{f(x) \cdot \Pi_{m=0}^{n-1} f_0(T^m x)} \le \frac{f(T^n x)}{f(x)} \kappa^{-n},$$

for $x \in W^n_{\xi,i}$, we have by Lemma 3.4(a),

$$f(W^n_{\xi,i}) \, ||DT^n|^{-1}|_{C^0(W^n_{\xi,i})} \le C_f^2 f(W_\xi) \kappa^{-n}. \tag{3.28}$$

Splitting up the Jacobians according to times k and $n-k$ and using (3.28) on the intervals of time $n-k$, we obtain,

$$\begin{aligned}
\sum_i f(W^n_{\xi,i})|W^n_{\xi,i}|^\gamma ||DT^n|^{-1} J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^0(W^n_{\xi,i})} &\le \sum_{i \in \mathcal{I}_n(W_\xi)} C f(W_\xi) \kappa^{-n} |W^n_{\xi,i}|^\gamma |J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^0(W^n_{\xi,i})} \\
&\quad + \sum_{k=1}^n \sum_{j \in L_k(W_\xi)} |W^k_{\xi,j}|^\gamma ||DT^k|^{-1} J_{W^k_{\xi,j}} T^k|_{\mathcal{C}^0(W^k_{\xi,j})} \left(\sum_{i \in \mathcal{I}_n(W^k_j)} C\kappa^{-(n-k)} \frac{|W^n_{\xi,i}|^\gamma}{|W^k_{\xi,j}|^\gamma} |J_{W^n_{\xi,i}} T^{n-k}|_{\mathcal{C}^0(W^n_{\xi,i})} \right) \\
&\le C f(W_\xi)|W_\xi|^\gamma (\theta_*^{1/s} \kappa^{-1})^n + \sum_{k=1}^n \sum_{j \in L_k(W_\xi)} |W^k_{\xi,j}|^\gamma ||DT^k|^{-1} J_{W^k_{\xi,j}} T^n|_{\mathcal{C}^0(W^k_{\xi,j})} C(\theta_*^{1/s} \kappa^{-1})^{n-k},
\end{aligned} \tag{3.29}$$

where we have used Lemma 3.3(a) on each of the terms involving $\mathcal{I}_n(W^k_{\xi,j})$ from time k to time n with $\varsigma = \gamma$ and $s = (1-\gamma_0)/(1-\gamma)$.

For each $k \ge 1$, since $|W^k_{\xi,j}| \ge \delta_0/3$, we have by bounded distortion Lemma 3.1,

$$\begin{aligned}
\sum_{j \in L_k(W_\xi)} |W^k_{\xi,j}|^\gamma ||DT^k|^{-1} J_{W^k_{\xi,j}} T^n|_{\mathcal{C}^0(W^k_{\xi,j})} &\le (1+2C_d) 3\delta_0^{\gamma-1} \sum_{j \in L_k(W_\xi)} \int_{W^k_{\xi,j}} |DT^k|^{-1} J_{W^k_{\xi,j}} T^k dm_W \\
&\le C\delta_0^{\gamma-1} \int_{W_\xi} |DT^{-k}| \, dm_W.
\end{aligned}$$

Putting this estimate together with (3.27) and (3.29) yields,

$$\left| \int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, dm_W \right| \le C|h|_w(|\psi|_\infty + H^p_n(\psi)) \Big[f(W_\xi)|W_\xi|^\gamma + \sum_{k=1}^n (\theta_*^{1/s} \kappa^{-1})^{n-k} \int_{W_\xi} |DT^{-k}| \, dm_W \Big]$$

for some uniform constant C. Thus

$$\begin{aligned}\Big|\int_{M_\ell} \mathcal{L}^n h\, \psi\circ T^{-n}\, dm\Big| &\le C|h|_w(|\psi|_\infty + H_n^p(\psi))\Big(\sum_j \int_{E_j} f(W_\xi)|W_\xi|^\gamma\, d\lambda(\xi) + \sum_{t>k_0}\int_{E_t} f(W_\xi)|W_\xi|^\gamma\, d\lambda(\xi)\\ &\quad + \sum_j\sum_{k=1}^n (\theta_*^{1/s}\kappa^{-1})^{n-k}\int_{B_j}|DT^{-k}|dm + \sum_{t>k_0}\sum_{k=1}^n(\theta_*^{1/s}\kappa^{-1})^{n-k}\int_{\mathbb{H}_t}|DT^{-k}|dm\Big)\\ &\le C|h|_w(|\psi|_\infty + H_n^p(\psi))\Big(\sum_j \lambda(E_j) + \sum_{t>k_0}\int_{E_t} f(W_\xi)|W_\xi|^\gamma\, d\lambda(\xi)\\ &\quad + \sum_{k=1}^n (\theta_*^{1/s}\kappa^{-1})^{n-k}\int_{M_\ell}|DT^{-k}|dm\Big).\end{aligned}$$

The first two sums are finite since there are only finitely many E_j and using **(H3)**(5) for $t > k_0$. Since there are only finitely many M_ℓ by assumption on $\mathcal{S}_0$, the first two sums remain finite when we sum over ℓ. For the third sum, we sum over ℓ and use the fact that $\int_M |DT^{-k}|dm = 1$ for each $k \ge 1$. Thus the third sum is uniformly bounded in n using the fact that $\theta_*^{1/s}\kappa^{-1} < 1$ by **(H1)** and the discussion after Proposition 2.3 since $\frac{1-\gamma_0}{1-\gamma} < \frac{1-\gamma_0}{1-\alpha}$. □

The next lemma is very similar to [GL, Proposition 4.1] and is used in the proof of Lemma 2.2 to show that the relevant embeddings are in fact injective.

Lemma 3.8. *The embedding $B_w \hookrightarrow (\mathcal{C}^p(M))'$ is injective.*

Proof. For $h \in \mathcal{C}^1(M)$ and $W \in \mathcal{W}^s$, we define

$$\langle D_W^p(h), \psi\rangle = \int_W h\psi\, dm_W, \quad \psi \in \mathcal{C}^p(M).$$

Since $|\langle D_W^p(h),\psi\rangle| \le |h|_w |W|^\gamma f(W)|\psi|_{\mathcal{C}^p(W)}$, $D_W^p(h)$ defines a distribution of order p on M, i.e., $D_W^p(h) \in (\mathcal{C}^p(M))'$. And since the map $h \to D_W^p(h)$ is continuous in the $|\cdot|_w$ -norm, it can be extended to $\mathcal{B}_w$.

We assume $|h|_w \neq 0$ and show that $h \neq 0$ as an element of $(\mathcal{C}^p(M))'$. Since $|h|_w \neq 0$, there exists $\psi \in \mathcal{C}^p(M)$ and $W \in \mathcal{W}^s$ such that $\langle D_W^p(h),\psi\rangle =: \delta > 0$. Since the map $W \to \langle D_W^p(h),\psi\rangle$ is continuous for $h \in \mathcal{C}^1(M)$, by density, it is continuous for all $h \in B_w$. Thus we can find an open set E foliated by curves $W' \in \mathcal{W}^s$ close to W such that $\langle D_{W'}^p(h),\psi\rangle \ge \delta/2$ for each $W' \subset E$.

We localize the support of ψ to this set as follows. We extend each stable curve W' in E by a length $\varepsilon > 0$ in either direction to form a larger set $E' \supset E$. We call these extended curves W'_ε. We multiply ψ by a smooth bump function φ such that $\varphi = 0$ on $M \setminus E'$ and $\varphi = 1$ on E. We choose φ so that $|\varphi\psi|_{\mathcal{C}^p(W'_\varepsilon)} \le C|\psi|_{\mathcal{C}^p(M)}\varepsilon^{-p}$, for some uniform constant C. Then

$$\begin{aligned}\langle D_{W'_\varepsilon}^p(h), \varphi\psi\rangle &= \langle D_{W'}^p(h),\psi\rangle + \langle D_{W'_\varepsilon\setminus W'}^p(h),\varphi\psi\rangle\\ &\ge \delta/2 - C|\psi|_{\mathcal{C}^p(M)}\varepsilon^{-p}|W'_\varepsilon\setminus W'|^\gamma \ge \delta/2 - C|\psi|_{\mathcal{C}^p(M)}\varepsilon^{\gamma-p}.\end{aligned}$$

This can be made larger than $\delta/4$ by choosing ε sufficiently small since $\gamma > p$ by definition of the norms.

Thus the function $\varphi\psi \in \mathcal{C}^p(M)$ satisfies $h(\varphi\psi) \neq 0$. We conclude that $h \neq 0$ as an element of $(\mathcal{C}^p(M))'$. □

We conclude this section by proving the following important fact regarding compactness.

Lemma 3.9. *The unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$.*

Proof. Let $0 < \varepsilon \le \varepsilon_0$ be fixed. Let $k_\varepsilon \in \mathbb{N}$ be the least integer k such that $1/k^{r_h} < \varepsilon$. We split M into two parts, $A = \cup_{k \le k_\varepsilon} H_k$ and $B = M \setminus A$. By **(H3)**(4), any $W \in \mathcal{W}^s$ such that $W \subset B$ must satisfy $|W| \le C_2 \varepsilon$.

Let $h \in \mathcal{C}^1(M)$ with $\|h\|_{\mathcal{B}} \le 1$. First we estimate the weak norm of h on curves W in B. If $W \subset \mathbb{H}_k$ for $k \ge k_\varepsilon$, and $|\psi|_{W,\gamma,p} \le 1$, then

$$|\int_W h\psi \, dm_W| \le \|h\|_s |\psi|_{W,\alpha,q} \le \|h\|_s |W|^\alpha f(W) |\psi|_{\mathcal{C}^q(W)} \le C\|h\|_s \varepsilon^{\alpha-\gamma}. \tag{3.30}$$

Now on A, notice that there exists a constant $D_\varepsilon > 1$ such that $1/f(W) \le D_\varepsilon$. Also, since A contains only finitely many homogeneity strips, we may choose finitely many charts χ_i as defined in Section 3.1. In each chart, the set of functions $F \in \Xi$ is compact in the $\mathcal{C}^1$-norm. Thus we may choose finitely many curves $W_i \in \mathcal{W}^s$ such that $\{W_i\}_{i=1}^{N_\varepsilon}$ forms an ε-covering of $\mathcal{W}^s|_A$ in the distance $d_{\mathcal{W}^s}$. Indeed, we choose each of the W_i to satisfy $|W_i| \ge \varepsilon$ since we may approximate the norm of h on any stable curve with length less than ε by 0 according to (3.30).

For each W_i, let I_i be the interval on the horizontal axis in the chart on which the corresponding function G_{F_i} is defined, i.e., $W_i = G_{F_i}(I_i)$. Since any ball of finite radius in the $\mathcal{C}^p$-norm is compactly embedded in $\mathcal{C}^q$, we may choose finitely many functions $\overline{\psi}_{i,j} \in \mathcal{C}^p(I_i)$ such that $\{\overline{\psi}_{i,j}\}_{j=1}^{L_\varepsilon}$ forms an ε-covering in the $\mathcal{C}^q(I_i)$-norm of the ball of radius $C_g D_\varepsilon \varepsilon^{-\gamma}$ in $\mathcal{C}^p(I_i)$, where C_g is from (3.17).

Now let $W = G_{F_W}(I_W) \in \mathcal{W}^s|_A$ with $|W| \ge \varepsilon$, and $\psi \in \mathcal{C}^p(W)$ with $|\psi|_{W,\gamma,p} \le 1$. We fix a chart and choose one of the curves $W_i = G_{F_i}(I_i)$ such that $d_{\mathcal{W}^s}(W_i, W) < \varepsilon$. Let $\overline{\psi} = \psi \circ G_{F_W}$ be the push down of ψ to I_W and note that $|\overline{\psi}|_{\mathcal{C}^p(I_W)} \le C_g f(W)^{-1} |W|^{-\gamma} \le C_g D_\varepsilon \varepsilon^{-\gamma}$.

Next let $I = I_i \cap I_W$ and choose $\overline{\psi}_{i,j} \in \mathcal{C}^p(I_i)$ such that $|\overline{\psi} - \overline{\psi}_{i,j}|_{\mathcal{C}^q(I)} \le \varepsilon$. Define $\psi_{i,j} = \overline{\psi}_{i,j} \circ G_{F_i}^{-1}$ to be the lift of $\overline{\psi}_{i,j}$ to W_i. Note that

$$|\psi_{i,j}|_{W_i,\gamma,p} \le 2C_g \frac{f(W_i)|W_i|^\gamma}{f(W)|W|^\gamma} \le 2C_g C_f C_w.$$

by Lemma 3.4 since W_i and W lie in the same homogeneity region and $|W| \ge \varepsilon$. Then normalizing ψ and $\psi_{i,j}$ by $2C_g C_f C_w$, we estimate

$$\left| \int_W h\psi \; dm_W - \int_{W_i} h\psi_{i,j} \; dm_W \right| \;\le\; \varepsilon^\beta \|h\|_u 2C_g C_f C_w.$$

We have proved that for each $0 < \varepsilon \le \varepsilon_0$, there exist finitely many bounded linear functionals $\ell_{i,j}$, $\ell_{i,j}(h) = \int_{W_i} h\psi_{i,j} dm_W$, such that

$$|h|_w \le \max_{i \le N_\varepsilon; \, j \le L_\varepsilon} \ell_{i,j}(h) + \varepsilon^\beta C\|h\|_u + \varepsilon^{\alpha-\gamma} C\|h\|_s \le \max_{i \le N_\varepsilon; \, j \le L_\varepsilon} \ell_{i,j}(h) + \varepsilon^\beta C c_u^{-1} \|h\|_{\mathcal{B}},$$

which implies the required compactness. □

4. Lasota-Yorke Estimates

Since by Lemma 2.2, $\mathcal{L}$ is continuous on $\mathcal{B}$, it suffices to prove Proposition 2.3 for $h \in \mathcal{C}^1(M)$.

4.1. Estimating the Weak Norm.

Let $h \in \mathcal{C}^1(M)$, $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^p(W)$ such that $|\psi|_{W,\gamma,p} \le 1$. Let W_i^n denote the elements of $\mathcal{G}_n(W)$ as defined in Section 3.3. For $n \ge 0$, we write,

$$\int_W \mathcal{L}^n h \, \psi \, dm_W = \sum_{W_i^n \in \mathcal{G}_n(W)} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|} \psi \circ T^n \, dm_W, \tag{4.1}$$

where $J_{W_i^n} T^n$ denotes the Jacobian of T^n along W_i^n.

Using the definition of the weak norm on each W_i^n, we estimate (4.1) by

$$|\int_W \mathcal{L}^n h \, \psi \, dm_W| \;\le\; \sum_{W_i^n \in \mathcal{G}_n} |h|_w ||DT^n|^{-1} J_{W_i^n} T^n|_{\mathcal{C}^p(W_i^n)} |\psi \circ T^n|_{\mathcal{C}^p(W_i^n)} f(W_i^n) \, |W_i^n|^\gamma. \tag{4.2}$$

By (3.26), we have $|\psi \circ T^n|_{\mathcal{C}^p(W_i^n)} \le C|\psi|_{\mathcal{C}^p(W)} \le C f(W)^{-1}|W|^{-\gamma}$. Using this estimate plus (3.25) in equation (4.2), we obtain

$$|\int_W \mathcal{L}^n h\, \psi\, dm_W| \;\le\; C|h|_w \sum_{W_i^n \in \mathcal{G}_n} \frac{|W_i^n|^\gamma}{|W|^\gamma} \cdot \frac{f(W_i^n)}{f(W)} \, ||DT^n|^{-1} J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)}.$$

Finally, using (3.28) we estimate

$$|\int_W \mathcal{L}^n h\, \psi\, dm_W| \le C|h|_w \kappa^{-n} \sum_{W_i^n \in \mathcal{G}_n} \left(\frac{|W_i^n|}{|W|} \right)^\gamma \cdot |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \le CC_5 |h|_w \kappa^{-n},$$

where in the last inequality we have used Lemma 3.3(b) with $\varsigma = \gamma$. Taking the supremum over all $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^p(W)$ with $|\psi|_{W,\gamma,p} \le 1$ yields (2.11).

4.2. **Estimating the Strong Stable Norm.** As before, let $h \in \mathcal{C}^1(M)$, $W \in \mathcal{W}^s$ and denote by W_i^n the elements of $\mathcal{G}_n(W)$. For $\psi \in \mathcal{C}^q(W)$ with $|\psi|_{W,\alpha,q} \le 1$, define $\overline{\psi}_i = |W_i^n|^{-1} \int_{W_i^n} \psi \circ T^n \, dm_W$. Following equation (4.1), we write

$$\int_W \mathcal{L}^n h\, \psi\, dm_W = \sum_i \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|} (\psi \circ T^n - \overline{\psi}_i)\, dm_W + \overline{\psi}_i \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|}\, dm_W. \tag{4.3}$$

To estimate the first term on the right hand side of (4.3), we first estimate $|\psi \circ T^n - \overline{\psi}_i|_{\mathcal{C}^q(W_i^n)}$. If $H_W^q(\psi)$ denotes the Hölder constant of ψ along W, then equation (3.26) implies $H_{W_i^n}^q(\psi \circ T^n - \overline{\psi}_i) \le C\Lambda^{-qn} H_W^q(\psi)$, since $\overline{\psi}_i$ is constant on W_i^n. To estimate the $\mathcal{C}^0$ norm, note that $\overline{\psi}_i = \psi \circ T^n(y_i)$ for some $y_i \in W_i^n$. Thus for each $x \in W_i^n$,

$$|\psi \circ T^n(x) - \overline{\psi}_i| = |\psi \circ T^n(x) - \psi \circ T^n(y_i)| \le H_{W_i^n}^q(\psi \circ T^n) |W_i^n|^q \le C H_W^q(\psi) \Lambda^{-qn}.$$

These estimates together with the fact that $|\varphi|_{W,\alpha,q} \le 1$ imply

$$|\psi \circ T^n - \overline{\psi}_i|_{\mathcal{C}^q(W_i^n)} \le C\Lambda^{-qn} |\psi|_{\mathcal{C}^q(W)} \le C\Lambda^{-qn} |W|^{-\alpha} f(W)^{-1}. \tag{4.4}$$

We apply (4.4), the distortion estimate (3.25) and the definition of the strong stable norm to the first term of (4.3),

$$\begin{aligned} |\sum_i \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|} (\psi \circ T^n - \overline{\psi}_i)\, dm_W| &\le C \sum_i \|h\|_s \frac{|W_i^n|^\alpha}{|W|^\alpha} \frac{f(W_i^n)}{f(W)} \left| \frac{J_{W_i^n} T^n}{|DT^n|} \right|_{C^0(W_i^n)} \Lambda^{-qn} \\ &\le\; C\Lambda^{-qn} \kappa^{-n} \|h\|_s \sum_i \frac{|W_i^n|^\alpha}{|W|^\alpha} |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \;\le C' \Lambda^{-qn} \kappa^{-n} \|h\|_s, \end{aligned} \tag{4.5}$$

where in the second line we have used (3.28) and Lemma 3.3(b) with $\varsigma = \alpha$.

For the second term on the right hand side of (4.3), we use the fact that $|\overline{\psi}_i| \le |W|^{-\alpha} f(W)^{-1}$ since $|\psi|_{W,\alpha,q} \le 1$. Recall the notation introduced before the statement of Lemma 3.3. Grouping the pieces $W_i^n \in \mathcal{G}_n(W)$ according to most recent long ancestors, we have

$$\begin{aligned} \sum_i \frac{1}{|W|^\alpha f(W)} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|}\, dm_W =& \sum_{k=1}^n \sum_{j \in L_k} \sum_{i \in \mathcal{I}_n(W_j^k)} \frac{1}{|W|^\alpha f(W)} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|}\, dm_W \\ &+ \sum_{i \in \mathcal{I}_n(W)} \frac{1}{|W|^\alpha f(W)} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|}\, dm_W \end{aligned}$$

where we have split up the terms involving $k = 0$ and $k \geq 1$. We estimate the terms with $k \geq 1$ by the weak norm and the terms with $k = 0$ by the strong stable norm,

$$\sum_i \frac{1}{|W|^\alpha f(W)} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|} \, dm_W \leq C \sum_{k=1}^n \sum_{j \in L_k} \sum_{i \in \mathcal{I}_n(W_j^k)} \frac{|W_i^n|^\gamma f(W_i^n)}{|W|^\alpha f(W)} |h|_w \left| \frac{J_{W_i^n} T^n}{|DT^n|} \right|_{\mathcal{C}^0(W_i^n)}$$
$$+ C \sum_{i \in \mathcal{I}_n(W)} \frac{|W_i^n|^\alpha f(W_i^n)}{|W|^\alpha f(W)} \|h\|_s ||DT^n|^{-1} J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)}.$$

As usual, by (3.28), the ratio of f's times $|DT^n|^{-1}$ is bounded by $C\kappa^{-n}$.

In the first sum above corresponding to $k \geq 1$, we split the Jacobians according to times k and $n-k$ and use Lemma 3.3(a) in each term from time k to time $n-k$,

$$\sum_{i \in \mathcal{I}_n(W_j^k)} \frac{|W_i^n|^\gamma}{|W_j^k|^\gamma} |J_{W_i^n} T^{n-k}|_{\mathcal{C}^0(W_i^n)} \leq C_4 \theta_*^{(n-k)/s},$$

where $s = \frac{1-\gamma_0}{1-\gamma}$. Using this estimate, we obtain,

$$\sum_{k=1}^n \sum_{j \in L_k} \sum_{i \in \mathcal{I}_n(W_j^k)} \frac{|W_i^n|^\gamma}{|W|^\alpha} \frac{|W_j^k|^\alpha}{|W_j^k|^\alpha} \frac{f(W_i^n)}{f(W)} \left| \frac{J_{W_i^n} T^n}{|DT^n|} \right|_{\mathcal{C}^0(W_i^n)}$$
$$\leq C \delta_0^{\gamma-\alpha} \kappa^{-n} \sum_{k=1}^n \sum_{j \in L_k} \frac{|W_j^k|^\alpha}{|W|^\alpha} |J_{W_j^k} T^k|_{\mathcal{C}^0(W_j^k)} \theta_*^{(n-k)/s},$$

since $|W_j^k| \geq \delta_0/3$. The last two sums are bounded independently of n and W by Lemma 3.3(b) with $\varsigma = \alpha$.

Finally, for the sum corresponding to $k = 0$, we have

$$\sum_{i \in \mathcal{I}_n(W)} \frac{|W_i^n|^\alpha}{|W|^\alpha} |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \leq C \theta_*^{n/s_0},$$

again using Lemma 3.3(a) with $s_0 = \frac{1-\gamma_0}{1-\alpha}$.

Gathering these estimates together, we obtain,

$$\sum_i \frac{1}{|W|^\alpha f(W)} \left| \int_{W_i^n} h |DT^n|^{-1} J_{W_i^n} T^n \, dm_W \right| \leq C \delta_0^{\gamma-\alpha} |h|_w \kappa^{-n} + C \|h\|_s \theta_*^{n/s_0} \kappa^{-n}. \tag{4.6}$$

Putting together (4.5) and (4.6) in (4.3) proves (2.12),

$$\|\mathcal{L}^n h\|_s \leq C(\Lambda^{-qn} + \theta_*^{n/s_0}) \kappa^{-n} \|h\|_s + C \delta_0^{\gamma-\alpha} \kappa^{-n} |h|_w.$$

4.3. **Estimating the Strong Unstable Norm.** Fix $\varepsilon \leq \varepsilon_0$ and consider two curves $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) \leq \varepsilon$. For $n \geq 1$, we describe how to partition $T^{-n} W^\ell$, $\ell = 1, 2$, into matched pieces U_j^ℓ to which we will apply the strong unstable norm $\| \cdot \|_u$ and unmatched pieces V_k^ℓ to which we will apply the strong stable norm $\| \cdot \|_s$.

Recall $\mathcal{S}_0^{\mathbb{H}} = \mathcal{S}_0 \cup (\cup_{k > k_0} S_k^H)$ and define $\mathcal{S}_{-n}^{\mathbb{H}} := \cup_{i=0}^n T^i(\mathcal{S}_0^{\mathbb{H}})$ to be the expanded singularity set for T^{-n} taking into account the boundaries of the homogeneity regions. Let ω be a connected component of $W^1 \setminus \mathcal{S}_{-n}^{\mathbb{H}}$. To each point $x \in T^{-n}\omega$, we associate an unstable curve γ_x (vertical in the chart) of length at most $C\Lambda^{-n}\varepsilon$ such that its image $T^n \gamma_x$, if not cut by a singularity or the boundary of a homogeneity strip, will have length $C\varepsilon$. By assumption **(H2)**, all the tangent vectors to $T^i \gamma_x$ lie in the unstable cone $C^u(T^i x)$ for each $i \geq 0$ so that they remain uniformly transverse to the stable cone and enjoy the uniform expansion given by **(H2)**(2).

Doing this for each connected component of $W^1 \setminus \mathcal{S}^{\mathbb{H}}_{-n}$, we subdivide $W^1 \setminus \mathcal{S}^{\mathbb{H}}_{-n}$ into a countable collection of subintervals of points for which $T^n\gamma_x$ intersects $W^2 \setminus \mathcal{S}^{\mathbb{H}}_{-n}$ and subintervals for which this is not the case. This in turn induces a corresponding partition on $W^2 \setminus \mathcal{S}^{\mathbb{H}}_{-n}$.

We denote by V^ℓ_k the pieces in $T^{-n}W^\ell$ which are not matched up by this process and note that the images $T^nV^\ell_k$ occur either at the endpoints of W^ℓ or because the vertical segment γ_x has been cut by a singularity. In both cases, the length of the curves $T^nV^\ell_k$ can be at most $C\varepsilon^{t_0}$ due to the type of tangency allowed between curves in $\mathcal{S}^{\mathbb{H}}_{-1}$ and the stable cone by **(H3)**(3),(4).

In the remaining pieces the foliation $\{T^n\gamma_x\}_{x\in T^{-n}W^1}$ provides a one-to-one correspondence between points in W^1 and W^2. We further subdivide these curves in W^ℓ in such a way that the lengths of their images under T^{-i} is less than δ_0 for each $0 \le i \le n$ and these subdivided pieces are pairwise matched by the foliation $\{T^n\gamma_x\}$. We call these matched pieces U^ℓ_j. Possibly changing $\delta_0/2$ to δ_0/C for some uniform constant $C > 0$ (depending only on the distortion constant and the angle between stable and unstable cones), in the definition of $\mathcal{G}_n(W)$, we can arrange it so that $U^\ell_j \subset W^{\ell,n}_i$ for some $W^{\ell,n}_i \in \mathcal{G}_n(W^\ell)$ and $V^\ell_k \subset W^{\ell,n}_{i'}$ for some $W^{\ell,n}_i \in \mathcal{G}_n(W^\ell)$ for all $j,k \ge 1$ and $\ell = 1,2$. There are at most one U^ℓ_j and two V^ℓ_k per $W^{\ell,n}_i \in \mathcal{G}_n(W^\ell)$.

In this way we write $W^\ell = (\cup_j T^nU^\ell_j) \cup (\cup_k T^nV^\ell_k)$. Note that the images $T^nV^\ell_k$ of the unmatched pieces must be short while the images of the matched pieces U^ℓ_j may be long or short. Recalling the notation of Section 3.1, we have arranged a pairing of the pieces U^k_j with the following property:

$$\begin{aligned} &\text{If } \ U^1_j = G_{F^1_j}(I_j) = \{\chi_{i_j}(x^1_j + (t, F_{U^1_j}(t))) : t \in I_j\}, \\ &\text{then } \ U^2_j = G_{F^2_j}(I_j) = \{\chi_{i_j}(x^2_j + (t, F_{U^2_j}(t))) : t \in I_j\}, \end{aligned} \tag{4.7}$$

so that the point $x = x^1_j + (t, F_{U^1_j}(t))$ in the chart is associated with the point $\bar{x} = x^2_j + (t, F_{U^2_j}(t))$ by the vertical segment $\chi^{-1}_{i_j}(\gamma_x)$ for each $t \in I_j$.

Given ψ_ℓ on W^ℓ with $|\psi_\ell|_{W^\ell,\gamma,p} \le 1$ and $d_q(\psi_1,\psi_2) \le \varepsilon$, with the above construction we must estimate

$$\begin{aligned} &\left| \int_{W^1} \mathcal{L}^n h\, \psi_1 \, dm_W - \int_{W^2} \mathcal{L}^n h\, \psi_2 \, dm_W \right| \\ &\qquad \le \sum_{\ell,k} \left| \int_{T^nV^\ell_k} \mathcal{L}^n h\, \psi_\ell \, dm_W \right| + \sum_j \left| \int_{T^nU^1_j} \mathcal{L}^n h\, \psi_1 \, dm_W - \int_{T^nU^2_j} \mathcal{L}^n h\, \psi_2 \, dm_W \right|. \end{aligned} \tag{4.8}$$

We do the estimate over the unmatched pieces V^ℓ_k first using the strong stable norm. To do this, we group pieces $T^nV^\ell_k$ in the following manner. We say $T^nV^\ell_k$ is created at time $0 \le t \le n-1$ if t is the first time that an endpoint of $T^{n-t}V^\ell_k$ is created by an intersection with $\mathcal{S}^{\mathbb{H}}_{-1}$. Note that due to the transversality conditions **(H3)**(3),(4), we have $|T^{n-t}V^\ell_k| \le C\varepsilon^{t_0}$, where C is a uniform constant. We set $A(t) = \{(k,\ell) : V^\ell_k \text{ created at time } t\}$. We will change variables to estimate the norm on $T^{n-t-1}V^\ell_k$ for $(k,\ell) \in A(t)$.

The expression we must estimate on unmatched pieces is

$$\left| \sum_{\ell,k} \int_{T^nV^\ell_k} \mathcal{L}^n h\, \psi \, dm_W \right| = \sum_{t=0}^{n-1} \sum_{(k,\ell)\in A(t)} \int_{T^{n-t-1}V^\ell_k} (\mathcal{L}^{n-t-1}h)\, |DT^{t+1}|^{-1} J_{T^{n-t-1}V^\ell_k} T^{t+1}\, \psi \circ T^{t+1} \, dm_W.$$

Note that by (3.26), $|\psi_\ell \circ T^{t+1}|_{\mathcal{C}^q(T^{n-t-1}V_k^\ell)} \le C|\psi_\ell|_{\mathcal{C}^p(W^\ell)} \le Cf(W^\ell)^{-1}|W^\ell|^{-\gamma}$. Fixing t, k and ℓ, we estimate following (4.5),

$$\begin{aligned}
&\int_{T^{n-t-1}V_k^\ell} (\mathcal{L}^{n-t-1}h)\, |DT^{t+1}|^{-1} J_{T^{n-t-1}V_k^\ell} T^{t+1}\, \psi \circ T^{t+1}\, dm_W \\
&\qquad \le \|\mathcal{L}^{n-t-1}h\|_s \frac{|T^{n-t-1}V_k^\ell|^\alpha}{|W^\ell|^\gamma} \frac{f(T^{n-t-1}V_k^\ell)}{f(W^\ell)} \left| \frac{|J_{T^{n-t-1}V_k^\ell}T^{n-t-1}}{|DT^{t+1}|^{-1}} \right|_{\mathcal{C}^0(T^{n-t-1}V_k^\ell)} \\
&\qquad \le C\kappa^{-n}\|h\|_s |T^{n-t-1}V_k^\ell|^{\alpha-\gamma} \frac{|T^{n-t-1}V_k^\ell|^\gamma}{|W^\ell|^\gamma} |J_{T^{n-t-1}V_k^\ell}T^{n-t-1}|_{\mathcal{C}^0},
\end{aligned} \tag{4.9}$$

where in the last line, we have used (3.28) as well as the bound $\|\mathcal{L}^i h\|_s \le C\kappa^{-i}\|h\|_s$ for any i from Section 4.2.

Now since $|T^{n-t}V_k^\ell| \le C\varepsilon^{t_0}$, by **(H3)**(1) we have $|T^{n-t-1}V_k^\ell| \le C\varepsilon^{\xi t_0}$. Also, we estimate over pieces $T^{n-t-1}V_k^\ell$ rather than $T^{n-t}V_k^\ell$ because we have created $T^{n-t}V_k^\ell$ due to an intersection with $\mathcal{S}_{-1}^{\mathbb{H}}$, but this is one step earlier than we would cut pieces for our generation $\mathcal{G}_t(W^\ell)$ as described in Section 3.3. There may be many pieces $T^{n-t}V_k^\ell$ for each connected component of $\mathcal{G}_t(W^\ell)$; however, there are at most two pieces $T^{n-t-1}V_k^\ell$, $(k,\ell) \in A(t)$ in each connected component of $\mathcal{G}_{t+1}(W^\ell)$ so that we can control the sum over these pieces via Lemma 3.3(b).

Using these facts together with (4.9), we estimate,

$$\begin{aligned}
|\sum_{\ell,k} \int_{T^nV_k^\ell} \mathcal{L}^n h\psi| &\le C\kappa^{-n}\|h\|_s \varepsilon^{\xi t_0(\alpha-\gamma)} \sum_{t=0}^{n-1} \sum_{(k,\ell)\in A(t)} \frac{|T^{n-t-1}V_k^\ell|^\gamma}{|W^\ell|^\gamma} |J_{T^{n-t-1}V_k^\ell}T^{n-t-1}|_{\mathcal{C}^0} \\
&\le Cn\kappa^{-n}\|h\|_s \varepsilon^{\xi t_0(\alpha-\gamma)},
\end{aligned} \tag{4.10}$$

where we have used Lemma 3.3(b) in the last line on the sum over each set $A(t)$. Now the exponent of ε is at least β since we chose $\beta < \xi t_0(\alpha - \gamma)$ in the definition of the norms.

The only pieces not covered by the above estimate are those pieces created at the endpoints of W^1 or W^2 (and not due to any singularity cuts). There are at most 2 such pieces and they each have length less than $C\varepsilon$ by definition of $d_{\mathcal{W}^s}(\cdot,\cdot)$. Thus we estimate directly on these pieces,

$$|\int_{T^nV_k^\ell} \mathcal{L}^n h\, \psi| \le \|\mathcal{L}^n h\|_s |T^nV_k^\ell|^\alpha f(T^nV_k^\ell) |\psi|_{\mathcal{C}^q(W^\ell)} \le C\kappa^{-n}\|h\|_s \varepsilon^{\alpha-\gamma} \frac{|T^nV_k^\ell|^\gamma}{|W^\ell|^\gamma} \frac{f(T^nV_k^\ell)}{f(W^\ell)} \tag{4.11}$$

and the two ratios are bounded since $T^nV_k^\ell \subset W^\ell$ and using Lemma 3.4(a). Since $\alpha - \gamma \ge \beta$, this completes the estimate on the unmatched pieces.

Next, we estimate the difference of matched pieces in (4.8). Recalling the notation defined by (4.7), on each U_j^2 we define

$$\phi_j = (|DT^n|^{-1} J_{U_j^1} T^n\, \psi_1 \circ T^n) \circ G_{F_j^1} \circ G_{F_j^2}^{-1}.$$

The function ϕ_j is well defined on U_j^2 and changing variables to integrate on U_j^ℓ, we must estimate,

$$\begin{aligned}
&\left| \int_{U_j^1} h|DT^n|^{-1} J_{U_j^1}T^n\, \psi_1 \circ T^n - \int_{U_j^2} h|DT^n|^{-1} J_{U_j^2}T^n\, \psi_2 \circ T^n \right| \\
&\le \left| \int_{U_j^1} h|DT^n|^{-1} J_{U_j^1}T^n\, \psi_1 \circ T^n - \int_{U_j^2} h\, \phi_j \right| + \left| \int_{U_j^2} h(\phi_j - |DT^n|^{-1} J_{U_j^2}T^n\, \psi_2 \circ T^n) \right|.
\end{aligned} \tag{4.12}$$

We estimate the first term on the right hand side of equation (4.12) using the strong unstable norm. The estimates (3.25), (3.28) and (3.26) imply that

$$\begin{aligned} &|\,|DT^n|^{-1}J_{U_j^1}T^n\cdot\psi_1\circ T^n|_{U_j^1,\gamma,p} = f(U_j^1)|U_j^1|^\gamma|\,|DT^n|^{-1}J_{U_j^1}T^n\cdot\psi_1\circ T^n|_{\mathcal{C}^p(U_j^1)} \\ &\le C\frac{f(U_j^1)}{f(W^1)}\cdot\frac{|U_j^1|^\gamma}{|W^1|^\gamma}\cdot|\,|DT^n|^{-1}J_{U_j^1}T^n|_{\mathcal{C}^0(U_j^1)} \le C\kappa^{-n}|J_{U_j^1}T^n|_{\mathcal{C}^0(U_j^1)}\frac{|U_j^1|^\gamma}{|W^1|^\gamma}. \end{aligned} \tag{4.13}$$

Similarly, since $\mathrm{Lip}(G_{F_j^1}\circ G_{F_j^2}^{-1})\le C_g$, where C_g is from (3.17),

$$|\phi_j|_{U_j^2,\gamma,p}\le C\frac{f(U_j^2)}{f(W^1)}\frac{|U_j^2|^\gamma}{|W^1|^\gamma}|\,|DT^n|^{-1}J_{U_j^1}T^n|_{\mathcal{C}^0(U_j^1)}\le C\kappa^{-n}|J_{U_j^1}T^n|_{\mathcal{C}^0(U_j^1)}\frac{|U_j^1|^\gamma}{|W^1|^\gamma},$$

where $\frac{f(U_j^2)}{f(U_j^1)}\le C_f$ by Lemma 3.4(a) since the two curves lie in the same homogeneity strip, and we have used the fact that $|U_j^2|\le C|U_j^1|$ due to the pairing (4.7). By the definition of ϕ_j and $d_q(\cdot,\cdot)$,

$$d_q(|DT^n|^{-1}J_{U_j^1}T^n\psi_1\circ T^n,\phi_j)=\left|\left[|DT^n|^{-1}J_{U_j^1}T^n\psi_1\circ T^n\right]\circ G_{F_j^1}-\phi_j\circ G_{F_j^2}\right|_{\mathcal{C}^q(I_j)} = 0.$$

Finally, we note that by Lemma 3.2, we have $d_{\mathcal{W}^s}(U_j^1,U_j^2)\le C_*\Lambda^{-n}d_{\mathcal{W}^s}(W^1,W^2)\le C_*\Lambda^{-n}\varepsilon=:\varepsilon_1$.

In view of (4.13), we renormalize the test functions by $R_j = C\kappa^{-n}|J_{U_j^1}T^n|_{\mathcal{C}^0(U_j^1)}\frac{|U_j^1|^\gamma}{|W^1|^\gamma}$. Then we apply the definition of the strong unstable norm with ε_1 in place of ε. Thus,

$$\begin{aligned} \sum_j\left|\int_{U_j^1}h|DT^n|^{-1}J_{U_j^1}T^n\,\psi_1\circ T^n-\int_{U_j^2}h\,\phi_j\right| &\le C\varepsilon_1^\beta\kappa^{-n}\|h\|_u\sum_j\frac{|U_j^1|^\gamma}{|W^1|^\gamma}|J_{U_j^1}T^n|_{\mathcal{C}^0(U_j^1)} \\ &\le C\|h\|_u\Lambda^{-n\beta}\kappa^{-n}\varepsilon^\beta \end{aligned} \tag{4.14}$$

where the sum is $\le C_5$ by Lemma 3.3(b) with $\varsigma=\gamma$ since there is at most one matched piece U_j^1 corresponding to each component of $T^{-n}W^1$, $W_i^{1,n}\in\mathcal{G}_n(W^1)$.

Now we estimate the second term on the right hand side of (4.12) using the strong stable norm,

$$\begin{aligned} &|\int_{U_j^2}h\left(\phi_j-|DT^n|^{-1}J_{U_j^2}T^n\psi_2\circ T^n\right)| \\ &\qquad\le\; C\|h\|_s|U_j^2|^\alpha f(U_j^2)\left|\phi_j-|DT^n|^{-1}J_{U_j^2}T^n\psi_2\circ T^n\right|_{\mathcal{C}^q(U_j^2)}. \end{aligned} \tag{4.15}$$

In order to estimate the $\mathcal{C}^q$-norm of the function in (4.15), we split it up into two differences. Since $\mathrm{Lip}(G_{F_j^\ell})$, $\mathrm{Lip}(G_{F_j^\ell}^{-1})\le C_g$, $\ell=1,2$, we write

$$\begin{aligned} &|\phi_j-(|DT^n|^{-1}J_{U_j^2}T^n)\cdot\psi_2\circ T^n|_{\mathcal{C}^q(U_j^2)} \\ &\le C\left|\left[(|DT^n|^{-1}J_{U_j^1}T^n)\cdot\psi_1\circ T^n\right]\circ G_{F_j^1}-\left[(|DT^n|^{-1}J_{U_j^2}T^n)\cdot\psi_2\circ T^n\right]\circ G_{F_j^2}\right|_{\mathcal{C}^q(I_j)} \\ &\le C\left|(|DT^n|^{-1}J_{U_j^1}T^n)\circ G_{F_j^1}\left[\psi_1\circ T^n\circ G_{F_j^1}-\psi_2\circ T^n\circ G_{F_j^2}\right]\right|_{\mathcal{C}^q(I_j)} \\ &\quad+C\left|\left[(|DT^n|^{-1}J_{U_j^1}T^n)\circ G_{F_j^1}-(|DT^n|^{-1}J_{U_j^2}T^n)\circ G_{F_j^2}\right]\psi_2\circ T^n\circ G_{F_j^2}\right|_{\mathcal{C}^q(I_j)} \\ &\le C|\,|DT^n|^{-1}J_{U_j^1}T^n|_{\mathcal{C}^0(U_j^1)}\left|\psi_1\circ T^n\circ G_{F_j^1}-\psi_2\circ T^n\circ G_{F_j^2}\right|_{\mathcal{C}^q(I_j)} \\ &\quad+Cf(W^2)^{-1}|W^2|^{-\gamma}\left|(|DT^n|^{-1}J_{U_j^1}T^n)\circ G_{F_j^1}-(|DT^n|^{-1}J_{U_j^2}T^n)\circ G_{F_j^2}\right|_{\mathcal{C}^q(I_j)}, \end{aligned} \tag{4.16}$$

where in the last step we have used (3.25). In order to bound these two terms, we prove the following lemma.

Lemma 4.1. *There exists $C > 0$ such that for each $j \geq 1$,*

(a) $|(|DT^n|^{-1} J_{U^1_j} T^n) \circ G_{F^1_j} - (|DT^n|^{-1} J_{U^2_j} T^n) \circ G_{F^2_j}|_{\mathcal{C}^q(I_j)} \leq C||DT^n|^{-1} J_{U^2_j} T^n|_{\mathcal{C}^0(U^2_j)} \varepsilon^{p_0 - q}$;

(b) $|\psi_1 \circ T^n \circ G_{F^1_j} - \psi_2 \circ T^n \circ G_{F^2_j}|_{\mathcal{C}^q(I_{r_j})} \leq C f(W^2)^{-1} |W^2|^{-\gamma} \varepsilon^{p-q}$.

We postpone the proof of the lemma to Section 4.3.1 and show how this completes the estimate on the strong unstable norm. Notice that $||DT^n|^{-1} J_{U^1_j} T^n|_{\mathcal{C}^0(U^1_j)} \leq C||DT^n|^{-1} J_{U^2_j} T^n|_{\mathcal{C}^0(U^2_j)}$ by (4.18) in the proof of Lemma 4.1(a). Then using Lemma 4.1 together with (4.16) yields by (4.15)

$$\begin{aligned}
\sum_j \Big| \int_{U^2_j} h(\phi_j - |DT^n|^{-1} J_{U^2_j} T^n \psi_2 \circ T^n) \, dm_W \Big| \\
&\leq C \|h\|_s \sum_j \frac{|U^2_j|^\alpha f(U^2_j)}{|W^2|^\gamma f(W^2)} \Big| |DT^n|^{-1} J_{U^2_j} T^n \Big|_{\mathcal{C}^0(U^2_j)} \varepsilon^{p-q} \\
&\leq C \|h\|_s \varepsilon^{p-q} \kappa^{-n} \sum_j \frac{|U^2_j|^\gamma}{|W^2|^\gamma} |J_{U^2_j} T^n|_{\mathcal{C}^0(U^2_j)},
\end{aligned} \tag{4.17}$$

where we have used (3.28) in the last step and the sum is finite by Lemma 3.3(b). This completes the estimate on the second term on the right hand side of (4.12). Now we use this bound, together with (4.10) and (4.14) to estimate (4.8)

$$\left| \int_{W^1} \mathcal{L}^n h \, \psi_1 \, dm_W - \int_{W^2} \mathcal{L}^n h \, \psi_2 \, dm_W \right| \leq C \kappa^{-n} (n \|h\|_s \varepsilon^{\xi t_0 (\alpha - \gamma)} + \|h\|_u \Lambda^{-n\beta} \varepsilon^\beta + \|h\|_s \varepsilon^{p-q}).$$

Since $\beta < \min\{p - q, \xi t_0(\alpha - \gamma)\}$, we may divide through by ε^β and take the appropriate suprema to complete the proof of (2.13).

4.3.1. *Proof of Lemma 4.1.* We recall the following general fact whose proof can be found in [DZ2, Lemma 4.3].

Lemma 4.2 ([DZ2])**.** *Let X be a metric space and choose $0 < r < s \leq 1$. Suppose $g_1, g_2 \in \mathcal{C}^s(X)$ satisfy $|g_1 - g_2|_{\mathcal{C}^0(X)} \leq D_1 \varepsilon^s$ for some constant $D_1 > 0$. Then*

$$|g_1 - g_2|_{\mathcal{C}^r(X)} \leq 3\varepsilon^{s-r} \max\{D_1, H^s(g_1) + H^s(g_2)\},$$

where $H^s(\cdot)$ denotes the Hölder constant of exponent s on X.

Proof of Lemma 4.1(a). Throughout the proof, for ease of notation we write J^n_k for $|DT^n|^{-1} J_{U^k_j} T^n$.

For any $t \in I_j$, $x = G_{F^1_j}(t)$ and $\bar{x} = G_{F^2_j}(t)$ lie on a common unstable curve γ_x (which is a vertical line segment in the chart). Note that $d_W(T^n x, T^n \bar{x}) \leq C\varepsilon$ since $T^n(x)$ and $T^n(\bar{x})$ lie on the element $T^n \gamma_x \in \mathcal{W}^u$ which intersects W^1 and W^2; this curve has length at most $C\varepsilon$ due to the uniform transversality of stable and unstable cones. By (3.2) and Lemma 3.2(b),

$$|J^n_1(x) - J^n_2(\bar{x})| \leq C |J^n_2|_{\mathcal{C}^0(U^2_j)} (d(T^n x, T^n \bar{x})^{p_0} + \theta(T^n x, T^n \bar{x})),$$

where $\theta(T^n x, T^n \bar{x})$ is the angle between the tangent line to W^1 at $T^n x$ and the tangent line to W^2 at $T^n \bar{x}$. Let $y \in W^2$ be the unique point in W^2 whose lift $\chi_i^{-1}(y)$ in the chart containing W^1 and W^2 lies on the same vertical segment as $\chi_i^{-1}(T^n x)$. Since by assumption $d_{\mathcal{W}^s}(W^1, W^2) \leq \varepsilon$, we have $\theta(T^n x, y) \leq \varepsilon$. Due to the uniform transversality of curves in $\mathcal{W}^u$ and $\mathcal{W}^s$ and the fact that W^1 and W^2 are graphs of C^2 functions with uniformly bounded C^2 norms, we have $\theta(y, T^n \bar{x}) \leq C\varepsilon$ and so $\theta(T^n x, T^n \bar{x}) \leq C\varepsilon$. Thus,

$$|J^n_1(x) - J^n_2(\bar{x})| \leq C \varepsilon^{p_0} |J^n_2|_{\mathcal{C}^0(U^2_j)}. \tag{4.18}$$

Noting that $\mathrm{Lip}(G_{F_j^\ell}) \le C_g$, $\ell = 1, 2$, (4.18) implies that

$$|J_1^n \circ G_{F_j^1} - J_2^n \circ G_{F_j^2}|_{\mathcal{C}^0(I_j)} \le C |J_2^n|_{\mathcal{C}^0(U_j^2)} \varepsilon^{p_0}.$$

Now the fact that $J_1^n \circ G_{F_j^1}, J_2^n \circ G_{F_j^2} \in \mathcal{C}^{p_0}(I_j)$ means we may apply Lemma 4.2 to their difference to conclude

$$|J_1^n \circ G_{F_j^1} - J_2^n \circ G_{F_j^2}|_{\mathcal{C}^q(I_j)} \le C(|J_1^n|_{\mathcal{C}^{p_0}(U_j^1)} + |J_2^n|_{\mathcal{C}^{p_0}(U_j^2)}) \varepsilon^{p_0 - q}.$$

The lemma follows since $|J_\ell^n|_{\mathcal{C}^{p_0}(U_j^\ell)} \le C|J_\ell^n|_{\mathcal{C}^0(U_j^\ell)}$ by (3.25) and $|J_1^n|_{\mathcal{C}^0(U_j^1)} \le C|J_2^n|_{\mathcal{C}^0(U_j^2)}$ by (4.18). □

Proof of Lemma 4.1(b). Let F_{W^ℓ} be the function whose graph $G_{W^\ell}(I_{W^\ell}) = W^\ell$, and set $g_j^\ell := G_{W^\ell}^{-1} \circ T^n \circ G_{F_j^\ell}$, $\ell = 1, 2$. Notice that since $\mathrm{Lip}(G_{W^\ell}^{-1})$, $\mathrm{Lip}(G_{F_j^\ell}) \le C_g$, and due to the uniform contraction along stable curves, we have $\mathrm{Lip}(g_j^\ell) \le C$, where C is independent of W^ℓ and j. We may assume that $g_j^\ell(I_j) \subset I_{W^1} \cap I_{W^2}$ since if not, by the uniform transversality of $C^u(x)$ and $C^s(x)$, we must be in a neighborhood of one of the endpoints of W^ℓ of length at most $C\varepsilon$; such short pieces may be estimated as in (4.11) using the strong stable norm. Thus

$$\begin{aligned} |\psi_1 \circ T^n \circ G_{F_j^1} - \psi_2 \circ T^n \circ G_{F_j^2}|_{\mathcal{C}^q(I_j)} \le |\psi_1 \circ G_{W^1} \circ g_j^1 - \psi_2 \circ G_{W^2} \circ g_j^1|_{\mathcal{C}^q(I_j)} \\ + |\psi_2 \circ G_{W^2} \circ g_j^1 - \psi_2 \circ G_{W^2} \circ g_j^2|_{\mathcal{C}^q(I_j)}. \end{aligned} \tag{4.19}$$

Using the above observation about g_j^1, we estimate the first term of (4.19) by

$$|\psi_1 \circ G_{W^1} \circ g_j^1 - \psi_2 \circ G_{W^2} \circ g_j^1|_{\mathcal{C}^q(I_j)} \le C|\psi_1 \circ G_{W^1} - \psi_2 \circ G_{W^2}|_{\mathcal{C}^q(g_j^1(I_j))} \le C\varepsilon, \tag{4.20}$$

by definition of $d_q(\psi_1, \psi_2)$.

To estimate the second term of (4.19), notice that for $t \in I_j$, $g_j^1(t) - g_j^2(t)$ measures the difference in the horizontal coordinates (in the chart) of $T^n \circ G_{F_j^1}(t)$ and $T^n \circ G_{F_j^2}(t)$. Since the distance between W^1 and W^2 is at most ε along vertical segments in the chart and the segment connecting $T^n \circ G_{F_j^1}(t)$ and $T^n \circ G_{F_j^2}(t)$ lies in the unstable cone of the chart containing W^1 and W^2, we have $|g_j^1 - g_j^2|_{\mathcal{C}^0(I_j)} \le C\varepsilon$, using the uniform transversality of stable and unstable cones. Thus for $t \in I_j$,

$$|\psi_2 \circ G_{W^2} \circ g_j^1(t) - \psi_2 \circ G_{W^2} \circ g_j^2(t)| \le C|\psi_2|_{\mathcal{C}^p} |g_j^1(t) - g_j^2(t)|^p \le C|\psi_2|_{\mathcal{C}^p} \varepsilon^p.$$

This estimate combined with Lemma 4.2 applied to $\psi_2 \circ G_{W^2} \circ g_j^1 - \psi_2 \circ G_{W^2} \circ g_j^2$, yields $|\psi_2 \circ G_{W^2} \circ g_j^1 - \psi_2 \circ G_{W^2} \circ g_j^2|_{\mathcal{C}^q(I_j)} \le C|\psi_2|_{\mathcal{C}^p} \varepsilon^{p-q}$. This together with (4.19) and (4.20) proves the lemma since $|\psi_2|_{\mathcal{C}^p(W^2)} \le f(W^2)^{-1}|W^2|^{-\gamma}$. □

5. Proof of Theorem 2.4 and Corollary 2.5

Recall that the Lasota-Yorke estimate (2.14) and the compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ proved in Lemma 3.9 imply via the standard Hennion argument that the spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is bounded by κ^{-1} and the essential spectral radius is bounded by $\sigma_0 < 1$ (see for example [HH]). We proceed to prove the following lemma characterizing the peripheral spectrum of $\mathcal{L}$ on $\mathcal{B}$.

Recall that $\mathcal{S}_0^{\mathbb{H}} = \mathcal{S}_0 \cup (\cup_{k > k_0} S_k^H)$ and $\mathcal{S}_{-n}^{\mathbb{H}} = \cup_{i=0}^n T^i(\mathcal{S}_0^{\mathbb{H}})$ denote the expanded singularity set for T^{-n} taking into account the boundaries of the homogeneity regions. $\mathcal{S}_n^{\mathbb{H}}$ is defined analogously. Also, $\mathbb{V}_\theta$ denotes the eigenspace of $\mathcal{L}$ associated with the eigenvalue $e^{2\pi i\theta}$ and Π_θ denotes the corresponding eigenprojector onto $\mathbb{V}_\theta$.

Lemma 5.1. *Let $\mathbb{V} = \oplus_\theta \mathbb{V}_\theta$. Then,*

(i) *the spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is 1;*
(ii) *$\mathcal{L}$ restricted to $\mathbb{V}$ has semi-simple spectrum (no Jordan blocks);*
(iii) *$\mathbb{V}$ consists of signed measures;*

(iv) *all measures in $\mathbb{V}$ are absolutely continuous with respect to $\overline{\mu} = \Pi_0 m$. Moreover, 1 is in the spectrum of $\mathcal{L}$.*

(v) *Let $\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}$ denote the ε-neighborhood of $\mathcal{S}^{\mathbb{H}}_{-1}$. Then for each $\mu \in \mathbb{V}$, there exists $C > 0$ such that for all $\varepsilon > 0$, we have $\mu(\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}) \le C\varepsilon^{\xi t_0(\alpha-\gamma)}$. In particular, $\mu(\cup_{n\in\mathbb{Z}} T^n(\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon n^{-2/\xi t_0(\alpha-\gamma)}})) \le C\varepsilon^{\xi t_0(\alpha-\gamma)}$ and $\mu(\mathcal{S}^{\mathbb{H}}_{\pm n}) = 0$.*

Proof. Items (ii)-(iv) are proved in [DZ1, Lemma 5.1] and their proofs remain unchanged here so we do not repeat them. We proceed to prove items (i) and (v).

(i) First note that the spectral radius must be at least 1: If it were smaller than 1, then since $m \in \mathcal{B}$, Lemma 3.7 would yield the following contradiction,

$$1 = m(1) = m(1 \circ T^n) = \mathcal{L}^n m(1) \le C\|\mathcal{L}^n m\|_{\mathcal{B}} \le C\|\mathcal{L}\|^n \|m\|_{\mathcal{B}} \xrightarrow[n\to\infty]{} 0.$$

To show the spectral radius of $\mathcal{L}$ is not more than 1, suppose $z \in \mathbb{C}$, $|z| > 1$, satisfies $\mathcal{L}h = zh$ for some $h \in \mathcal{B}$, $h \ne 0$. For $\psi \in \mathcal{C}^p(M)$, Lemma 3.7 implies,

$$|h(\psi)| \le |z^{-n}\mathcal{L}^n h(\psi)| = |z|^{-n}|h(\psi \circ T^n)| \le |z|^{-n} C|h|_w(|\psi|_\infty + H^p_n(\psi \circ T^n)) \xrightarrow[n\to\infty]{} 0,$$

since $H^p_n(\psi \circ T^n) \le C\Lambda^{-pn}|\psi|_{\mathcal{C}^p(M)}$ by (3.26), contradicting the assumption on h.

(v) Let $\mu \in \mathbb{V}$ and fix $\varepsilon > 0$. Let $\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}$ denote the ε-neighborhood of $\mathcal{S}^{\mathbb{H}}_{-1}$ and let h_k be a sequence of $\mathcal{C}^1$ functions converging to μ in $\mathcal{B}$; then since $\mathcal{L}$ is bounded, $\mathcal{L}h_k$ converges to $\mathcal{L}\mu$ in $\mathcal{B}$. It follows from arguments similar to the proofs of Lemmas 3.5 and 3.6 that $(\mathcal{L}h_k)_\varepsilon(\psi) := \mathcal{L}h_k(1_{\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}}\psi)$ belongs to $\mathcal{B}_w$ due to the transversality and types of tangencies permitted by **(H3)** between curves in $\mathcal{S}^{\mathbb{H}}_{-1}$ and the stable cone. Then, for $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^p(W)$, $|\psi|_{W,\gamma,p} \le 1$,

$$\int_W (\mathcal{L}h_k)_\varepsilon \psi \, dm_W = \int_W \mathcal{L}h_k \cdot 1_{\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}} \psi \, dm_W = \sum_i \int_{W^1_i \cap T^{-1}\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}} h_k |DT|^{-1} J_{W^1_i}T \psi \circ T dm_W.$$

Notice that since $W^1_i \in \mathcal{G}_1(W)$ are created by intersections of W with $\mathcal{S}^{\mathbb{H}}_{-1}$, it follows that there are at most two connected components in each $W^1_i \cap T^{-1}\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}$ and $|TW^1_i \cap \mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}| \le C\varepsilon^{t_0}$ due to **(H3)**(3),(4). Consequently, we estimate the above expression as in (4.9) and (4.10),

$$\begin{aligned}\left|\int_W (\mathcal{L}h_k)_\varepsilon \psi \, dm_W\right| &\le C\|h_k\|_s \kappa^{-1} \sum_i \frac{|W^1_i \cap T^{-1}\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}|^\alpha}{|W|^\gamma} |J_{W^1_i}T|_{\mathcal{C}^0(W^1_i)} \\ &\le C\varepsilon^{\xi t_0(\alpha-\gamma)}\|h_k\|_s \sum_i \frac{|W^1_i|^\gamma}{|W|^\gamma}|J_{W^1_i}T|_{\mathcal{C}^0(W^1_i)} \le C\|h_k\|_s \varepsilon^{\xi t_0(\alpha-\gamma)},\end{aligned}$$

where we have used **(H3)**(1) to pass from $|TW^1_i \cap \mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}|$ to $|W^1_i \cap T^{-1}\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}|$. Similarly, $(\mathcal{L}h_k)_\varepsilon$ is a Cauchy sequence in $\mathcal{B}_w$ and so must converge to $(\mathcal{L}\mu)_\varepsilon(\psi) := \mathcal{L}\mu(1_{\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon}}\psi)$. Then by the above estimate, we have $|\mathcal{L}\mu(\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon})| \le C\|\mu\|_s \varepsilon^{\xi t_0(\alpha-\gamma)}$. But since $\mathcal{L}\mu = z\mu$ for some $z \in \mathbb{C}$, $|z| = 1$, we have the same bound for $\mu(\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon})$.

In particular, this implies $\mu(\mathcal{S}^{\mathbb{H}}_{-1}) = 0$. Then using repeatedly the fact that $\mathcal{L}^n\mu = z^n\mu$, $|z| = 1$, and since $\mathcal{S}^{\mathbb{H}}_{-n} = \cup_{i=0}^n T^i\mathcal{S}^{\mathbb{H}}_{-1}$ and $T^{-n}\mathcal{S}^{\mathbb{H}}_{-n} = \mathcal{S}^{\mathbb{H}}_n$, we conclude $\mu(\mathcal{S}^{\mathbb{H}}_{-n}) = 0$ for each $n \ge 0$. Moreover, we have $\mu(\cup_{n\in\mathbb{Z}} T^n(\mathcal{S}^{\mathbb{H}}_{-1,\varepsilon n^{-2/\xi t_0(\alpha-\gamma)}})) \le C\varepsilon^{\xi t_0(\alpha-\gamma)} \sum_{n\in\mathbb{Z}} |n|^{-2} \le C\varepsilon^{\xi t_0(\alpha-\gamma)}$. □

Further information about the measures corresponding to the peripheral spectrum of $\mathcal{L}$ can be proved using similar techniques as in Lemma 5.1: In other words, they are proved using properties of the Banach spaces we have defined without relying on the specifics of T. We summarize these results in our next lemma, which we state without proof since the proof can be found in [DL, Lemmas 5.5 and 5.7].

Lemma 5.2. (i) *There exist a finite number of* $q_k \in \mathbb{N}$ *such that the spectrum of* $\mathcal{L}$ *on the unit disk is* $\cup_k\{e^{2\pi i \frac{p}{q_k}} : 0 \le p < q_k,\, p \in \mathbb{N}\}$*. In addition, the set of ergodic probability measures absolutely continuous with respect to* $\overline{\mu}$ *form a basis of* $\mathbb{V}_0$*.*

(ii) T *admits only finitely many physical probability measures and they belong to* $\mathbb{V}_0$*.*

(iii) *The ergodic decomposition with respect to Lebesgue and with respect to* $\overline{\mu}$ *coincide. In addition, the ergodic decomposition with respect to Lebesgue corresponds to the supports of the physical measures.*

The only properties of T that are used in the proof of the preceding lemma in [DL] are the invertibility of T and the items in Lemma 5.1. Lemmas 5.1 and 5.2 complete the proof of items (1)-(3) of Theorem 2.4.

Item (4) follows immediately from Lemma 5.1(iv), since if $(T, \overline{\mu})$ is ergodic, there can be no other ergodic invariant measure absolutely continuous with respect to $\overline{\mu}$.

To see that $(T^n, \overline{\mu})$ ergodic for all $n \in \mathbb{N}$ implies a spectral gap for $\mathcal{L}$, assume there exists $\nu \in \mathbb{V}_\theta$ for some $\theta \neq 0$. By Lemma 5.2(i), it must be that $\theta = p/q$ for some $p, q \in \mathbb{N}$ so that ν is an invariant measure for T^q. But $\overline{\mu}$ is also an invariant measure for T^q and ν is absolutely continuous with respect to $\overline{\mu}$ by Lemma 5.1(iv). This contradicts the fact that $(T^q, \overline{\mu})$ is ergodic. Thus $\mathcal{L}$ has no other eigenvalues on the unit circle and so by quasi-compactness, $\mathcal{L}$ has a spectral gap on $\mathcal{B}$.

The spectral gap implies that the spectral projectors Π_θ are all zero except for Π_0 which can be recharacterized by $\Pi_0 h = \lim_{n\to\infty} \mathcal{L}^n h$ for all $h \in \mathcal{B}$. It thus follows that any distribution $h \in \mathcal{B}$ with $h(1) = 1$ satisfies $\lim_{n\to\infty} \|\mathcal{L}^n h - \overline{\mu}\|_{\mathcal{B}} \le C\|h\|_{\mathcal{B}}(\sigma')^n$, where $\sigma' < 1$ is the spectral radius of $\mathcal{L} - \Pi_0$ on $\mathcal{B}$. This completes the proof of item (5) of the theorem.

5.1. **Decay of Correlations.** In this section, we prove items (6) and (7) of Theorem 2.4 under the assumption that $\mathcal{L}$ has a spectral gap. In order to discuss correlations and the limit theorems of Corollary 2.5, we need the following multiplier property for functions with reasonable discontinuities.

Lemma 5.3. *Let* $\mathcal{P}$ *be a countable partition of* M *that satisfies the assumptions of Lemma 3.5 and suppose in addition that there is a uniform bound* N_1 *on the number of* $P \in \mathcal{P}$ *that each homogeneity region* $\mathbb{H}_k$ *can intersect.*

Let $\lambda > \max\{\beta/(1-\beta), p\}$*. If* ϕ *is a function on* M *such that* $\sup_{P\in\mathcal{P}} |\phi|_{\mathcal{C}^\lambda(P)} < \infty$ *and* $h \in \mathcal{B}$*, then* $\phi h \in \mathcal{B}$*. Moreover,* $\|\phi h\|_{\mathcal{B}} \le C\|h\|_{\mathcal{B}} \sup_{P\in\mathcal{P}} |\phi|_{\mathcal{C}^\lambda(P)}$ *for some* $C > 0$ *independent of* ϕ *and* h*.*

Proof. Let $\mathcal{P}$ and ϕ be as in the statement of the lemma. By density, it suffices to prove the lemma for $h \in C^1(M)$. By Lemma 3.5, $\phi h \in \mathcal{B}$. We proceed to estimate its norm. For brevity, we write

$$|\phi|_{\mathcal{C}^\lambda(\mathcal{P})} = \sup_{P\in\mathcal{P}} |\phi|_{\mathcal{C}^\lambda(P)}.$$

To estimate the strong stable norm, we fix $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^q(W)$ such that $|\psi|_{W,\alpha,q} \le 1$. For each $P_i \in \mathcal{P}$, set $W_i = W \cap P_i$. Then

$$\Big|\int_W \phi h\, \psi\, dm_W\Big| = \Big|\sum_i \int_{W_i} \phi h\, \psi\, dm_W\Big| \le \sum_i \|h\|_s |W_i|^\alpha f(W_i) |\phi|_{\mathcal{C}^q(W_i)} |\psi|_{\mathcal{C}^q(W_i)} \le C_f N_1 K \|h\|_s |\phi|_{\mathcal{C}^\lambda(\mathcal{P})},$$

where we have used the assumptions on ∂P_i to bound the maximum number of W_i by $N_1 K$, and Lemma 3.4(a) to bound $f(W_i)/f(W)$.

Now to estimate the strong unstable norm of ϕh, we let $\varepsilon \le \varepsilon_0$ and choose $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) < \varepsilon$. For $\ell = 1, 2$, let $\psi_\ell \in \mathcal{C}^p(W^\ell)$ such that $|\psi_\ell|_{W^\ell,\gamma,p} \le 1$ and $d_q(\psi_1, \psi_2) \le \varepsilon$.

Recalling the notation of Section 3.1, we write $W^\ell = G_{F^\ell}(I_{W^\ell})$, $\ell = 1, 2$. We subdivide each curve W^ℓ into matched and unmatched pieces, similar to those in Section 4.3. To each point $x \in W^1$, we attach a vertical (in the chart) line segment γ_x, centered at x of length 2ε. We define $U^\ell_j \subset W^\ell$ to be the maximal connected curves for which U^ℓ_j belongs to a single element $P \in \mathcal{P}$

and the family $\{\gamma_x\}_{x \in U_j^\ell}$ intersects W^2 but does not intersect ∂P for any $P \in \mathcal{P}$. We label by $V_i^\ell \subset W^\ell$ the remaining maximal pieces for which there is no matching by the vertical segments γ_x. We also require each V_i^ℓ to be contained in a single $P \in \mathcal{P}$. Note that there are at most $2N_1K + 2$ unmatched pieces and at most N_1K matched pieces by assumption on $\mathcal{P}$. Also, due to the weak transversality of ∂P with $C^s(x)$, we have $|V_i^\ell| \le C_t \varepsilon^{t_0}$ for each ℓ, i and a uniform constant C_t.

We define $\varphi = (\phi\psi_1) \circ G_{F^1} \circ G_{F^2}^{-1}$ and note that φ is well defined on each matched piece U_j^2. We must estimate

$$
\begin{aligned}
|\int_{W^1} \phi h\, \psi_1 \, dm_W - \int_{W^2} \phi h \, \psi_2 \, dm_W| &= \sum_{i,\ell} \int_{V_i^\ell} \phi h \, \psi_\ell \, dm_W \\
&\quad + \sum_j \left(\int_{U_j^1} \phi h \, \psi_1 \, dm_W - \int_{U_j^2} h\varphi \, dm_W \right) + \int_{U_j^2} h(\varphi - \phi\psi_2) \, dm_W.
\end{aligned}
\tag{5.1}
$$

The first sum on the right hand side of (5.1) over unmatched pieces is estimated by,

$$
\begin{aligned}
|\sum_{i,\ell} \int_{V_i^\ell} \phi h \, \psi_\ell \, dm_W| &\le \sum_{i,\ell} \|h\|_s |V_i^\ell|^\alpha f(V_i^\ell) |\phi|_{\mathcal{C}^q(V_i^\ell)} |\psi_\ell|_{\mathcal{C}^q(V_i^\ell)} \\
&\le (2N_1K + 2) \|h\|_s |\phi|_{\mathcal{C}^\lambda(\mathcal{P})} C_t \varepsilon^{t_0(\alpha - \gamma)}.
\end{aligned}
\tag{5.2}
$$

Next we estimate the difference over matched pieces in (5.1). By construction, $d_{\mathcal{W}^s}(U_j^1, U_j^2) \le \varepsilon$ since $U_j^\ell \subseteq W^\ell$. Moreover, letting I_j denote the common t-interval over which U_j^1 and U_j^2 are both defined, we have

$$
d_q(\phi\psi_1, \varphi) = |(\phi\psi_1) \circ G_{F^1} - \varphi \circ G_{F^2}|_{\mathcal{C}^q(I_j)} = 0.
$$

Note that $|\phi\psi_1|_{U_j^1, \gamma, p} \le C |\phi|_{\mathcal{C}^\lambda(\mathcal{P})}$ for some uniform constant C since $U_j^1 \subseteq W^1$ and by Lemma 3.4(a). Similarly, since $G_{F^1} \circ G_{F^2}^{-1}$ has bounded C^1-norm, we have $|\varphi|_{U_j^2, \gamma, p} \le C |\phi|_{\mathcal{C}^\lambda(\mathcal{P})}$. Renormalizing the test functions, we apply the definition of the strong unstable norm to estimate

$$
|\sum_j \int_{U_j^1} \phi h \, \psi_1 \, dm_W - \int_{U_j^2} h\varphi \, dm_W| \le N_1 K \varepsilon^\beta \|h\|_u C |\phi|_{\mathcal{C}^\lambda(\mathcal{P})}.
\tag{5.3}
$$

Finally, we estimate the third sum on the right hand side of (5.1) using the strong stable norm.

$$
|\sum_j \int_{U_j^2} h(\varphi - \phi\psi_2) \, dm_W| \le \sum_j \|h\|_s |\varphi - \phi\psi_2|_{\mathcal{C}^q(U_j^2)} |U_j^2|^\alpha f(U_j^2).
$$

Again using that G_{F^2} has bounded C^1-norm, we estimate

$$
|\varphi - \phi\psi_2|_{\mathcal{C}^q(U_j^2)} \le C |(\phi\psi_1) \circ G_{F^1} - (\phi\psi_2) \circ G_{F^2}|_{\mathcal{C}^q(I_j)}.
$$

For $t \in I_j$, we have

$$
|(\phi\psi_1) \circ G_{F^1}(t) - (\phi\psi_2) \circ G_{F^2}(t)| \le |\phi|_\infty |\psi_1 \circ G_{F^1}(t) - \psi_2 \circ G_{F^2}(t)| + |\psi_2|_{\mathcal{C}^0(W^2)} |\phi \circ G_{F^1}(t) - \phi \circ G_{F^2}(t)|.
$$

The first difference above is bounded by ε due to the assumption $d_q(\psi_1, \psi_2) \le \varepsilon$. The second difference is bounded by $|\phi|_{\mathcal{C}^\gamma(\mathcal{P})} \varepsilon^\lambda$. Now using Lemma 4.2, we conclude

$$
|\varphi - \phi\psi_2|_{\mathcal{C}^q(U_j^2)} \le C |W^2|^{-\gamma} f(W^2)^{-1} |\phi|_{\mathcal{C}^\lambda(\mathcal{P})} \varepsilon^{\lambda - q}.
\tag{5.4}
$$

Putting together (5.2), (5.3) and (5.4) with (5.1), we have

$$
|\int_{W^1} \phi h \, \psi_1 \, dm_W - \int_{W^2} \phi h \, \psi_2 \, dm_W| \le C |\phi|_{\mathcal{C}^\gamma(\mathcal{P})} (\|h\|_s \varepsilon^{t_0(\alpha-\gamma)} + \|h\|_u \varepsilon^\beta + \|h\|_s \varepsilon^{\lambda - q}),
$$

for some uniform constant C depending on N_1 and K. This completes the estimate on the strong unstable norm since $\beta \le \min\{t_0(\alpha - \gamma), p - q\}$ and $\lambda > q$. □

Now given ϕ as in Lemma 5.3 and $\psi \in \mathcal{C}^p(T^{-k}\mathcal{W}^s)$ for some $k \geq 0$, we define their correlation function by

$$C_{\phi,\psi}(n) := \overline{\mu}(\phi\,\psi \circ T^n) - \overline{\mu}(\phi)\overline{\mu}(\psi).$$

Define $\overline{\mu}_\phi = \phi\overline{\mu}$. By Lemma 5.3, we have $\overline{\mu}_\phi \in \mathcal{B}$. Thus $\Pi_0\overline{\mu}_\phi = \overline{\mu}(\phi)\overline{\mu}$ and so by Lemma 3.7,

$$|\overline{\mu}(\phi\,\psi \circ T^n) - \overline{\mu}(\phi)\overline{\mu}(\psi)| = |(\mathcal{L}^n\overline{\mu}_\phi - \overline{\mu}(\phi)\overline{\mu})(\psi)| \leq C\|\mathcal{L}^n\overline{\mu}_\phi - \overline{\mu}(\phi)\overline{\mu}\|_{\mathcal{B}}(|\psi|_\infty + H^p_k(\psi))$$

and the exponential rate of convergence is given by the spectral radius of $\mathcal{L} - \Pi_0$ on $\mathcal{B}$. The proof of item (6) of Theorem 2.4 is completed by noting that $\|\overline{\mu}_\phi\|_{\mathcal{B}} \leq C|\phi|_{\mathcal{C}^\lambda(\mathcal{P})}$ by Lemma 5.3.

To prove item (7), for $\phi, \psi \in \mathcal{C}^\lambda(M)$, we define the Fourier transform of the correlation function,

$$\hat{C}_{\phi,\psi}(z) := \sum_{n\in\mathbb{Z}} z^n C_{\phi,\psi}(n).$$

The importance of this function stems from the connection between its poles and the Ruelle resonances, which are in principal measurable in physical systems, [Ru1, Ru2, PP1, PP2, L2].

Given the spectral picture we have established, it follows by standard arguments that the function is convergent in a neighborhood of $|z| = 1$ and admits a meromorphic extension in the annulus $\{z \in \mathbb{C} : \sigma_0 < |z| < \sigma_0^{-1}\}$ where σ_0 is from (2.14). It follows that the poles of the correlation function are in a one-to-one correspondence (including multiplicity) with the spectrum of $\mathcal{L}$ outside the disk of radius σ_0.

5.2. **Proof of Corollary 2.5.** Let $\mathcal{P}$ be a partition of M satisfying the assumptions of Lemma 5.3 and fix $\lambda > \max\{p, \beta/(1-\beta)\}$. Let $g : M \to \mathbb{R}^d$ be a vector-valued function such that each component g_i satisfies $|g_i|_{\mathcal{C}^\lambda(\mathcal{P})} < \infty$, $i = 1, \dots, d$. Define $|g|_{\mathcal{C}^\lambda(\mathcal{P})} = \big(\sum_{i=1}^d |g_i|^2_{\mathcal{C}^\lambda(\mathcal{P})}\big)^{1/2}$. For $z \in \mathbb{C}^d$, we define the generalized transfer operator $\mathcal{L}_{zg}$ on $\mathcal{B}$ by

$$\mathcal{L}_{zg}h(\psi) = h(e^{z\cdot g}\psi \circ T) \quad \text{for } \psi \in \mathcal{C}^p(\mathcal{W}^s).$$

The proofs of the limit theorems follow from the fact that $\mathcal{L}_{zg}$ is an analytic perturbation of $\mathcal{L} = \mathcal{L}_0$ for small $|z|$.

Lemma 5.4. *For $g : M \to \mathbb{R}^2$ with $|g|_{\mathcal{C}^\lambda(\mathcal{P})} < \infty$, the map $z \mapsto \mathcal{L}_{zg}$ is analytic for all $z \in \mathbb{C}^d$.*

Proof. Fix $z \in \mathbb{C}^d$ and define the operator $\mathcal{P}_n h = \mathcal{L}((z\cdot g)^n h)$ for $h \in \mathcal{B}$. By Lemma 5.3, $(z\cdot g)^n h \in \mathcal{B}$ and

$$\|\mathcal{P}_n h\|_{\mathcal{B}} = \|\mathcal{L}((z\cdot g)^n h)\|_{\mathcal{B}} \leq C\|\mathcal{L}\|\|h\|_{\mathcal{B}}|z\cdot g|^n_{\mathcal{C}^\lambda(\mathcal{P})} \leq C\|h\|_B|z|^n|g|^n_{\mathcal{C}^\lambda(\mathcal{P})}.$$

This implies that the operator $\sum_{n=0}^\infty \frac{1}{n!}\mathcal{P}_n$ is well-defined on $\mathcal{B}$ and equals $\mathcal{L}_{zg}$ since

$$\sum_{n=0}^\infty \frac{1}{n!}\mathcal{P}_n h(\psi) = h\left(\sum_{n=0}^\infty \frac{(z\cdot g)^n}{n!}\psi \circ T\right) = h(e^{z\cdot g}\psi \circ T) = \mathcal{L}_{zg}h(\psi), \text{ for } \psi \in \mathcal{C}^p(\mathcal{W}^s),$$

and we know the sum converges for all $z \in \mathbb{C}^d$. □

It follows from the analyticity of $z \mapsto \mathcal{L}_{zg}$ that both the discrete spectrum and the corresponding spectral projectors of $\mathcal{L}_{zg}$ vary smoothly with z [Ka]. Since $\mathcal{L}_0$ has a spectral gap, so does $\mathcal{L}_{zg}$ for $|z|$ sufficiently small.

Proof of Corollary 2.5(a). The proof of this using Lemma 5.4 is precisely the same as the proof of [DZ1, Theorem 2.6(a)] and is omitted.

Proof of Corollary 2.5(b). In the current setting, the vector-valued almost-sure invariance principle follows from the abstract results of Gouëzel [G]. We fix $g \in \mathcal{C}^\lambda(\mathcal{P})$, taking values in $\mathbb{R}^d$, and distribute $(g \circ T^j)_{j\in\mathbb{N}}$ according to a (not necessarily invariant) probability measure $\nu \in \mathcal{B}$.

For $n \in \mathbb{N}$, letting $z_j = it_j$, $t_j \in \mathbb{R}^d$, $j = 1, \dots n$, we see that $\mathcal{L}_{itg}$ codes the characteristic function of the process $(g \circ T^j)$ in the sense of [G, Section 2.1], i.e., $\nu(e^{i\sum_{j=0}^n t_j\cdot g\circ T^j}) = \mathcal{L}_{it_ng}\cdots\mathcal{L}_{it_0g}\nu(1)$.

Lemma 5.4 implies in particular that $\mathcal{L}_{itg}$ satisfies the assumptions of strong continuity in [G, Section 2.2] so we may apply [G, Theorem 2.1] to conclude the vector-valued almost-sure invariance principle.

The error exponent $r > 1/4$ in the statement of the corollary is justified by [G, eq. (1.2)] since our observables $g \circ T^j$ are in $L^\infty(M)$ for each $j \in \mathbb{N}$.

6. Dispersing Billiards with Corner Points

Recall from Section 2.4.1 that Q denotes the dispersing billiard table with corner points and $M = \cup_{i=1}^{b_0}(\Gamma_i \times [-\pi/2, \pi/2])$ denotes the phase space of the billiard map $\mathcal{F}$ in canonical coordinates (r, φ). For any $x \in M$, we denote by $\tau(x)$ the time of the first (non-tangential) collision of the trajectory starting at x under the billiard flow.

The new phenomenon for billiards with corner points (compared to the periodic Lorentz gas with finite horizon) is the existence of series of finite consecutive reflections near a corner point. During those series, the free paths are short, i.e. $\tau(x) \approx 0$, and so the expansion of unstable vectors is weak. Let us fix a sufficiently small $\epsilon_1 > 0$, and call a series of consecutive reflections a corner series if they all occur in the ϵ_1-neighborhood of one corner point. We recall two facts found in [BSC1, BSC2] which we shall use.

- (F1) The number of reflections in any corner series is uniformly bounded above by $m_0 = \left[\frac{\pi}{\theta_0}\right] + 1$, where $\theta_0 > 0$ is the minimum angle of intersection of the corner points. Thus there exists a constant $\tau' > 0$ such that for each $x \in M$ there is an $i \in \{0, ..., m_0 - 1\}$ such that $\tau(\mathcal{F}^i(x)) > \tau'$.
- (F2) Each corner series contains at most one grazing reflection, and that reflection is necessarily the first or the last one in the series. There exists a constant $c_0 > 0$ such that in each corner series of length $\ell + 1$, $\mathcal{F}^i x = (r_i, \varphi_i), 0 \le i \le \ell$, we have $\cos(\varphi_i) \ge c_0$ for all φ_i, except possibly one, and that exceptional one is either $i = 0$ or $i = \ell$. Corner series in which the first reflection is grazing are called *left-singular* and those in which the last reflection is grazing are called *right-singular*. Corner series with no grazing reflections are called *regular*.

Let $\hat{r}_i$, $i = 1, \cdots i_0$ be the r-coordinates of the corner points of ∂Q. We denote by $V_0 := \{(r, \varphi) : r = \hat{r}_i, i = 1, \ldots, i_0\}$ the collisions at the corners, and by $S_0^H = \{(r, \varphi) : \varphi = \pm\pi/2\}$ the grazing collisions (following the notation of Section 2.1). Let $\mathcal{S}_0 = S_0^H \cup V_0$. Note that $\mathcal{S}_0$ is a finite set of smooth curves.

Since the table lies in a compact region on $\mathbb{R}^2$, the free path function τ is bounded. Thus since we have also assumed that the scatterers have strictly positive curvature $\mathcal{K}(x)$ at each $x \in M$, there exist constants $\mathcal{K}_{\min}, \mathcal{K}_{\max}, \tau_{\max}$ such that

$$0 < \mathcal{K}_{\min} \le \mathcal{K}(x) \le \mathcal{K}_{\max} \quad \text{and} \quad 0 \le \tau(x) \le \tau_{\max}, \, \forall x \in M.$$

For the purposes of checking assumptions **(H1)**-**(H5)** of Section 2.1, we work with higher iterates of $\mathcal{F}$. We will choose this higher iterate large enough that the expansion needed for **(H2)**(2) as well as **(H5)** both hold.

We first establish some facts regarding the hyperbolicity of $\mathcal{F}$.

6.1. **Hyperbolicity.**

We begin by defining stable and unstable cones explicitly. The derivative $D\mathcal{F}$ at the point $x = (r, \varphi) \in M$ is the 2×2 matrix,

$$D\mathcal{F}(x) = -\frac{1}{\cos\varphi_1}\begin{pmatrix} \tau\mathcal{K} + \cos\varphi & \tau \\ \mathcal{K}_1(\tau\mathcal{K} + \cos\varphi) + \mathcal{K}\cos\varphi_1 & \tau\mathcal{K}_1 + \cos\varphi_1 \end{pmatrix} \tag{6.1}$$

where $x_1 = \mathcal{F}(x) = (r_1, \varphi_1)$ and $\mathcal{K}_1 = \mathcal{K}(r_1)$ [CM1, eq (2.26)]. Thus for any nonzero vector $dx = (dr, d\varphi) \in C^u(x)$, the slope of $dx_1 = (dr_1, d\varphi_1) = D\mathcal{F}(x)\, dx$ satisfies

$$\frac{d\varphi_1}{dr_1} = \mathcal{K}_1 + \frac{\cos\varphi_1}{\tau + \frac{\cos\varphi}{\mathcal{K} + \frac{d\varphi}{dr}}}. \tag{6.2}$$

If $\tau(\mathcal{F}^{-1}x) \geq \tau'$ (resp. $\tau(x) \geq \tau'$), we define the unstable (resp. stable) cone at x by,

$$\begin{aligned} C^u(x) &:= \{(dr, d\varphi) \in \mathcal{T}_x M \,:\, \mathcal{K}(x) \leq d\varphi/dr \leq \mathcal{K}(x) + 1/\tau'\} \quad \text{and} \\ C^s(x) &:= \{(dr, d\varphi) \in \mathcal{T}_x M \,:\, -\mathcal{K}(x) \geq d\varphi/dr \geq -\mathcal{K}(x) - 1/\tau'\}, \end{aligned} \tag{6.3}$$

so that the slopes are uniformly bounded above by $\mathcal{K}_{\max} + 1/\tau'$. The expression (6.2) implies that C^u is strictly invariant under $D\mathcal{F}$ while C^s is strictly invariant under $D\mathcal{F}^{-1}$ at such points.

For points where $\tau(\mathcal{F}^{-1}x) < \tau'$, i.e. during a corner series, we proceed differently. Suppose x_i, $i = 0, \dots \ell$, is a corner series for $\mathcal{F}$. Note that $C^u(x_0)$ is defined by (6.3) since $\tau(\mathcal{F}^{-1}x_0) \geq \tau'$. If x_0 begins a right-singular or regular corner series, then $\cos\varphi(x_i) \geq c_0$, $i = 0, \dots, \ell - 1$ so that by (6.2), we estimate inductively along the corner series,

$$\frac{d\varphi_i}{dr_i} \leq \mathcal{K}_{\max} + \frac{\mathcal{K}_{\max} + \frac{d\varphi_{i-1}}{dr_{i-1}}}{c_0} \leq \mathcal{K}_{\max} + \frac{2m_0 K_{\max} + 1}{c_0^{m_0}\tau'}, \quad i = 1, \dots \ell.$$

Thus the slopes of the $D\mathcal{F}^i$ images of vectors in $C^u(x_0)$ remain uniformly bounded above during regular and right-singular corner series so we may define $C^u(x)$ using this uniform upper bound in place of $\mathcal{K} + 1/\tau'$ at such points. Note that the lower bound in the cone remains always $\mathcal{K}$.

If x_0 begins a left-singular corner series for $\mathcal{F}$, we define $C^u(x_i)$ according to (6.3) with $\tau(\mathcal{F}^{-1}x)$ in place of τ' and there is no upper bound on the slopes in these cones. We define stable cones for corner series in the analogous way, but interchanging the role of left and right-singular corner series. By a similar argument to above, the slopes in $C^s(x)$ remain uniformly bounded above during regular and left-singular corner series and lose the uniform upper bound during right-singular corner series (where now we replace τ' by $\tau(x)$ in (6.3)). Thus the angles between stable and unstable cones are uniformly bounded away from zero on $M \setminus \mathcal{S}_0$, where $\mathcal{S}_0$ is specified below (see also [Ch1, Section 9]). Equation (6.2) implies that $C^u(x)$ defined this way is strictly invariant under $D\mathcal{F}$ and analogously $C^s(x)$ is strictly invariant under $D\mathcal{F}^{-1}$. Our piecewise definitions of stable and unstable cones result in cone fields that have finitely many domains of continuity; however, since the cones are strictly invariant, we may smooth them out so that they are indeed continuous on each component of $M \setminus \mathcal{S}_0$.

Next we study the expansion factor for vectors $dx = (dr, d\varphi)$ in the Euclidean norm, $\|dx\| = \sqrt{dr^2 + d\varphi^2}$. Recall τ' and m_0 from (F1) and define

$$\Lambda_0 := (1 + \tau'\mathcal{K}_{\min})^{1/m_0} > 1, \qquad \text{and} \qquad n_0 := \left\lceil \frac{\ln(1 + \mathcal{K}_{\min}^{-2})}{2\ln\Lambda_0} \right\rceil. \tag{6.4}$$

Given $x = (r, \varphi) \in M$, label $x = x_0 = (r_0, \varphi_0)$, $x_{-i} = (r_{-i}, \varphi_{-i}) = \mathcal{F}^{-1}x_{-i+1}$, $i = 1, \dots n$. The analogue of (6.1) for $D\mathcal{F}^{-1}$ yields,

$$D\mathcal{F}^{-1}(x) = -\frac{1}{\cos\varphi_{-1}} \begin{pmatrix} \tau_{-1}\mathcal{K} + \cos\varphi & -\tau_{-1} \\ -\mathcal{K}_{-1}(\tau_{-1}\mathcal{K} + \cos\varphi) - \mathcal{K}\cos\varphi_{-1} & \tau_{-1}\mathcal{K}_{-1} + \cos\varphi_{-1} \end{pmatrix}, \tag{6.5}$$

where we use the subscript -1 to denote the relevant quantities at $x_{-1} = \mathcal{F}^{-1}x$. Now for any nonzero vector $(dr, d\varphi) \in \mathcal{C}^s(x)$, we use this to estimate,

$$|d\varphi_{-1}| = \Big(\mathcal{K}_{-1}\frac{\tau_{-1}\mathcal{K} + \cos\varphi}{\cos\varphi_{-1}} + \mathcal{K}\Big)|dr| + \Big(\frac{\tau_{-1}\mathcal{K}_{-1}}{\cos\varphi_{-1}} + 1\Big)|d\varphi| \geq \Big(\frac{\tau_{-1}\mathcal{K}_{-1}}{\cos\varphi_{-1}} + 1\Big)|d\varphi|.$$

This implies that for any $n \geq m_0 + n_0$,

$$\begin{aligned} \frac{\|D\mathcal{F}^{-n}dx\|}{\|dx\|} &= \sqrt{\frac{dr_{-n}^2 + d\varphi_{-n}^2}{dr^2 + d\varphi^2}} \geq \frac{|d\varphi_{-n}|}{\sqrt{dr^2 + d\varphi^2}} \\ &\geq (1 + \mathcal{K}^{-2})^{-1/2} \prod_{i=1}^{n} \left(1 + \frac{\tau(x_{-i})\mathcal{K}(x_{-i})}{\cos\varphi_{-i}}\right) \geq \frac{(1 + \tau'\mathcal{K}_{\min})^{[n/m_0]}}{(1 + \mathcal{K}_{\min}^{-2})^{1/2}} \\ &\geq \Lambda_0^{n-m_0+1}(1 + \mathcal{K}_{\min}^{-2})^{-1/2} \geq \Lambda_0^{n-m_0-n_0} \end{aligned} \tag{6.6}$$

where we have used the assumption that within every m_0-iterates, at least one collision x_i satisfies $\tau(x_i) > \tau'$, and n_0 is defined in (6.4). This implies that $D\mathcal{F}^{-n}$ eventually expands stable vectors uniformly.

We now choose $n_1 > m_0 + n_0$ sufficiently large to be able to apply [DT, Main Theorem]. Define $T := \mathcal{F}^{n_1}$. Below we will show that T satisfies the conditions **(H1)-(H5)**, and enjoys the spectral properties proved in Theorem 2.4. We will then extend these properties to $\mathcal{F}$.

6.2. **Smoothness and Singularities for** T**.** Since $T = \mathcal{F}^{n_1}$, T preserves the same smooth invariant measure μ as $\mathcal{F}$, and thus

$$|\det D_xT| = \cos\varphi(x)/\cos\varphi(Tx).$$

This verifies **(H1)** with $f = \cos\varphi$ and $f_0 \equiv \kappa = 1$.

Let $\mathcal{S}_0 = S_0^H \cup V_0$ be defined as above. Then $T^{\pm 1}$ lacks smoothness on the set $\mathcal{S}_{\pm 1} := \cup_{i=0}^{m_1} \mathcal{F}^{\mp i}\mathcal{S}_0$. In general, denote

$$\mathcal{S}_{\pm n} = \cup_{i=0}^{nm_1} \mathcal{F}^{\mp i}\mathcal{S}_0.$$

For each $n \in \mathbb{Z}$, $T^n : M \setminus \mathcal{S}_n \to M \setminus \mathcal{S}_{-n}$ is a $\mathcal{C}^2$ diffeomorphism.

To control distortion, we define homogeneity strips $\mathbb{H}_k$, as in Section 2.1 and following [BSC1]. For $k \geq k_0$, denote by

$$S_{\pm k}^H = \{(r,\varphi) : |\varphi| = \pm\pi/2 \mp k^{-2}\} \text{ and } \mathcal{S}_0^{\mathbb{H}} = \mathcal{S}_0 \cup (\cup_{k \geq k_0}^{\infty} S_{\pm k}^H), \tag{6.7}$$

and let $\mathbb{H}_k$ denote the region between S_k^H and S_{k+1}^H, so that $r_h = 3$. We place two restrictions on k_0: (1) k_0 is chosen large enough compared to c_0^{-1} from (F2) so that a corner series does not involve any homogeneity strips except perhaps at the first or the last reflection; (2) k_0 is chosen large enough to apply [DT, Main Theorem] to T. This choice of k_0 and the results of [DT] guarantee that T satisfies **(H5)** with $\gamma_0 = 0$ and the adapted norm $\|\cdot\|_*$ taken to be the Euclidean norm. This also fixes the choice of δ_0 in the definition of $\mathcal{W}^s$.

We set $\mathcal{S}_{\pm n}^{\mathbb{H}} = \cup_{i=0}^{nm_1} \mathcal{F}^{\mp i}\mathcal{S}_0^{\mathbb{H}}$ and call this the expanded singularity set for $T^{\pm n}$. As before, we call a curve *homogeneous* if it lies entirely in one of the homogeneity strips $\mathbb{H}_k$.

The time-reversibility of T implies that $\mathcal{S}_{-n}$ and $\mathcal{S}_n$ are symmetric about $\varphi = 0$ in M. Moreover the set $\mathcal{S}_n \setminus \mathcal{S}_0$ is a union of compact smooth stable curves for $n \geq 1$ and unstable curves for $n \leq -1$. Since V_0 consists of a finite number of vertical lines in M and since τ is bounded, it follows that there are only finitely many singular curves in $\mathcal{F}^iV_0$ for each $i \geq 0$, all of which are unstable curves. Similarly, $\mathcal{F}^i\mathcal{S}_0^{\mathbb{H}}$ comprises finitely many smooth unstable curves for each i. Thus $\mathcal{S}_{-1}$ comprises a finite number of smooth compact curves. Indeed, each smooth curve in $\mathcal{S}_{-1} \setminus \mathcal{S}_0$ terminates on a smooth curve in $\mathcal{S}_{-1}$ and is contained in one monotonically increasing continuous curve which stretches all the way from $\varphi = -\pi/2$ to $\varphi = \pi/2$. This property is often referred to as *continuation of singularity lines*. Similarly, $\mathcal{S}_n$ consists of a finite number of smooth, compact curves for each n.

Now since $C^u(x)$ and $C^s(x)$ as we have defined them are invariant under $D\mathcal{F}$ and $D\mathcal{F}^{-1}$ respectively, they are also invariant under DT and DT^{-1}. This verifies **H2**(1). Moreover, defining $\Lambda := \Lambda_0^{m_1-m_0-n_0} > 1$, (6.6) implies $\|DT^{-1}(x)dx\| \geq \Lambda\|dx\|$ for $dx \in C^s(x)$. This, and its symmetric counterpart for the unstable cone, verifies item (2) of **(H2)** with the norm $\|\cdot\|_*$ taken to be the Euclidean norm.

It follows from (6.6) and the fact that we have chosen k_0 large compared to c_0^{-1} from fact (F2) that there exists $B_1 \geq 1$ such that for $x \in M \setminus \mathcal{S}_1$,

$$\frac{1}{B_1 \cos\varphi(\mathcal{F}^{-1}x)} \leq \frac{\|D\mathcal{F}^{-1}(x)v\|}{\|v\|} \leq \frac{B_1}{\cos\varphi(\mathcal{F}^{-1}x)} \qquad \text{for } v \in C^s(x). \tag{6.8}$$

(See [Ch1, Lemma 9.1].) The analogous fact holds for $D\mathcal{F}$. Accordingly, since $C^{-1}k^{-2} \leq \cos\varphi(x) \leq Ck^{-2}$ for some uniform constant C and $x \in \mathbb{H}_k$, if $\mathcal{F}^{-1}W \subset \mathbb{H}_k$, we must have $|\mathcal{F}^{-1}W| \leq Ck^{-3}$ and,[12]

$$|\mathcal{F}^{-1}W| \leq C|W|k^2 \leq C|W||\mathcal{F}^{-1}W|^{-2/3} \quad \Longrightarrow \quad |\mathcal{F}^{-1}W| \leq C|W|^{3/5}. \tag{6.9}$$

Iterating this equation yields $|T^{-1}W| \leq C|W|^{\xi}$ with $\xi = (3/5)^{n_1}$, verifying **(H3)**(1).

Item (2) of **(H3)** is automatic since as already described above, $\mathcal{S}_{-1}$, the singularity set for T, comprises finitely many curves. To check item (3) in **(H3)**, note that except at corner series, the stable and unstable cones are bounded away from both the vertical and horizontal directions as explained above. Since the angles between stable and unstable cones are uniformly bounded away from zero and curves in $\mathcal{S}_{-n} \setminus \mathcal{S}_0$ are unstable curves for $n \geq 1$, they are uniformly transverse to the stable cone. Thus it remains to check that curves in $\mathcal{S}_0$ satisfy **(H3)**(3). S_0^H is uniformly transverse to the stable cone since the stable cone is bounded away from the horizontal. Near V_0, however, stable curves may be arbitrarily close to vertical during right-singular corner series.

It is proved in [BSC1, Lemma 2.7] (see also [Ch1, Section 9]) that if $x = (r, \varphi)$ is contained in a stable curve W and $(dr, d\varphi)$ is the tangent vector to W at x, then

$$\frac{d\varphi}{dr} \leq \frac{C}{|r - r_0|^{1/2}} \tag{6.10}$$

where (r_0, φ_0) is the endpoint of W closest to x. Thus $|\varphi - \varphi_0| \leq 2C|r - r_0|^{1/2}$ so that any ε-neighborhood of V_0 contains a length of at most $C'\varepsilon^{1/2}$ along W, which is what we need to establish **(H3)**(3) with $t_0 = 1/2$.

Item (4) of **(H3)** follows immediately since all the homogeneity curves S_k^H, $k \geq k_0$, are horizontal lines while $C^s(x)$ is bounded away from the horizontal. Moreover, any stable curve $W \subset \mathbb{H}_k$ satisfies $|W| \leq Ck^{-3}$ and we have chosen $r_h = 3$.

Finally, the required series in **(H3)**(5) converges since it is dominated by $\sum_{k \geq k_0} k^{-2-3\epsilon} < \infty$ for all $\epsilon > 0$.

6.3. **Distortion Bounds.** Since the map T has bounded Jacobian in the vicinity of the singular curves $V_0 \cup T^{-1}V_0$, it satisfies the same distortion bound estimates as for billiards derived from a Lorentz gas with finite horizon. Indeed it was proved in [BSC1, BSC2, Ch1] that there exist invariant families $\mathcal{W}^s$ and $\mathcal{W}^u$, which contain all homogeneous stable and unstable curves respectively, with length less than some positive constant δ. In addition, by choosing a bound on the curvature of these curves to be sufficiently large, we ensure that these families $\mathcal{W}^s$ and $\mathcal{W}^u$ are invariant in the sense described in **(H4)**.

To establish (2.3) of **(H4)**, we establish it for $\mathcal{F}$ and then note that it can be extended to any iterate of $\mathcal{F}$ (and thus to T) using the uniform hyperbolicity in the cones. From (6.5) we see that $1/\cos\varphi(\mathcal{F}^{-1}x)$ is unbounded, τ is bounded and Hölder continuous with exponent 1/2 and all other functions in $D_x\mathcal{F}^{-1}$ are bounded and smooth. Thus to establish (2.3) for $D_x\mathcal{F}^{-1}$, it suffices to establish the analogous distortion bound for $1/\cos\varphi(\mathcal{F}^{-1}x)$.

[12]We obtain a better estimate than the usual $|\mathcal{F}^{-1}W| \leq C|W|^{1/2}$ because we require that $\mathcal{F}^{-1}W$ lie in a single homogeneity strip, while the usual estimate does not use this fact. See [CM1].

Now let $W \in \mathcal{W}^s$ be such that $\mathcal{F}^{-1}W \in \mathcal{W}^s$ and $\mathcal{F}^{-1}W \subset \mathbb{H}_k$. Then using (6.8) and the fact that $|T^{-1}W| \le Ck^{-3}$, we have for any $x, y \in W$,

$$\begin{aligned}\left|\frac{1}{\cos\varphi(\mathcal{F}^{-1}x)} - \frac{1}{\cos\varphi(\mathcal{F}^{-1}y)}\right| &\le \frac{1}{\cos\varphi(\mathcal{F}^{-1}x)\cos\varphi(\mathcal{F}^{-1}y)}|\cos\varphi(\mathcal{F}^{-1}y) - \cos\varphi(\mathcal{F}^{-1}x)| \\ &\le \frac{Ck^2}{\cos\varphi(\mathcal{F}^{-1}x)} d_W(\mathcal{F}^{-1}x, \mathcal{F}^{-1}y) \le \frac{C}{\cos\varphi(\mathcal{F}^{-1}x)} d_W(\mathcal{F}^{-1}x, \mathcal{F}^{-1}y)^{1/3}.\end{aligned} \tag{6.11}$$

A similar bound holds for $x, y \in W \in \mathcal{W}^u$ such that $\mathcal{F}^{-1}W \in \mathcal{W}^u$, but with $d_W(x,y)^{1/3}$ in place of $d_W(\mathcal{F}^{-1}x, \mathcal{F}^{-1}y)^{1/3}$. This establishes (2.3) of **(H4)** with $p_0 = 1/3$.

Similarly, since $|D_xT| = \cos\varphi(x)/\cos\varphi(Tx)$, (2.4) of **(H4)** holds using the same estimate of $1/\cos\varphi$ as above. This completes the verification of **(H1)-(H5)** for T.

6.4. A Spectral Gap for $\mathcal{L}_T$ and $\mathcal{L}_F$.

Since we have verified **(H1)-(H5)** for T and have fixed the values for Λ, r_h, ξ, k_0, γ_0 and δ_0, we may also fix $\theta_* < 1$ from (2.6). We now choose the values for the parameters $\alpha, \beta, \gamma, p, q, \varepsilon_0$ and c_u appearing in the norms subject to the constraints given in Section 2.3. This fixes the Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$.

With this choice of parameters, $\mathcal{L}_T$, the transfer operator associated to T, is well defined on $\mathcal{B}$ and $\mathcal{B}_w$ and we may apply both Proposition 2.3 and Theorem 2.4 to $\mathcal{L}_T$.

In order to conclude quasi-compactness and the same characterization of the spectrum for $\mathcal{L}_\mathcal{F}$, the transfer operator associated to $\mathcal{F}$, we must show that $\mathcal{L}_\mathcal{F}$ is bounded as an operator on $\mathcal{B}$. This plus the fact that $\mathcal{L}_T = \mathcal{L}_\mathcal{F}^{n_1}$ will be enough to apply Theorem 2.4 to $\mathcal{L}_\mathcal{F}$ with essential spectral radius increased by the exponent $1/n_1$.

Proposition 6.1. *There exists $C > 0$ such that for all $h \in \mathcal{B}$,*

$$\|\mathcal{L}_\mathcal{F} h\|_s \le C\|h\|_s \qquad \text{and} \qquad \|\mathcal{L}_\mathcal{F} h\|_u \le C(\|h\|_u + \|h\|_s).$$

Following the notation of Section 3.3, for $W \in \mathcal{W}^s$, let W_i denote the components of $\mathcal{F}^{-1}W$ belonging to $\mathcal{G}_1(W)$, i.e. each W_i is a stable homogeneous curve of length less than or equal to δ_0 on which $\mathcal{F}$ is smooth. In order to prove the proposition, we will need the following lemma.

Lemma 6.2. *There exists $B_2 \ge 1$ such that for all $W \in \mathcal{W}^s$ and $\varsigma \in [0,1]$, we have*

$$\sum_{W_i \in \mathcal{G}_1(W)} \frac{|W_i|^\varsigma}{|W|^\varsigma} |J_{W_i}\mathcal{F}|_{\mathcal{C}^0(W_i)} \le B_2.$$

Proof. Since $\mathcal{S}_{-1}$ comprises finitely many curves, there exists $N \in \mathbb{N}$ such that given $W \in \mathcal{W}^s$, W may be subdivided into at most N connected components by $\mathcal{S}_{-1}$. Each of these components after iteration by $\mathcal{F}^{-1}$ may in turn be cut either by the boundaries of homogeneity strips or may be subdivided to have length at most δ_0 in the process of creating $\mathcal{G}_1(W)$. Since any piece that is subdivided artificially lies in one homogeneity region, bounded distortion implies that the sum over minimum contractions on such pieces is bounded by a constant depending only on the distortion. On the other hand, the expansion for a curve landing in $\mathbb{H}_k$ is given by (6.8) as $\ge B_1^{-1}k^{-2}$. Thus,

$$\sum_{W_i \in \mathcal{G}_1(W)} |J_{W_i}\mathcal{F}|_{\mathcal{C}^0(W_i)} \le N(C + B_2^{-1} \sum_{k \ge k_0} k^{-2}) \le C'. \tag{6.12}$$

Now $|\mathcal{F}W_i|/|W_i| \le |J_{W_i}\mathcal{F}|_{\mathcal{C}^0} \le C|\mathcal{F}W_i|/|W_i|$ by bounded distortion. Thus given any $0 \le \varsigma \le 1$, we have

$$\begin{aligned}\sum_{W_i\in\mathcal{G}_1(W)} \frac{|W_i|^\varsigma}{|W|^\varsigma}|J_{W_i}\mathcal{F}|_{\mathcal{C}^0(W_i)} &\le C \sum_{W_i\in\mathcal{G}_1(W)} \frac{|W_i|^\varsigma}{|W|^\varsigma}\frac{|\mathcal{F}W_i|}{|W_i|} = C \sum_{W_i\in\mathcal{G}_1(W)} \frac{|W|^{1-\varsigma}}{|W_i|^{1-\varsigma}}\frac{|\mathcal{F}W_i|}{|W|} \\ &\le C\left(\sum_{W_i\in\mathcal{G}_1(W)} \frac{|\mathcal{F}W_i|}{|W_i|}\right)^{1-\varsigma} \le C(C')^{1-\varsigma},\end{aligned} \tag{6.13}$$

where we have used Jensen's inequality and the fact that $\sum_i |\mathcal{F}W_i| = |W|$. □

Proof of Proposition 6.1. We essentially must redo the estimates of Sections 4.2 and 4.3, but for just one iterate of $\mathcal{L}_\mathcal{F}$ and without requiring any contraction in the norm.

Estimate on the strong stable norm. Let $h \in \mathcal{C}^1(M)$, $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^q(W)$ with $|\psi|_{W,\alpha,q} \le 1$. Then following (4.2), we estimate

$$\begin{aligned}|\int_W \mathcal{L}^n h\,\psi\,dm_W| \;&\le \sum_{W_i\in\mathcal{G}_1} \|h\|_s ||D\mathcal{F}|^{-1}J_{W_i}\mathcal{F}|_{\mathcal{C}^q(W_i)}|\psi\circ\mathcal{F}|_{\mathcal{C}^q(W_i)}\cos(W_i)\,|W_i|^\alpha \\ &\le C\|h\|_s \sum_i \frac{|W_i|^\alpha}{|W|^\alpha}|J_{W_i}\mathcal{F}|_{\mathcal{C}^0(W_i)} \le C\|h\|_s B_3,\end{aligned} \tag{6.14}$$

for some uniform constant C where we have used (3.25), (3.26) and (3.28) to simplify the expression in the second line and Lemma 6.2 with $\varsigma = \alpha$ in the last step. Taking the appropriate suprema yields the required bound on $\|\mathcal{L}_\mathcal{F} h\|_s$.

Estimate on the strong unstable norm. Given $\varepsilon \le \varepsilon_0$, let $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) < \varepsilon$. For $\ell = 1, 2$, let $\psi_\ell \in \mathcal{C}^p(W^\ell)$ such that $|\psi_\ell|_{W^\ell,\gamma,p} \le 1$ and $d_q(\psi_1,\psi_2) \le \varepsilon$.

Following Section 4.3, we partition $\mathcal{F}^{-1}W^\ell$ into matched pieces U_j^ℓ and unmatched pieces V_k^ℓ using a smooth foliation $\{\gamma_x\}_{x\in U_j^1}$ of homogeneous unstable curves. The precise characterization of matched pieces given by (4.7) applies to the matched pieces U_j^ℓ. By (6.10), the length of the unmatched curves $\mathcal{F}V_k^\ell$ is at most $C\varepsilon^{1/2}$.

Now following (4.8), (4.9) and (4.10), we estimate the norm on unmatched pieces first,

$$\sum_{\ell,k}\left|\int_{V_k^\ell} h|D\mathcal{F}|^{-1}J_{V_k^\ell}\mathcal{F}\psi_\ell\circ\mathcal{F}\,dm_W\right| \le C\varepsilon^{\xi(\alpha-\gamma)/2}\|h\|_s\sum_{\ell,k}\frac{|V_k^\ell|^\gamma}{|W^\ell|^\gamma}|J_{V_k^\ell}\mathcal{F}|_{\mathcal{C}^0(V_k^\ell)} \tag{6.15}$$

and the sum is finite by Lemma 6.2 with $\varsigma = \gamma$.

To estimate the difference on matched pieces, we follow (4.12) to write

$$\begin{aligned}&\left|\int_{U_j^1} h|D\mathcal{F}|^{-1}J_{U_j^1}\mathcal{F}\,\psi_1\circ\mathcal{F}\,dm_W - \int_{U_j^2} h|D\mathcal{F}|^{-1}J_{U_j^2}\mathcal{F}\,\psi_2\circ\mathcal{F}\,dm_W\right| \\ &\le \left|\int_{U_j^1} h|D\mathcal{F}|^{-1}J_{U_j^1}\mathcal{F}\,\psi_1\circ\mathcal{F} - \int_{U_j^2} h\,\phi_j\right| + \left|\int_{U_j^2} h(\phi_j - |D\mathcal{F}|^{-1}J_{U_j^2}\mathcal{F}\,\psi_2\circ\mathcal{F})\right|,\end{aligned} \tag{6.16}$$

where $\phi_j = (|D\mathcal{F}|^{-1}J_{U_j^1}\mathcal{F}\,\psi_1\circ\mathcal{F})\circ G_{F_j^1}\circ G_{F_j^2}^{-1}$ is well defined on U_j^2 due to the pairing given by (4.7).

We estimate the first term on the right hand side of (6.16) using the strong unstable norm. Notice that since we are only applying one iterate of $\mathcal{F}$ and due to bounded distortion, we have

$d_{\mathcal{W}^s}(U_j^1, U_j^2) \le C d_{\mathcal{W}^s}(W^1, W^2) \le C\varepsilon$. Also, using the normalization of the test functions due to (4.13), we follow (4.14) to estimate

$$\sum_j \left| \int_{U_j^1} h |D\mathcal{F}|^{-1} J_{U_j^1}\mathcal{F}\, \psi_1 \circ \mathcal{F} - \int_{U_j^2} h\, \phi_j \right| \le C\varepsilon^\beta \|h\|_u, \tag{6.17}$$

where we have again used Lemma 6.2 to bound the sum.

Finally, we estimate the second term on the right hand side of (6.16) using the strong stable norm, following (4.15),

$$\left| \int_{U_j^2} h(\phi_j - |D\mathcal{F}|^{-1} J_{U_j^2}\mathcal{F} \psi_2 \circ \mathcal{F}) \right| \le\; C\|h\|_s |U_j^2|^\alpha \cos U_j^2 \left| \phi_j - |D\mathcal{F}|^{-1} J_{U_j^2}\mathcal{F}\, \psi_2 \circ \mathcal{F} \right|_{\mathcal{C}^q(U_j^2)}.$$

We split up the estimate on the $\mathcal{C}^q$-norm of the test function following (4.16). Then, since the proof of Lemma 4.1 goes through essentially unchanged with $\mathcal{F}$ in place of T^n (except that we lose contraction), we estimate,

$$\sum_j \left| \int_{U_j^2} h(\phi_j - |D\mathcal{F}|^{-1} J_{U_j^2}\mathcal{F}\psi_2 \circ \mathcal{F}) \right| \le C\|h\|_s \varepsilon^{p-q} \sum_j \frac{|U_j^2|^\gamma}{|W^2|^\gamma} |J_{U_j^2}\mathcal{F}|_{\mathcal{C}^0(U_j^2)}, \tag{6.18}$$

and again the sum is finite by Lemma 6.2.

Now we bring together the estimates in (6.15), (6.17) and (6.18) to conclude,

$$\left| \int_{W^1} \mathcal{L}_{\mathcal{F}} h\, \psi_1\, dm_W - \int_{W^2} \mathcal{L}_{\mathcal{F}} h\, \psi_2\, dm_W \right| \;\le\; C\left(\|h\|_s \varepsilon^{\xi(\alpha-\gamma)/2} + \|h\|_u \varepsilon^\beta + \|h\|_s \varepsilon^{p-q} \right).$$

Since $\beta < \min\{p-q, \xi(\alpha-\gamma)/2\}$, we divide through by ε^β and take the appropriate suprema to complete the estimate on the unstable norm. □

It follows from Proposition 6.1 that $\mathcal{L}_{\mathcal{F}}$ is bounded and therefore quasi-compact on $\mathcal{B}$ due to the quasi-compactness of $\mathcal{L}_T = \mathcal{L}_{\mathcal{F}}^{n_1}$. Thus the spectrum of $\mathcal{L}_{\mathcal{F}}$ on $\mathcal{B}$ is characterized by items (1)-(3) of Theorem 2.4. In particular, each element of its peripheral spectrum in $\mathcal{B}$ is absolutely continuous with respect to $\overline{\mu} := \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathcal{L}_{\mathcal{F}}^i m$, where m here denotes Lebesgue measure on M.

However, it is well known that $\mathcal{F}$ preserves the smooth invariant measure $\mu = c\cos\varphi dm$, where c is a normalizing constant. Since $\cos\varphi \in \mathcal{C}^1(M)$, we have $\mu \in \mathcal{B}$ by Lemma 3.5 so that μ is absolutely continuous with respect to $\overline{\mu}$. But since the support of μ is all of M, we must have $\mu = \overline{\mu}$. In addition, the mixing properties of $\mathcal{F}$ imply that 1 is the only eigenvalue on the unit circle and that μ is its unique probability measure. Thus, $\mathcal{L}_{\mathcal{F}}$ enjoys a spectral gap on $\mathcal{B}$ and items (4)-(7) of Theorem 2.4 apply. As a consequence, the limit theorems of Corollary 2.5 hold for $\mathcal{F}$.

7. Reduced maps for two types of billiards with focusing boundaries

In this section we turn to the two specific classes of billiards that were studied in [CZ4] and introduced in Section 2.4.2: Non-smooth stadia and certain types of Bunimovich tables containing circular arcs greater than a semicircle. Both billiards were first studied by Bunimovich in [Bu], where hyperbolicity and ergodicity were proved. Recall the hyperbolic set $\bar{M}$ defined by (2.15) and denote by $R : \bar{M} \to \mathbb{Z}^+$ the first return time to $\bar{M}$. We will work exclusively with the induced map $T := \mathcal{F}^R$. Although the billiard maps $\mathcal{F}$ exhibit only polynomial decay of correlations, it was shown in [CZ4] that T exhibits exponential decay of correlations and we will show here that the associated transfer operator $\mathcal{L}_T$ has a spectral gap on $\mathcal{B}$, proving Theorem 2.7.

Verifying properties **(H1)**-**(H5)** for T will proceed similarly to Section 6, but is simpler in this case since we will not prove any results for $\mathcal{L}_F$ so the process of translating results for $\mathcal{L}_T$ into results for $\mathcal{L}_{\mathcal{F}}$ carried out in Section 6.4 is not needed here. Below we will use the notation $A \sim B$ if there exists $C > 1$ such that $C^{-1} \le \frac{A}{B} \le C$.

7.1. **Nonsmooth stadia.** Note that for this type of billiard, the phase space $\bar{M}$ is contained in the two rectangles $\Gamma_i \times [-\pi/2, \pi/2]$, $i = 1, 2$, where each Γ_i corresponds to one of the two circular arcs in ∂Q.

As noted in Section 2.4.2, $T = \mathcal{F}^R$ preserves the conditional measure μ_0 since its Jacobian satisfies $|DT(x)| = \cos\varphi(x)/\cos\varphi(Tx)$ so that **(H1)** is satisfied as in Section 6.

The hyperbolicity of the reduced map T, i.e. **(H2)**, was verified in [CM1, CZ2]. Indeed, these references focus on the dynamics of unstable curves mapped forward under T, while below we focus on the the dynamics of stable curves mapped under T^{-1}, but by symmetry, these properties are identical.

Following [CM1, Section 8.4], we first define stable and unstable cones in $\bar{M}$, see (2.15), by

$$\bar{C}^s(x) = \{(dr, d\varphi) : 0 \le d\varphi/dr \le -\mathcal{K}(x)\}, \qquad C^u(x) = \{(dr, d\varphi) : \mathcal{K}(x) \le d\varphi/dr \le 0\} \tag{7.1}$$

where $\mathcal{K}(x) = -1/\mathbf{r}_i$ for $x \in \Gamma_i$ and $\mathbf{r}_i$ is the radius of the circular arc Γ_i. Below we will narrow $\bar{C}^s(x)$ somewhat in order to ensure uniform transversality of our stable curves with $\mathcal{S}_{-1}$ and of $C^s(x)$ with $C^u(x)$. Thus stable curves are increasing and unstable curves are decreasing (precisely the opposite of what occurs for dispersing billiards). The uniform hyperbolicity required for **(H2)**(1) and (2) follows from the strictly negative curvature of the circular arcs and the fact that the free flight time for the inverse return map T^{-1} is uniformly bounded away from zero, as we have assumed that the tables satisfy Bunimovich's Defocusing Mechanism, see [CM1, Chapter 8].

Note that by (7.1), the cones C^u and $\bar{C}^s$ share the same boundary $d\varphi/dr = 0$. To guarantee the uniform transversal property of the cones, we define a smaller stable cone field C^s such that the boundary in the direction of $(dr, 0)$ in $\bar{C}^s$ is replaced by $D_{x_1}\mathcal{F}^{-1}(dr_1, 0)$, with $(r_1, \varphi_1) = x_1 := \mathcal{F}x$. More precisely, note that by (6.5), if $d\varphi_1/dr_1 = 0$ and $(dr, d\varphi) = D_{x_1}\mathcal{F}^{-1}(dr_1, 0)$, then

$$\frac{d\varphi}{dr} = g_1(x) := -\mathcal{K}(r)\frac{\tau(x) + \cos\varphi/\mathcal{K}(r) + \cos\varphi_1/\mathcal{K}(r_1)}{\tau(x) + \cos\varphi_1/\mathcal{K}(r_1)} = -\mathcal{K}(r)\left(1 + \frac{\cos\varphi/\mathcal{K}(r)}{\tau(x) + \cos\varphi_1/\mathcal{K}(r_1)}\right).$$

When $x_1 = \mathcal{F}x$ lies on a different focusing boundary than x, by the Bunimovich defocusing mechanism, we know that there exists $c_0 > 0$ such that $\tau(x) + \cos\varphi/\mathcal{K}(r) + \cos\varphi_1/\mathcal{K}(r_1) > c_0$, and also $\tau(x) \ge \tau(x) + \cos\varphi_1/\mathcal{K}(r_1) \ge \tau(x)/2$ [CM1, Section 8.4]. This implies that

$$g_1(x) \ge \frac{c_0|\mathcal{K}|_{\min}}{\tau_{\max}},$$

remembering that $\mathcal{K}(r) < 0$.

Since we have chosen $\bar{M}$ to consist of only *last* collisions with focusing arcs (in forward time), the only other possibility for $x \in \bar{M}$ is that $\mathcal{F}x$ lies on a flat segment of the boundary. In this case, we have $g_1(x) = -\mathcal{K}(x)$, again using (6.5).

Now setting $c_1 = \frac{1}{2}\min\left\{\frac{c_0|\mathcal{K}|_{\min}}{\tau_{\max}}, |\mathcal{K}|_{\min}\right\}$, we define the narrower stable cones by,

$$C^s(x) = \{(dr, d\varphi) : c_1 \le d\varphi/dr \le -\mathcal{K}(x)\}. \tag{7.2}$$

By the above discussion, they satisfy $D_xT^{-1}\bar{C}^s(x) \subset C^s(T^{-1}x)$, wherever D_xT^{-1} is defined. With this definition of $C^s(x)$ and $C^u(x)$, the stable and unstable cones are uniformly transverse to one another as required. This completes the verification of **(H2)**.

We now describe the precise structure of the singularity sets in order to verify **(H3)**. We set $\mathcal{S}_0 = \partial\bar{M}$ and let $\mathcal{S}_{\pm 1} = \mathcal{S}_0 \cup (\cup_{i=0}^R \mathcal{F}^{\mp 1}\mathcal{S}_0)$ denote the singularity sets for $T^{\pm 1}$.

Curves in $\mathcal{S}_{-1} \setminus \mathcal{S}_0$ are decreasing and those in $\mathcal{S}_1 \setminus \mathcal{S}_0$ are increasing, although the slopes of curves in $\mathcal{S}_{-1} \setminus \mathcal{S}_0$ get arbitrary close to horizontal near $\varphi = \pm\frac{\pi}{2}$. Also, by our choice of the phase space for the reduced map, the singular set $\mathcal{S}_{-1}$ is symmetric about $\varphi = 0$ with the singular set of the forward reduced map studied in [CZ1]. More precisely, on each rectangle, $\Gamma_i \times [-\pi/2, \pi/2]$, $i = 1, 2$, the map T^{-1} has two types of sequences of singularity curves converging to 4 accumulation points in $\bar{M}$, x_i, $i = 1, \dots 4$.

The first type accumulates near x_1 and x_2, which have $\varphi(x_1) = \pi/2$ and $\varphi(x_2) = -\pi/2$; they are generated by trajectories nearly "sliding" along the circular arcs. Let us denote one such sequence in $\mathcal{S}_{-1}$ accumulating on x_i by $\{S_{i,n}\}_{n\in\mathbb{N}}$. We denote the region between $S_{i,n}$ and $S_{i,n-1}$ by $M_{i,n}$. A point $x \in M_{i,n}$ undergoes n consecutive nearly sliding reflections along one arc before colliding with another part of the boundary. A smooth stable curve W is such that $T^{-1}W$ that completely crosses M_n, then $T^{-n}W \cap M_n$ has (Euclidean) length $\sim n^{-2}$ [CZ4].

It follows from [CM1, eq. (8.23)] that the expansion satisfies

$$\frac{\|D_xT^{-1}v\|}{\|v\|} \sim \frac{1}{\cos\varphi(T^{-1}x)} \sim n \quad \text{for } v \in C^s(x) \tag{7.3}$$

whenever $T^{-1}x \in M_{i,n}$. Indeed, in the Euclidean metric, it follows from [CM1, Section 8.9] that there is expansion of order n when $T^{-1}W$ lands in $M_{i,n}$ (due to (7.3)), and there is another order n expansion when W maps out of $M_{i,n}$. This is because once we fix a non-smooth stadium, there exists a choice of n_0 large enough so that a sequence of sliding collisions on one arc landing in $\cup_{n\geq n_0} M_{i,n}$ is not followed by another sequence of sliding collisions on the other arc, i.e. it must land ouside the set $\cup_{n\geq n_0} M_{i,n}$, $i = 1, 2$, on the other arc.

Note that due to (7.3), we do not have bounded distortion for curves landing across multiple $M_{i,n}$, $i = 1, 2$. In order to control this distortion, we define homogeneity curves to coincide with a subset of the singularity curves $\{S_{i,n}\}$. Specifically, for $k \geq k_0$ to be chosen later, we define $S^H_{i,k} = S_{i,n_k}$ when $n_k \approx k^2$ so that $S^H_{i,k}$ is approximately distance k^{-2} from x_i.[13] Let $\mathbb{H}_{i,k}$ be the region between $S^H_{i,k}$ and $S^H_{i,k+1}$ and note that $\mathbb{H}_{i,k}$ contains at most $2k+1$ cells $M_{i,n}$, $n \approx k^2, \dots, k^2 + 2k$. Thus we subdivide $T^{-1}W$ according to the singularity curves S_{i,n_k}, $k \geq k_0$ one step earlier than it would be cut by the dynamics. The remaining $S_{i,n}$ cut $T^{-1}W$ when it leaves under a second iterate of T^{-1}. We do not introduce any other artificial cuts.

The second type of singular curves accumulate near the other two points x_3, x_4 that are located on the two lines $\varphi = \pm\varphi_0$. They are generated by trajectories experiencing many bounces off the two straight sides of the stadium. As before, we denote these two sequences as $\{S_{i,n}\}_{n\in\mathbb{N}}$, $i = 3, 4$, accumulating on x_i and let $M_{i,n}$ denote the connected region bounded by the adjacent curves $S_{i,n}, S_{i,n-1}$ in $M \setminus \mathcal{S}_{-1}$. Points in $M_{i,n}$ experience exactly n reflections off the straight sides before landing on the opposite arc of ∂Q. Again let W be a stable curve passing through x_i and crossing $S_{i,n}$, for all $n \geq n_0$. As shown in [CZ4], the length of $W_n := W \cap M_{i,n}$ satisfies $|W_n| \sim n^{-2}$. For any $x \in W_n$, its Jacobian satisfies $J_W T^{-1}(x) \sim n$. There is no need for any homogeneity strips near x_3 and x_4.

From the constructions and facts recalled above, it is clear that **(H3)**(1) is satisfied with $\xi = 1/2$, by an estimate similar to (6.9).[14] Using the above facts and since the boundary of each cell $M_{i,n}$ is comprised of 4 smooth curves, **(H3)**(2) is satisfied. By the definition of $C^s(x)$ in (7.2), **(H3)**(3) is satisfied with $t_0 = 1$ since curves in $\mathcal{S}_{-1}$ are decreasing curves, while stable curves are increasing and bounded away from the horizontal.

(H3)(4) is also satisfied since the boundaries of the homogeneity strips have been chosen to coincide with the curves S_{i,n_k}, $i = 1, 2$, $k \geq k_0$, which are uniformly transverse to the stable cones. Moreover, $r_h = 3$. Item (5) of **(H3)** follows immediately since the series is dominated by $\sum_{k\geq k_0} k^{-2-3\epsilon} < \infty$ for all $\epsilon > 0$ using the fact that $f = \cos\varphi \approx k^{-2}$ on each $\mathbb{H}_k$.

For **(H4)**, the existence of invariant families of stable and unstable curves follows from [CZ2] or [CM1, Section 8.10]. Note that the curvature bounds proved there do not depend on the particulars of smooth versus nonsmooth stadia. To establish (2.3) of **(H4)**, for $T^{-1}x$ landing near x_1 and x_2, we need to use similar arguments as we did in Section 6, because $\|D_xT^{-1}\| \sim 1/\cos\varphi(T^{-1}x)$ by

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[13] In fact, $n_k = k^t$ for any $t > 1$ would work as well for the convergence of the series (7.4) and (7.5).

[14] We get $\xi = 3/5$ for curves landing in one of the $\mathbb{H}_{i,k}$ and $\xi = 1/2$ for curves starting in one of the $M_{i,n}$, and use the lesser of the two exponents.

(7.3). An estimate similar to (6.11) yields the required distortion bound with $p_0 = 1/3$, due to the spacing defined by $r_h = 3$. The cases when $x \in M_{i,n}$ are addressed by the distortion bounds in [CM1, Section 8.12] and yield $p_0 = 1/2$, so we take the lesser of the two exponents for the value of p_0. Similarly, (2.4) of **(H4)** holds with $p_0 = 1/3$. This completes the verification of **(H1)-(H4)** for T.

It remains to verify **(H5)**. In particular, we want to emphasize that (2.5) only holds with $\gamma_0 > 0$. Any stable curve $W \in \mathcal{W}^s$ for which $T^{-1}W$ is cut into an unbounded number of short stable curves must be in one of three places: (1) $T^{-1}W$ lands near one of the accumulating singular points x_1 or x_2 as described above; (2) W lies in one of the homogeneity regions $\mathbb{H}_{i,k}$; or (3) W lies near one of the accumulation points x_3 or x_4. We proceed to prove (2.5) for each of these cases.

First we address those curves landing near x_i, $i = 1, 2$, which are cut according to our homogeneity curves $S^H_{i,k} = S_{i,n_k}$, $k \geq k_0$ upon landing. Suppose that $T^{-1}W$ intersects $\mathbb{H}_{i,k}$, $k \geq k_0$. Setting $V_k = T^{-1}W \cap \mathbb{H}_{i,k}$ and $W_k = TV_k$, we have $|V_k| \sim k^{-3}$, $J_{V_k}T \sim k^{-2}$ by (7.3) and so necessarily $|W_k| \sim k^{-5}$ by bounded distortion. Thus this series satisfies the traditional one-step expansion with $\gamma_0 = 0$,

$$\sum_{k \geq k_0} \frac{|W_k|}{|V_k|} \leq \sum_{k \geq k_0} Ck^{-2} \leq Ck_0^{-1}, \tag{7.4}$$

for some uniform constant C depending on the table. This can be made less than 1 by choosing k_0 large. This implies (2.5) for any $\gamma_0 \in (0, 1)$ via the Hölder inequality, so we are still free to choose γ_0.

Next we consider the case when $W \subset \mathbb{H}_{i,k}$, $i = 1, 2$. Since $\mathbb{H}_{i,k}$ contains at most $2k + 1$ cells $M_{i,n}$, as described above, $T^{-1}W$ will be cut into at most $2k + 1$ pieces by the singularity curves. Note that by choosing k_0 sufficiently large, we can guarantee that $T^{-1}W$ does not intersect any homogeneity strips associated with any of the other components of the phase space, so there is no additional cutting to take into consideration. Setting $W_n = W \cap M_{i,n}$ and $V_n = T^{-1}W_n$, we have $|W_n| \sim n^{-2}$ and $|V_n| \sim n^{-1}$ as described above. Letting n_k denote the index of $S_{i,n}$ coinciding with $S^H_{i,k}$ ($n_k \sim k^2$) we estimate the sum required for (2.5) with $\gamma_0 = 0$, by

$$\sum_{j=0}^{2k} \frac{|W_{n_k+j}|}{|V_{n_k+j}|} \leq Ckn_k^{-1} \leq Ck^{-1}. \tag{7.5}$$

As in (7.4), this can be made small by choosing k_0 large and implies (2.5) for any $\gamma_0 \in (0, 1)$.

Finally, we consider the case when W is cut by singular curves close to the line $\varphi = \varphi_0$ (i.e. close to the singular points x_3 and x_4 described above). Let $x_{-1} = T^{-1}x$ and for $v \in \mathcal{T}_xW$, we denote $v_{-1} = DT^{-1}v$. For $x \in \bar{M}$, define $\tau^R(x) = \tau(x) + \ldots + \tau(\mathcal{F}^{R-1}x)$ to be the time between the collisions at x and $Tx = \mathcal{F}^R x$.

Since this estimate is more delicate than the one-step expansion near x_1 and x_2, we will use a special scaled norm on the tangent space, defined as follows. Let $A = \cup_{i=3,4} \cup_{n \geq n_0} M_{i,n}$ and for $v = (dr, d\varphi) \in C^s(x)$, define $\|v\|_* = |dr|$ when $x \in A$, and $\|v\|_* = B_s|dr|$ when $x \in A^c$, for some constant B_s to be determined later. Since the slopes of vectors in $C^s(x)$ are bounded away from $\pm\infty$, we can extend this norm to be uniformly equivalent to the Euclidean norm in the tangent space at x.

We choose n_0 large enough such that for any $x \in A$, $T^{-1}x$ belongs to A^c and $T^{-1}x$ does not lie in any of the homogeneity regions $\mathbb{H}_{i,k}$, $k \geq k_0$. The main reason that we can guarantee this is because $\varphi_0 > 0$. Note that this scaling does not affect our previous estimates in (7.4) and (7.5) since the neighborhoods defined by $M_{i,n}$ with $n \geq n_0$ for each of the x_i do not map to one another under one iterate of T^{-1} in non-smooth stadia.

Recall that for $x \in M_{i,n}$, $i = 3, 4$, its trajectory hits the other arc after n collisions with the flat boundary under the original billiard map. Thus its free path satisfies $\tau^n(x_{-1}) = 2n \cos \varphi_0 / |\mathcal{K}| + \mathcal{O}(1)$, where $\mathcal{O}(1) \le C$, for some uniform positive constant $C > 0$ depending on the table.

According to (7.2) for $v \in C^s(x)$, $d\varphi/dr > 0$. Thus for $x \in M_{i,n}$, using (8.3) and (8.21) in [CM1], for $v \in C^s(x)$, the expansion factor satisfies

$$\frac{\|v_{-1}\|_*}{\|v\|_*} = \frac{B_s \cos\varphi}{\cos\varphi_{-1}} \cdot \left(\frac{\tau^R(x_{-1})(|\mathcal{K}(x)| - d\varphi/dr)}{\cos\varphi} + 1 \right) = \frac{B_s \tau^n(x_{-1})|\mathcal{K}(x)|}{\cos\varphi_0} + \mathcal{O}(1) = 2B_s n + \mathcal{O}(1),$$

where we have used the fact that φ_{-1} is approximately φ_0 for $x \in M_{i,n}$ with n large, and $\cos\varphi_0$ is bounded away from 0.

Let $V_n = T^{-1} W_n$, where $W_n = W \cap M_{i,n}$ as before. The distortion bound on W yields,

$$(2B_s n + \mathcal{O}(1)) e^{-C_d |V_n|^{1/2}} \le \frac{|V_n|_*}{|W_n|_*} \le (2B_s n + \mathcal{O}(1)) e^{C_d |V_n|^{1/2}}, \tag{7.6}$$

where $|V_n|_*$ is the length of V_n measured in the adapted metric. If necessary, we increase n_0 sufficiently so that $C_0 := e^{C_d |V_{n_0}|^{1/2}} < 2$.

Due to the facts from [CZ4] recalled earlier about the spacing and Jacobian on $M_{i,n}$, there exists a uniform constant $a > 0$, such that $|V_n|_* = a n^{-1} + \mathcal{O}(n^{-2})$. This implies that if $W \in \mathcal{W}^s$ lies entirely in A, we have

$$|W|_* = \sum_{n > n_0} |W_n|_* = \sum_{n > n_0} |V_n|_* \cdot \frac{|W_n|_*}{|V_n|_*} \ge \frac{a}{B_s n_0} + \mathcal{O}(n_0^{-2}).$$

Using this and again (7.6), for any $\gamma_0 \in (0, 1)$, the one-step expansion estimate holds:

$$\sum_{n=n_0}^{\infty} \left(\frac{|V_n|_*}{|W|_*} \right)^{\gamma_0} \frac{|W_n|_*}{|V_n|_*} \le \vartheta(\gamma_0) := \frac{1}{B_s^{1-\gamma_0} \gamma_0} + \mathcal{O}(n_0^{-1}). \tag{7.7}$$

According to **(H5)**, we must choose $\gamma_0 \in (0, 1/r_h) = (0, 1/3)$. For definiteness and with (2.7) in mind, we choose $\gamma_0 = 1/4$. Thus choosing $B_s = 7$ and n_0 sufficiently large, we can make $\vartheta < 1$. Note that the above expression diverges when $\gamma_0 = 0$, which is the traditional one-step expansion.

The definition of $\| \cdot \|_*$ increases expansion by a factor of $B_s = 7$ when mapping from A to A^c, but decreases expansion by a factor of $1/7$ when mapping from A^c to A. In order to overcome this contraction factor, we formulate the following complexity assumption on the stadium.

Let $\Lambda_0 = 1 + \tau^R_{\min}/\mathbf{r}_{\max}$, where $\mathbf{r}_{\max} = \max\{\mathbf{r}_1, \mathbf{r}_2\}$ and $\tau^R_{\min} = \min_{x \in \bar{M}}\{\tau^R(x)\} > 0$. We assume there exists $n_1 > 0$ such that

$$\Lambda_0^{n_1} > 7 \text{ and } A \cap (\cup_{i=1}^{n_1} T^{-i} A) = \emptyset. \tag{7.8}$$

Note that this assumption can easily be satisfied for nonsmooth stadia by choosing geometric parameters so that $\Lambda_0 > 2$, which forces $n_1 = 3$. Then typically, the first three iterates of the orbits of x_i, $i = 1, 2$, are disjoint. Thus choosing n_0 sufficiently large we can guarantee (7.8).

This assumption guarantees that enough expansion builds up for T^{-n_1} to overcome the factor of $1/7$ encountered when mapping from A^c to A. Thus the expansion for T^{-n_1} dominates the complexity and so **(H5)** is satisfied for W close to x_3 and x_4. The choice of n_0 also fixes the value of δ_0 in the definition of $\mathcal{W}^s$.

This completes the required verification of **(H1)**-**(H5)** and so completes the proof of Theorem 2.7 via Theorem 2.4 for the reduced map T^{n_1}. To pass from T^{n_1} to T, simply note that by (7.4), (7.5) and (7.7), the one-step expansion is uniformly finite for T even if it is not contracting. Thus $\|\mathcal{L}_T\|_{\mathcal{B}}$ is finite as explained in Section 6.4 and $\mathcal{L}_T$ inherits the spectral gap from $\mathcal{L}_{T^{n_1}} = \mathcal{L}_T^{n_1}$.

7.2. **Bunimovich tables.** The verification for this class of billiards proceeds much as the class above, except that due to the nature of the table and the unspecified location of corner points (where the smooth arcs comprising the boundary ∂Q terminate), the necessary complexity assumption requires more conditions than (7.8).

Since the tables we consider have both dispersing and focusing boundaries (recall that we have assumed that there are no flat boundaries in our Bunimovich tables), they are treated by a combination of the techniques described for stadia in Section 7.1 and for dispersing billiards in Section 6. In particular, the stable and unstable cones are defined separately on these two types of boundaries as described in each of those two sections, using (6.3) and (7.2). Given our work in Section 7.1, we define homogeneity strips on the dispersing boundaries as in (6.7) with exponent $r_h = 3$; on the focusing boundaries we choose them to coincide with a subset of the singularity curves $S_{i,n}$ with exponent $r_h = 3$ near $\varphi = \pm\pi/2$ as described in Section 7.1. The stable/unstable cones can be defined as in (7.2). We will not repeat the verification of **(H1)-(H4)** as above, but instead focus on two main points: the verification of **(H3)** and **(H5)**. We recall the structure of the singularity sets for the return map $T = \mathcal{F}^R$ from [CZ4].

On each component $\Gamma_i \times [-\pi/2, \pi/2] \subset M$ corresponding to a focusing arc Γ_i, there are once again 4 accumulation points for the singularity set $\mathcal{S}_{-1}$ of T^{-1}, which we shall denote by x_i, $i = 1, \dots 4$ as in the previous section. The first two of these points, x_1 and x_2 are created by the same "sliding" trajectories as x_1 and x_2 described in Section 7.1 and the analysis of expansion factors is the same. A similar analysis of expansion factors holds at dispersing boundaries as in (7.4) since we have chosen $r_h = 3$ and the expansion upon landing near a dispersing boundary is also of order $1/\cos\varphi(T^{-1}x)$. Thus the series required for **(H5)** can be made arbitrarily small as in (7.4) and (7.5) by choice of k_0 for all stable curves landing on dispersing boundaries and on focusing boundaries near x_1 and x_2.

The second two points x_3 and x_4 lie on the line $\varphi = 0$ and are created by trajectories which run near one the diameters of the circular arc. Such trajectories make successive bounces across Γ_i while rotating slowly around the circle until they reach an opening through which they escape to collide with a different arc.

As before, denote by $\{S_{i,n}\}_{n\in\mathbb{N}}$ the sequence of curves in $\mathcal{S}_{-1}$ accumulating on x_i, $i = 3, 4$, and by $M_{i,n}$ the connected region in $\bar{M} \setminus \mathcal{S}_{-1}$ bounded by $S_{i,n}$ and $S_{i,n-1}$. The curves $S_{i,n}$ are distance of order n^{-1} from x_i and are uniformly transverse to the stable cone. Thus any stable curve W crossing $\bar{M}_{i,n}$ satisfies $|W_n| := |W \cap M_{i,n}| = an^{-2} + \mathcal{O}(n^{-3})$. In addition, the expansion factor on $W \cap M_{i,n}$ under T^{-1} is $4n\mathbf{r} + \mathcal{O}(1)$, where $\mathbf{r}$ is the radius of the large arc [CZ4]. As before, let $V_n := T^{-1}W_n$.

As before, define $A = \cup_{i=3,4} \cup_{n\geq n_0} M_{i,n}$ and $A^c = \bar{M} \setminus A$. As in Section 7.1, we define the scaled norm $\|v\|_*$ for any tangent vector in $\mathcal{T}_x M$. Now by repeating the same calculation as in (7.7), we can prove the one step expansion estimate **(H5)** for $W \in \mathcal{W}^s$, $W \subset A$, with $\vartheta < 1$, $B_s = 7$ and $\gamma_0 = 1/4$.

In order to address more general W, we need to resort to a higher iterate of T and formulate our complexity assumption following [CZ1]. We split the curves in $\mathcal{S}_{-1}$ into primary and secondary singularities. The secondary singularities are all those curves $S_{i,n}$ with $n > n_0$ for some n_0 chosen below. In addition, at dispersing boundaries, we consider all the boundaries of homogeneity strips to be secondary singularities. The primary singularities are the finitely many remaining curves in $\mathcal{S}_{-1}$ and are denoted by $\mathcal{S}^P_{-1}$. Define $\mathcal{S}^P_{-n} = \cup_{i=0}^n T^i(\mathcal{S}^P_{-1})$ to be the set of primary singularity curves for T^{-n} and let $K_{P,n}$ denote the minimum number of curves in $\mathcal{S}^P_{-n}$ which intersect at any one point of M. The assumption on complexity for the Bunimovich table is then two-fold.

(1) There exists $n_0 > 0$ sufficiently large and $n_1 \in \mathbb{N}$ such that

$$\Lambda^{n_1} > 7 \;\text{ and }\; A \cap (\cup_{i=1}^{n_1} T^{-i} A) = \emptyset \tag{7.9}$$

where Λ is the minimum expansion factor in the p-metric for stable vectors under DT^{-1}.

(2) There exists $n_2 > 0$ such that

$$K_{P,n_2} < \Lambda^{n_2}. \tag{7.10}$$

Note that (7.9) is the same as (7.8) and guarantees that the expansion in the scaled metric has a chance to recover when mapping from A^c to A. It is easily satisfied if the orbits of the singular points x_i are disjoint for the first several iterates. On the other hand, (7.10) is a complexity condition which is necessary due to the indeterminate location of corner points on the Bunimovich table. It now follows from [CZ1, Theorem 12] and [CZ4] that T^{n_3} satisfies **(H5)** for some $n_3 > 0$.

Finally, we check that **(H3)**(1)-(5) are satisfied for this class of tables. As in Section 7.1, **(H3)**(1) is satisfied for T with $\xi = 1/2$, again using (6.9), and for T^{n_3} with $\xi \le 2^{-n_3}$. Also since the boundary of each cell $M_{i,n}$ is comprised of 4 smooth curves and the maximum length of a stable curve in $M_{i,n}$ goes to zero with n, **(H3)**(2) is satisfied. By the definition of $C^s(x)$ in (6.3) and (7.2), **(H3)**(3) is satisfied with $t_0 = 1$ as before. **(H3)**(4) is also satisfied on focusing boundaries with $r_h = 3$ since the boundaries of the homogeneity strips have been chosen to coincide with the curves S_{i,n_k}, $i = 1, 2$, $k \ge k_0$, which are uniformly transverse to the stable cones. On dispersing boundaries, the transversality is also uniform as described in Section 6 and $r_h = 3$ as well. Item (5) of **(H3)** follows immediately since the series is dominated by $\sum_{k \ge k_0} k^{-2-3\epsilon} < \infty$ for all $\epsilon > 0$ on both focusing and dispersing boundaries, using the fact that $f = \cos\varphi \approx k^{-2}$ on each $\mathbb{H}_k$.

Having verified **(H1)**-**(H5)**, we may conclude a spectral gap for $\mathcal{L}_{T^{n_3}} = \mathcal{L}_T^{n_3}$ and since $\|\mathcal{L}_T\|_{\mathcal{B}}$ is finite even when **(H5)** is not contracting as explained in Section 7.1, the spectral gap follows for $\mathcal{L}_T$ as well, completing the proof of Theorem 2.7.

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