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SPECTRAL ANALYSIS OF NON-SELF-ADJOINT ELLIPTIC OPERATORS

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1. Introduction

Many important problems of mathematical physics lead to analysis of the differential equation

$$\sum_{k=0}^{n} A_k \frac{\partial^k}{\partial t^k} u = f, \qquad \text{in } \Omega, \tag{1}$$

where A_k are symmetric positive definite elliptic operators of order 2m. When dealing with analysis of these equations we assume that Ω - the domain of definition is bounded and $\partial\Omega$ - the boundary is sufficiently smooth. We consider homogeneous boundary conditions and non-homogeneous initial conditions.

When applying Laplace transform we arrive at

$$A(p)\tilde{u} = \sum_{k=0}^{n} p^k A_k \tilde{u} = \tilde{f}^*, \tag{2}$$

where a tilde denotes the Laplace transform and $\tilde{f}^*$ includes initial conditions. The operator A(p) is a complex symmetric non-self-adjoint elliptic operator.

For analysis of equations (2) we have introduced [1 - 2] spaces of analytic functions valued in Sobolev spaces, which are isomorphic to weighted anisotropic Sobolev spaces convenient for analysis of equations (1).

Now we shall deal with spectral analysis of complex symmetric operators and show that it is possible to obtain similar results on existence of eigenvalues and completeness of sets of eigenvectors as in the case of symmetric compact operators.

2. Spectral analysis

Operators A(p) are complex symmetric operators. Thus it holds $A^*(p) = A(\bar{p})$ and

$$(Ax,\bar{x}) = (x,\bar{A}\bar{x}) \ . \tag{3}$$

When $A_k A_l \neq A_l A_k$ i.e. when operatore A_k are noncommutative

$$AA^* \neq A^*A \tag{4}$$

and A(p) is a nonnormal operator. Thus for their analysis it is not possible to apply the spectral theory of symmetric compact operators. However it is possible to generalize some of its results.

I.C. Gokhberg and M.G. Krein [3] delt with the spectral analysis of (1) from the point of view of a nonlinear eigenvalue problem

$$\sum_{k=0}^{n} \lambda^k A_k e = 0 \ . \tag{5}$$

When applying this approach we cannot use valuable results of the linear spectral theory.

Therefore for the problem under consideration we define a linear eigenvalue problem considering the equation

$$A(p)e(p) = \sum_{k=0}^{n} p^k A_k e(p) = \lambda(p)e(p) \ , \tag{6}$$

where $\lambda(p)$ for which the solutions of (6) exist are eigenvalues and the corresponding solutions $e(p)$ are eigenvectors of (6). Both eigenvalues and eigenvectors are in general functions of the parameter p. Eigenvalues in the sense of (5) are values of p for which

$$\lambda(p) = 0 \tag{7}$$

and the corresponding values of $e(p)$ are eigenvectors of (5).

For nonnegative real values of p A(p) is a symmetric positive definite elliptic operator. Thus it has discrete spectrum and a complete pairwise orthogonal set of eigenvectors. Then there exists a neighbourhood Ω_{p_1} of the positive real semiaxis p_1, where A(p) has the compact inverse $B(p) = A^{-1}(p)$ and $B_1(p) = \text{Re } B(p)$ and $B_2(p) = \text{Im } B(p)$ are positive symmetric compact operators.

The we can prove:

Theorem 1. The operator $B(p) = A^{-1}(p)$ has at least one nonzero eigenvalue and its eigenvalues and eigenvectors are solutions of the variational problem

$$\min \max \left[|(B\ e,\bar{e})| - |\mu||(e,\bar{e})| \right] , \quad \mu = 1/\lambda \ . \tag{8}$$

Proof: As $B_1(p)$ and $B_2(p)$ are positive operators the trace of B(p) is not equal to zero. Therefore B(p) is not a quasi-nilpotent operator and has at least one nonzero eigenvalue. Further the Gateaux derivative of (8) yields the condition

$$\frac{1}{|(B\,e,\overline{e})|}[(B\,e,\overline{h})(\overline{B}\,\overline{e},e) + (B\,e,\overline{e})(\overline{B}\,\overline{e},h)] - \quad (9)$$

$$- |\mu|\frac{1}{|(e,\overline{e})|}[(e,\overline{h})(\overline{e},e) + (e,\overline{e})(\overline{e},h)] = 0$$

What is fulfilled by

$$B\,e = \mu e \,. \quad (10)$$

Analysis of the second Gateaux derivative shows that (10) is a saddle point of (8).

Theorem 2. Eigenvectors of a complex symmetric operator B(p) and eigenvectors of its adjoint $B^*(p) = B(\overline{p})$ form biorthogonal systems which can be biorthonormalized.

Proof: For $\mu_k \neq \mu_l$ it holds $\mu_k(e_k,\overline{e}_l) = (Ae_k,\overline{e}_l)$ and $\mu_l(e_k,\overline{e}_l) = \mu_l(e_l,\overline{e}_k) = (Ae_l,\overline{e}_k) = (Ae_k,\overline{e}_l)$. Then

$$(\mu_k - \mu_l)(e_k,\overline{e}_l) = 0 \,. \quad (11)$$

Hence for $\mu_k \neq \mu_l$ $(e_k,\overline{e}_l) = 0$ and eigenvalues $e_k,\overline{e}_l$ form biorthogonal systems.

Points p, where it holds $(e(p), \overline{e(p)}) = 0$, will be called exceptional points of the operator B(p). We can prove:

Theorem 3. Symmetric complex compact operators $B(p) = A^{-1}(p)$ are semisimple with exception of exceptional points.

Proof: We shall make the proof for an eigenvelue of the multiplicity two. In this case the Jordan canonical form will be

$$Be_1 = \mu e_1 + e_2 \,,$$
$$Be_2 = \mu e_2 \,. \quad (12)$$

After biorthogonalization $x_2 = e_2$, $x_1 = k_1 e_1 + k_2 e_2$ we arrive at

$$Bx_1 = \mu_1 x_1 + a x_2 \ ,$$
$$Bx_2 = \mu x_2 \ . \qquad (1\,3)$$

Multiplying the first equation (1 3) by x_2 and the second one by x_1 we arrive at

$$(Ax_1, \overline{x}_2) = a\ (x_2, \overline{x}_2)\ ,$$
$$(Ax_2, \overline{x}_1) = 0\ , \qquad (1\,4)$$

what can be fulfilled only when $(x_2, \overline{x}_2) = (e_2, \overline{e}_2) = 0$. In a similar way we can prove our assertion also for eigenvalues of higher multiplicity.

This theorem holds also for complex symmetric matrices. When the eigenvector e_n belonging to the eigenvalue λ of the multiplicity n fulfil the condition $(e_n, \overline{e}_n) \neq 0$ the corresponding canonical form is diagonal and the matrix is simple. J. H. Wilkinson has shown an example of a complex symmetric matrix, which cannot be diagonalized. It is [4]

$$A = \begin{bmatrix} 2i & 1 \\ 1 & 0 \end{bmatrix} . \qquad (1\,5)$$

This matrix has a two-fold eigenvalue $\lambda = i$ and the eigenvector $e_2 = [1, \ -i]$, thus $(e_2, \overline{e}_2) = 0$ and according to the above results the matrix cannot be diagonalized.

Then similarly as in the case of symmetric compact operators we can construct a complete system of eigenvectors. It holds:

Theorem 4. Operators $B(p) = A^{-1}(p)$ and $A(p)$ have with exception of exceptional points a countable complete set of eigenvectors $e_1, e_2, e_3, \ldots$ biorthogonal or biorthonormal to the complex conjugate set of eigenvectors of the adjoint operators $\overline{B(p)}$ and $\overline{A(p)}$ corresponding to eigenvalues $\mu_1, \mu_2, \mu_3, \ldots$ (resp. $\lambda_k = 1/\mu_k$) with $|\mu_1| \geq |\mu_2| \geq |\mu_3| \geq \ldots$ such that for $f = Bh$ we have

$$f = \sum_k (f, \overline{e}_k) e_k = \sum_k (f, e_k) \overline{e}_k \ , \qquad (1\,6)$$

what corresponds to covariant and contravariant expansions of vectors, respectively. Then it holds

$$\|f\|^2 = \sum_k (f, \bar{e}_k)(\bar{f}_k, \bar{e}_k) \ . \tag{17}$$

The proof is similar to that for symmetric compact operators.

At exceptional points it is necessary to replace a basis with the eigenvector by an other biorthonormal basis of the subspace correspon- ding to the multiple eigenvalue.

Finally we can prove the basic theorem on analycity of eigen-values and eigenvectors of A(p).

Theorem 5. Suppose that $A(p) = A_0 + pA_1 + p^2A_2 + \ldots + p^nA_n$, where A_k are positive definite elliptic operators. Suppose that λ is an eigenvalue of multiplicity m of the operator A(p) at p_0, where p_0 assumes real nonnegative vaues. Then there exist ordinary power series $\lambda_1(p - p_0), \ldots, \lambda_m(p - p_0)$ and power series in Hilbert space $e_1(p - p_0)$, $\ldots, e_m(p - p_0)$ all convergent in a neighbourhood of p_0, which satisfy the following conditions:

1. $e_i(p - p_0)$ is an eigenvector of A(p) belonging to the ei-genvalue $\lambda_i(p - p_0)$, i.e.

$$A(p)e_i(p - p_0) = \lambda_i(p - p_0)e_i(p - p_0), \quad i = 1, \ldots, m \ , \tag{18}$$

$\lambda_i(0) = \lambda$, $i = 1, \ldots, m$ and the eigenvectors $e_i(p - p_0)$ form with eigenvectors $\bar{e}_j(p - p_0)$ of $\overline{A(p)}$ biorthonormal sets, i.e.

$$(e_i(p - p_0), \bar{e}_j(p - p_0)) = \delta_{ij}, \quad i,j = 1, \ldots, m, \tag{19}$$

2. There exists such a neighbourhood of λ and a positive number ρ such that the spectrum of $C(p - p_0) = A(p)$ for p with $|p - p_0| < \rho$ consists exactly of the points $\lambda_1(p - p_0), \ldots, \lambda_m(p - p_0)$.

Proof can be done by a generalization of results of E.Rellich [5]. F. Rellich proved such theorem for an operator $A(\varepsilon)$ for small real values of ε. He restricted himself to orthonormal systems of eigen-vectors. Then scalar product of analytic functions are analytic only at real values of the parameter ε and the Weierstrass preparation theorem can be applied only to real values of ε. Introducing of biorthonormal sets of eigenfunctions and scalar products $(f, \bar{f})$ enables to apply the Weierstrass preparations theorem also to complex values of p.

Moreover after introducing biorthonormal sets of eigenvectors it is possible to generalize the proof also for complex values of p_0. Similarly it is possible to generalize other theorems of F.Rellich.

References

[1] Brilla,J., *New functional spaces and linear nonstationary problems of mathematical physics*, Proceedings of Equadiff 5, Bratislava 1981, Teubner, Leipzig 1982, 64-71.

[2] Brilla,J., *Novye funkcional'nye prostranstva i linejnye nestacionarnye problemy matematičeskoj fiziki*, Proceedings of the 7th Soviet-Czechoslovak Conference, Yerevan State University 1982, 49-58.

[3] Gokhberg,I.C., Krein, M.G., *Vvedenie v teoriju linejnych nesamosoprjazhennych operatorov*, Nauka, Moskva, 1965.

[4] Wilkinson,J.H., *The algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965.

[5] Rellich,F., *Perturbation Theory of Eigenvalue Problems*, Gordon and Breach, New York - London - Paris, 1969.