Spectral Analysis of OFDM signals and its Improvement by Polynomial Cancellation Coding

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***Abstract*:** In orthogonal frequency division multiplexing (OFDM) systems, the subcarriers are generated by an inverse discrete Fourier transform (DFT). OFDM has a relatively slow spectral rolloff. In the literature, the spectra of these subcarriers are often represented by a sequence of *sinc* functions with the same polarity. This does not explain the intercarrier interference (ICI) cancellation properties of polynomial cancellation coding (PCC) in OFDM. In this paper, the Fourier transform of the subcarriers are derived and the results show that spectra of these subcarriers are more accurately represented by a sequence of *sinc* functions with alternating polarity. The derived representation is consistent with the ICI cancellation properties of PCC. This paper also gives the derivation of spectral rolloff as a function of frequency and the number of subcarriers in OFDM and PCC-OFDM. It is shown that the spectral rolloff of OFDM signals is improved by PCC and the resultant OOB power is much lower than in normal OFDM.

1. Introduction

Orthogonal frequency division multiplexing (OFDM) is a modulation technique used in many new digital data transmission systems such as digital video broadcasting (DVB), digital audio broadcasting (DAB) and wireless local area networks (WLANs). However OFDM suffers from high sensitivity to frequency errors and high peak-to-average power ratio (PAPR). Moreover OFDM has a relatively large out-of-band (OOB) spectrum [1]. The OOB power should be minimized to avoid interference between adjacent broadcast channels.

In OFDM, the subcarriers are generated using an *N*-point inverse discrete Fourier transform (DFT). The spectrum of each subcarrier decreases according to a *sinc* function [2]. The *sinc* functions have sidelobes that are relatively large and do not decay quickly with frequency. As a result, the spectral rolloff of OFDM signals is slow. The OOB power is not low enough for many OFDM application systems. One of the simple and effective ways to reduce the OOB power is to use windowing. This eliminates the sharp transitions at symbol boundaries in the time domain signal and results in more rapid spectral rolloff. However windowing reduces the delay spread tolerance [2]. It is also possible to use filtering techniques, however filtering techniques are more complex to implement than windowing ones and filtering may distort the wanted signal [2].

Polynomial cancellation coding (PCC) is a technique that makes OFDM much less sensitive to frequency errors and more tolerant to multipath with large delay spreads [3]. An OFDM system with PCC is called a polynomial cancellation coded OFDM (PCC-OFDM). In this paper it is shown that PCC also improves the overall power spectrum of the OFDM signal. Accurate frequency domain representation of OFDM subcarriers are presented and the relationship of the spectral rolloff and frequency is given for the power spectrum in the case of both OFDM and PCC-OFDM. In addition, the effect of PCC on the time domain signal is discussed.

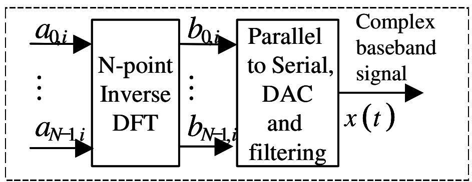


Figure 1. An OFDM transmitter

2. OFDM and its power spectrum

In a normal OFDM signal, the complex values modulating the subcarriers in each symbol period are statistically independent of each other. They are also independent of the values modulating any subcarrier in any previous or subsequent symbol period. As a result the power spectrum of the overall signal can be found by summing the power spectra of all individual subcarriers for any symbol period.

Figure 1 shows a typical OFDM transmitter. The complex baseband signal $x(t)$ is given by

$$x(t)=\sum_{i=-\infty}^{\infty}\sum_{k=-N/2}^{N/2-1} a_{k,i}\exp j\left(2\pi f_k\left(t+\frac{T}{2}-iT\right)\right) \quad (1)$$

where T is the symbol period, $f_k = k/T$ is the frequency of the *k*-th subcarrier and $a_{k,i}$ is the complex data symbol modulating the *k*-th subcarrier in the *i*–th symbol period. $x(t)$ is a sample function of a random process. The power spectral density of a random process is by definition [4]

$$S_x(f)=E\left\{\lim_{T_{AV}\to\infty}\frac{\left|X_{T_{AV}}(f)\right|^2}{T_{AV}}\right\} = \lim_{T_{AV}\to\infty}\frac{E\left\{X_{T_{AV}}(f)X^*_{T_{AV}}(f)\right\}}{T_{AV}}, \quad (2)$$

where T_{AV} is the period over which the power spectral density is being calculated and the expectation is over all the possible sample functions. To apply this to OFDM, take the case where T_{AV} is a large number of complete symbol periods, say from $t=-mT-T/2$ to $t=mT+T/2$ with large *m*. The expectation is then over all the possible values of $a_{k,i}$. For zero mean independent random variables, $a_{k,i}$'s, with variance $E\left\{|a|^2\right\}$, the power spectrum of an OFDM signal can be shown to be

$$S_x(f)=E\left\{|a|^2\right\}\sum_{k=0}^{N-1}\frac{\left|X_k(f)^2\right|}{T} \quad (3)$$

where $\left|X_k(f)^2\right|/T$ is the power spectrum of the *k*-th subcarrier.

Now the power spectra of the individual subcarriers can be found by first calculating the continuous Fourier transform of the *k*-th subcarrier in the symbol period from $t=-T/2$ to $t=T/2$. The time domain representation of this subcarrier is given by

$$x_k(t)=\frac{1}{\sqrt{N}}\exp\left(j2\pi f_k\left(t+\frac{T}{2}\right)\right), \quad -\frac{T}{2}\le t\le\frac{T}{2}. \quad (4)$$

The extra factor of $1/\sqrt{N}$ in (4) is to normalize the signal so that the total power in the transmitted signal independent of the number of subcarriers in the system. The continuous Fourier transform of $x_k(t)$ is given by

$$X_k(f)=\frac{1}{\sqrt{N}}T\left(-1^k\right)\left[\frac{\sin(\pi(fT-k))}{\pi(fT-k)}\right] \quad (5)$$

This has the familiar $\sin(x)/x$ form given in the literature. However the $(-1)^k$ factor shows that rather than the spectra of subcarriers being a sequence of $\sin(x)/x$ functions of the same polarity as is usually given in the literature, they have the alternating positive and negative peaks. Figure 2 shows the Fourier transforms of four subcarriers. Note that the Fourier transforms for adjacent subcarriers have alternately positive and negative peak values. These alternating signs are important in understanding how PCC results in an improved overall power spectrum.

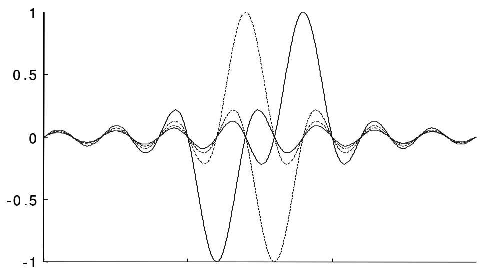


Figure 2. Fourier Transforms of four adjacent subcarriers in an OFDM signal.

In a recent paper [5], the Fourier transform of an OFDM subcarrier has been derived. However in that work a symbol period of $[0,T]$ is considered. This results in a complex Fourier transform which does not clearly reveal the alternating peak structure of subcarriers and the cancellation properties of PCC. It also gives an apparent doubling in the zero-crossing properties of the transform. The paper also included a simple analysis of PCC-OFDM for case of two weighted subcarriers, but did not point out the dependence of the spectrum on N or f.

Typically in an OFDM system the bandwidth and the average power of the transmitted signal is fixed. The number of subcarriers can be chosen by the designer. The bandwidth of the OFDM signal depends on the sampling period T_S and $T=NT_S$. Substituting $T=NT_S$ in (5) gives

$$X_k(f) = \sqrt{N}T_S(-1)^k\left[\frac{\sin(\pi(fT_S N - k))}{\pi(fT_S N - k)}\right] \quad (6)$$

And finally, the power spectrum for the subcarrier is given by

$$S_k(f) = T_S\left[\frac{\sin(\pi(fT_S N - k))}{\pi(fT_S N - k)}\right]^2 \quad (7)$$

The spectrum of a subcarrier falls of as $1/(f^2N^2)$. The overall spectrum being the sum of spectrum of N subcarriers, the OOB spectrum of the complete OFDM signal falls off as $1/(f^2N)$. Even for large N, the OOB power is high and a relatively large guard band must be left between different OFDM signals. However a large N results in high PAPR and increased sensitivity to frequency errors [2].

Figure 3 shows the overall power spectra for an OFDM signal for the cases of OFDM systems with 32 and 256 subcarriers. Note the relatively gradual decrease of the OOB power spectrum with frequency.

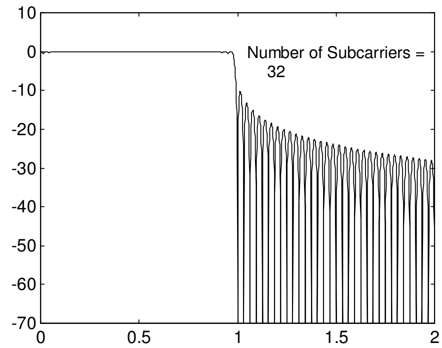


Figure 3(a). Power spectrum of an OFDM signal with 32 subcarriers

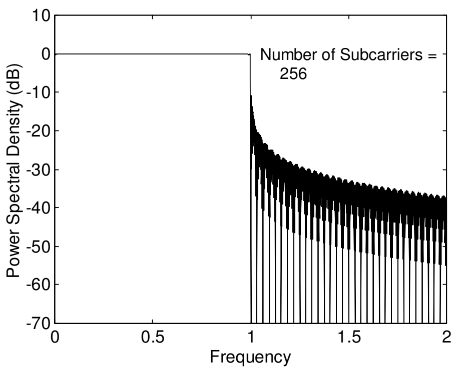


Figure 3(b). Power spectrum for OFDM signal with 256 subcarriers

3. Power Spectrum of PCC-OFDM

In PCC-OFDM, independent data to be transmitted are mapped onto the weighted groups of subcarriers rather than individual subcarriers [3]. For example applying PCC to pairs of adjacent subcarriers gives $a_0 = -a_1$, $a_2 = -a_3$, and so on. Adjacent pairs must have relative weightings $+1$ and -1. In the general case for groups of m subcarriers the relative weightings are given by the coefficients of the polynomial $(1-x)^{m-1}$.

The overall power spectrum of the transmitted signal in PCC-OFDM is obtained by first finding the power spectra of all the groups of weighted subcarriers and then by adding the power spectra of these groups. Considering the case of pairs of weighted subcarriers in a group, the Fourier transform the pair of adjacent subcarriers, k-th and $(k+1)$ th, is given by

$$X_k(f) - X_{k+1}(f) = \sqrt{N}T_S\cos(\pi k)\left[\frac{\sin(\pi(fT_S N - k))}{\pi(fT_S N - k)}\right] - \sqrt{N}T_S\cos(\pi(k+1))\left[\frac{\sin(\pi(fT_S N - k - 1))}{\pi(fT_S N - k - 1)}\right] \quad (8)$$

Taking the case of k even and after some manipulation this can be simplified to

$$X_k(f) - X_{k+1}(f) = \left[\frac{-\sqrt{N}T_S\sin(\pi(fT_S N - k))}{\pi(fT_S N - k)(fT_S N - k - 1)}\right] \quad (9)$$

And the power spectrum of the weighted subcarrier pair is given by

$$S_{k,k+1}(f) = T_s\left[\frac{\sin(\pi(fT_s N - k))}{\pi(fT_s N - k)(fT_s N - k - 1)}\right]^2 \quad (10)$$

The Fourier transform of each weighted pair of subcarriers falls off with an envelope that depends on $1/(f^2N^2)$ which is faster than the case of OFDM. The overall OOB spectrum of the complete PCC OFDM signal falls off as $1/(f^4N^3)$.

By similar calculation it can be shown that if weighted groups of three subcarriers are used, the out-of-band spectrum falls off as $1/(f^6N^5)$. For the general case of groups of m weighted subcarriers the out-of-band spectrum falls off as $1/\left(f^{2m}N^{2m-1}\right)$.

Figure 4 shows the Fourier Transforms of groups of two and three subcarriers. The rapid roll-off achieved by PCC can also be understood with reference to Figure 2 where it can be seen that adjacent subcarriers have sidelobes which are very similar in value, and thus if adjacent subcarriers are weighted with opposite values the sidelobes will to a large extent cancel each other out. As a result, the frequency sidelobes of the weighted subcarriers flatten out quickly.

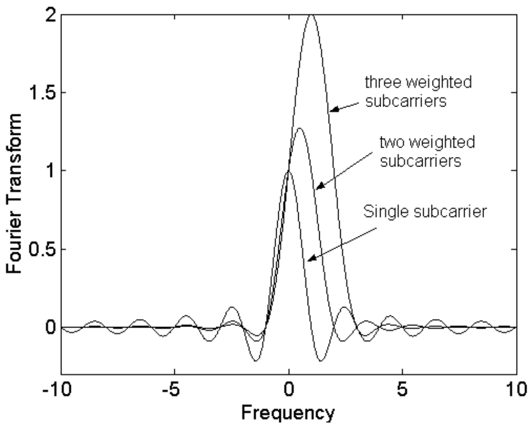


Figure 4. Fourier transforms of weighted groups of subcarriers

Figures 5 shows the overall power spectra for PCC OFDM for a number of values of m and N. We can see that the first lobe of the power spectrum in PCC-OFDM is less than in OFDM. The first lobe of the spectrum in PCC-OFDM is almost 10 dB less when $m = 2$ and 15 dB less when $m = 3$. The spectral rolloff of OOB spectrum is also faster in PCC-OFDM than in OFDM.

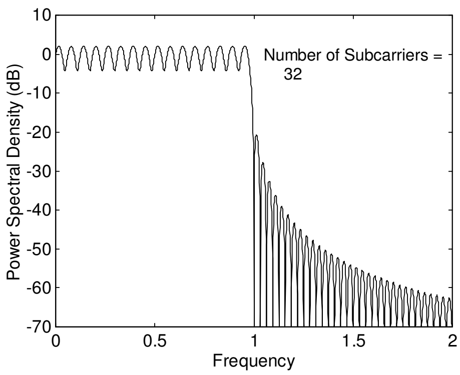
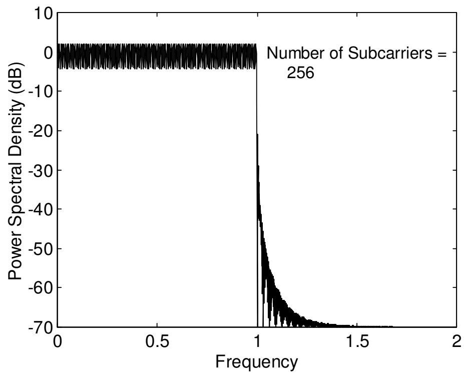
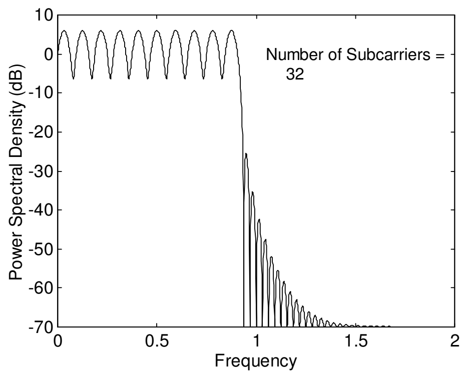


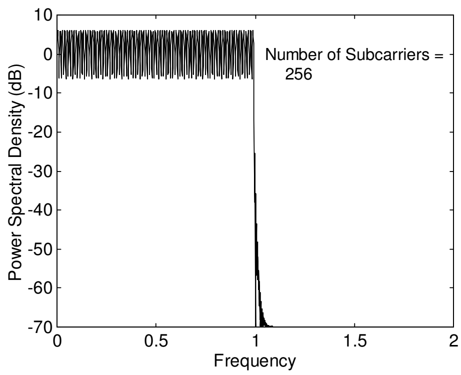
Figure 5(a). Modulation in pairs, 32 subcarriers



(b) Modulation in pairs, 256 subcarriers



(c) Modulation in groups of three, 32 subcarriers



(d) Modulation in groups of three, 256 subcarriers

One of the major disadvantages of PCC is that it causes the spectral efficiency to be reduced by half. However most of it will be compensated by elimination of cyclic prefix [3]. In addition the coding redundancy required for PCC-OFDM will be a less than in OFDM to achieve the same error performance [6].

4. Windowing effect of PCC in the time domain

In the previous section it is shown that PCC improves the rolloff of OFDM power spectrum and reduces the OOB power. This can also be explained by the considering the effect of PCC in the time domain. For weighted groups of two subcarriers, we have $a_{2k+1} = -a_{2k}$. The l-th output component of the transmitter inverse DFT due to the pair of inputs a_{2k} and a_{2k+1} is given by

$$b_l = a_{2k} \exp\left(\frac{j2\pi kl}{T}\right)\left(1-\exp\left(\frac{j2\pi l}{T}\right)\right) \quad (11)$$

This shows that PCC with data mapped onto weighted subcarrier pairs is equivalent to the windowing of the outputs of the inverse DFT with nonzero inputs on the even numbered subcarriers. The windowing function is complex and is given by $\left(1-\exp\left(j2\pi l/T\right)\right)$. The plot of this window is shown in Figure 6. The plot shows that window is very smooth at the symbol boundaries. The real and imaginary part of this signals are also smooth. This will reduce the OOB power in PCC-OFDM that might otherwise have been caused by the sharp transitions at symbol boundaries. Thus an equivalent system can be built using time domain windowing as shown in Figure 7. In the case of PCC with three weighted subcarriers in a group, it can be easily shown that the windowing function is given by $\left(1-\cos\left(2\pi l/T\right)\right)$. This is real and is a Hanning window.

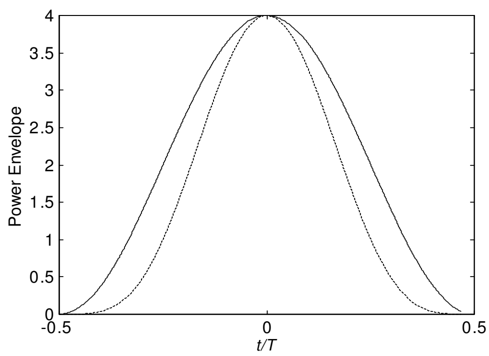


Figure 6. Variation in the power envelope across a symbol period for PCC OFDM. Solid line $m = 2$, dotted line $m = 3$.

5. Conclusion

The Fourier transform of an OFDM subcarrier is derived and is shown to be of the form $(-1)^k \sin(x)/x$, where k represents the subcarrier index. Therefore polarity of a particular subcarrier depends on k. The alternating sign on the adjacent subcarriers is consistent with the ICI cancellation properties of PCC in OFDM. The spectral analysis of OFDM is extended to PCC-OFDM. It is shown that PCC improves the spectral rolloff of OFDM signals. The improvement in the spectral rolloff of PCC-OFDM is discussed in the time domain. The effect of PCC in the time domain is windowing. This will result in most of the signal energy being concentrated at the middle of symbol period and smoothing of signals at the symbol boundaries. This will reduce the OOB power in PCC-OFDM.

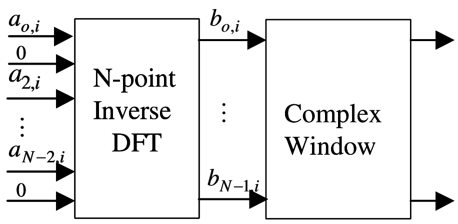


Figure 7. Time domain equivalent to weighting pairs of subcarriers

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6. References:

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