SPECTRAL ANALYSIS OF SEMIGROUPS AND GROWTH-FRAGMENTATION EQUATIONS

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Abstract. The aim of this paper is twofold:

(1) On the one hand, the paper revisits the spectral analysis of semigroups in a general Banach space setting. It presents some new and more general versions, and provides comprehensible proofs, of classical results such as the spectral mapping theorem, some (quantified) Weyl's Theorems and the Krein-Rutman Theorem. Motivated by evolution PDE applications, the results apply to a wide and natural class of generators which split as a dissipative part plus a more regular part, without assuming any symmetric structure on the operators nor Hilbert structure on the space, and give some growth estimates and spectral gap estimates for the associated semigroup. The approach relies on some factorization and summation arguments reminiscent of the Dyson-Phillips series in the spirit of those used in [87, 82, 48, 81].

(2) On the other hand, we present the semigroup spectral analysis for three important classes of "growth-fragmentation" equations, namely the cell division equation, the self-similar fragmentation equation and the McKendrick-Von Foerster age structured population equation. By showing that these models lie in the class of equations for which our general semigroup analysis theory applies, we prove the exponential rate of convergence of the solutions to the associated remarkable profile for a very large and natural class of fragmentation rates. Our results generalize similar estimates obtained in [94, 68] for the cell division model with (almost) constant total fragmentation rate and in [18, 17] for the self-similar fragmentation equation and the cell division equation restricted to smooth and positive fragmentation rate and total fragmentation rate which does not increase more rapidly than quadratically. It also improves the convergence results without rate obtained in [79, 34] which have been established under similar assumptions to those made in the present work.

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1. Introduction

This paper deals with the study of decay properties for C_0-semigroup of bounded and linear operators and their link with spectral properties of their generator in a Banach framework as well as some applications to the long time asymptotic of growth-fragmentation equations.

1.1. **Spectral analysis of semigroups.** The study of spectral property of (unbounded) operators and of C_0-semigroups of operators has a long history which goes back (at least) to the formalization of functional analysis by D. Hilbert [57, 58] at the beginning of the 20th century for the first issue and surely before the modern theoretical formalization of C_0-semigroups of operators in general Banach spaces impulsed by E. Hille and K. Yosida [59, 112, 60] in the 1940's for the second issue. For a given operator Λ on a Banach space X which generates a C_0-semigroup $S_\Lambda(t) = e^{t\Lambda}$ of bounded operators, the two following issues are of major importance:

- describe its spectrum, the set of its eigenvalues and the associated eigenspaces;
- prove the spectral mapping theorem

$$\Sigma(e^{t\Lambda})\backslash\{0\} = e^{t\Sigma(\Lambda)}, \tag{1.1}$$

and deduce the asymptotical behaviour of trajectories associated to the semigroup.

Although it is well-known that the first issue can be a complicated task and the second issue is false in general (see [61] or [33, Section IV.3.a] for some counterexamples), there exists some particular classes of operators (among which the class of compact and self-adjoint operators in a Hilbert space) for which these problems can be completely solved. In the present paper, motivated by evolution Partial

Differential Equations applications and inspired by the recent paper [48] (see also [87, 82, 81]), we identify a class of operators which split as

$$\Lambda = \mathcal{A} + \mathcal{B}, \tag{1.2}$$

where $\mathcal{A}$ is "much more regular than $\mathcal{B}$" and $\mathcal{B}$ has some dissipative property (and then a good localization of its spectrum) for which a positive answer can be given. The dissipative property assumption we make can be formulated in terms of the time indexed family of iterated time convolution operators $(\mathcal{A}S_{\mathcal{B}})^{(*k)}(t)$ in the following way

(H1) for some $a^* \in \mathbb{R}$ and for any $a > a^*$, $\ell \geq 0$, there exists a constant $C_{a,\ell} \in (0,\infty)$ such that the following growth estimate holds

$$\forall\, t \geq 0, \qquad \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t)\|_{\mathscr{B}(X)} \leq C_{a,\ell}\, e^{at}. \tag{1.3}$$

On the other hand, we make the key assumption that some iterate enough time convolution enjoys the growth and regularizing estimate:

(H2-3) there exist an integer $n \geq 1$ such that for any $a > a^*$, there holds

$$\forall\, t \geq 0, \qquad \|(\mathcal{A}S_{\mathcal{B}})^{(*n)}(t)\|_{\mathscr{B}(X,Y)} \leq C_{a,n,Y}\, e^{at}, \tag{1.4}$$

or

$$\forall\, t \geq 0, \qquad \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*n)}(t)\|_{\mathscr{B}(X,Y)} \leq C_{a,n,Y}\, e^{at}, \tag{1.5}$$

for some suitable subspace $Y \subset X$ and a constant $C_{a,n,Y} \in (0,\infty)$.

In assumption **(H2-3)** we will typically assume that $Y \subset D(\Lambda^\zeta)$, $\zeta > 0$, and/or $Y \subset X$ with compact embedding.

Roughly speaking, for such a class of operators, we will obtain the following set of results:

• *Spectral mapping theorem.* We prove a partial, but principal, spectral mapping theorem which asserts that

$$\Sigma(e^{t\Lambda}) \cap \Delta_{e^{at}} = e^{t\Sigma(\Lambda)\cap\Delta_a}, \qquad \forall\, t \geq 0,\ \forall\, a > a^*, \tag{1.6}$$

where we define the half-plane $\Delta_\alpha := \{\xi \in \mathbb{C};\ \Re e\xi > \alpha\}$ for any $\alpha \in \mathbb{R}$. Although (1.6) is less accurate than (1.1) it is strong enough to describe the semigroup evolution at first order in many situations. In particular, it implies that the spectral bound $s(\Lambda)$ and the growth bound $\omega(\Lambda)$ coincide if they are at the right hand side of a^*, or in other words

$$\max(s(\Lambda), a^*) = \max(\omega(\Lambda), a^*), \tag{1.7}$$

and it gives even more accurate asymptotic informations on the semigroup whenever $\Sigma(\Lambda) \cap \Delta_{a^*} \neq \emptyset$.

• *Weyl's Theorem.* We prove some (quantified) version of the Weyl's Theorem which asserts that the part of the spectrum $\Sigma(\Lambda) \cap \Delta_{a^*}$ consists only of *discrete eigenvalues* and we get some informations on the localization and number of eigenvalues as well as some estimates on the total dimension of the associated sum of eigenspaces.

• *Krein-Rutman Theorem.* We prove some (possibly quantified) version of the Krein-Rutman Theorem under some additional (strict) positivity hypothesizes on the generator Λ and the semigroup S_Λ.

Let us describe our approach in order to get the above mentioned *"spectral mapping theorem"*, this one being the key result in order to get our versions of Weyl's Theorem and Krein-Rutman Theorem. Following [87, 82, 48] (and many authors before!), the spectral analysis of the operator Λ with splitting structure (1.2) is performed by writing the resolvent factorization identity (with our definition of the resolvent in (2.1))

$$\mathcal{R}_\Lambda(z) = \mathcal{R}_\mathcal{B}(z) - \mathcal{R}_\mathcal{B}(z)\mathcal{A}\mathcal{R}_\Lambda(z) \tag{1.8}$$

as well as

$$\mathcal{R}_\Lambda(z) = \mathcal{R}_\mathcal{B}(z) - \mathcal{R}_\Lambda(z)\mathcal{A}\mathcal{R}_\mathcal{B}(z) \tag{1.9}$$

or an iterative version of (1.9), and by exploiting the information that one can deduce from (1.3), (1.4) and (1.5) at the level of the resolvent operators.

At the level of the semigroup, (1.8) yields the Duhamel formula

$$S_\Lambda = S_\mathcal{B} + S_\mathcal{B} * (\mathcal{A}S_\Lambda),$$

in its classic form, and (1.9) yields the Duhamel formula

$$S_\Lambda = S_\mathcal{B} + S_\Lambda * (\mathcal{A}S_\mathcal{B})$$

in a maybe less standard form (but also reminiscent of perturbation theory).

On the other hand, iterating one of the above identities, it is well-known since the seminal works by Dyson and Phillips [31, 96], that S_Λ can be expended as the Dyson-Phillips series

$$S_\Lambda = \sum_{\ell=0}^{\infty} (-1)^\ell \, S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*\ell)},$$

as soon as the left hand side series converges, and that matter has not an easy answer in general. The summability issue of the Dyson-Phillips series can be circumvented by considering the stopped Dyson-Phillips series

$$S_\Lambda = \sum_{\ell=0}^{n} (-1)^\ell \, S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*\ell)} + (-1)^{n+1} \, S_\Lambda * (\mathcal{A}S_\mathcal{B})^{(*n)}. \tag{1.10}$$

It is an usual trick in order to establish eventual norm continuity (see [15] and the references therein) and it has been also recently used in [48, 81] in order to enlarge or to shrink the functional space where the semigroup S_Λ satisfies some spectral gap estimate.

In the present work, and in the case $\Delta_{a^*} \cap \Sigma(\Lambda) = \{\lambda\}$, $\lambda \in \mathbb{C}$ semisimple eigenvalue and $a^* < a < \Re e\lambda$ in order to make the discussion simpler, our spectral mapping theorem simply follows by using the classical representation of a semigroup by means of the inverse Laplace transform of the resolvent as yet established by Hille in [59], and then writing

$$\begin{aligned} S_\Lambda(t) \quad &= \quad \Pi_{\Lambda,\lambda}\, e^{\lambda t} + \sum_{\ell=0}^{N} (-1)^\ell S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*\ell)}(t) \\ &\quad + \frac{i}{2\pi}\, e^{at} \int_{a-i\infty}^{a+i\infty} e^{zt}\, (-1)^{N+1}\, \mathcal{R}_\Lambda(z)\, (\mathcal{A}\mathcal{R}_\mathcal{B}(z))^N \, dz \end{aligned} \tag{1.11}$$

for N large enough, in such a way that each term is appropriately bounded, and where $\Pi_{\Lambda,\lambda}$ stands for the projector on the (finite dimensional) eigenspace associated to λ. That formula immediately implies the spectral mapping theorem (1.6),

and more importantly for us, it generalizes the Liapunov result [70] about the asymptotic behaviour of trajectories to that class of equations. It is worth emphasizing that our result characterizes the class of operators with "separable spectrum" for which the partial spectral mapping theorem holds (in the sense that we exhibit a condition which is not only sufficient but also necessary!) and then, in some sense, we prove for *general semigroup* the spectral mapping theorem known for *analytic semigroup* or more generally *eventually continuous semigroup* [95, 61] and for *general Banach space* the partial spectral mapping theorem in a *Hilbert space* framework that one can deduce from the Gearhart-Prüss Theorem [41, 98].

With such a representation formula at hand, the precise analysis of the semigroup S_Λ reduces to the analysis of the spectrum of Λ at the right hand side of a, for any $a > a^*$. In other words, the next fundamental issue consists in describing the part of the spectrum $\Sigma(\Lambda) \cap \Delta_a$ in order to take advantage of the information given by (1.6). The simplest situation is when $\Sigma(\Lambda) \cap \Delta_a$ only contains eigenvalues with finite (algebraical) multiplicity which is the situation one gets when one can apply Weyl's Theorem [111] (see also [63, Theorem IV.5.35]). In our second main result in an abstract setting, we recover Voigt's version [107] of Weyl's Theorem (for which we give a comprehensive and elementary proof) and we deduce a characterization of semigroup in a general Banach space for which the partial spectral mapping theorem holds with finite and discrete eigenspectrum in Δ_a. We must emphasize that our proof is very simple (it exclusively uses the Fredholm alternative [38] in it most basic form) and in particular does not use the essential spectrum set. Moreover we are able to formulate a quantified version of the Weyl's theorem in the sense that we provide a bound on the total dimension of the eigenspaces associated to the discrete eigenvalues which lie in $\Sigma(\Lambda) \cap \Delta_a$.

In order to describe in a more accurate way the part of the spectrum $\Sigma(\Lambda)$, one of the most popular technics is to use a self-adjointness argument for the operator Λ as a infinite dimensional generalization of the symmetric structure of matrix. That implies $\Sigma(\Lambda) \subset \mathbb{R}$, and together with Weyl's Theorem that leads to a completely satisfactory description of the spectrum operator and the dynamics of the associated semigroup. One of the most famous application of that strategy is due to Carleman [22] who carried on with the study of the linearized space homogeneous Boltzmann equation initiated by Hilbert and who obtained the spectral gap for the associated operator by combining Weyl's Theorem together with the symmetry of the operator [57] and the regularity of the gain term [58] (see also the works by Grad [43, 44] on the same issue, and the work by Ukai [103] on the more complicated space non-homogeneous setting). It is worth emphasizing that this kind of hilbertian arguments have been recently extended to a class of operators, named as *"hypocoercive operators"*, which split as a self-adjoint partially coercive operator plus an anti-adjoint operator. For such an operator one can exhibit an equivalent Hilbert norm which is also a Liapunov function for the associated evolution dynamics and then provides a spectral gap between the first eigenvalue and the remainder of the spectrum. We refer the interested reader to [104] for a pedagogical introduction as well as to [56, 53, 88, 54, 55, 105, 29, 28] for some of the original articles.

In the seminal work [87], C. Mouhot started an abstract theory of "*enlargement* of the functional space of spectral analysis of operators" which aim to carry on the spectral knowledge on an operator Λ and its associated semigroup S_Λ in some

space E (typically a "small Hilbert space" in which some self-adjointeness structure can be exploited) to another larger general Banach space $\mathcal{E} \supset E$. It is worth emphasizing that the "*enlargement* of functional spaces" trick for spectral analysis is reminiscent of several anterior works on Boltzmann equations and on Fokker-Planck type equations where however the arguments are intermingled with some nonlinear stability arguments [4, 5, 110] and/or reduced to some particular evolution PDE in some situation where explicit eigenbasis can be exhibited [14, 40]. While [87] was focused on the linearized space homogeneous Boltzmann operator and the results applied to sectorial operators, the "*extension* theory" (we mean here *"enlargement"* or *"shrinkage"* of the functional space) has been next developed in a the series of papers [82, 48, 81] in order to deal with non-sectorial operators. A typical result of the theory is that the set $\Sigma(\Lambda) \cap \Delta_a$ does not change when the functional space on which Λ is considered changes. We do not consider the *extension* issue, which is however strongly connected to our approach, in the present work and we refer to the above mentioned articles for recent developments on that direction.

Let us also mention that one expects that the spectrum $\Sigma(\Lambda)$ of Λ is close to the spectrum $\Sigma(\mathcal{B})$ of $\mathcal{B}$ if $\mathcal{A}$ is "small". Such a *"small perturbation method"* is an efficient tool in order to get some informations on the spectrum of an operator Λ "in a perturbation regime". It has been developped in [96, 61, 106, 63], and more recently in [1, 15]. Again, we do not consider that *"small perturbation"* issue here, but we refer to [82, 83, 6, 102] where that kind of method is investigated in the same framework as the one of the abstract results of the present paper.

Last, we are concerned with a positive operator Λ defined on a Banach lattice X and the associated semigroup S_Λ as introduced by R.S. Phillips in [97]. For a finite dimensional Banach space and a strictly positive matrix the Perron-Frobenius Theorem [92, 39] state that the eigenvalue with largest real part is unique, real and simple. Or in other words, there exists $a \in \mathbb{R}$ such that $\Sigma(\Lambda) \cap \Delta_a = \{\lambda\}$ with $\lambda \in \mathbb{R}$ a simple eigenvalue. In a infinite dimensional Banach space the Krein-Rutman Theorem [67] establishes the same result for a class of Banach lattices and under convenient strict positivity and compactness assumptions on Λ. The Krein-Rutman Theorem is then extended to broader classes of Banach lattices and broader classes of operator in many subsequent works, see e.g. [46, 47, 3]. We present a very natural and general version of the abstract Krein-Rutman Theorem on a Banach lattice assuming that additionally to the precedent splitting structure the operator Λ satisfies a weak and a strong maximum principle. Our result improves the known versions of Krein-Rutman Theorem in particular because we weaken the required compactness assumption made on (the resolvent of) Λ. Moreover, our result is quite elementary and self-contained.

Let us stress again that our approach is very similar to the "*extension* of the functional space of spectral analysis of operators" and that our starting point is the work by Mouhot [87] where (1.9) is used in order to prove an enlargement of the operator spectral gap for the space homogeneous linearized Boltzmann equation. Because of the self-adjointeness structure of the space homogeneous linearized Boltzmann equation one can conclude thanks to classical arguments (namely the operator $\mathcal{B}$ is sectorial and the remaining term in (1.11) with $N = 1$ converges, see [91] or [33, Corollary IV.3.2 & Lemma V.1.9]). Our approach is in fact reminiscent of the huge number of works on the spectral analysis of operator which attend to take advantage of a splitting structure (1.2) and then consider Λ as a (compact,

small) perturbation of $\mathcal{B}$. One of the main difference with the classical approach introduced by Hilbert and Weyl is that when one usually makes the decomposition

$$\Lambda = \mathcal{A}_0 + \mathcal{B}_0$$

where $\mathcal{B}_0$ is dissipative and $\mathcal{A}_0$ is $\mathcal{B}_0$-compact, we remark that $\mathcal{A}_0 = \mathcal{A} + \mathcal{A}^c$ with $\mathcal{A}$ "smooth" and $\mathcal{A}^c$ small (it is the way that one uses to prove that $\mathcal{A}_0$ is compact") and we write

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{B} := \mathcal{A}^c + \mathcal{B}_0.$$

In such a way, we get quite better "smoothing properties" on $\mathcal{A}$ (with respect to $\mathcal{A}_0$) without loosing to much on the "dissipative property" of $\mathcal{B}$ (with respect to $\mathcal{B}_0$). Of course, the drawback of the method is that one has to find an appropriate splitting (1.2) for the operator as well as an appropriate space X for which one is able to prove the estimates (1.3), (1.4) and (1.5). However, the efficiency of the method is attested by the fact that it has been successfully used for several evolution PDEs such as the space homogenous and space nonhomogeneous elastic Boltzmann equations in [87, 48], the space homogenous and space nonhomogeneous inelastic Boltzmann equations in [82, 102], some Fokker-Planck type equations in [48, 6, 81, 32], the Landau equation in [23] and the growth-fragmentation equation in [18, 17].

1.2. **Growth-fragmentation equations.** The second aim of the paper is to establish the long time asymptotic for the solutions of some growth-fragmentation equations as a motivation, or an illustration, of the abstract theory developped in parallel. We then consider the growth-fragmentation equation

$$\partial_t f = \Lambda f := \mathcal{D} f + \mathcal{F} f \quad \text{in} \quad (0,\infty) \times (0,\infty). \tag{1.12}$$

Here $f = f(t,x) \geq 0$ stands for the *number density of cells* (or *particles, polymers, organisms, individuals*), where $t \geq 0$ is the *time* variable and $x \in (0,\infty)$ is the *size* (or *mass, age*) variable. In equation (1.12) the *growth* operator writes

$$(\mathcal{D} f)(x) := -\tau(x)\partial_x f(x) - \nu(x)\, f(x) \tag{1.13}$$

where the (continuous) function $\tau : [0,\infty) \to \mathbb{R}$ is the drift speed (or *growth rate*), and we will choose $\tau(x) = 1$ or $\tau(x) = x$ in the sequel, and the function $\nu : [0,\infty) \to [0,\infty)$ is a damping rate. The drift and damping term $\mathcal{D}$ models the growth (for particles and cells) or the aging (for individuals) and the death which can be represented by the scheme

$$\{x\} \overset{e^{-\nu(x)}}{\longrightarrow} \{x + \tau(x)\, dx\}.$$

In the other hand, the *fragmentation* operator $\mathcal{F}$ is defined by

$$(\mathcal{F} f)(x) := \int_x^\infty k(y,x) f(y) dy - K(x) f(x) \tag{1.14}$$

and the fragmentation kernel k is relied to the total rate of fragmentation K by

$$K(x) = \int_0^x k(x,y)\, \frac{y}{x}\, dy. \tag{1.15}$$

The fragmentation operator $\mathcal{F}$ models the division (breakage) of a single *mother particle* of size x into two or more pieces (*daughter particles, offsprings*) of size

$x_i \geq 0$, or in other words, models the event

$$\{x\} \ \xrightarrow{k}\ \{x_1\} + ... + \{x_i\} + ... , \tag{1.16}$$

in such a way that the mass is conserved

$$x = \sum_i x_i, \quad 0 \leq x_i \leq x.$$

It is worth emphasizing that the above mass conservation at the microscopic level is rendered by equation (1.15) at the statistical level.

In order to simplify the presentation we will only consider situations where the size repartition of offsprings is invariant by the size scaling of the mother particle, or more precisely that there exists a function (or abusing notations, a measure) $\wp : [0,1] \to \mathbb{R}_+$ such that

$$k(x,y) = K(x)\,\kappa(x,y), \quad \kappa(x,y) = \wp(y/x)/x, \tag{1.17}$$

as well as, in order that the compatibility relation (1.15) holds,

$$\int_0^1 z\,\wp(dz) = 1. \tag{1.18}$$

Here $\kappa(x,.)$ represent the probability density of the distribution of daughter particles resulting of the breakage of a mother particle of size $x \in (0,\infty)$ and the assumption (1.17) means that this probability density is invariant by scaling of the size. As a first example, when a mother particle of size x breaks into two pieces of exact size $\sigma\, x$ and $(1-\sigma)x$, $\sigma \in (0,1)$, the associated kernel is given by

$$\kappa(x,y) = \delta_{y=\sigma x}(dy) + \delta_{y=(1-\sigma)x}(dy) = \frac{1}{\sigma}\delta_{x=\frac{y}{\sigma}}(dx) + \frac{1}{1-\sigma}\delta_{x=\frac{y}{1-\sigma}}(dx), \tag{1.19}$$

or equivalently $\wp = \delta_\sigma + \delta_{1-\sigma}$. In the sequel, we will also consider the case when $\wp$ is a smooth function. In any cases, we define

$$z_0 := \inf \operatorname{supp} \wp \in [0,1). \tag{1.20}$$

The evolution equation (1.12) is complemented with an initial condition

$$f(0,.) = f_0 \quad \text{in} \quad (0,\infty), \tag{1.21}$$

and a boundary condition that we will discuss for each examples presented below.

Instead of trying to analyze the most general growth-fragmentation equation, we will focus our study on some particular but relevant classes of models, namely the cell division equation with constant growth rate, the self-similar fragmentation equation and the age structured population equation.

1.2.1. **Example 1. Equal mitosis equation.** We consider a population of cells which divide through a binary fragmentation mechanism with equal size offsprings, grow at constant rate and are not damped. The resulting evolution equation is the *equal mitosis equation* which is associated to the operator $\Lambda = \mathcal{D} + \mathcal{F}$, where $\mathcal{D}$ is defined by (1.13) with the choice $\tau = 1$ and $\nu = 0$, and where $\mathcal{F}$ is equal mitosis operator defined by (1.14) with the following choice of fragmentation kernel

$$k(x,y) \quad = \quad 2\,K(x)\,\delta_{x/2}(dy) = 4\,K(x)\,\delta_{2y}(dx). \tag{1.22}$$

Equivalently, k is given by (1.17) with $\wp(dz) := 2\,\delta_{z=1/2}$. The equal mitosis equation then writes

$$\frac{\partial}{\partial t} f(t,x) + \frac{\partial}{\partial x} f(t,x) + K(x) f(t,x) = 4K(2x)\, f(t,2x), \tag{1.23}$$

and it is complemented with the boundary condition

$$f(t,0) = 0. \tag{1.24}$$

As its name suggests, such an equation appears in the modeling of cells division when mitosis occurs (see [11, 101, 76, 10] and the references therein for linear models as well as [51, 10] for more recent nonlinear models for tumor growth) but also appears in telecommunications systems to describe some internet protocols [7].

We assume that the total fragmentation rate K is a nonnegative C^1 function defined on $[0,\infty)$ which satisfies the positivity assumption

$$\exists\, x_0 \geq 0, \quad K(x) = 0 \ \forall x < x_0, \quad K(x) > 0 \ \forall\, x > x_0, \tag{1.25}$$

as well as the growth assumption

$$K_0\, x^\gamma\, \mathbf{1}_{x \geq x_1} \leq K(x) \leq K_1 \ \max(1, x^\gamma), \tag{1.26}$$

for $\gamma \geq 0$, $x_0 \leq x_1 < \infty$, $0 < K_0 \leq K_1 < \infty$.

There is no law of conservation for the equal mitosis equation. However, by solving the dual eigenvalue-eigenfunction problem

$$\mathcal{D}^*\phi + \mathcal{F}^*\phi = \lambda\phi, \quad \lambda \in \mathbb{R}, \quad \phi \geq 0, \ \phi \not\equiv 0, \tag{1.27}$$

one immediately observes that any solution f to the equal mitosis equation (1.23) satisfies

$$\int_0^\infty f(t,x)\,\phi(x)\,dx = e^{\lambda t}\int_0^\infty f_0\,\phi(x)\,dx.$$

The first eigenvalue λ corresponds to an exponential growth rate of (some average quantity of) the solution. In an ecology context λ is often called the *Malthus parameter* or the *fitness* of the cells/organisms population. In order to go further in the analysis of the dynamics, one can solve the primal eigenvalue-eigenfunction problem

$$\mathcal{D}f_\infty + \mathcal{F}f_\infty = \lambda f_\infty, \quad f_\infty \geq 0, \ f_\infty \not\equiv 0, \tag{1.28}$$

which provides a Malthusian profile f_∞ which is a remarkable solution in the sense that $f(t,x) := e^{\lambda t}\, f_\infty(x)$ is a solution to the the equal mitosis equation (1.23). We may then expect that (1.28) captures the main features of the model, and more precisely that

$$f(t,x)e^{-\lambda t} = f_\infty(x) + o(1) \quad \text{as} \quad t \to \infty. \tag{1.29}$$

We refer to [27, 76, 94, 79, 93, 68] (and the references therein) for results about the existence of solutions to the primal and dual eigenvalue problems (1.27) and (1.28) as well as for results on the asymptotic convergence (1.29) without rate or with exponential rate.

1.2.2. **Generalization of example 1. Smooth cell-division equation.** We generalize the previous equal mitosis model by considering the case of a general fragmentation operator (1.14) where the total fragmentation rate K still satisfies (1.25) and (1.26) and where however we restrict ourself to the case of a smooth size offsprings distribution $\wp$, more precisely, we assume

$$\wp_0' := \int_0^1 |\wp'(z)|\,dz < \infty. \tag{1.30}$$

We will sometimes make the additional strong positivity assumption

$$\wp(z) \geq \wp_* \ \forall\, z \in (0,1), \quad \wp_* > 0, \tag{1.31}$$

or assume the additional monotony condition and constant number of offsprings condition

$$\frac{\partial}{\partial y}\int_0^x \kappa(y,z)\,dz \le 0, \quad n_F := \int_0^x \kappa(x,z)\,dz > 1, \quad \forall\, x,y \in \mathbb{R}_+. \tag{1.32}$$

Observe that this monotony condition is fulfilled for the equal and unequal mitosis kernel (1.19) and for the smooth distribution of offsprings functions $\wp(z) := c_\theta\, z^\theta$, $\theta > -1$, see [68].

The smooth cell-division equation reads

$$\frac{\partial}{\partial t} f(t,x) + \frac{\partial}{\partial x} f(t,x) = (\mathcal{F} f(t,.))(x), \tag{1.33}$$

and it is complemented again with the boundary condition (1.24) and the initial condition (1.21). We call the resulting model as the *smooth cell-division equation*. While the general fragmentation operator is used for a long time in physics in order to modelize the dynamics of cluster breakage (and it is often associated to the coagulation operator which modelizes the opposite agglomeration mechanism, see [75]) it is only more recently that it appears (in a chemical or biology context) associated as here with the drift operator $\mathcal{D} = -\tau(x)\partial_x$ (see [52]). The general fragmentation operator is used in order to take into unequal cell-division according to experimental evidence [64, 90]. In recent years, the above smooth cell-division equation (1.33) also appears in many works on the modeling of proteins [99, 45, 21, 20].

Concerning the mathematical analysis of the *smooth cell-division equation*, and in particular the long time behaviour of solutions, a similar picture as for the equal mitosis equation is expected and some (at least partial) results have been obtained in [78, 79, 77, 68, 30, 18, 19, 8].

With the above notations and for latter reference, we introduce the critical exponent $\alpha^* \ge 1$ uniquely implicitly defined by the equation

$$\wp_{\alpha^*} = K_0/K_1 \in (0,1], \quad \wp_\alpha := \int_0^1 z^\alpha\, \wp(dz). \tag{1.34}$$

1.2.3. **Example 2. Self-similar fragmentation equation.** We consider now the pure fragmentation equation associated to power law total fragmentation rate and then to a fragmentation rate

$$K(x) = x^\gamma, \quad \gamma > 0, \quad k(x,y) = K(x)\,\wp(y/x)/x, \tag{1.35}$$

where $\wp$ is a continuous function satisfying (1.30). By analogy with the probabilistic name for the associated Markov process, see e.g. [12], we call that model the *"self-similar fragmentation equation"*.

The pure fragmentation model is then obtained for $\tau = \nu \equiv 0$ in (1.12) and therefore reads

$$\partial_t g = \tilde{\mathcal{F}}\, g$$

where $\tilde{\mathcal{F}}$ is a temporary notation for the fragmentation operator associated to the kernel $\tilde{k} := k/\gamma$. This equation arises in physics to describe fragmentation processes. We refer to [37, 113, 13] for the first study and the physics motivations, to [12] and the references therein for a probabilistic approach

For this equation the only steady states are the Dirac masses, namely $x\, g(t,x) = \rho\, \delta_{x=0}$. On the other hand, if g is a solution to the pure fragmentation equation,

we may introduce the rescaled density f defined by

$$f(t,x) = e^{-2t} g\left(e^{\gamma\, t} - 1, x e^{-t}\right), \tag{1.36}$$

which is a solution to the fragmentation equation in self-similar variables (see for instance [34])

$$\frac{\partial}{\partial t} f = \mathcal{F}\, f - x\, \frac{\partial}{\partial x} f - 2f. \tag{1.37}$$

This is a mass preserving equation with no detailed balance condition. However, one can show that

$$\Lambda f_\infty = 0, \quad \Lambda^* \phi = 0$$

with $\phi(x) = x$ and for some positive self-similar profile f_∞. Equation (1.37) falls in the class of growth-fragmentation equation with first eigenvalue $\lambda = 0$ because of the conservation of the mass. As for the cell-division equation one expects that the remarkable (self-similar profile) solution f_∞ is attractive and that (1.29) holds again. Existence of the self-similar profile f_∞ and convergence (without rate) to this one has been established in [34, 79] while a rate of convergence for $\gamma \in (0,2)$ has been proved in [18, 17].

1.2.4. **Example 3. Age structured population equation.** We consider an age structured population of individuals which age, die and give birth, and which is described by the density $f(t,x)$ of individuals with age $x \geq 0$ at time $t \geq 0$. The very popular associated evolution PDE is

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial x}(\tau(x) f) = -\nu(x)\, f, \quad f(t, x=0) = \int_0^\infty K(y) f(t,y) dy, \tag{1.38}$$

and it is communally attributed to A. McKendrik [74] and H. von Foerster [108] (although the dynamics of age structured population has been anteriorly developed by A. Lotka and F. Sharpe [72] and before by L. Euler, as well as by P.H. Leslie [69] in a discrete time and age framework). In the sequel we call that model as the *age structured population equation*. Let us also mention that nonlinear versions of equation (1.38) which take into account possibly overcrowding effect can be found in the work of Gurtin and MacCamy [50].

In equation (1.38) the function K corresponds to the birth rate, the function ν to the death rate and the function τ to the aging rate (so that $\tau \equiv 1$). Notice that the age structured population equation can be seen as a particular example of the growth-fragmentation equation (1.12)–(1.14) making the following choice for k :

$$k(x,y) = K(x)\, [\delta(y = x) + \delta(y = 0)], \tag{1.39}$$

which corresponds to the limit case $\sigma = 0$ in (1.19). In order to simply the presentation we make the assumptions

$$\tau = \nu = 1, \quad 0 \leq K \in C_b^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+), \quad \|K\|_{L^1} > 1. \tag{1.40}$$

The expected longtime behaviour of solutions to the age structured population equation is the same as the one described for a general growth-fragmentation equation, in particular the long time converge (1.29) with exponential rate is known to hold. Here, the mathematical analysis is greatly simplified by remarking that the offsprings number satisfies a (Volterra) delay equation (the so-called renewal equation) which in turns can be handled through a direct Laplace transform as first shown by W. Feller [35], and developped later in [36, 109, 76, 62]. Let us also

mention that the long time converge (1.29) can also be obtained by entropy method [84, 79].

1.2.5. **Main result.** Let us introduce the functional spaces in which we will work. For any exponent $p \in [1,\infty]$ and any nonnegative weight function ξ, we denote by $L^p(\xi)$ the Lebesgue space $L^p(\mathbb{R}_+;\xi\,dx)$ or $L^p(\mathbb{R};\xi\,dx)$ associated to the norm

$$\|u\|_{L^p(\xi)} := \|u\,\xi\|_{L^p},$$

and we simply use the shorthand L^p_α for the choice $\xi(x) := \langle x\rangle^\alpha$, $\alpha \in \mathbb{R}$, $\langle x\rangle^2 := 1+|x|^2$, as well as the shorthand $\dot L^p_\alpha$ for the choice $\xi(x) := |x|^\alpha$, $\alpha \in \mathbb{R}$.

Theorem 1.1. *Consider the growth-fragmentation equations with the corresponding structure assumption and boundedness of coefficients as presented in the preceding sections and define the functional space X as follows:*

- **(1)** *Cell-division equation: take $X = L^1_\alpha$, $\alpha > \alpha^*$, where $\alpha^* \geq 1$ is defined in* (1.34)*;*
- **(2)** *Self-similar fragmentation equation: take $X = \dot L^1_\alpha \cap \dot L^1_\beta$, $0 \leq \alpha < 1 < \beta$;*
- **(3)** *Age structured population equation: take $X = L^1$.*

There exists a unique couple (λ, f_∞) with $\lambda \in \mathbb{R}$ and $f_\infty \in X$ solution to the stationary equation

$$\mathcal{F} f_\infty - \tau\partial_x f - \nu f_\infty = \lambda\, f_\infty, \quad f_\infty \geq 0, \quad \|f_\infty\|_X = 1. \tag{1.41}$$

There exists $a^{**} < \lambda$ and for any $a > a^{**}$ there exists C_a such that for any $f_0 \in X$, the associated solution $f(t) = e^{\Lambda t} f_0$ satisfies

$$\|f(t) - e^{\lambda t}\,\Pi_{\Lambda,\lambda} f\|_X \leq C\, e^{at}\, \|f_0 - e^{\lambda t}\,\Pi_{\Lambda,\lambda} f_0\|_X, \tag{1.42}$$

where $\Pi_{\Lambda,\lambda}$ is the projection on the remarkable solutions line vect(f_∞) which is given by

$$\Pi_{\Lambda,\lambda} h = \langle \phi, h\rangle\, f_\infty$$

where $\phi \in X'$ is the unique positive and normalized solution to the dual first eigenvalue problem

$$\mathcal{F}^*\phi + \tau\partial_x f - \nu\phi = \lambda\,\phi, \quad \phi \geq 0, \quad \langle \phi, f_\infty\rangle = 1. \tag{1.43}$$

Moreover, an explicit bound on the spectral gap $\lambda - a^{**}$ is available for

- **(i)** *the cell-division equation with constant total fragmentation rate $K \equiv K_0$ on $(0,\infty)$, $K_0 > 0$, and a fragmentation kernel which satisfies the monotony condition and constant number of offsprings condition* (1.32)*;*
- **(ii)** *the self-similar fragmentation equation with smooth and positive offsprings distribution in the sense that* (1.30) *and* (1.31) *hold.*

Let us make some comments about the above result.

Theorem 1.1 generalizes, improves and unifies the long time asymptotic convergence with exponential rate results which were known only for particular cases of growth fragmentation equation, namely for the cell division model with (almost) constant total fragmentation rate and monotonous offspring distribution in [94, 68] and the for the self-similar fragmentation equation and the cell division equation restricted to smooth and positive fragmentation rate and total fragmentation rate which does not increase more rapidly than quadratically in [18, 17]. The rate of convergence (1.42) is proved under similar hypothesizes as in [79, 34] where such a convergence is established without rate. While it has been established in [94, 68] a

similar L^1 decay as (1.42) when the initial datum is bounded for a stronger norm than the L^1 norm, and it was conjectured that the additional boundedness on the initial datum is necessary, Theorem 1.1 shows that it is not the case and that the additional stronger norm boundedness assumption made in [94, 68] can be removed (although the norm used in the present paper is different from (and stronger than) those used in [94, 68] the fundamental point is that the norm measuring the distance between the solution and its limit is the same for positive time and at initial time). Let us emphasize that we do not claim that Theorem 1.1 is new for the age structured population equation. However, we want to stress here that our proof of the convergence (1.42) is similar for all these growth-fragmentation equations while the previous proofs of convergence results with rate were very different for the three subclasses of models. It is also likely that our approach can be generalized to larger classes of growth operator and of fragmentation kernel as those considered in [77, 30, 19, 8] for instance. However, for the sake of simplicity we have not followed that line of research here. It is finally worth noticing that our result excludes the two "degenerate equations" which are the equal self-similar fragmentation equation associated to equal mitosis offspring distribution and the age structured population equation associated to deterministic birth rate $K(z) := K\,\delta_{z=L}$.

Let us make some comments about the different methods of proof which may be based on linear tools (Laplace transform, Eigenvalue problem, suitable weak distance, semigroup theory) and nonlinear tools (existence of self-similar profile by fixed point theorems, GRE and E-DE methods).

• *Direct Laplace transform method.* For the age structured population equation a direct Laplace transform analysis can be performed at the level of the associated renewal equation and leads to an exact representation formula which in turns implies the rate of convergence (1.42) (see [35, 36, 62]).

• *PDE approach via compactness and GRE methods.* Convergence results (without rate) has been proved in [84, 34, 79] for a general class of growth-fragmentation equation which is basically the same class as considered in the present paper thanks to the use of the so called *general relative entropy method*. More precisely, once existence of a solution $(\lambda, f_\infty, \phi)$ to the primal eigenvalue problem (1.41) and dual eigenvalue problem (1.43) is established, we refer to [94, 19, 77, 30] where those problems are tackled, one can easily compute the evolution of the generalized relative entropy $\mathcal{J}$ defined thanks to

$$\mathcal{J}(f) := \int_0^\infty j(f/f_\infty)\, f_\infty\, \phi\, dx$$

for a convex and non negative function $j : \mathbb{R} \to \mathbb{R}$ and for a generic solution $f = f(t)$ to the growth-fragmentation equation. One can then show the (at least formal) identity

$$\mathcal{J}(f(t)) + \int_0^t \mathcal{D}_{\mathcal{J}}(f(s))\, ds = \mathcal{J}(f(0)) \qquad \forall\, t \geq 0, \tag{1.44}$$

where $\mathcal{D}_{\mathcal{J}} \geq 0$ is the associated generalized dissipation of relative entropy defined by

$$\mathcal{D}_{\mathcal{J}}(f) := \int_0^\infty \int_0^\infty k(x_*, x) \mathbf{1}_{x_* \geq x} \left[j(u) - j(u_*) - j'(u_*)\,(u - u_*) \right] f_\infty\, \phi_*\, dx dx_*,$$

with the notations $u := f/f_\infty$ and $h_* = h(x_*)$. Identity (1.44) clearly implies that $\mathcal{J}$ is a Liapunov functional for the growth fragmentation dynamics which in turns implies the long time convergence $f(t) \to f_\infty$ as $t \to \infty$ (without rate) under positivity assumption on the kernel k or the associated semigroup.

• *Suitable weak distance.* In order to circumvent the possible weak information given by $\mathcal{D}_{\mathcal{J}}$ in the case of cell-division models when k is not positive (for the equal mitosis model for instance) an exponential rate of convergence for an (almost) constant total rate of fragmentation has been established by Perthame and co-authors in [94, 68] by the mean of the use of a suitable weak distance.

• *Entropy - dissipation of entropy (E-DE) method.* In the case of the strong positivity assumption (1.31) an entropy - dissipation of entropy method has been implemented in [18, 8] where the inequality $\mathcal{D}_{\mathcal{J}} \geq c\,\mathcal{J}$ is proved for $j(s) = (s-1)^2$. The Gronwall lemma then straightforwardly implies a rate of convergence in a weighted Lebesgue L^2 framework.

• *Semigroup methods.* Semigroups technics have been widely used since the first works by Metz, Diekmann and Webb on the age structured equation and the equal mitosis equation in a bounded size setting [109, 76], see also [33] and the references therein for more recent papers. For a general fragmentation kernel (including the smooth cell-division equation and the self-similar fragmentation equation) the *enlargement of semigroup spectral analysis* has been used in [18, 17] in order to extends to a L^1 framework some of the convergence (with rate) results proved in [18] in a narrow weighted L^2 framework. Let us conclude that our approach is clearly a semigroup approach where the novelty comes from the fact that we are able to prove that the considered growth-fragmentation operator falls in the class of operators with splitting structure as described in the first part of the introduction and for which our abstract Kren-Rutmann Theorem applies. Our approach is then completely linear, very accurate and still very general. The drawback is the use of the abstract semigroup framework and some complex analysis tool (and in particular the use of the Laplace transform and inverse Laplace transform at the level of the abstract (functional space) associated evolution equation).

The outline of the paper is as follows. In the next section, we establish the partial spectral mapping theorem in an abstract framework. In section 3, we establish two versions of the Weyl's Theorem in an abstract framework and we then verify that they apply to the growth-fragmentation equations. The last section is devoted to the proof of the Krein-Rutman Theorem and to the end of the proof of Theorem 1.1.

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2. Spectral mapping for semigroup generators

2.1. **Notations and definitions.** There are many textbooks addressing the theory of semigroups since the seminal books by Hille and Philipps [60, 61] among

them the ones by Kato [63], Davies [26], Pazy [91], Arendt et al [3] and more recently by Engel and Nagel [33] to which we refer for more details. In this section we summarize some basic definitions and facts on the analysis of operators in a abstract and general Banach space picked up from above mentioned books as well as from the recent articles [48, 80]. It is worth mentioning that we adopt the sign convention of Kato [63] on the resolvent operator (which is maybe opposite to the most widespread convention), see (2.1). For a given real number $a \in \mathbb{R}$, we define the half complex plane

$$\Delta_a := \{z \in \mathbb{C},\, \Re e\, z > a\}.$$

For some given Banach spaces $(E, \|\cdot\|_E)$ and $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ we denote by $\mathscr{B}(E, \mathcal{E})$ the space of bounded linear operators from E to $\mathcal{E}$ and we denote by $\|\cdot\|_{\mathscr{B}(E,\mathcal{E})}$ or $\|\cdot\|_{E\to\mathcal{E}}$ the associated norm operator. We write $\mathscr{B}(E) = \mathscr{B}(E, E)$ when $E = \mathcal{E}$. We denote by $\mathscr{C}(E, \mathcal{E})$ the space of closed unbounded linear operators from E to $\mathcal{E}$ with dense domain, and $\mathscr{C}(E) = \mathscr{C}(E, E)$ in the case $E = \mathcal{E}$. We denote by $\mathscr{K}(E, \mathcal{E})$ the space of compact linear operators from E to $\mathcal{E}$ and $\mathscr{K}(E) = \mathscr{K}(E, E)$ in the case $E = \mathcal{E}$.

We denote by $\mathscr{G}(X)$ the space of generators of a C_0-semigroup. For $\Lambda \in \mathscr{G}(E)$ we denote by $S_\Lambda(t) = e^{t\,\Lambda}$, $t \geq 0$, the associated semigroup, by $\mathrm{D}(\Lambda)$ its domain, by $\mathrm{N}(\Lambda)$ its null space, by

$$M(\Lambda) = \cup_{\alpha\geq 1} N(\Lambda^\alpha)$$

its algebraic null space, and by $\mathrm{R}(\Lambda)$ its range. For any given integer $k \geq 1$, we define $\mathrm{D}(\Lambda^k)$ the Banach space associated with the norm

$$\|f\|_{\mathrm{D}(\Lambda^k)} = \sum_{j=0}^{k} \|\Lambda^j f\|_E.$$

For $\Lambda \in \mathscr{G}(E)$, we denote by $\Sigma(\Lambda)$ its spectrum, so that for any $z \in \mathbb{C}\backslash\Sigma(\Lambda)$ the operator $\Lambda - z$ is invertible and the resolvent operator

$$\mathcal{R}_\Lambda(z) := (\Lambda - z)^{-1} \tag{2.1}$$

is well-defined, belongs to $\mathscr{B}(E)$ and has range equal to $D(\Lambda)$. We then define the spectral bound $s(\Lambda) \in \mathbb{R} \cup \{-\infty\}$ by

$$s(\Lambda) := \sup\{\Re e\, \xi;\ \xi \in \Sigma(\Lambda)\}$$

and the growth bound $\omega(\Lambda) \in \mathbb{R} \cup \{-\infty\}$ by

$$\omega(\Lambda) := \inf\{b \in \mathbb{R};\ \exists\, M_b \text{ s.t. } \|S_\Lambda(t)\|_{\mathscr{B}(X)} \leq M_b\, e^{bt}\ \forall\, t \geq 0\},$$

and we recall that $s(\Lambda) \leq \omega(\Lambda)$ as a consequence of Hille's identity [59]: for any $\xi \in \Delta_{\omega(\Lambda)}$ there holds

$$-\, R_\Lambda(\xi) = \int_0^\infty S_\Lambda(t)\, e^{-\xi t}\, dt, \tag{2.2}$$

where the RHS integral normally converges.

We say that Λ is *a-hypo-dissipative* on X if there exists some norm $|||\cdot|||_X$ on X equivalent to the initial norm $\|\cdot\|_X$ such that

$$\forall\, f \in D(\Lambda),\ \exists\, \varphi \in F(f) \ \text{ s.t. } \ \Re e\, \langle \varphi, (\Lambda - a)\, f\rangle \leq 0, \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket for the duality in X and X^* and $F(f) \subset X^*$ is the dual set of f defined by

$$F(f) = F_{|||\cdot|||}(f) := \left\{\varphi \in X^*;\ \langle\varphi, f\rangle = |||f|||_X^2 = |||\varphi|||_{X^*}^2\right\}.$$

We just say that Λ is *hypo-dissipative* if Λ is a-hypo-dissipative for some $a \in \mathbb{R}$. From the Hille-Yosida Theorem it is clear that any generator $\Lambda \in \mathscr{G}(X)$ is an hypo-dissipative operator and that

$$\omega(\Lambda) := \inf\{b \in \mathbb{R};\ \Lambda \text{ is } b\text{-hypo-dissipative}\}.$$

A spectral value $\xi \in \Sigma(\Lambda)$ is said to be isolated if

$$\Sigma(\Lambda) \cap \{z \in \mathbb{C},\ |z-\xi| \le r\} = \{\xi\} \quad \text{for some } r > 0.$$

In the case when ξ is an isolated spectral value we may define $\Pi_{\Lambda,\xi} \in \mathscr{B}(E)$ the spectral projector by the Dunford integral

$$\Pi_{\Lambda,\xi} := \frac{i}{2\pi} \int_{|z-\xi|=r'} \mathcal{R}_\Lambda(z)\, dz, \tag{2.4}$$

with $0 < r' < r$. Note that this definition is independent of the value of r' as the application

$$\mathbb{C} \setminus \Sigma(\Lambda) \to \mathscr{B}(E), \quad z \to \mathcal{R}_\Lambda(z)$$

is holomorphic. It is well-known that $\Pi^2_{\Lambda,\xi} = \Pi_{\Lambda,\xi}$ so that $\Pi_{\Lambda,\xi}$ is effectively a projector, and its range $R(\Pi_{\Lambda,\xi}) = \overline{M(\Lambda-\xi)}$ is the closure of the algebraic eigenspace associated to ξ.

We recall that $\xi \in \Sigma(\Lambda)$ is said to be an eigenvalue if $N(\Lambda-\xi) \neq \{0\}$. The range of the spectral projector is finite-dimensional if and only if there exists $\alpha_0 \in \mathbb{N}^*$ such that

$$N(\Lambda-\xi)^\alpha = N(\Lambda-\xi)^{\alpha_0} \neq \{0\} \quad \text{for any} \quad \alpha \ge \alpha_0,$$

so that

$$\overline{M(\Lambda-\xi)} = M(\Lambda-\xi) = N((\Lambda-\xi)^{\alpha_0}) \quad \text{and} \quad N(\Lambda-\xi) \neq \{0\}.$$

In that case, we say that ξ is a discrete eigenvalue, written as $\xi \in \Sigma_d(\Lambda)$. For any $a \in \mathbb{R}$ such that

$$\Sigma(\Lambda) \cap \Delta_a = \{\xi_1, \dots, \xi_J\}$$

where $\xi_1, \dots, \xi_J$ are distinct discrete eigenvalues, we define without any risk of ambiguity

$$\Pi_{\Lambda,a} := \Pi_{\Lambda,\xi_1} + \dots \Pi_{\Lambda,\xi_J}.$$

For some given Banach spaces X_1, X_2, X_3 and some given functions

$$S_1 \in L^1(\mathbb{R}_+; \mathscr{B}(X_1, X_2)) \quad \text{and} \quad S_2 \in L^1(\mathbb{R}_+; \mathscr{B}(X_2, X_3)),$$

we define the convolution $S_2 * S_1 \in L^1(\mathbb{R}_+; \mathscr{B}(X_1, X_3))$ by

$$\forall\, t \ge 0, \quad (S_2 * S_1)(t) := \int_0^t S_2(s)\, S_1(t-s) ds.$$

When $S_1 = S_2$ and $X_1 = X_2 = X_3$, we define recursively $S^{(*1)} = S$ and $S^{(*\ell)} = S * S^{(*(\ell-1))}$ for any $\ell \ge 2$.

For a generator L of a semigroup such that $\omega(L) < 0$ we define the fractional powers $L^{-\eta}$ and L^η for $\eta \in (0,1)$ by Dunford formulas [65, 66], see also [33, section II.5.c],

$$L^{-\eta} := c_{-\eta} \int_0^\infty \lambda^{-\eta}\, \mathcal{R}_L(\lambda)\, d\lambda, \quad L^\eta := c_\eta \int_0^\infty \lambda^{\eta-1}\, L\, \mathcal{R}_L(\lambda)\, d\lambda, \tag{2.5}$$

for some constants $c_\eta, c_{-\eta} \in \mathbb{C}^*$. The operator $L^{-\eta}$ belongs to $\mathscr{B}(X)$ and, denoting $X_\eta := R(L^{-\eta})$, the operator L^η is the unbounded operator with domain $D(\Lambda^\eta) =$

X_η and defined by $L^\eta = (L^{-\eta})^{-1}$. We also denote $X_0 = X$ and $X_1 = D(L)$. Moreover, introducing the J-method interpolation norm

$$\|f\|_{\tilde{X}_\eta} := \inf\Big\{ \sup_{t>0} \|t^{-\eta}\, J(t, g(t))\|_X;\ g \,\text{such that}\, f = \int_0^\infty g(t)\, \frac{dt}{t} \Big\}$$

with $J(t,g) := \max(\|g\|_X, t\|Lg\|_X)$ and the associated Banach space $\tilde{X}_\eta$ which corresponds to the interpolation space $S(\infty, -\eta, X_0; \infty, 1-\eta, X_1)$ of Lions and Peetre [71] defined by

$$\tilde{X}_\eta := \{f \in X;\ \|f\|_{\tilde{X}_\eta} < \infty\},$$

the following inclusions

$$X_\eta \subset \tilde{X}_\eta \subset X_{\eta'} \tag{2.6}$$

hold with continuous embedding for any $0 < \eta' < \eta < 1$. Let us emphasize that the first inclusion follows from the second inclusion in [66, Proposition 2.8] and [66, Theorem 3.1] while the second inclusion in (2.6) is a consequence of the first inclusion in [66, Proposition 2.8] together with [66, Theorem 3.1] and the classical embedding $S(\infty, -\theta, X_0; \infty, 1-\eta, X_1) \subset S(1, -\theta', X_0; \infty, 1-\theta', X_1)$ whenever $X_1 \subset X_0$ and $0 < \theta' < \theta < 1$.

2.2. **An abstract spectral mapping theorem.** We present in this section a *"principal spectral mapping theorem"* for a class of semigroup generators Λ on a Banach space which split as a hypodissipative part $\mathcal{B}$ and a "more regular part" $\mathcal{A}$ as presented in the introduction. In order to do so, we introduce a more accurate version of the growth and regularizing assumption **(H2-3)**, namely

(H2) there exist $\zeta \in (0,1]$ and $\zeta' \in [0,\zeta)$ such that $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded and there exists an integer $n \geq 1$ such that for any $a > a^*$, there holds

$$\forall\, t \geq 0, \qquad \|S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*n)}(t)\|_{\mathscr{B}(X, D(\Lambda^\zeta))} \leq C_{a,n,\zeta}\, e^{at} \tag{2.7}$$

for a constant $C_{a,n,\zeta} \in (0,\infty)$.

Theorem 2.1. *Consider X a Banach space, Λ the generator of a semigroup $S_\Lambda(t) = e^{t\Lambda}$ on X, two real numbers $a^*, a' \in \mathbb{R}$, $a^* < a'$, and assume that the spectrum $\Sigma(\Lambda)$ of Λ satisfies the following separation condition*

$$\Sigma(\Lambda) \subset \Delta^c_{a^*} \cup \Delta_{a'}. \tag{2.8}$$

The following growth estimate on the semigroup

(1) *there exists a projector $\Pi \in \mathscr{B}(X)$ satisfying $\Lambda\Pi = \Pi\Lambda$, $\Lambda_1 := \Lambda_{|X_1} \in \mathcal{B}(X_1)$, $X_1 := R\Pi$, $\Sigma(\Lambda_1) \subset \Delta_{a^*}$ and for any real number $a > a^*$ there exists a constant C_a such that*

$$\forall\, t \geq 0, \quad \Big\| e^{t\,\Lambda}(I - \Pi) \Big\|_{\mathscr{B}(X)} \leq C_a\, e^{a\,t}, \tag{2.9}$$

is equivalent to the following splitting structure hypothesis

(2) *there exist two operators $\mathcal{A}, \mathcal{B} \in \mathscr{C}(X)$, such that $\Lambda = \mathcal{A} + \mathcal{B}$ and hypothesizes* **(H1)** *and* **(H2)** *are met.*

Moreover, under assumption **(2)**, *for any $a > a^*$ there exists a computable constant $M = M(a, \mathcal{A}, \mathcal{B})$ such that*

$$\Sigma(\Lambda) \cap \Delta_a \subset B(0, M). \tag{2.10}$$

Remark 2.2. **(a)** Theorem 2.1 gives a characterization (and thus a criterium with the conditions **(H1)** and **(H2)**) for an operator Λ to satisfy a partial (but principal) spectral mapping theorem under the only additional assumption that the spectrum satisfies a separation condition.

(b) Hypothesis **(H1)** holds for any $\ell \in \mathbb{N}$ when it is true for $\ell = 0$ (that is $(\mathcal{B}-a)$ is hypodissipative in X for any real number $a > a^*$) and $\mathcal{A} \in \mathscr{B}(X)$.

(c) As a consequence of point **(b)**, the implication **(1)** $\Rightarrow$ **(2)** is just immediate by taking $n := 1$, $\mathcal{A} := \Lambda\Pi$, $\mathcal{B} := \Lambda - \mathcal{A}$.

(d) We believe that the implication **(2)** $\Rightarrow$ **(1)** is new. It can be seen as a condition under which a *"spectral mapping theorem for the principal part of the spectrum holds"* in the sense that (1.6) holds. Indeed, defining $\Lambda_0 := \Lambda(I - \Pi)$, for any $a > a^*$ there holds $\Sigma(\Lambda) = \Sigma(\Lambda_0) \cup \Sigma(\Lambda_1)$, $\Sigma(e^{t\Lambda}) = \Sigma(e^{t\Lambda_0}) \cup \Sigma(e^{t\Lambda_1})$, $\Sigma(e^{t\Lambda_1}) = e^{t\Sigma(\Lambda_1)}$ (because $\Lambda_1 \in \mathscr{B}(X_1)$), $\Sigma(\Lambda_0) \subset \Delta_a$ (by hypothesis) and $\Sigma(e^{t\Lambda_0}) \subset \Delta_{e^{at}}$ (from the conclusion (2.9)). In particular, under assumption **(2)** the spectral bound $s(\Lambda)$ and the growth bound $\omega(\Lambda)$ coincide if they are at the right hand side of a^*, or in other words (1.7) holds.

(e) When $a^* < 0$ the above result gives a characterization of uniformly exponentially stable semigroup (see e.g. [33, Definition V.1.1]) in a Banach space framework. More precisely, when assumption **(2)** holds with $a^* < 0$, there holds

$$S_\Lambda \text{ is uniformly exponentially stable iff } s(\Lambda) < 0.$$

That last assertion has to be compared to the Gearhart-Prüss Theorem which gives another characterization of uniformly exponential stability in a Hilbert framework, see [41, 98, 3] as well as [33, Theorem V.1.11] for a comprehensive proof.

(f) Although the splitting condition in **(2)** may seem to be strange, it is in fact quite natural for many partial differential operators, including numerous cases of operators which have not any self-adjointness property, as that can be seen in the many examples studied in [87, 82, 83, 48, 6, 18, 17, 23, 32, 102, 24]. It is also worth emphasizing that our result does not require any kind of "regularity" on the semigroup as it is usually the case for spectral mapping theorem. In particular, we do not require that the semigroup is eventually norm continuous as in [61] or [33, Theorem IV.3.10].

(g) For a sectorial operator $\mathcal{B}$, hypothesis **(H1)** holds for any operator $\mathcal{A}$ which is $\mathcal{B}$-bounded. More precisely, in a Hilbert space framework, for a "hypoelliptic operator $\mathcal{B}$ of order ζ" and for a $\mathcal{B}^{\zeta'}$-bounded operator $\mathcal{A}$, with $\zeta' \in [0, \zeta)$, in the sense that for any $f \in D(\mathcal{B})$

$$(\mathcal{B}f, f) \le -a\,\|\mathcal{B}^\zeta f\|^2 + C\|f\|^2 \quad \text{and} \quad \|\mathcal{A}f\| \le C\,(\|\mathcal{B}^{\zeta'} f\| + \|f\|),$$

then **(H1)** holds and **(H2)** holds with $n = 0$. A typical example is $\mathcal{B} = -\Delta$ and $\mathcal{A} = a(x) \cdot \nabla$ in the space $L^2(\mathbb{R}^d)$ with $a \in L^\infty(\mathbb{R}^d)$. In that case, Theorem 2.1 is nothing but the classical spectral mapping theorem which is known to hold in such a sectorial framework. We refer to [91, section 2.5] and [33, Section II.4.a] for an introduction to sectorial operators (and analytic semigroups) as well as [33, Section IV.3.10] for a proof of the spectral mapping theorem in that framework.

(h) Condition **(H2)** is fulfilled if $\mathcal{A}$ and $\mathcal{B}$ satisfy **(H1)** as well as

$$t \mapsto \|(\mathcal{A}S_{\mathcal{B}})^{(*n)}(t)\|_{\mathscr{B}(X,D(\Lambda^{\varsigma}))}\, e^{-at} \in L^p(0,\infty)$$

for any $a > a*$ and for some $p \in [1,\infty]$.

Proof of Theorem 2.1. We only prove that **(2)** implies **(1)** since the reverse implication is evident (see remark 2.2-**(c)**).
Step 1. Let us fix $a \in (a^*, a')$. From classical complex analysis arguments (see [48, Proof of Theorem 2.13] or [91, Chapter 1]) for any $f \in D(\Lambda)$ and $t \geq 0$, we have

$$S_\Lambda(t)f = S_{\Lambda_1}(t)\,\Pi f + \mathcal{T}(t)f, \tag{2.11}$$

where

$$\mathcal{T}(t) := \lim_{M'\to\infty} \frac{i}{2\pi} \int_{a-iM'}^{a+iM'} e^{zt}\,\mathcal{R}_\Lambda(z)\,dz, \tag{2.12}$$

and the term $\mathcal{T}(t)f$ might be only defined as a semi-convergent integral. From the splitting $\Lambda = \mathcal{A} + \mathcal{B}$, we have

$$\mathcal{R}_\Lambda(z) = \mathcal{R}_{\mathcal{B}}(z) - \mathcal{R}_\Lambda(z)\,\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z)$$

on the open region of $\mathbb{C}$ where $\mathcal{R}_\Lambda$ and $\mathcal{R}_{\mathcal{B}}$ are well defined functions (and thus analytic functions), and iterating the above formula, we get

$$\mathcal{R}_\Lambda(z) = \sum_{\ell=0}^{N} (-1)^\ell \mathcal{R}_{\mathcal{B}}(z)\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^\ell + \mathcal{R}_\Lambda(z)(-1)^{N+1}(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^{N+1} \tag{2.13}$$

for any integer $N \geq 0$. Gathering (2.11), (2.12) and (2.13), we obtain the representation formula

$$\begin{aligned} S_\Lambda(t)f \quad &= \quad S_{\Lambda_1}(t)\,\Pi f + \sum_{\ell=0}^{N} (-1)^\ell S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t)f \\ &+ \quad \lim_{M'\to\infty} \frac{i}{2\pi} \int_{a-iM'}^{a+iM'} e^{zt}\,(-1)^{N+1}\,\mathcal{R}_\Lambda(z)\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^{N+1} f\,dz \end{aligned} \tag{2.14}$$

for any $f \in D(\Lambda)$ and $t \geq 0$.
Step 2. In order to conclude we have to explain why the last term in (2.14) is appropriately bounded for N large enough. We define

$$\mathcal{U}(z) := \mathcal{R}_{\mathcal{B}}(z) - \cdots + (-1)^n \mathcal{R}_{\mathcal{B}}(z)\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^n \tag{2.15}$$

and

$$\mathcal{V}(z) := (-1)^{n+1}(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^{n+1} \tag{2.16}$$

where $n \geq 1$ is the integer given by assumption **(H2)**. With $N = n$, we may rewrite the identity (2.13) as

$$\mathcal{R}_\Lambda(z)(I - \mathcal{V}(z)) = \mathcal{U}(z), \tag{2.17}$$

and then deduce

$$\mathcal{W}(z) := \mathcal{R}_\Lambda(z)\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^{N+1} = \mathcal{U}(z) \sum_{\ell=0}^{\infty} \mathcal{V}(z)^\ell\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^{N+1}, \tag{2.18}$$

under the condition that the RHS series converges.

Step 3. We recall the notations $X_s = D(\Lambda^s)$. On the one hand, from (2.7) we deduce that

$$\|\mathcal{R}_{\mathcal{B}}(z)\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^n\|_{X\to X_\zeta} \le C'_{a,n}$$

for any $a > a^*$, for any $z \in \Delta_a$ and a constant $C'_{a,n}$. As a consequence, writing

$$\mathcal{R}_{\mathcal{B}}(z)^{1-\varepsilon}\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^n = (\mathcal{B}-z)^\varepsilon\,[\mathcal{R}_{\mathcal{B}}(z)\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^n],$$

we get

$$\|\mathcal{R}_{\mathcal{B}}(z)^{1-\varepsilon}\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^n\|_{X\to X_{\zeta-\varepsilon}} \le C_{a,n} \tag{2.19}$$

for any $\varepsilon \in [0,\zeta]$.

On the other hand, we claim that

$$\|\mathcal{R}_{\mathcal{B}}(z)^\varepsilon\|_{\tilde X_{\zeta-\varepsilon}\to \tilde X_{\zeta'}} \le C_{a,\varepsilon}/\langle z\rangle^{\varepsilon\,(\zeta-\zeta'-\varepsilon)} \tag{2.20}$$

for any $a > a^*$, $0 \le \varepsilon < \zeta - \eta'$, for any $z \in \Delta_a$ and a constant $C_{a,\varepsilon}$. First, from the resolvent identity

$$\mathcal{R}_{\mathcal{B}}(z) = z^{-1}\,(\mathcal{R}_{\mathcal{B}}(z)\mathcal{B} - I),$$

we have for $i,j = 0,1$, $i \ge j$,

$$\|\mathcal{R}_{\mathcal{B}}(z)\|_{X_i\to X_j} \le C^{1,a}_{i,j}(z) \quad \forall\, z \in \Delta_a$$

with $C^{1,a}_{0,0}(z) = C^{1,a}_{1,1}(z) = C^{1,a}$ and $C^{1,a}_{1,0}(z) = C^{1,a}_{1,0}/\langle z\rangle$, $C^{1,a}_{i,j} \in \mathbb{R}_+$. Next, thanks to the first representation formula in (2.5), we clearly have

$$\|\mathcal{R}_{\mathcal{B}}(z)^\varepsilon\|_{X_1\to X_0} \le C(a,\varepsilon)\int_0^\infty \frac{\lambda^{-\varepsilon}}{\langle z\rangle + \lambda}\,d\lambda \le C^{\varepsilon,a}_{1,0}(z) := \frac{C^{\varepsilon,a}_{1,0}}{\langle z\rangle^\varepsilon}.$$

Last, thanks to the interpolation theorem [71, Théorème 3.1] we get for $0 \le \theta < \theta' < 1$

$$\|\mathcal{R}_{\mathcal{B}}(z)^\varepsilon\|_{\tilde X_{\theta'}\to \tilde X_\theta} \le \|\mathcal{R}_{\mathcal{B}}(z)^\varepsilon\|^\theta_{X_1\to X_1}\,\|\mathcal{R}_{\mathcal{B}}(z)^\varepsilon\|^{\theta'-\theta}_{X_1\to X_0}\,\|\mathcal{R}_{\mathcal{B}}(z)^\varepsilon\|^{1-\theta'}_{X_0\to X_0}$$

from which we conclude to (2.20). Writing now

$$\mathcal{R}_{\mathcal{B}}(z)\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^n = \mathcal{R}_{\mathcal{B}}(z)^\varepsilon\,[\mathcal{R}_{\mathcal{B}}(z)^{1-\varepsilon}\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^n]$$

with the optimal choice $\varepsilon = (\zeta-\zeta')/2$, we finally deduce from (2.19), (2.20) and the inclusions (2.6) that

$$\|\mathcal{R}_{\mathcal{B}}(z)\,(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^n\|_{X\to X_{\zeta'}} \le \frac{C_a}{\langle z\rangle^\alpha}, \quad \alpha := \frac{(\zeta-\zeta')^2}{8} \in (0,1).$$

Because $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded we conclude with

$$\|\mathcal{V}(z)\|_{\mathscr{B}(X)} \le C_a\,\frac{1}{\langle z\rangle^\alpha}, \tag{2.21}$$

and then in particular, for M large enough,

$$z = a + iy,\ |y| \ge M \quad \Rightarrow \quad \|\mathcal{V}(z)\|_{\mathscr{B}(X)} \le \frac{1}{2}. \tag{2.22}$$

Step 4. In the one hand, defining $N := ([1/\alpha]+1)(n+1)-1$ and $\beta := ([1/\alpha]+1)\alpha > 1$, we get from (2.21) that

$$\|(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^{N+1}\|_{\mathscr{B}(X)} \le \|\mathcal{V}(z)\|^{[1/\alpha]+1}_{\mathscr{B}(X)} \le \frac{C}{\langle y\rangle^\beta} \tag{2.23}$$

for any $z = a + iy$, $|y| \geq M$. In the other hand, we get from (2.22) that the series term in (2.18) is normally convergent uniformly in $z = a + iy$, $|y| \geq M$. All together, we obtain

$$\begin{aligned} \|\mathcal{W}(z)\|_{\mathscr{B}(X)} &\leq \|\mathcal{U}(z)\|_{\mathscr{B}(X)} \Big\|\sum_{\ell=0}^{\infty} \mathcal{V}(z)^{\ell}\Big\|_{\mathscr{B}(X)} \|(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^{N+1}\|_{\mathscr{B}(X)} \\ &\leq \frac{C}{|y|^{\beta}} \end{aligned} \tag{2.24}$$

for any $z = a + iy$, $|y| \geq M$.

In order to estimate the last term in (2.14), we write

$$\begin{aligned} &\Big\|\lim_{M'\to\infty} \frac{i}{2\pi} \int_{a-iM'}^{a+iM'} e^{zt}\, \mathcal{W}(z)\, dz\Big\|_{\mathscr{B}(X)} \\ &\quad \leq \frac{e^{at}}{2\pi} \int_{a-iM}^{a+iM} \|\mathcal{R}_{\Lambda}(z)\|_{\mathscr{B}(X)}\, \|(\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^{N+1}\|_{\mathscr{B}(X)}\, dy \\ &\qquad + \frac{e^{at}}{2\pi} \int_{\mathbb{R}\setminus[-M,M]} \|\mathcal{W}(a+iy)\|_{\mathscr{B}(X)}\, dy, \end{aligned}$$

where the first integral is finite thanks to $\Sigma(\Lambda) \cap [a - iM, a + iM] = \emptyset$ and (2.23), and the second integral is finite because of (2.24). □

We give now two variants of Theorem 2.1 which are sometimes easier to apply. First, for a generator Λ of a semigroup S_{Λ} with the splitting structure (1.2) we introduce an alternative growth assumption to **(H1)**, namely

(H1′) for some $a^* \in \mathbb{R}$ and for any $a > a_*$ the operator $\mathcal{B} - a$ is hypodissipative and there exists a constant $C_a \in (0, \infty)$ such that

$$\int_0^{\infty} e^{-a\,s}\, \|\mathcal{A}\, S_{\mathcal{B}}(s)\|_{\mathscr{B}(X)}\, ds \leq C_a. \tag{2.25}$$

Remark 2.3. Estimate (2.25) is reminiscent of the usual condition under which for a given generator $\mathcal{B}$ of a semigroup $S_{\mathcal{B}}$ the perturbed operator $\mathcal{A} + \mathcal{B}$ also generates a semigroup (see [95], [85], [106, condition (1.1)]). Here however the condition $C_a < 1$ is not required since we have already made the assumption that S_{Λ} exists.

It is clear by writing

$$S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)} = [S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell-1)}] * (\mathcal{A}S_{\mathcal{B}})$$

and performing an iterative argument, that **(H1′)** implies **(H1)**. We then immediately deduce from Theorem 2.1 a first variant:

Corollary 2.4. *Consider X a Banach space, Λ the generator of a semigroup $S_{\Lambda}(t) = e^{t\Lambda}$ on X and a real number $a^* \in \mathbb{R}$. Assume that the spectrum $\Sigma(\Lambda)$ of Λ satisfies the separation condition* (2.8) *and that there exist two operators $\mathcal{A}, \mathcal{B} \in \mathscr{C}(X)$ such that $\Lambda = \mathcal{A} + \mathcal{B}$ and hypothesises* **(H1′)** *and* **(H2)** *are met. Then the conclusions* (2.9) *and* (2.10) *in Theorem 2.1 hold.*

Next, for a generator Λ of a semigroup S_{Λ} with the splitting structure (1.2) we introduce the alternative growth and regularizing assumptions to **(H1′)** and **(H2)**, namely

(H1″) for some $a^* \in \mathbb{R}$, $\zeta \in (0,1]$ and for any $a > a_*$, $\zeta' \in [0,\zeta]$ the operator $\mathcal{B} - a$ is hypodissipative and assumption **(H1)** or **(H1′)** also hold with $\mathcal{B}(X)$ replaced by $\mathcal{B}(D(\Lambda^{\zeta'}))$;

(H2″) there exist an integer $n \geq 1$ and a real number $b > a^*$ such that

$$\forall\, t \geq 0, \qquad \|(\mathcal{A}S_{\mathcal{B}})^{(*n)}(t)\|_{\mathscr{B}(X,D(\Lambda^\zeta))} \leq C_{b,n,\zeta}\, e^{bt},$$

for a constant $C_{b,n,\zeta} \in (0,\infty)$.

Our second variant of Theorem 2.1 is the following.

Corollary 2.5. *Consider X a Banach space, Λ the generator of a semigroup $S_\Lambda(t) = e^{t\Lambda}$ on X and a real number $a^* \in \mathbb{R}$. Assume that the spectrum $\Sigma(\Lambda)$ of Λ satisfies the separation condition* (2.8) *and that there exist two operators $\mathcal{A}, \mathcal{B} \in \mathscr{C}(X)$ such that $\Lambda = \mathcal{A} + \mathcal{B}$ and hypothesises* **(H1″)** *and* **(H2″)** *are met. Then the conclusions* (2.9) *and* (2.10) *in Theorem 2.1 hold.*

Corollary 2.5 is an immediate consequence of Theorem 2.1 or Corollary 2.4 together with the following simple variant of [48, Lemma 2.15] which makes possible to deduce the more accurate regularization and growth condition **(H2)** from the rough regularization and growth condition **(H2″)**.

Lemma 2.6. *Consider two Banach spaces E and $\mathcal{E}$ such that $E \subset \mathcal{E}$ with dense and continuous embedding. Consider some operators $\mathcal{L}$, $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{E}$ such that $\mathcal{L}$ splits as $\mathcal{L} = \mathcal{A} + \mathcal{B}$, some real number $a^* \in \mathbb{R}$ and some integer $m \in \mathbb{N}^*$. Denoting with the same letter $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}$ the restriction of these operators on E, we assume that the two following dissipativity conditions are satisfied:*

(i) *for any $a > a^*$, $\ell \geq m$, $X = E$ and $X = \mathcal{E}$, there holds*

$$\|(\mathcal{A}\, S_{\mathcal{B}})^{(*\ell)}\|_{\mathscr{B}(X)} \leq C_{a,\ell}\, e^{at};$$

(ii) *for some constants $b \in \mathbb{R}$, $b > a^*$, and $C_b \geq 0$, there holds*

$$\|(\mathcal{A}\, S_{\mathcal{B}})^{(*m)}\|_{\mathscr{B}(\mathcal{E},E)} \leq C_b\, e^{bt}.$$

Then for any $a > a^*$, there exist some constructive constants $n = n(a) \in \mathbb{N}$, $C_a \geq 1$ such that

$$\forall\, t \geq 0 \qquad \|(\mathcal{A}S_{\mathcal{B}})^{(*n)}(t)\|_{\mathscr{B}(\mathcal{E},E)} \leq C_a\, e^{a\,t}.$$

Proof of Lemma 2.6. We fix $a > a^*$ and $a' \in (a^*, a)$, and we note $T := (\mathcal{A}\, S_{\mathcal{B}})^{(*m)}$. For $n = p\,m$, $p \in \mathbb{N}^*$, we write

$$\begin{aligned}(\mathcal{A}S_{\mathcal{B}})^{(*n)}(t) &= T^{(*p)}(t)\\ &= \int_0^t \mathrm{d}t_{p-1} \int_0^{t_{p-1}} \mathrm{d}t_{p-2} \ldots \int_0^{t_2} \mathrm{d}t_1\, T(\delta_p) \ldots T(\delta_1)\end{aligned}$$

with

$$\delta_1 = t_1, \quad \delta_2 = t_2 - t_1, \ \ldots, \ \delta_{p-1} = t_{p-1} - t_{p-2} \text{ and } \ \delta_p = t - t_{p-1}.$$

For any $p \geq 1$, there exist at least one increments δ_r, $r \in \{1, ..., p\}$, such that $\delta_r \leq t/p$. Using **(ii)** in order to estimate $\|T(\delta_r)\|_{\mathscr{B}(\mathcal{E},E)}$ and **(i)** in order to bound

the other terms $\|T(\delta_q)\|_{\mathscr{B}(X)}$ in the appropriate space $X = \mathcal{E}$ or $X = E$, we have

$$\begin{aligned}
&\|T(\mathcal{A}S_{\mathcal{B}})^{(*n)}(t)\|_{\mathscr{B}(\mathcal{E},E)} \leq \\
&\leq \int_0^t dt_{p-1} \int_0^{t_{p-1}} \mathrm{d}t_{p-2} \ldots \int_0^{t_2} \mathrm{d}t_1 C'_b\, e^{b\,\delta_r} \prod_{q\neq r} C_{a',m}\, e^{a'\,\delta_q} \\
&\leq C'_b\, C_{a',m}^{p-1}\, e^{a'\,t}\, e^{b\,t/p} \int_0^t \mathrm{d}t_{p-1} \int_0^{t_{p-1}} \mathrm{d}t_{p-2} \ldots \int_0^{t_2} \mathrm{d}t_1 \\
&= C\, t^{p-1}\, e^{(a'+b/p)\,t} \leq C'\, e^{at},
\end{aligned}$$

by taking p (and then n) large enough so that $a' + b/p < a$. □

3. Weyl's Theorem for semigroup generators

3.1. **An abstract semigroup Weyl's Theorem.** We present in this section a version of Weyl's Theorem about compact perturbation of dissipative generator in the spirit of the preceding spectral mapping theorem. For that purpose, we introduce the growth and compactness assumption

(H3) for the same integer $n \geq 1$ as in assumption **(H2)** and with the same notations, the time indexed family of operators $(\mathcal{A}S_{\mathcal{B}})^{(*(n+1))}(t)$ satisfies the growth and compactness estimate

$$\forall\, a > a^*, \quad \int_0^\infty \|(\mathcal{A}S_{\mathcal{B}})^{(*(n+1))}(t)\|_{\mathscr{B}(X,Y)}\, e^{-at}\, dt \leq C''_{n+1,a},$$

for some constant $C''_{n+1,a} \geq 0$ and some (separable) Banach space Y such that $Y \subset X$ with compact embedding.

Theorem 3.1. *Consider X a Banach space, Λ the generator of a semigroup $S_\Lambda(t) = e^{t\Lambda}$ on X and a real number $a^* \in \mathbb{R}$. The following quantitative growth estimate on the semigroup*

(1) *for any $a > a^*$ there exist an integer $J \in \mathbb{N}$, a finite family of distinct complex numbers $\xi_1, ..., \xi_J \in \bar{\Delta}_a$, some finite rank projectors $\Pi_1, \ldots \Pi_J \in \mathscr{B}(X)$ and some operators $T_j \in \mathscr{B}(R\Pi_j)$, satisfying $\Lambda\Pi_j = \Pi_j\Lambda = T_j\Pi_j$, $\Sigma(T_j) = \{\xi_j\}$, in particular*

$$\Sigma(\Lambda) \cap \bar{\Delta}_a = \{\xi_1, \ldots, \xi_J\} \subset \Sigma_d(\Lambda),$$

and a constant C_a such that

$$\forall\, t \geq 0, \quad \Big\| e^{t\,\Lambda} - \sum_{j=1}^{J} e^{t\,T_j}\, \Pi_j \Big\|_{\mathscr{B}(X)} \leq C_a\, e^{a\,t}, \tag{3.1}$$

is equivalent to the following splitting structure of the generator

(2) *there exist two operators $\mathcal{A}, \mathcal{B} \in \mathscr{C}(X)$ such that $\Lambda = \mathcal{A} + \mathcal{B}$ and hypothesises* **(H1)**, **(H2)** *and* **(H3)** *are met.*

Remark 3.2. **(a)** When $\Sigma(\Lambda) \cap \Delta_{a^*} \neq \emptyset$ the above theorem gives a description of the principal asymptotic behaviour of the C_0-semigroup S_Λ, namely it states that it is essentially compact (see e.g. [42, Definition 2.1]). As a matter of fact, if conditions **(H1)**, **(H2)** and **(H3)** hold with $a^* < 0$, then the C_0-semigroup S_Λ is either uniformly exponentially stable or essentially compact.

(b) We adopt the convention $\{\xi_1, \dots, \xi_J\} = \emptyset$ if $J = 0$.

(c) In the case when $\mathcal{B}$ is sectorial and $\mathcal{A}$ is $\mathcal{B}$-compact, Theorem 3.1 is nothing but the classical Weyl's Theorem on the spectrum [111] or [63, Theorem IV.5.35] combined with the spectral mapping theorem, see [33, Section IV.3.10] for instance.

(d) Assumption (ii) is similar to the definition in [107] of the fact that *"$\mathcal{A}$ is $\mathcal{B}$-power compact"*, but it is written at the semigroup level rather than at the resolvent level. Under such a power compact hypothesis, Voigt establishes a generalisation of Weyl's Theorem. His proof uses the analytic property of the resolvent function $\mathcal{R}_\Lambda(z)$ obtained by Ridav and Vidav in [100] as we present here. As a matter of fact, Theorem 3.1 is a simple consequence of [107, Theorem 1.1] together with Theorem 2.1. However, for the sake of completeness we give an elementary proof of Theorem 3.1 (which consists essentially to prove again [107, Theorem 1.1] without the help of [100] and to applied Theorem 2.1).

Proof of Theorem 3.1. The cornerstone of the proof is the use of the identity

$$\mathcal{R}_\Lambda(z)(I - \mathcal{V}(z)) = \mathcal{U}(z)$$

established in (2.17), where we recall that the functions $\mathcal{U}$ and $\mathcal{V}$ defined by (2.15) and (2.15) are analytic functions on Δ_{a^*} with values in $\mathscr{B}(X)$. Moreover, we have

$$\mathcal{V}(z) \in \mathscr{B}(X, Y) \subset \mathscr{K}(X) \quad \forall\, z \in \Delta_a \tag{3.2}$$

because of assumption **(ii)** and

$$\|\mathcal{V}(z)\|_{\mathscr{B}(X)} \le 1/2 \quad \forall\, z \in \Delta_a \cap B(0, M)^c \tag{3.3}$$

for some $M = M(a)$ thanks to (2.21).

Step 1. We prove that $\Sigma(\Lambda) \cap \Delta_a$ is finite for any $a > a^*$.

Let us fix $\xi \in \Delta_a$ and define $C(z) := I - \mathcal{V}(z)$, $C_0 := C(\xi)$. Because of (3.2) and thanks to the Fredholm alternative [38] (see also [16, Theorème VI.6] for a modern and comprehensible statement and proof), there holds

$$R(C_0) = N(C_0^*)^\perp, \quad \dim N(C_0) = \dim N(C_0^*) := N \in \mathbb{N}.$$

If $N \ge 1$, we introduce $(f_1, \dots, f_N)$ a basis of the null space $N(C_0)$ and $(\varphi_1, \dots, \varphi_N)$ a family of independent linear forms in X' such that $\varphi_i(f_j) = \delta_{ij}$. Similarly, we introduce $(\psi_1, \dots, \psi_N)$ a basis of the null space $N(C_0^*)$ and $(g_1, \dots, g_N)$ a family of independent linear vectors in X such that $\psi_i(g_j) = \delta_{ij}$. Then defining the projectors on X

$$\pi_0 := \sum_{i=1}^n f_i\, \varphi_i \quad \text{and} \quad \pi_1 := \sum_{i=1}^n g_i\, \psi_i,$$

and $X_1 := \pi_0(X)$, $X_0 := (I - \pi_0)(X)$, $Y_1 := \pi_1(X)$, $Y_0 := (I - \pi_1)(X)$, we have

$$X = X_0 \oplus X_1, \;\; X_1 = N(C_0) = \mathrm{vect}(f_1, \dots, f_N),$$

and

$$X = Y_0 \oplus Y_1, \;\; Y_0 = R(C_0), \;\; Y_1 = \mathrm{vect}(g_1, \dots, g_N).$$

On the one hand, $C_0 : X_0 \to Y_0$ is bijective by definition, and the family of linear mappings

$$D_0(z) := C_0^{-1}\, (I - \pi_1)\, C(z)_{|X_0} : X_0 \to X_0,$$

is an analytic function with respect to the parameter $z \in \Delta_a$ and satisfies $D_0(\xi) = I_{X_0}$. We deduce that $D_0(z)$ is also invertible for z belonging to a neighbourhood $B(\xi, r_0(\xi))$ of ξ, $r_0(\xi) > 0$, and then

$$(I - \pi_1)\, C(z)_{|X_0} : X_0 \to Y_0 \text{ is invertible for any } z \in B(\xi, r_0(\xi)). \tag{3.4}$$

On the other hand, the family of linear mappings

$$D_1(z) := \pi_1\, C(z)_{|Y_1} : Y_1 \to Y_1,$$

is an analytic function with respect to the parameter $z \in \Delta_a$ and $D_1(z)$ is invertible for any $z \in \Delta_a \cap B(0, M')^c$ for M' large enough, because $\mathcal{V}(z) \to 0$ as $\Im m z \to \infty$ from (2.21). Since Y_1 is finite dimensional we may define $z \mapsto \det(D_1(z))$, which is an analytic function on Δ_a and satisfies $\det(D_1(z)) \neq 0$ for any $z \in \Delta_a \cap B(0, M')^c$. As a consequence, $z \mapsto \det(D_1(z))$ has some isolate zeros and since $\det(D_1(z_0)) = 0$ there exists a neighbourhood $B(\xi, r_1(\xi))$ of ξ, $r_1(\xi) > 0$, so that $\det(D_1(z)) \neq 0$ for any $z \in B(z_0, r_1(\xi))\backslash\{\xi\}$ from which we deduce

$$\pi_1\, C(z)_{|Y_1} : Y_1 \to Y_1 \text{ is invertible for any } z \in B(\xi, r_1(\xi))\backslash\{\xi\}. \tag{3.5}$$

Gathering (3.4) and (3.5), we get that $C(z) : X \to Y_0 \oplus Y_1 = X$ is surjective for any $z \in B(\xi, r(\xi))\backslash\{\xi\}$, $r(\xi) := \min(r_0(\xi), r_1(\xi)) > 0$, and then bijective thanks to the Fredholm alternative.

By compactness of the set $\bar{\Delta}_a \cap B(0, M')$, we may cover that set by a finite number of balls $B(\xi_j, r(\xi_j))$, $1 \leq j \leq J$, so that $C(z) = I - \mathcal{V}(z)$ is invertible for any $z \in \Omega := \Delta_a\backslash\{\xi_1, ..., \xi_J\}$. As a consequence, the function

$$\mathcal{W}(z) := \mathcal{U}(z)\,(I - \mathcal{V}(z))^{-1}$$

is well defined and analytic on Ω and coincide with $\mathcal{R}_\Lambda(z)$ on the half plane Δ_b, where $b = \omega(\Lambda)$ is the growth bound of S_Λ. These facts immediately imply that $\mathcal{R}_\Lambda = \mathcal{W}$ on Ω and then

$$\Sigma(\Lambda) \cap \Delta_a \subset \{\xi_1, ..., \xi_J\} = \Delta_a\backslash\Omega.$$

Step 2. We prove that $\Sigma(\Lambda) \cap \Delta_{a^*} \subset \Sigma_d(\Lambda)$. Indeed, for any $\xi \in \Sigma(\Lambda) \cap \Delta_{a^*}$, we may write with the notations of step 1

$$\begin{aligned} \Pi_{\Lambda,\xi}(I - \pi_1) &= \int_{|z-\xi|=r(\xi)/2} \mathcal{R}_\Lambda(z)(I - \pi_1) \\ &= \int_{|z-\xi|=r(\xi)/2} \mathcal{U}(z)\,(I - \mathcal{V}(z))^{-1}(I - \pi_1) = 0, \end{aligned}$$

where in the last line we have used that $I - \mathcal{V}(z) : X_0 \to Y_0$ is bijective for any $z \in B(\xi, r(\xi))$ so that $(I - \mathcal{V}(z))^{-1}(I - \pi_1)$ is well defined and analytic on $B(\xi, r(\xi))$. We deduce that

$$\Pi_{\Lambda,\xi} = \Pi_{\Lambda,\xi}\pi_1,$$

and then

$$\dim R(\Pi_{\Lambda,\xi}) \leq \dim R(\pi_1) = N.$$

That precisely means $\xi \in \Sigma_d(\Lambda)$.

Step 3. The fact that S_Λ satisfies the growth estimate (3.1) is an immediate consequence of Theorem 2.1. □

3.2. **A quantified version of the Weyl's Theorem.** We present now a quantified version of Weyl's Theorem 3.1.

Theorem 3.3. *Consider X a Banach space, Λ the generator of a semigroup $S_\Lambda(t) = e^{t\Lambda}$ on X, a real number $a^* \in \mathbb{R}$ and assume that there exist two operators $\mathcal{A}, \mathcal{B} \in \mathscr{C}(X)$ such that $\Lambda = \mathcal{A} + \mathcal{B}$ and the hypothesises* **(H1)**, **(H2)** *and* **(H3)** *of Theorem 3.1 are met. Assume furthermore that*

$$\int_0^\infty \|e^{-at}\,\mathcal{A} S_{\mathcal{B}}(t)\|_{\mathscr{B}(X,Y)}\,dt \le C_a, \tag{3.6}$$

for some constant $C_a \in \mathbb{R}_+$, and there exists a sequence of N dimensional range increasing projector π_N and a sequence of positive real numbers $\varepsilon_N \to 0$ such that

$$\forall\, f \in Y \quad \|\pi_N^\perp f\|_X \le \varepsilon_N \|f\|_Y. \tag{3.7}$$

For any $a > a^*$, there exists a integer n^* (which depends on a constructive way on a, (π_N), (ε_N) and the constants involved in the assumptions **(H1)**, **(H2)**, **(H3)** *and* (3.6)*) such that*

$$\sharp(\Sigma(\Lambda) \cap \Delta_a) \le n^*, \quad dim\, R(\Pi_{\Lambda,a}) \le n^*. \tag{3.8}$$

We assume moreover **(H1″)** *and $\|(S_{\mathcal{B}}\mathcal{A})^{(*n)}\|_{\mathscr{B}(X,D(\Lambda^\varsigma))} e^{-at} \in L^1(0,\infty)$ for any $a > a^*$. Then, for any $a > a^*$, there exists a constant C_a' such that for any Jordan basis $(g_{i,j})$ associated to the eigenspace $R\Pi_{\Lambda,a}$ there holds*

$$\|g_{i,j}\|_X = 1, \quad \|g_{i,j}\|_{D(\Lambda^\varsigma)\cap Y} \le C_a'. \tag{3.9}$$

Proof of Theorem 3.3. Step 1. Let us fix $a > a^*$ and let us define for any $z \in \Delta_a$ the compact perturbation of the identity

$$\Phi(z) := I + \mathcal{A}\, R_{\mathcal{B}}(z) : X \to X.$$

On the one hand, because of (3.6), we know that there exists a constant C_a such that

$$\forall\, z \in \Delta_a \qquad \|\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z)\|_{\mathscr{B}(X,Y)} \le C_a,$$

and then

$$\|\pi_N^\perp \mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z)\|_{\mathscr{B}(X)} \le \varepsilon_N\, C_a < 1$$

for N large enough. We deduce from the above smallness condition that

$$I + \pi_N^\perp \mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z) : R\pi_N^\perp \to R\pi_N^\perp$$

is a an isomorphism for any $z \in \Delta_a$. On the other hand, thanks to the Fredholm alternative, it is clear that $\Phi(z)$ is invertible if, and only if, $\pi_N \Phi(z)$ has maximal rank N.

Step 2. For a given basis $(g_1, ..., g_N)$ of $R\pi_N$ we denote by $\pi_{N,i}$ a projection on $\mathbb{C}\, g_i$, $1 \le i \le N$, and we define $\Phi_{N,i}(z) := \pi_{N,i}\Phi(z)$. For a given $i \in \{1, ..., N\}$ and if $z_i \in \Delta_a$ satisfies $\Phi_{N,i}(z_i) = 0$ we have

$$\pi_{N,i} + \pi_{N,i}\,\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z_i) = 0$$

and

$$\Phi_{N,i}(z) = \pi_{N,i} + \pi_{N,i}\,\mathcal{A} \sum_{n=0}^\infty \mathcal{R}_{\mathcal{B}}(z_i)^{n+1}\,(z - z_i)^n \quad \forall\, z \in \Delta_a \cap B(z_i, r),$$

with $r := \|\mathcal{R}_{\mathcal{B}}(z_i)\|^{-1} \geq C_a^{-1} > 0$. From the two last equations, we deduce

$$\Phi_{N,i}(z) = -\pi_{N,i} \sum_{n=1}^{\infty} \mathcal{R}_{\mathcal{B}}(z_i)^n \, (z - z_i)^n \quad \forall\, z \in \Delta_a \cap B(z_i, r_i)$$

and we observe that $\operatorname{rank} \Phi_{N,i}(z) = 1$ for any $z \in B(z_i, r)$, $z \neq z_i$. As a consequence, in any ball B of radius r, $\operatorname{rank} \pi_N \Phi(z) = N$ for any $z \in B \cap \Delta_a$ except at most N points z_1, ..., $z_N \in B$, and the total dimension of the "defect of surjectivity" $\operatorname{codim} R(\pi_N \Phi(z_i)) = \dim \Pi_{\Lambda, z_i}$ is at most N. Covering the region $\bar{\Delta}_a \cap B(0, M)$ by $(1 + 2M/r)^2$ balls of radius $r \in (C_a^{-1}, M)$, we see that (3.8) holds with $n^* := (1 + 2MC_a)^2 N$.

Step 3. Consider a Jordan basis $(g_{j,m})$ associated to an eigenvalue $\xi \in \Sigma_d(\Lambda) \cap \Delta_a$, and then defined by

$$\Lambda g_{j,m} = \xi g_{j,m} + g_{j,m-1}.$$

We write

$$g_{j,m} = \mathcal{R}_{\mathcal{B}}(\xi) g_{j,m-1} - \mathcal{R}_{\mathcal{B}}(\xi) \mathcal{A} g_{j,m},$$

and iterating the formula

$$g_{j,m} = \sum_{\ell=0}^{L-1} (-1)^\ell \mathcal{R}_{\mathcal{B}}(\xi) (\mathcal{A}\, \mathcal{R}_{\mathcal{B}}(\xi))^\ell g_{j,m-1} + (-1)^L (\mathcal{R}_{\mathcal{B}}(\xi) \mathcal{A})^n g_{j,m},$$

from which we easily conclude that (3.9) holds. □

3.3. **The equal mitosis and smooth cell-division equations.** In this paragraph we are concerned with the equal mitosis equation (1.23) and the cell division equation (1.33) with smooth offsprings distribution, so that we consider the operator

$$\Lambda f(x) := -\frac{\partial}{\partial x} f(x) - K(x) f(x) + (\mathcal{F}^+ f)(x)$$

where K satisfies (1.25) and (1.26) and the gain part $\mathcal{F}^+$ of the fragmentation operator is defined by

$$\mathcal{F}^+ f(x) := \int_x^\infty k(y, x)\, f(y)\, dy \tag{3.10}$$

with k satisfying (1.17) and $\wp = 2\delta_{1/2}$ (in the equal mitosis case) or $\wp$ is a function which satisfies (1.18) and (1.30) (in the case of the smooth cell-division equation). We recall that the operator is complemented with a boundary condition (1.24).

We fix $\alpha > \alpha^*$, with $\alpha^* > 1$ defined thanks to equation (1.34), and we define $K_0' := K_0 - \wp_\alpha K_1 > 0$ as well as the critical abscissa $a^* \in \mathbb{R}$ when $K(0) = 0$ or $\wp = 2\delta_{1/2}$ by

$$a^* := -K_0' \text{ if } \gamma = 0, \qquad a^* := -\infty \text{ if } \gamma > 0.$$

When $K(0) > 0$, the positivity conditions (1.25) and (1.26) imply that there exists a constant $K_* > 0$ such that

$$K(x) \geq K_* \qquad \forall\, x \geq 0. \tag{3.11}$$

When $K(0) > 0$ and $\wp$ satisfies (1.30), we then define the critical abscissa by

$$a^* := -\min(K_0', K_*) \text{ if } \gamma = 0, \qquad a^* := -K_* \text{ if } \gamma > 0.$$

We perform the spectral analysis of Λ in the Banach space $X = L^1_\alpha$. It is worth emphasizing that we classically have (and that is also a straightforward consequence of the following Lemmas)

$$D(\Lambda) = \{f \in L^1_{\alpha+\gamma},\ \partial_x f \in L^1_\alpha,\ f(0) = 0\}.$$

As a first step in the proof of Theorem 1.1 we have

Proposition 3.4. *Under the above assumptions and notations, the conclusions of Theorem 3.1 hold for the cell-division semigroup in L^1_α for any $\alpha > \alpha^*$ and for any $a > a^*$. Moreover the conclusions of Theorem 3.3 hold under the additional assumptions that $K(0) = 0$ and $\wp$ satisfies the smoothness condition* (1.30).

In order to establish Proposition 3.4 we will introduce an adequate splitting $\Lambda = \mathcal{A} + \mathcal{B}$ and we prove that $\mathcal{A}$ and $\mathcal{B}$ satisfy conditions **(H1)**, **(H2)** and **(H3)** (or one of the "prime" variants of them).

Taking one real number $R \in [1, \infty)$ to be chosen later, we define

$$K_R := K\,\chi_R \quad K^c_R := K\,\chi^c_R,$$

where $\chi_R(x) = \chi(x/R)$, $\chi^c_R(x) = \chi^c(x/R)$, χ being the Lipschitz function defined on $\mathbb{R}_+$ by $\chi(0) = 1$, $\chi' = -\mathbf{1}_{[1,2]}$ and $\chi^c = 1 - \chi$, as well as

$$\mathcal{F}^+_R = \int_x^\infty k_R(y,x)\,f(y)\,dy, \quad \mathcal{F}^{+,c}_R = \int_x^\infty k^c_R(y,x)\,f(y)\,dy,$$

and then

$$\mathcal{A} = \mathcal{A}_R = \mathcal{F}^+_R, \quad \mathcal{B} = \mathcal{B}_R = -\frac{\partial}{\partial x} - K(x) + \mathcal{F}^{+,c}_R$$

so that $\Lambda = \mathcal{A} + \mathcal{B}$.

Lemma 3.5. *(1) For any $0 \le \alpha \le 1 \le \beta$ there holds $\mathcal{A} \in \mathscr{B}(\dot L^1_\alpha, \dot L^1_\alpha \cap \dot L^1_\beta)$.*

(2) For the mitosis operator, there holds $\mathcal{A} \in \mathscr{B}(W^{1,1}, W^{1,1}_\alpha)$ for any $\alpha \ge 0$. In particular $\mathcal{A} \in \mathscr{B}(D(\Lambda^\eta))$ for $\eta = 0, 1$;

(3) Under assumption (1.30) *on $\wp$ there holds $\mathcal{A} \in \mathscr{B}(\dot L^1_{\beta-1}, \dot W^{1,1}_\beta)$ for any $\beta \ge 0$, where*

$$\dot W^{1,1}_\beta := \{f \in L^1_{loc}(0,\infty);\ \partial_x f \in \dot L^1_\beta\}.$$

(4) Under assumption (1.30) *on $\wp$ and the additional assumption $K(0) = 0$, there holds $\mathcal{A} \in \mathscr{B}(L^1_\alpha, W^{1,1}_\alpha)$ for any $\alpha \ge 0$.*

Proof of Lemma 3.5. We split the proof into four steps.
Step 1. Fix $0 \le f \in \dot L^1_\alpha$ as well as $\alpha' \ge \alpha$. We compute

$$\begin{aligned}
\|\mathcal{A}f\|_{\dot L^1_{\alpha'}} &= \int_0^\infty f(x) \int_0^x k_R(x,y)\,y^{\alpha'}\,dy\,dx \\
&= \wp_{\alpha'} \int_0^\infty f(x)\,K_R(x)\,x^{\alpha'}\,dx \\
&\le \wp_{\alpha'}\,K_1\,\langle R\rangle^{\gamma+\alpha'-\alpha} \int_0^{2R} f(x)\,x^\alpha\,dx,
\end{aligned}$$

so that $\mathcal{A} \in \mathscr{B}(\dot L^1_\alpha, \dot L^1_{\alpha'})$.
Step 2. For the mitosis operator, we have

$$\partial_x \mathcal{A}f = 8\,(\partial_x K_R)(2x)\,f(2x) + 8\,K_R(2x)\,(\partial_x f)(2x),$$

and similar estimates as above leads to the bound

$$\|\partial_x \mathcal{A}_R\, f\|_{L^1_\alpha} \le 4\,\langle R\rangle^\alpha\, \|K\|_{W^{1,\infty}(0,2R)}\, \|f\|_{W^{1,1}}.$$

for any $\alpha \ge 0$ and $f \in W^{1,1}$, from which (2) follows.
Step 3. Under the regularity assumption (1.30), there holds

$$\partial_x \mathcal{A} f = \int_x^\infty K_R(y)\, \wp'(x/y)\, y^{-2}\, f(y)\, dy - K_R(x)\, \wp(1)\, f(x)/x,$$

so that

$$\|\partial_x \mathcal{A} f\|_{\dot L^1_\beta} \le \big\{\wp(1) + \wp'_\beta\big\} K_1\, \langle R\rangle^\gamma \int_0^{2R} f(x)\, x^{\beta-1}\, dx,$$

and we conclude with $\mathcal{A} \in \mathscr{B}(\dot L^1_{\beta-1}, \dot W^{1,1}_\beta)$. □
Step 4. With the assumption of point (4) and recalling that K is C^1, there holds $\|K(x)/x\|_{L^\infty(0,2R)}$ for any $R \in (0,\infty)$ and then from the above expression of $\partial_x \mathcal{A} f$, we get

$$\|\partial_x \mathcal{A} f\|_{L^1_\alpha} \quad \le \quad \big\{\wp(1) + \wp'_0\big\}\, \|K(x)/x\|_{L^\infty(0,2R)} \|f\|_{L^1_\alpha}$$

for any $f \in L^1_\alpha$ and $\alpha \ge 0$. □

Lemma 3.6. *For any $a > a^*$ there exists $R^*(a) > 0$ such that the operator $\mathcal{B}_R$ is a-hypodissipative in L^1_α for any $R \in (R^*(a), \infty)$.*

Proof of Lemma 3.6. We introduce the primitive functions

$$\mathcal{K}(z) := \int_0^z K(u)\, du, \quad \mathcal{K}(z_1, z_2) := \mathcal{K}(z_2) - \mathcal{K}(z_1) \tag{3.12}$$

and, for any given $a \in (a^*, 0]$, we define the space

$$\mathcal{E} := L^1(\phi), \quad \phi(x) := \phi_0(x)\, \mathbf{1}_{x \le x_2} + \phi_\infty(x)\, \mathbf{1}_{x \ge x_2},$$

where

$$\phi_0(x) := \frac{e^{\mathcal{K}(x) + a\, x}}{e^{\mathcal{K}(x_2) + a\, x_2}}, \quad \phi_\infty(x) := \frac{x^\alpha}{x_2^\alpha},$$

and where

$$x_2 \;:=\; \max(1, x_1, 2\alpha/(a + K_0')) \quad \text{if } \gamma = 0; \tag{3.13}$$

$$x_2 \;:=\; \max(1, x_1, [3\alpha/(x_1 K_0')]^{1/\gamma}, [-3a/K_0']^{1/\gamma}) \quad \text{if } \gamma > 0. \tag{3.14}$$

Consider $f \in D(\Lambda)$ a real-valued function and let us show that for any $a > a^*$ and any $R > R^*(a)$, $R^*(a)$ to be chosen later,

$$\int_0^\infty \mathrm{sign}(f(x))\, \mathcal{B} f(x)\, \phi(x)\, dx \le a \|f\|_{\mathcal{E}}. \tag{3.15}$$

Since the case of complex-valued functions can be handled in a similar way and since the norm $\|\cdot\|_{\mathcal{E}}$ is clearly equivalent to $\|\cdot\|_{L^1_\alpha}$, that will end the proof.

One the one hand, we have

$$\begin{aligned} \|\mathcal{F}^{+,c} f\|_{L^1(\phi_0\, \mathbf{1}_{x \le x_2})} \quad &= \quad \int_R^\infty K_R^c(x)\, f(x) \int_0^{x \wedge x_2} \wp(y/x)\, \phi_0(y)\, \frac{dy}{x}\, dx \\ &\le \quad \eta(x_2/R) \int_R^\infty K_R^c(x)\, f(x)\, dx, \end{aligned}$$

with

$$\eta(u) := \big(\sup_{[0,x_2]} \phi_0\big)\Big(\int_0^u \wp(z)\,dz\Big),$$

so that, performing one integration by part, we calculate

$$\begin{aligned}
&\int_0^{x_2} \mathrm{sign}(f(x))\,\mathcal{B}f(x)\,\phi_0(x)\,dx = \\
&\quad = \int_0^{x_2} \{-K(x)|f(x)| - \partial_x|f(x)|\}\,\phi_0(x)\,dx + \int_0^{x_2} \mathrm{sign}(f)\,(\mathcal{F}^{+,c}f)\,\phi_0\,dx \\
&\quad \le -|f(x_2)| + a\int_0^{x_2} |f(x)|\,\phi_0(x)\,dx + \eta(x_2/R)\int_R^\infty K_R^c(x)\,f(x)\,dx. \qquad (3.16)
\end{aligned}$$

On the other hand, performing one integration by part again, we compute

$$\begin{aligned}
&\int_{x_2}^\infty \mathrm{sign}(f(x))\,\mathcal{B}_R f(x)\,\phi_\infty(x)\,dx = \\
&\le |f(x_2)| + \int_{x_2}^\infty |f(x)|\,\{-K\phi_\infty + \partial_x\phi_\infty + K_R^c(x)\int_0^x \wp(\frac{y}{x})\,\phi_\infty(y)\,\frac{dy}{x}\}\,dx \\
&\le |f(x_2)| + \int_{x_2}^\infty |f(x)|\,[(\wp_\alpha K_1 - K_0)\,x^\gamma + \alpha/x]\,\phi_\infty(x)\,dx \\
&\le |f(x_2)| + \int_{x_2}^\infty |f(x)|\,\Big\{a + \theta\,x^\gamma\Big\}\phi_\infty(x)\,dx, \qquad (3.17)
\end{aligned}$$

with $\theta := -(a + K_0')/2$ if $\gamma = 0$ and $\theta := -K_0'/3$ if $\gamma > 0$ thanks to the choice of x_2. Gathering (3.16) and (3.17), and taking R^* large enough in such a way that $\eta(x_2/R^*)\,K_1 + \theta \le 0$, we get that $\mathcal{B}_R$ is a-dissipative in $\mathcal{E}$. □

Lemma 3.7. *The operator Λ generates a C_0-semigroup on L^1_α .*

Proof of Lemma 3.7. Thanks to Lemmas 3.5 and 3.6, we have that Λ is b-dissipative in $\mathcal{E}$ with $b := \|\mathcal{A}_R\|_{\mathscr{B}(\mathcal{E})} + a$. On the other hand, one can show that $R(\Lambda - b) = X$ for b large enough and conclude thanks to Lumer-Phillips Theorem ([73] or [91, Theorem I.4.3]) that Λ generates a C_0-semigroup. Equivalently, one can argue as in [34, Proof of Theorem 3.2] by introducing an approximation sequence of bounded total fragmentation rates (K^n) and proving that for any fixed initial datum $f_0 \in X$ the associated sequence of solutions (f^n) (constructed by a mere Banach fixed point Theorem in $C([0,T];\mathcal{E})$, $\forall\, T > 0$) is a Cauchy sequence. That establishes that for any $f_0 \in \mathcal{E}$ there exists a unique solution $f \in C(\mathbb{R}_+;\mathcal{E})$ to the Cauchy problem associated to the operator Λ and then that Λ generates a C_0-semigroup. □

We define

$$Y_r := \{f \in W^{r,1}_\alpha \cap L^1_{\alpha+\gamma+1}(\mathbb{R});\ \mathrm{supp} f \subset [0,\infty)\}, \quad r \in [0,1],$$

as a family of interpolating spaces between $Y_0 = L^1_{\alpha+\gamma+1} \supset X$ and $Y_1 \supset D(\Lambda)$.

Lemma 3.8. *If $\wp$ satisfies* (1.30) *and $K(0) = 0$, for any $a > a^*$ there exists a constant C_a such that*

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{\mathscr{B}(X,Y_1)} \le C_a\,e^{at}. \qquad (3.18)$$

If $\wp$ satisfies (1.30) *and $K(0) \neq 0$, for any $a > a^*$ and any $r \in [0,1)$ there exists a constant $C_{a,r}$ such that*

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*2)}(t)\|_{\mathscr{B}(X,Y_r)} \le C_{a,r}\,e^{at}. \qquad (3.19)$$

Proof of Lemma 3.8. Extending by 0 a function $g \in L^1(\mathbb{R}_+)$ outside of $[0,\infty)$, we may identify $L^1(\mathbb{R}_+) = \{g \in L^1(\mathbb{R});\ \operatorname{supp} g \subset [0,\infty)\}$ and extending $k = k(x,y)$ by 0 outside of $\{(x,y);\ 0 \le y \le x\}$, we may consider $S_{\mathcal{B}}(t)$ and $\mathcal{A}$ as operators acting on $L^1(\mathbb{R})$ which preserve the support $[0,\infty)$ (they then also act on $L^1(\mathbb{R}_+)$).

Step 1. Assume first $K(0) = 0$. From Lemma 3.5-(4) and Lemma 3.6, for any $f \in L^1_\alpha$, we get

$$\|\partial_x(\mathcal{A}S_{\mathcal{B}}(t)f)\|_{L^1_\alpha} \le C\,\|S_{\mathcal{B}}(t)f\|_{L^1_\alpha} \le C_a\, e^{at}\|f\|_{L^1_\alpha}.$$

From Lemma 3.5-(1) and Lemma 3.6 we get a similar estimate on $\|\mathcal{A}S_{\mathcal{B}}(t)\|_{\mathscr{B}(L^1_\alpha, L^1_{\alpha+\gamma+1})}$ and that ends the proof of (3.18).

Step 2. We assume now $K(0) \neq 0$. We introduce the notations

$$\mathcal{B}_0 f(x) := -\frac{\partial}{\partial x} f(x) - K(x) f(x)$$

and then the shorthands $\mathcal{A}^c := \mathcal{F}^{+,c}$, $U := \mathcal{A}S_{\mathcal{B}}$, $U^c := \mathcal{A}^c S_{\mathcal{B}}$, $U_0 := \mathcal{A}S_{\mathcal{B}_0}$ and $U_0^c := \mathcal{A}^c S_{\mathcal{B}_0}$, where $S_{\mathcal{B}}$ (resp. $S_{\mathcal{B}_0}$) is the semigroup associated to the generator $\mathcal{B}$ (resp. $\mathcal{B}_0$) complemented with the boundary condition (1.24). Thanks to the Duhamel formula

$$S_{\mathcal{B}} = S_{\mathcal{B}_0} + S_{\mathcal{B}_0} * \mathcal{A}^c S_{\mathcal{B}}, \tag{3.20}$$

we have

$$U = U_0 + U_0 * U^c. \tag{3.21}$$

From the explicit representation formula

$$(S_{\mathcal{B}_0}(t)f)(x) = e^{-\mathcal{K}(x-t,x)} f(x-t) \quad \forall\, t \ge 0,\ \forall\, x \in \mathbb{R}, \tag{3.22}$$

where $\mathcal{K}$ is the primitive function defining in (3.12), we get

$$U_0(t)f(x) = \int_x^\infty k_R(y,x)\, e^{-\mathcal{K}(y-t,y)} f(y-t)\, dy,$$

so that

$$\begin{aligned}\partial_x[U_0(t)f(x)] \quad &= \quad \int_x^\infty K_R(y)\, \wp'(x/y)\, y^{-2}\, e^{-\mathcal{K}(y-t,y)} f(y-t)\, dy \\ &\quad -\frac{K_R(x)}{x}\, \wp(1)\, e^{-\mathcal{K}(x-t,x)} f(x-t).\end{aligned}$$

Using that $\mathcal{K}(y-t,y) \ge K_* t$ for any $y \ge t$, we have

$$e^{-\mathcal{K}(y-t,y)}|f(y-t)| \le e^{at}\,|f(y-t)| \quad \forall\, t \ge 0,\ \forall\, y \in \mathbb{R},$$

and we deduce

$$\begin{aligned}\|\partial_x[U_0(t)f]\|_{L^1_\alpha} \quad &\le \quad \wp(1)\int_0^{2R} K_R(x)\,\langle x\rangle^\alpha \frac{|f(x-t)|}{(x-t)+t}\, dx\, e^{at} \\ &\quad + \int_0^{2R} K_R(x) \int_0^x |\wp'(y/x)|\,\langle y\rangle^\alpha \frac{dy}{x}\frac{|f(x-t)|}{(x-t)+t}\, dx\, e^{at} \\ &\le \quad \Big(\wp(1) + \wp_0'\Big) K_1\,\langle 2R\rangle^{\gamma+\alpha}\frac{e^{at}}{t}\int_0^{2R}|f(x)|dx.\end{aligned}$$

Since clearly (from the explicit representation formula (3.22) for instance) the same estimate holds for $\|U_0(t)\|_{\mathscr{B}(L^1,L^1_{\alpha+\gamma+1})}$ we get by interpolation and for any $r\in[0,1]$

$$\|U_0(t)\|_{\mathscr{B}(L^1,Y_r)}\le C_a\,\frac{e^{at}}{t^r}.$$

Thanks to the identity (3.21), the fact that $\mathcal{A}^c\in\mathscr{B}(L^1_\gamma,L^1)$ and Lemma 3.6, we deduce for $r\in[0,1)$

$$\begin{aligned}\|U(t)f\|_{Y_r} &\le C_a\,\frac{e^{at}}{t^r}\,\|f\|_{L^1}+\int_0^t C_a\,\frac{e^{a(t-s)}}{(t-s)^r}\,\|\mathcal{A}^cS_{\mathcal{B}}(s)f\|_{L^1}\\ &\le C_{a,r}\,\frac{e^{at}}{t^r}\,\|f\|_{L^1_{\max(\alpha,\gamma)}}.\end{aligned}$$

On the other hand, from Lemma 3.5-(1) and Lemma 3.6, we know that

$$\|U(t)f\|_{L^1_{\alpha+\gamma}}\le\|\mathcal{A}\|_{\mathscr{B}(L^1_\alpha,L^1_{\alpha+\gamma})}\,\|S_{\mathcal{B}}(t)f\|_{L^1_\alpha}\le C_ae^{at}\|f\|_{L^1_\alpha}.$$

These two estimates together imply (3.19). □

Lemma 3.9. *For the equal mitosis equation, there holds*

$$\|(\mathcal{A}\,S_{\mathcal{B}})^{(*2)}(t)\|_{\mathscr{B}(\mathcal{E},Y_1)}\le C\,t\quad\forall\,t\ge0 \tag{3.23}$$

for some constant C which only depends on K through its norm $\|K\|_{W^{1,\infty}(0,R)}$ where R is defined in Lemma 3.6.

Proof of Lemma 3.9. Thanks to the Duhamel formula (3.20) and the iterated Duhamel formula

$$S_{\mathcal{B}}=S_{\mathcal{B}_0}+S_{\mathcal{B}_0}*\mathcal{A}^cS_{\mathcal{B}_0}+S_{\mathcal{B}_0}*\mathcal{A}^cS_{\mathcal{B}_0}*\mathcal{A}^cS_{\mathcal{B}},$$

we have

$$U=U_0+U_0*U^c\quad\text{and}\quad U=U_0+U_0*U_0^c+U_0*U_0^c*U^c,$$

from which we finally deduce

$$U^{*2}=U_0^{*2}+U_0^{*2}*U^c+U_0*U_0^c*U+U_0*U_0^c*U^c*U. \tag{3.24}$$

From the explicit representation formula (3.22), we get

$$(U_0(t)f)(x)\;=\;4\,K_R(2x)\,e^{-\mathcal{K}(2x-t,2x)}f(2x-t)$$

as well as

$$(U_0^c(t)f)(x)\;=\;4\,K_R^c(2x)\,e^{-\mathcal{K}(2x-t,2x)}f(2x-t).$$

We then easily compute

$$\begin{aligned}(U_0^{*2}(t)f)(x)&=\int_0^t(U_0(t-s)U_0(s)f)(x)\,ds\\ &=4\,K_R(2x)\int_0^t e^{-\mathcal{K}(2x-t+s,2x)}(U_0(s)f)(2x-t+s)\,ds\\ &=16\,K_R(2x)\int_0^t K_R(2(2x-t+s))\\ &\qquad e^{-\mathcal{K}(2x-t+s,2x)-\mathcal{K}(4x-2t+s,2(2x-t+s))}f(4x-2t+s)\,ds\\ &=16\,K_R(2x)\int_{u_0}^{u_1}K_R(2u-4x+2t)\,e^{-\Theta(u)}f(u)\,du\end{aligned} \tag{3.25}$$

with $u_0 := (4x-2t)\wedge(2x-t+R/2)$, $u_1 := (4x-t)\wedge(2x-t+R/2)$ and $\Theta(u) := \mathcal{K}(u-2x+t,2x)+\mathcal{K}(u,2u-4x+2t)$. Similarly, we have

$$(U_0 * U_0^c(t)f)(x) = 16\,K_R(2x)\int_{u_0}^{u_1} K_R^c(2u-4x+2t)\,e^{-\Theta(u)}f(u)\,du. \tag{3.26}$$

The two last terms are clearly more regular (in x) than the initial function f. Moreover, extending by 0 the function f outside of $[0,\infty)$, we see thanks to (3.22) that $S_{\mathcal{B}_0}(t)f$, and then $U_0^{*2}(t)f$ and $U_0 * U_0^c(t)f$, are well defined as functions in $L^1(\mathbb{R})$. Using the lower bound $\Theta \geq 0$ and performing some elementary computations, we easily get from (3.24), (3.25) and (3.26) the estimate

$$\|\partial_x[(\mathcal{A}S_B)^{(*2)}(t)f]\|_{L^1(\mathbb{R})} \leq C_R\,t\,\|f\|_{L^1}, \quad \forall\, t\geq 0,$$

which ends the proof of (3.23) since then $(\mathcal{A}S_B)^{(*2)}(t)f \in C(\mathbb{R})$ and $\mathrm{supp}\,(\mathcal{A}S_B)^{(*2)}(t)f \subset [0,\infty]$ imply $(\mathcal{A}S_B)^{(*2)}(t)f(0)=0$. □

With all the estimates established in the above lemmas, we are able to present the

Proof of Proposition 3.4. We just have to explain why the hypothesis of Theorem 3.1 are satisfied in each cases. Hypothesis **(H1)** is an immediate consequence of Lemma 3.5-(1) and Lemma 3.6 together with Remark 2.2-(b). For that last claim we use that $Y_r \subset D(\Lambda^{r'})$ for any $0\leq r'<r<1$ thanks to the classical interpolation theory, see (2.6), [66] and [71].

In the case when $\wp$ satisfies (1.30), Lemma 3.8 and Remark 2.2-(h) imply that hypothesises **(H2)** and **(H3)** are met with $n=1$, $\zeta=1$ and $\zeta'=0$ in the case $K(0)=0$ and are met with $n=2$, $\zeta\in(0,1)$ and $\zeta'=0$ in the case $K(0)\neq 0$. Also notice that the additional assumptions of Theorem 3.3 are met in the case that $K(0)=0$.

For the equal mitosis equation, Lemma 3.5-(2) and Lemma 3.6 imply that assumption **(H1)** also holds with $\mathscr{B}(X)$ replaced by $\mathscr{B}(D(\Lambda))$ and $\mathscr{B}(X,L^1_{\alpha+1})$, while Lemma 3.9 implies that **(H2″)** holds with $n=2$, $\zeta=1$, $\zeta'=0$ and any $b>0$, so that we can apply Corollary 2.5 (assumption **(H2)** holds with n large enough thanks to Lemma 2.6). Finally, hypothesis **(H3)** follows from **(H2)** and the fact that $D(\Lambda)\cap L^1_{\alpha+1}\subset X$ compactly. □

3.4. **Quantified Weyl's Theorem for the self-similar fragmentation equation.** In this paragraph we are concerned with the self-similar fragmentation equation (1.37), so that we consider the operator

$$\Lambda f(x) := -x\,\frac{\partial}{\partial x}f(x) - 2\,f(x) - K(x)f(x) + (\mathcal{F}^+f)(x)$$

where $K(x)=x^\gamma$ and $\mathcal{F}^+$ is defined in (3.10) with k satisfying (1.17), (1.18) and (1.30).

We perform the spectral analysis of Λ in the Banach space $X := \dot{L}^1_\alpha \cap \dot{L}^1_\beta$ for $0\leq\alpha<1<\beta<\infty$ and we set $a^* := \alpha-1\in[-1,0)$. It is worth emphasizing that

$$D(\Lambda) = \{f\in \dot{L}^1_\alpha\cap\dot{L}^1_{\beta+\gamma},\ \partial_x f\in \dot{L}^1_{\alpha+1}\cap\dot{L}^1_{\beta+1}\}.$$

As a first step in the proof of Theorem 1.1 we have

Proposition 3.10. *Under the above assumptions and notations, the conclusions of Theorem 3.3 hold for the self-similar fragmentation semigroup in $\dot L^1_\alpha \cap \dot L^1_\beta$ for $0 \le \alpha < 1 < \beta < \infty$ and for any $a > a^*$.*

In order to establish Proposition 3.10 we will introduce an adequate splitting $\Lambda = \mathcal{A} + \mathcal{B}$ and we prove that $\mathcal{A}$ and $\mathcal{B}$ satisfy conditions **(H1)**, **(H2)**, **(H3)** with $n = 1$ as well as (3.6) with $Y \subset D(\Lambda)$.

We introduce the following splitting inspired from [17]. With the notations of the preceding section, we define

$$k^s(x,y) := k(x,y)\,\chi^c_\delta(x)\,\chi^c_\varepsilon(y)\,\chi_R(y)$$

for $0 < \varepsilon \le \delta/2 \le 1$, $R \ge 2$ to be specified latter, and then

$$k^c := k^{c,1} + k^{c,2} + k^{c,3}$$

with $k^{c,1}(x,y) = k(x,y)\,\chi_\delta(x)$, $k^{c,2}(x,y) = \chi^c_\delta(x)\,\chi^c_R(y)$ and $k^{c,3}(x,y) = k(x,y)$ $\chi^c_\delta(x)\,\chi_\varepsilon(y)$. We then define

$$\mathcal{A}f(x) = \int_x^\infty k^s(y,x)\,f(y)\,dy, \quad \mathcal{A}^{c,i}f(x) = \int_x^\infty k^{c,i}(y,x)\,f(y)\,dy,$$

and then

$$\mathcal{A}^c = \mathcal{A}^{c,1} + \mathcal{A}^{c,2} + \mathcal{A}^{c,3}, \quad \mathcal{B}_0 = -x\,\frac{\partial}{\partial x} - 2 - K(x), \quad \mathcal{B} = \mathcal{B}_0 + \mathcal{A}^c$$

so that $\Lambda = \mathcal{A} + \mathcal{B}$.

Lemma 3.11. *For any $0 \le \alpha' \le 1 \le \beta'$, there holds $\mathcal{A} \in \mathscr{B}(\dot L^1_1, \dot W^{1,1}_{\alpha'} \cap \dot W^{1,1}_{\beta'})$ and $\mathcal{A}^{c,i} \in \mathscr{B}(\dot L^1_{\alpha'+\gamma}, \dot L^1_{\alpha'})$ for $i = 1,2,3$, with*

$$\begin{aligned}
\|\mathcal{A}^{c,1}f\|_{\dot L^1_{\alpha'}} &\le \wp_{\alpha'}\,\delta^\gamma\,\|f\|_{\dot L^1_{\alpha'+\gamma}},\\
\|\mathcal{A}^{c,2}f\|_{\dot L^1_{\alpha'}} &\le \wp_{\alpha'}\int_R^\infty f(x)\,x^{\alpha'+\gamma}\,dx,\\
\|\mathcal{A}^{c,3}f\|_{\dot L^1_{\alpha'}} &\le \|\wp(z)\,z^{\alpha'}\|_{L^1(0,\varepsilon/\delta)}\,\|f\|_{\dot L^1_{\alpha'+\gamma}},
\end{aligned}$$

Proof of Lemma 3.11. For $0 \le f \in \dot L^1_{\alpha'+\gamma}$, we compute

$$\begin{aligned}
\|\mathcal{A}^{c,3}f\|_{\dot L^1_{\alpha'}} &= \int_0^\infty f(x)\,x^\gamma\,\chi^c_\delta(x)\int_0^x \wp(y/x)\,y^{\alpha'}\chi_\varepsilon(y)\,\frac{dy}{x}\,dx\\
&\le \|\wp(z)\,z^{\alpha'}\|_{L^1(0,2\varepsilon/\delta)}\|f\|_{\dot L^1_{\alpha'+\gamma}},
\end{aligned}$$

and that establishes the last claim. For the other claims we refer to Lemma 3.5 and [17, 3. Proof of the main theorem] where very similar estimates have been proved. □

Lemma 3.12. *For any $a > a^*$ there exist $R, \delta, \varepsilon > 0$ such that the operator $\mathcal{B}$ is a-hypodissipative in X.*

Proof of Lemma 3.12. We define the space

$$\mathcal{E} := L^1(\phi), \quad \phi(x) := x^\alpha + \eta\,x^\beta, \quad \eta > 0.$$

Consider $f \in D(\Lambda)$ a real-valued function and let us show that for any $a > a^*$ and

for suitable $R \in (1,\infty)$, $\delta \in (0,1)$, $\varepsilon \in (0,\delta/2)$ to be chosen later,

$$\int_0^\infty \operatorname{sign}(f(x))\, \mathcal{B}f(x)\, \phi(x)\, dx \le a\|f\|_{\mathcal{E}}. \tag{3.27}$$

Since the case of complex-valued functions can be handled in a similar way and since the norm $\|\cdot\|_{\mathcal{E}}$ is clearly equivalent to $\|\cdot\|_X$, that will end the proof.

From the identity

$$\int_0^\infty \operatorname{sign}(f(x))\, \mathcal{B}_0 f(x)\, x^r\, dr = \int_0^\infty ((r-1)x^r - x^{r+\gamma})\, |f|\, dx$$

and the following inequality which holds for $\eta > 0$ small enough

$$\eta\,(\beta - 1 - a)\, x^\beta \le \frac{1+a-\alpha}{2} x^\alpha + \eta\, \frac{1-\wp_\beta}{2}\, x^{\beta+\gamma} \quad \forall\, x > 0,$$

we readily deduce

$$\int_0^\infty \operatorname{sign}(f(x))\, \mathcal{B}_0 f(x)\, \phi(x)\, dx \le a \int_0^\infty |f|\, \phi\, dx + \int_0^\infty (\frac{\alpha - a - 1}{2} x^\alpha - \eta\, \frac{\wp_\beta + 1}{2} x^{\beta+\gamma})\, |f|\, dx.$$

On the other hand, we know from Lemma 3.11 that

$$\begin{aligned} \|\mathcal{A}^c f\|_{\mathcal{E}} \quad &\le \quad \wp_\alpha \delta^\gamma\, \|f\|_{\mathcal{E}} + (\frac{\wp_\alpha}{R^{\beta-\alpha}} + \eta\, \wp_\beta) \int_R^\infty x^{\beta+\gamma} |f| \\ &\quad + \|\wp(z)\, z^\alpha\|_{L^1(0,2\varepsilon/\delta)}\, \|f\|_{\mathcal{E}}. \end{aligned}$$

We then easily conclude to (3.27) putting together these two estimates and choosing R large enough, δ small enough, and then ε/δ small enough. □

Proof of Proposition 3.10. We just have to explain why the hypothesis of Theorem 3.3 are satisfied. Hypothesis **(H1)** is an immediate consequence of Lemma 3.11-(1), Lemma 3.12 and Remark 2.2-(b).

We define

$$Y := \{f \in W^{1,1}(\mathbb{R});\ \operatorname{supp} f \subset [\varepsilon, 2R]\}$$

endowed with the norm $\|\cdot\|_{W^{1,1}}$. We also define on X the projection π_N on the $2N+1$ dimension subspace

$$R\pi_N := \{\mathbf{1}_{[0,2R]}\, p,\ p \in \mathbb{P}_{2N}(\mathbb{R})\},$$

where $\mathbb{P}_{2N}$ stands for the set of polynomials of degree less that $2N$, thanks to

$$(\pi_N f)(x) := \chi_\varepsilon^c(x)\, \chi_{2R}(x)\, (p_{N,R} * f)(x), \quad p_{N,R}(x) := p(x/(4R))/(4R)$$

where p_N stands for the Bernstein polynomial $p_N(x) := \alpha_N\, (1-x^2)^N$ with α_N such that $\|p_N\|_{L^1(-1,1)} = 1$. By very classical approximation arguments, we have

$$\|f - \pi_N f\|_X \le \frac{C_R}{\sqrt{N}}\, \|f\|_{W^{1,1}} \quad \forall\, f \in Y,$$

so that (3.7) is fulfilled.

Finally, Lemma 3.11-(1) and Lemma 3.12 implies that

$$\|\mathcal{A}S_{\mathcal{B}}(t) f\|_Y \le C_a\, e^{at}\, \|f\|_X$$

for any $t \ge 0$, $f \in X$, $a > a^*$, from which we deduce **(H2)** with $n = 1$ thanks to Remark 2.2-(h) and the fact that $Y \subset D(\Lambda)$, **(H3)** since Y is compactly embedded in X and (3.6). □

3.5. **A remark on the age structured population equation.** The aim of this short section is to present a quantified version of the Weyl's Theorem for the age structured population equation (1.38) in the simple case when $\tau = \nu = 1$ and $K \in C_b(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$. More precisely, we consider the evolution equation

$$\partial_t f = \Lambda f = \mathcal{A} f + \mathcal{B} f \tag{3.28}$$

with $\mathcal{A}$ and $\mathcal{B}$ defined on $M^1(\mathbb{R})$ by

$$\begin{aligned} (\mathcal{A} f)(x) &:= \delta_{x=0} \int_0^\infty K(y)\, f(y)\, dy \\ (\mathcal{B} f)(x) &:= -\partial_x f(x) - f(x), \end{aligned}$$

and it is worth emphasising that the boundary condition in (1.38) has been equivalently replaced by the term $\mathcal{A} f$ involving a Dirac mass in $x = 0$. We perform the spectral analysis of Λ in the space $L^1(\mathbb{R}_+)$ as well as in the space $X := M^1(\mathbb{R})$ of bounded measures endowed with the total variation norm. In that last functional space, the domain is the space $BV(\mathbb{R})$ of functions with bounded variation.

Lemma 3.13. *In $X = M^1(\mathbb{R})$, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy*
(i) $\mathcal{A} \in \mathscr{B}(X, Y)$ where $Y = \mathbb{C}\,\delta_0 \subset X$ with compact embedding;
(ii) $S_{\mathcal{B}}(t)$ is -1-dissipative;
(iii) the family of operators $S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{B}}(t)$ satisfy

$$\|(S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{B}})(t)\|_{X \to D(\Lambda)} \le C\, e^{-t} \qquad \forall\, t \ge 0.$$

Proof of Lemma 3.13. We clearly have $\mathcal{A} \in \mathscr{B}(X, Y)$ because $K \in C_b(\mathbb{R}_+)$ and the -1-dissipativity of $S_{\mathcal{B}}$ follows from the explicit formula

$$S_{\mathcal{B}}(t) f(x) = f(x - t)\, e^{-t}\, \mathbf{1}_{x - t \ge 0}.$$

We next prove (iii). We write

$$\mathcal{A} S_{\mathcal{B}}(t) f = \delta_{x=0} \int_0^\infty K(y + t)\, f(y)\, dy\, e^{-t},$$

next

$$\begin{aligned} (S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{B}})(t) f &= \int_0^t S_{\mathcal{B}}(s)\, \mathcal{A} S_{\mathcal{B}}(t - s) f\, ds \\ &= \int_0^t \delta_{x-s=0} e^{-s}\, \mathbf{1}_{x-s \ge 0} \int_0^\infty K(y + t - s)\, f(y)\, dy\, e^{-(t-s)}\, ds \\ &= e^{-t}\, \mathbf{1}_{x \le t} \int_0^\infty K(y + t - x)\, f(y)\, dy, \end{aligned}$$

and then

$$\begin{aligned} \partial_x (S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{B}})(t) f &= -e^{-t}\, \delta_{x=t} \int_0^\infty K(y + t - x)\, f(y)\, dy \\ &\quad - e^{-t}\, \mathbf{1}_{x \le t} \int_0^\infty K'(y + t - x)\, f(y)\, dy. \end{aligned}$$

As a consequence, we deduce

$$\|\partial_x (S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{B}})(t) f\|_{M^1} \le \|K\|_{W^{1,\infty}}\, e^{-t}\, \|f\|_{M^1}$$

and a similar estimate for $\|(S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{B}})(t) f\|_{M^1}$. □

As a first step in the proof of Theorem 1.1 for the age structured population equation, we have

Proposition 3.14. *Under the above assumptions and notations, the conclusions of Theorem 3.3 hold for the age structured population semigroup S_Λ in $M^1(\mathbb{R})$ and in $L^1(\mathbb{R}_+)$ for any $a > -1$. Moreover, any eigenvalue $\xi \in \Sigma_d(\Lambda) \cap \Delta_{-1}$ is algebraically simple.*

Proof of Proposition 3.14. In order to prove the result in $X = M^1(\mathbb{R}_+)$, we just have to explain again why the hypothesis of Theorem 3.3 are satisfied . Conditions **(H1)**, **(H3)** and (3.6) are immediate consequences of Lemma 3.13-(i) & (ii) together with Remark 2.2-(b). We refer to [49, 25, 76] and the references therein for the existence theory in $L^1(\mathbb{R}_+)$ (which extends without difficulty to $M^1(\mathbb{R}_+)$) for the semigroup S_Λ. Hypothesis **(H2)** with $n = 1$ is nothing but Lemma 3.13-(iii). Finally, taking up again the proof of Theorem 3.3 and using the additional fact that $\dim Y = 1$, we easily get the algebraic simplicity of the eigenvalues $\xi \in \Sigma(\Lambda) \cap \Delta_{-1}$.

We may then extend easily the spectral analysis performed in $M^1(\mathbb{R}_+)$ to the functional space $L^1(\mathbb{R}_+)$. Indeed, for a function $f \in L^1$, Theorem 3.3 implies that

$$\|e^{t\Lambda} f - \sum_{j=1}^{J} e^{t\xi_j} \Pi_j f\|_{M^1} \leq C_a \, e^{at} \, \|f\|_{M^1}$$

for any $a > -1$ and $t \geq 0$. Because S_Λ is well defined in L^1 and the domain of Λ as an operator in $M^1(\mathbb{R}_+)$ is $BV(\mathbb{R}_+) \subset L^1(\mathbb{R}_+)$, all the terms involved in the above expression belong to L^1 and we can replace the norms $\|.\|_{M^1}$ by the norms $\|.\|_{L^1}$. □

4. POSITIVE SEMIGROUP AND KREIN-RUTMAN THEOREM

4.1. **The Krein-Rutman Theorem in an abstract setting.** In this section we consider a "Banach lattice of functions" X. We recall that a Banach lattice is a Banach space endowed with an order denoted by $\geq$ (or $\leq$) such that the following holds:

- The set $X_+ := \{f \in X;\ f \geq 0\}$ is a nonempty convex closed cone.
- For any $f \in X$, there exist some unique $f_\pm \in X_+$ such that $f = f_+ - f_-$, we then denote $|f| := f_+ + f_- \in X_+$.
- For any $f, g \in X$, $0 \leq f \leq g$ implies $\|f\| \leq \|g\|$.

We may define a dual order $\geq$ (or $\leq$) on X' by writing for $\psi \in X'$

$$\psi \geq 0 \ \ (\text{or } \psi \in X'_+) \quad \text{iff} \quad \forall\, f \in X_+ \ \ \langle \psi, f \rangle \geq 0,$$

so that X' is also a Banach lattice.

We then restrict our analysis to the case when X is a"space of functions". The examples of spaces we have in mind are the space of Lebesgue functions $X = L^p(\mathscr{U})$, $1 \leq p < \infty$, $\mathscr{U} \subset \mathbb{R}^d$ borelian set, the space $X = C(\mathscr{U})$ of continuous functions on a compact set $\mathscr{U}$ and the space $X = C_0(\mathscr{U})$ of uniformly continuous functions defined on an open set $\mathscr{U} \subset \mathbb{R}^d$ and which tend to 0 at the infinity. For any element (function) f in such a "space of functions" X, we may define without difficulty the composition functions $\theta(f)$ and $\theta'(f)$ for $\theta(s) = |s|$ and $\theta(s) = s_\pm$ as well as the support $\mathrm{supp} f$ as a closed subset of $\mathscr{U}$. Although we believe that our results extend to a broader class of Banach lattices, by now and in order to avoid technicality, we will restrict ourself to these examples of "space of functions" without specifying anymore but just saying that we consider a "Banach lattice of functions" (and we

refer to the textbook [3] for possible generalisation). As a first consequence of that choice, we may then obtain a nice and simple property on the generator of a positive semigroup.

Definition 4.1. Let us consider a Banach lattice X and a generator Λ of a semigroup S_Λ on X.

(a) - We say that the semigroup S_Λ is positive if $S_\Lambda(t)f \in X_+$ for any $f \in X_+$ and $t \geq 0$.

(b) - We say that a generator Λ on X satisfies Kato's inequalities if the inequality

$$\forall\, f \in D(\Lambda) \quad \Lambda\theta(f) \geq \theta'(f)\,\Lambda f \tag{4.1}$$

holds for $\theta(s) = |s|$ and $\theta(s) = s_+$.

(c) - We say that $-\Lambda$ satisfies a "weak maximum principle" if for any $a > \omega(\Lambda)$ and $g \in X_+$ there holds

$$f \in D(\Lambda) \text{ and } (-\Lambda + a)f = g \quad \text{imply} \quad f \geq 0. \tag{4.2}$$

(d) - We say that the opposite of the resolvent is a positive operator if for any $a > \omega(\Lambda)$ and $g \in X_+$ there holds $-R_\Lambda(a)g \in X_+$.

Here the correct way to understand Kato's inequalities is

$$\forall\, f \in D(\Lambda),\ \ \forall\, \psi \in D(\Lambda^*) \cap X'_+ \quad \langle \theta(f), \Lambda^*\psi\rangle \geq \langle \theta'(f)\,\Lambda f, \psi\rangle.$$

It is well known (see [89] and [2, Remark 3.10] and the textbook [3, Theorems C.II.2.4, C.II.2.6 and Remark C-II.3.12]) that the generator Λ of a positive semigroup S_Λ on one of our *Banach lattice space of functions* satisfies Kato's inequalities (4.1). It is also immediate from the Hille's identity (2.2) that (a) implies (d) and then (c) in the general Banach lattice framework. For a broad class of spaces X the properties (a), (b), (c) and (d) are in fact equivalent and we refer again to the textbook [3] for more details on that topics.

Last, we need some strict positivity notion on X and some strict positivity (or irreducibility) assumption on S_Λ that we will formulate in term of *"strong maximum principle"*. It is worth mentioning that we have not assumed that X_+ has nonempty interior, so that the strict positivity property cannot be defined using that interior set (as in Krein-Rutman's work [67]). However, we may define the strict order $>$ (or $<$) on X by writing for $f \in X$

$$f > 0 \quad \text{iff} \quad \forall\, \psi \in X_+\backslash\{0\}\ \ \langle \psi, f\rangle > 0,$$

and similarly a strict order $>$ (or $<$) on X' by writing for $\psi \in X'$

$$\psi > 0 \quad \text{iff} \quad \forall\, g \in X'_+\backslash\{0\}\ \ \langle \psi, g\rangle > 0.$$

It is worth emphasising that from the Hahn-Banach Theorem, for any $f \in X_+$ there exists $\psi \in X'_+$ such that $\|\psi\|_{X'} = 1$ and $\langle \psi, f\rangle = \|f\|_X$ from which we easily deduce that

$$\forall\, f, g \in X, \quad 0 \leq f < g \quad \text{implies} \quad \|f\|_X < \|g\|_X. \tag{4.3}$$

Definition 4.2. We say that $-\Lambda$ satisfies a "strong maximum principle" if for any given $f \in X$ and $\mu \in \mathbb{R}$, there holds

$$|f| \in D(\Lambda)\backslash\{0\} \ \text{ and } \ (-\Lambda + \mu)|f| \geq 0 \quad \text{imply} \quad f > 0 \ \text{ or } \ f < 0.$$

We can now state the following version of the Krein-Rutman Theorem in an general and abstract setting.

Theorem 4.3. *We consider a semigroup generator Λ on a "Banach lattice of functions" X, and we assume that*

(1) *Λ satisfies the assumptions of the semigroup Weyl's Theorem 3.1 for some $a^* \in \mathbb{R}$;*

(2) *there exist $b > a^*$ and $\psi \in D(\Lambda^*) \cap X'_+ \backslash \{0\}$ such that $\Lambda^* \psi \geq b\,\psi$;*

(3) *S_Λ is positive (and Λ satisfies Kato's inequalities);*

(4) *$-\Lambda$ satisfies a strong maximum principle.*

Defining $\lambda := s(\Lambda)$, there holds

$$a^* < \lambda = \omega(\Lambda) \quad \text{and} \quad \lambda \in \Sigma_d(\Lambda),$$

and there exists $0 < f_\infty \in D(\Lambda)$ and $0 < \phi \in D(\Lambda^*)$ such that

$$\Lambda f_\infty = \lambda\, f_\infty, \quad \Lambda^* \phi = \lambda\, \phi, \quad R\Pi_{\Lambda,\lambda} = \mathit{Vect}(f_\infty),$$

and then

$$\Pi_{\Lambda,\lambda} f = \langle f, \phi \rangle\, f_\infty \quad \forall\, f \in X.$$

Moreover, there exist $\alpha \in (a^*, \lambda)$ and $C > 0$ such that for any $f_0 \in X$

$$\|S_\Lambda(t) f - e^{\lambda t}\, \Pi_{\Lambda,\lambda} f_0\|_X \leq C\, e^{\alpha t}\, \|f_0 - \Pi_{\Lambda,\lambda} f_0\|_X \qquad \forall\, t \geq 0.$$

Remark 4.4. (a) Theorem 4.3 generalises the Perron-Frobenius Theorem [92, 39] for strictly positive matrix Λ, the Krein-Rutman Theorem [67] for irreducible, positive and compact semigroup on a Banach lattice with non empty interior cone and the Krein-Rutman Theorem variant [3, Corollary III-C.3.17] (see also [46] for the original proof) for irreducible, positive, eventually norm continuous semigroup with compact resolvent in a general Banach lattice framework. We also refer to the recent books [86, 9] and the references therein for more recent developments on the theory of positive operators. The main novelty here is that with assumptions **(1)** and **(2)** we do not ask for the semigroup to be eventually norm continuous and we only ask power compactness on the decomposition $\mathcal{A}$ and $\mathcal{B}$ of the operator Λ instead of compactness on the resolvent $\mathcal{R}_\Lambda$.

(b) Condition **(2)** is necessary because the requirement **(1)** only implies the needed compactness and regularity on the iterated operator $(\mathcal{A}\,\mathcal{R}_\mathcal{B}(z))^n$ for $z \in \Delta_{a^*}$. Condition **(2)** can be removed (it is automatically verified) if condition **(1)** holds for any $a^* \in \mathbb{R}$.

(c) In a general Banach lattice framework and replacing the strong maximum principle hypothesis **(4)** by the more classical irreducibility assumption on the semigroup (see for instance [3, Definition C-III.3.1]) Theorem 4.3 is an immediate consequence of Theorem 3.1 together with [3, Theorem C-III.3.12] (see also [26, 46, 47]). We have no idea whether the strong maximum principle and the irreducibility are equivalent assumptions although both are related to strict positivity of the semigroup or the generator. Anyway the strong maximum principle for the operator $-\Lambda$ is a very natural notion and hypothesis from our PDE point of view and that is the reason why we have chosen to presented the statement of Theorem 4.3 in that way. Moreover, the proofs in [3, part C] are presented in the general framework of Banach lattices and positive or reducible semigroups (no compactness assumption is required in the statement of [3, Theorem C-III.3.12]) so that quite abstract arguments are used during the proof. We give below a short,

elementary and somewhat self-contained proof of Theorem 4.3, and thus do not use [3, Theorem C-III.3.12].

Proof of Theorem 4.3. We split the proof into five steps.

Step 1. On the one hand, let us fix $0 \le f_0 \in D(\Lambda)$ such that $C := \langle f_0, \psi\rangle > 0$ which exists by definition of ψ. Then denoting $f(t) := S_\Lambda(t) f_0$, we have

$$\frac{d}{dt}\langle f(t), \psi\rangle = \langle \Lambda f(t), \psi\rangle = \langle f(t), \Lambda^* \psi\rangle \ge b\, \langle f(t), \psi\rangle,$$

which in turns implies

$$\langle f(t), \psi\rangle \ge C\, e^{bt} \quad \forall\, t \ge 0. \tag{4.4}$$

On the other hand, from Theorem 2.1 we know that $\omega(\Lambda) \le \max(a^*, \lambda)$. As a consequence, if $\lambda < b$ for any $a \in (\max(a^*, \lambda), b)$ there exists $C_a \in (0, \infty)$ such that

$$\langle f(t), \psi\rangle \le \|\psi\|_{X'}\, \|f(t)\|_X \le C_a\, \|\psi\|_{X'}\, e^{at}\, \|f_0\|_X.$$

That would be in contradiction with (4.4). We conclude that $a^* < b \le \lambda = \omega(\Lambda)$.

Step 2. We prove that there exists $f_\infty \in X$ such that

$$\|f_\infty\| = 1, \;\; f_\infty > 0, \;\; \Lambda f_\infty = \lambda f_\infty. \tag{4.5}$$

Thanks to the Weyl's Theorem 3.1 we know that for some $a < 0$

$$\Sigma(\Lambda) \cap \Delta_a = \cup_{j=1}^J \{\xi_j\} \subset \Sigma_d(\Lambda), \quad \Re e \xi_j = \lambda,$$

with $J \ge 1$. We introduce the Jordan basis $\mathscr{V} := \{g_{1,1}, ..., g_{J,L_J}\}$ of Λ in the invariant subspace $R\Pi_{\Lambda,a}$ as the family of vectors

$$g_{j,\ell} \ne 0, \quad \Lambda g_{j,\ell} = \xi_j g_{j,\ell} + g_{j,\ell-1} \quad \forall\, j \in \{1, ..., J\},\; \forall\, \ell \in \{1, ..., L_j\},$$

with the convention $g_{j,k} = 0$ if $k \le 0$ or $k \ge L_j + 1$, as well as the projectors (associated to the basis $\mathscr{V}$)

$$\Pi_{j,\ell} := \text{projection on}\, g_{j,\ell}, \quad \Pi_k := \text{projection on}\, \mathrm{Vect}(g_{j,k},\, 1 \le j \le J\}.$$

For any fix $g \in \mathscr{V}$ we write $g = g^1 - g^2 + ig^3 - ig^4$ with $g^\alpha \ge 0$ and we remark that there exists $\alpha \in \{1, ..., 4\}$ and $k_0 \ge 1$ such that $\Pi_{k_0} g^\alpha = \Pi_{\Lambda,a} g^\alpha \ne 0$. We then define $k^* = k^*(g^\alpha) := \max\{k;\; \Pi_k g^\alpha \ne 0\}$, that set being not empty (since it contains k_0). We may split the semigroup as

$$e^{\Lambda t} g^\alpha \;\; = \;\; \sum_j \sum_\ell e^{\Lambda t}\, \Pi_{j,\ell} g^\alpha + e^{\Lambda t}\, (I - \Pi_{\Lambda,a}),$$

with $\Pi_{j,\ell} g^\alpha = (\pi_{j,\ell} g^\alpha)\, g_{j,\ell}$, $\pi_{j,\ell} g^\alpha \in \mathbb{C}$, and

$$e^{\Lambda t}\, g_{j,\ell} = e^{\xi_j t} g_{j,\ell} + ... + t^{\ell-1}\, e^{\xi_j t} g_{j,1}.$$

Using the positivity assumption **(3)** and keeping only the leading order term in the above expressions, we have thanks to Theorem 3.1

$$0 \le \frac{1}{t^{k^*-1}} e^{(\Lambda-\lambda)t} g^\alpha = \sum_{j=1}^J (\pi_{j,k^*} g^\alpha) e^{(\xi_j - \lambda)t} g_{j,1} + o(1).$$

There exist a sequence (t_n) which tends to infinity and some complex numbers $z_j \in \mathbb{C}$, $|z_j| = 1$, such that passing to the limit in the above expression, we get

$$0 \le \sum_{j=1}^J (\pi_{j,k^*} g^\alpha)\, z_j\, g_{j,1} =: g_\infty.$$

Because of the choice of α and the fact that the vectors $g_{j,1}$, $1 \leq j \leq J$, are independent, we have then $g_\infty \in R\Pi_{\Lambda,a} \cap X_+ \backslash \{0\}$. Making act again the semigroup, we get

$$0 \leq e^{\Lambda t} g_\infty = \sum_{j=1}^{J} e^{\xi_j t} [(\pi_{j,k^*} g^\alpha)\, z_j\, g_{j,1}] \quad \forall\, t \geq 0,$$

which in particular implies

$$\sum_{j=1}^{J} \Im m \big\{ e^{\xi_j t} [(\pi_{j,k^*} g^\alpha)\, z_j\, g_{j,1}] \big\} = 0 \quad \forall\, t \geq 0,$$

and then $\pi_{j,k^*} g^\alpha = 0$ if $\Im m\, \xi_j \neq 0$. As a conclusion, we have proved that there exists $g_\infty \in N(\Lambda - \lambda) \cap X_+ \backslash \{0\}$. Together with the strong maximum principle we conclude that $f_\infty := g_\infty / \|g_\infty\|$ satisfies (4.5).

Moreover, the above argument for any $g = g_{j,L_j}$ associated to $\xi_j \neq \lambda$ and for any $\alpha \in \{1, ..., 4\}$ such that $\Pi_{j,L_j} g^\alpha \neq 0$ implies that $\pi_{j,k^*} g^\alpha = 0$, or in other words

$$\max_{\xi_j \neq \lambda} L_j < \max_{\xi_j = \lambda} L_j. \tag{4.6}$$

Step 3. We prove that there exists $\phi \in X'$ such that

$$\phi > 0, \ \Lambda^* \phi = \lambda \phi. \tag{4.7}$$

We define S_Λ^* the dual semigroup associated to S_Λ and we emphasize that it is not necessarily strongly continuous (for the norm in X') but only weakly continuous (for the weak topology $\sigma(X', X)$). However, introducing the splitting

$$S_\Lambda^* = (S_\Lambda (I - \Pi_{\Lambda,a}))^* + (S_\Lambda \Pi_{\Lambda,a})^*$$

and observing that

$$\|(S_\Lambda (I - \Pi_{\Lambda,a}))^*\|_{\mathscr{B}(X)} \leq C_a\, e^{at}$$

for some $a < \lambda$, the same finite dimension argument as in Step 2 implies that there exists $\phi \in N(\Lambda^* - \lambda) \cap X' \backslash \{0\}$.

Let us prove the strict positivity property. For $a > s(\Lambda)$ and $g \in X_+ \backslash \{0\}$, thanks to the weak and strong maximum principles **(3)** and **(4)**, there exists $0 < f \in X$ such that

$$(-\Lambda + a) f = g.$$

As a consequence, we have

$$\begin{aligned} \langle \phi, g \rangle &= \langle \phi, (-\Lambda + a) f \rangle \\ &= \langle (a - \Lambda^*) \phi, f \rangle = (a - \lambda) \langle \phi, f \rangle > 0. \end{aligned}$$

Since $g \in X_+$ is arbitrary, we deduce that $\phi > 0$. That concludes the proof of (4.7).

Step 4. We prove that $N(\Lambda - \lambda) = \mathrm{vect}(f_\infty)$. Consider a normalized eigenfunction $f \in X^{\mathbb{R}} \backslash \{0\}$ associated to the eigenvalue λ. First we observe that from Kato's inequality

$$\lambda |f| = \lambda f\, \mathrm{sign}(f) = \Lambda f\, \mathrm{sign}(f) \leq \Lambda |f|.$$

That inequality is in fact an equality, otherwise we should have

$$\lambda \langle |f|, \phi \rangle \neq \langle \Lambda |f|, \phi \rangle = \langle |f|, \Lambda^* \phi \rangle = \lambda \langle |f|, \phi \rangle,$$

and a contradiction. As a consequence, $|f|$ is a solution to the eigenvalue problem $\lambda |f| = \Lambda |f|$ so that the strong maximum principle assumption **(4)** implies $f > 0$ or

$f<0$, and without lost of generality we may assume $f>0$. Now, we write thanks to Kato's inequality again

$$\lambda(f-f_\infty)_+ = \Lambda(f-f_\infty)\,\mathrm{sign}_+(f-f_\infty) \le \Lambda(f-f_\infty)_+,$$

and for the same reason as above that last inequality is in fact an inequality. Since $(f-f_\infty)_+ = |(f-f_\infty)_+|$, the strong maximum principle implies that either $(f-f_\infty)_+=0$, or in other words $f\le f_\infty$, either $(f-f_\infty)_+>0$ or in other words $f>f_\infty$. Thanks to (4.3) and to the normalization hypothesis $\|f\|=\|f_\infty\|=1$ the second case in the above alternative is not possible. Repeating the same argument with $(f_\infty-f)_+$ we get that $f_\infty\le f$ and we conclude with $f=f_\infty$. For a general eigenfunction $f\in X^{\mathbb{C}}$ associated to the eigenvalue λ we may introduce the decomposition $f=f_r+if_i$ and we immediately get that $f_\alpha\in X^{\mathbb{R}}$ is an eigenfunction associated to λ for $\alpha=r,i$. As a consequence of what we have just established, we have $f_\alpha=\theta_\alpha f_\infty$ for some $\theta_\alpha\in\mathbb{R}$ and we conclude that $f=(\theta_r+i\theta_i)\,f_\infty\in\mathrm{vect}(f_\infty)$ again.

Step 5. We first claim that λ is algebraically simple. Indeed, if it is not the case, there would exist $f\in X^{\mathbb{R}}$ such that $\Lambda f=\lambda f+f_\infty$ and then

$$\lambda\langle f,\phi\rangle = \langle f,\Lambda^*\phi\rangle = \langle \Lambda f,\phi\rangle = \langle \lambda f+f_\infty,\phi\rangle,$$

which in turns implies $\langle f_\infty,\phi\rangle=0$ and a contradiction. With the notations of step 2 and thanks to (4.6), that implies that for any $\xi_j\ne\lambda$ there holds $L_j<1$ or in other words $\Sigma(\Lambda)\cap\Delta_a=\{\lambda\}$. We conclude the proof by using the semigroup Weyl's Theorem 3.1. □

4.2. **Krein-Rutman Theorem for the growth-fragmentation equations.** .

We start with the cell-division equation for which we apply the Krein-Rutman Theorem in the Banach lattice $L^1((x_0z_0,\infty);\langle x\rangle^\alpha dx)$, $\alpha>1$, where z_0 is defined by (1.20), instead of L^1_α in order that the operator Λ enjoys a strong maximum principle.

We have proved **(1)** in Proposition 3.4. We have $\Lambda^*1=K(x)\,(1-\wp_0)\ge 0$ so that **(2)** holds with $b=0>a^*$. The weak maximum principle **(3)** is an immediate consequence of Kato's inequalities which in turn follows from the fact that $\mathcal{F}^+\theta(f)\ge\theta'(f)\mathcal{F}^+f$ *a.e* for any $f\in D(\Lambda)$, $\theta(s)=|s|$ and $\theta(s)=s_+$. The strong maximum principle **(4)** follows from the fact that the equation

$$|f|\in D(\Lambda)\backslash\{0\}\ \text{ and }\ (-\Lambda+\mu)|f|\ge 0$$

may be rewritten as

$$-\partial_x|f|+(K(x)+\mu)|f|\ge\mathcal{F}^+|f|\ge 0,$$

and we conclude as in the [79, proof of Theorem 3.1] that the continuous function $|f|$ does not vanish on (x_0z_0,∞), so that $f>0$ on (x_0z_0,∞) or $f<0$ on (x_0z_0,∞).

For the self-similar equation the proof is exactly the same by applying the Krein-Rutman Theorem in the Banach lattice $L^1(\mathbb{R}_+;(x^\alpha+x^\beta)dx)$, $0\le\alpha<1<\beta<\infty$. Let us just emphasize that condition **(1)** has been proved in Proposition 3.10 and that condition **(2)** with $b=0>a^*$ follows from the fact that $\Lambda^*\phi=0$ for the positive function $\phi(x)=x$. We refer to [34, Section 3] for the proof of the weak and strong maximum principle **(3)** and **(4)**.

Finally, for the age structured population equation we apply the Krein-Rutman Theorem in the Banach lattice $L^1(\mathbb{R}_+)$. As in [84, Appendix], we observe that,

denoting by $\lambda > -1$ the real number such that

$$\int_0^\infty K(x)\, e^{-(1+\lambda)\, x}\, dx = 1$$

which exists thanks to condition (1.40), the function

$$\psi(x) := e^{(1+\lambda)x} \int_x^\infty K(y)\, e^{-(1+\lambda)\, y}\, dx$$

is a solution to the dual eigenvalue problem

$$\Lambda^* \psi = \partial_x \psi - \psi + K(x)\psi(0) = \lambda \psi, \quad 0 \le \psi \in L^\infty(\mathbb{R}_+)$$

so that in particular **(2)** holds with $\lambda > a^* = -1$. Condition **(1)** has been proved in Proposition 3.14 and the proof of the weak and strong maximum principle **(3)** and **(4)** is classical.

4.3. **Quantified spectral gap theorem for the cell division equation with constant total fragmentation rate.** We consider the very particular case of the cell-division equation with constant total fragmentation rate and fragmentation kernel which furthermore fulfils condition (1.32) for which we can give an accurate longtime behaviour asymptotic and answer to a question formulated in [94, 68]. We then consider the equation

$$\partial_t f + \partial_x f + K_0 f = K_0 \int_x^\infty \kappa(y,x)\, f(y)\, dy \tag{4.8}$$

with vanishing boundary condition (1.24), where $K_0 > 0$ is a constant and κ satisfies (1.32). In such a situation, we have the following accurate description of the spectrum.

Proposition 4.5. *The first eigenvalue is given by $\lambda = s(\Lambda) := (n_F - 1)K_0$ with n_F defined in* (1.32)*. On the other hand, for any $a^{**} \in (-K_0, (n_F-1)K_0)$ and any $\alpha > \alpha^*$, with α^* large enough (but explicit), Theorem 1.1 holds in L^1_α for any $a > a^{**}$, which in particular gives an explicit bound on the spectral gap $\lambda - \sup \Re e \Sigma(\Lambda) \backslash \{\lambda\}$.*

We use the following extension (shrinkage) of the functional space of the semigroup decay proved in [81].

Theorem 4.6 (Extension of the functional space of the semigroup decay)**.** *Let E and $\mathcal{E}$ be two Banach spaces such that $E \subset \mathcal{E}$ with dense and continuous embedding, and consider L the generator of a semigroup $S_L(t) := e^{tL}$ on E, $\mathcal{L}$ the generator of a semigroup $S_{\mathcal{L}}(t) := e^{t\mathcal{L}}$ on $\mathcal{E}$ with $\mathcal{L}_{|E} = L$.*
We assume that there exist two operators $\mathcal{A}, \mathcal{B} \in \mathscr{C}(\mathcal{E})$ such that

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \;\; L = A + B, \;\; A = \mathcal{A}_{|E}, \;\; B = \mathcal{B}_{|E},$$

and a real number $a \in \mathbb{R}$ such that there holds:

- **(i)** *$(B - a)$ is hypodissipative on E, $(\mathcal{B} - a)$ is hypodissipative on $\mathcal{E}$;*
- **(ii)** *$A \in \mathscr{B}(E)$, $\mathcal{A} \in \mathscr{B}(\mathcal{E})$;*
- **(iii)** *there is $n \ge 1$ such that*

$$\left\| (\mathcal{A} S_{\mathcal{B}})^{(*n)}(t) \right\|_{\mathscr{B}(\mathcal{E},E)} \le C'_a\, e^{at}.$$

The following equivalence holds:

(1) *There exists a finite rank projector* $\Pi_L \in \mathscr{B}(E)$ *which commutes with* L *and satisfy* $\Sigma(L_{|\Pi_L}) = \{0\}$*, so that the semigroup* $S_L = e^{tL}$ *satisfies the growth estimate*

$$\forall\, t \geq 0, \quad \|S_L(t) - \Pi_L\|_{\mathscr{B}(E)} \leq C_{L,a}\, e^{a\,t} \tag{4.9}$$

for any $a > a^*$ *and some constant* $C_{L,a} > 0$*;*

(2) *There exists a finite rank projector* $\Pi_{\mathcal{L}} \in \mathscr{B}(\mathcal{E})$ *which commutes with* $\mathcal{L}$ *and satisfy* $\Sigma(\mathcal{L}_{|\Pi_{\mathcal{L}}}) = \{0\}$*, so that the semigroup* $S_{\mathcal{L}} = e^{t\mathcal{L}}$ *satisfies the growth estimate*

$$\forall\, t \geq 0, \quad \|S_{\mathcal{L}}(t) - \Pi_{\mathcal{L}}\|_{\mathscr{B}(\mathcal{E})} \leq C_{\mathcal{L},a}\, e^{a\,t} \tag{4.10}$$

for any $a > a^*$ *and some constant* $C_{\mathcal{L},a} > 0$*.*

Proof of Proposition 4.5. Step 1. We recall some fact presented in [94, 68]. We introduce the rescaled function $g(t,x) := f(t,x)\, e^{-\lambda t}$ and the associated rescaled equation

$$\partial_t g + \partial_x g + n_F K_0 g = K_0 \int_x^\infty \kappa(y,x)\, g(y)\, dy \tag{4.11}$$

with vanishing boundary condition (1.24) and initial condition $g(0) = f_0$. We observe that the number of particles

$$\int_0^\infty g(t,x)\, dx$$

is conserved. One can then show using the Brouwer-Tykonov Theorem that there exists a steady state f_∞ by proceeding exactly as for the self-similar fragmentation equation [34, Section 3] (see also [94, 68] where other arguments are presented). Existence of the steady state f_∞ is also given by the Krein-Rutman Theorem presented in section 4.2. This steady sate corresponds to the first eigenfunction associated to the first eigenvalue $(n_F - 1)\, K_0$ of the cell-division equation (4.8).

Anyway, under assumption (1.32), it has been shown during the proof of [94, Theorem 1.1] and [68, Theorem 1.1] that the solution g to (4.11) satisfies

$$\|g(t) - \langle f_0\rangle\, f_\infty\|_{-1,1} \leq e^{-\lambda t} \|f_0 - \langle f_0\rangle\, f_\infty\|_{-1,1}$$

where for any $f \in L^1_1$ with mean 0 we have defined

$$\|f\|_{-1,1} := \int_0^\infty \Big| \int_0^x f(y)\, dy \Big|\, dx.$$

Step 2. For the mitosis equation, we introduce the splitting $\Lambda = \mathcal{A} + \mathcal{B}$ where

$$\mathcal{A} := \mathcal{F}_R^+, \quad \mathcal{B} := -\partial_x - n_F\, K_0 + \mathcal{F}_R^{+,c},$$

with the notations of section 3.3. We define $E := L^1_\alpha$, $\alpha > 1$, and for any $f \in L^1_\alpha$ with mean 0 we define

$$\|f\|_{-1,\alpha} := \int_0^\infty \Big| \int_0^x f(y)\, dy \Big|\, \langle x\rangle^{\alpha-1}\, dx,$$

as well as $\mathcal{E}$ the Banach space obtained by completion of L^1_α with respected to the norm $\|\cdot\|_{-1,\alpha}$. We clearly have that $\mathcal{A}$ satisfies **(ii)** in both spaces E and $\mathcal{E}$ and that $\mathcal{B}$ satisfies **(i)** in the space E for any $a > a^*$, $a^* := -K_0(1 - \wp_\alpha)$ thanks to Lemma 3.6. We claim that $\mathcal{B}$ also satisfies **(i)** in the space $\mathcal{E}$ for any $\alpha > \log_2 3$ and any $a > a' := 2K_0\,(3 \times 2^{-\alpha} - 1)$. In order to prove that last claim we proceed

along the line of the proof of [94, Theorem 1.1]. For any $g \in \mathcal{E}$, we introduce the notations

$$g(t) := e^{\mathcal{B}t} g, \quad G(t,x) := \int_0^x g(t,u)\,du,$$

and we compute

$$\begin{aligned}\partial_t G + \partial_x G + 2\,K_0 G &= -2K_0 \int_x^\infty \chi_R^c(2y)\,\partial_y G(t,2y)\,dy \\ &= 2K_0 \chi_R^c(2x)\,G(t,2x) + 4K_0 \int_x^\infty (\chi_R^c)'(2y)\,G(t,2y)\,dy.\end{aligned}$$

Similarly as in the proof of Lemma 3.6, we define

$$\phi(x) := \mathbf{1}_{[0,x_2]} + \frac{x^{\alpha-1}}{x_2^{\alpha-1}}\,\mathbf{1}_{[x_2,\infty)}$$

and we compute for $R \geq x_2$

$$\begin{aligned}\partial_t \int_0^\infty |G|\,\phi + \int_0^\infty |G|\,\phi\,\frac{\alpha-1}{x}\,\mathbf{1}_{x\geq x_2} + 2\,K_0 \int_0^\infty |G|\,\phi \\ \leq \frac{K_0}{2^{\alpha-1}} \int_0^\infty |G|\,\phi\,\chi_R^c + \frac{K_0}{2^{\alpha-1}} \int_0^\infty |(\chi_R^c)'|\,x\,|G|\,\phi\,dx \\ \leq 3\,\frac{K_0}{2^{\alpha-1}} \int_0^\infty |G|\,\phi,\end{aligned}$$

where in the last line we have used that $|x\,(\chi_R^c)'| \leq 2$ by definition of χ_R^c. We conclude by taking $x_2 = R$ large enough.

Moreover, we claim that

$$\left\| (\mathcal{A}S_{\mathcal{B}})^{(*2)}(t) \right\|_{\mathscr{B}(\mathcal{E},E)} \leq C'\,(1+t)\,e^{-\mu t}, \tag{4.12}$$

with $\mu := n_F K_0 = 2K_0$. In order to prove estimate (4.12), as in the proof of Lemma 3.9 and with the same notations, we compute starting from (3.25)

$$\begin{aligned}U_0^{(*2)}(t)g(x) &= 16K_0^2 \chi_R(2x)\,e^{-\mu t} \int_{u_0}^{u_1} \chi_R(2u-4x+2t)g(u)\,du \\ &= 16K_0^2 \chi_R(2x)\,e^{-\mu t} \Big\{ \Big[\chi_R(2u-4x+2t)G(u)\Big]_{u=u_0}^{u=u_1} - \frac{2}{R}\int_{u_0}^{u_1} \chi_R'(2u-4x+2t)G(u)\,du \Big\},\end{aligned}$$

and we get then for any $\beta \geq 0$

$$\|U_0^{(*2)}(t)g\|_{L^1_\beta} \leq C\,(1+t)\,e^{-\mu t}\,\|G\|_{L^1}.$$

We have a similar estimate for $U_0 * U_0^c$ and we then obtain (4.12) thanks to formula (3.24). We conclude by using the shrinkage of functional space result stated in Theorem 4.6, and we get for possible definition of α^* the unique real number such that $3\wp_{\alpha^*} - n_F = [a^{**} - (n_F-1)K_0]/K_0$.

Step 3. For the cell division equation with smooth offsprings distribution $\wp$ we can proceed along the line of [68, Theorem 1.1] and of step 2. We introduce the same splitting as for the mitosis equation and we work in the same spaces. We clearly have again that $\mathcal{A}$ satisfies **(ii)** in both spaces E and $\mathcal{E}$ and $\mathcal{B}$ satisfies **(i)** in E. We

claim that $\mathcal{B}$ also satisfies **(i)** in the space $\mathcal{E}$ for any $a > a' \in (-n_F K_0, 0)$ and any $\alpha > \alpha'$ where $\alpha' > 1$ is such that $a'/K_0 = -n_F + \frac{4}{\alpha'} n_F + \wp_{\alpha'-1}$. Indeed, we have

$$\begin{aligned} \partial_t G + \partial_x G + n_F\, K_0 G &= -K_0 \int_x^\infty \int_y^\infty \kappa_R^c(z,y)\, \partial_z G(z)\, dz dy \\ &= -K_0 \int_x^\infty \beta_R^c(z,x)\, G(z)\, dz, \end{aligned}$$

with

$$\begin{aligned} \beta_R^c(z,x) &:= -\frac{\partial}{\partial z} \int_x^\infty \kappa_R^c(z,y)\, dy \\ &= -n_F(x/z)\, (\chi_R^c(z))' + \chi_R^c(z)\, \beta(z,x), \end{aligned}$$

where

$$\beta(z,x) := -\frac{\partial}{\partial z} \int_x^\infty \kappa(z,y)\, dy, \quad n_F(u) := \int_u^1 \wp(u')\, du',$$

and where $\kappa_R^c(x,y) = \chi_R^c(x)\, \kappa(x,y)$ is defined on $\mathbb{R}_+^2$ by extended it to 0 outside of the set $\{(x,y) \in \mathbb{R}^2;\, 0 < y < x\}$. On the one hand, we have

$$\begin{aligned} \Phi_1(z) &:= \int_0^z n_F(x/z)\, |(\chi_R^c(z))'|\, \phi(x)\, dx \\ &\le n_F\, |(\chi_R^c(z))'|\, (x_2 + z^\alpha/\alpha) \le \frac{4}{\alpha}\, n_F\, \phi(z) \end{aligned}$$

for any $z \ge 0$ if $x_2 \le R^\alpha/\alpha$. On the other hand, we have

$$\begin{aligned} \Phi_2(z) &:= \int_0^z \chi_R^c(z)\, \beta(z,x)\, \phi(x)\, dx \\ &\le \chi_R^c(z) \Big\{ \eta(x_2/R) + \wp_{\alpha-1}\, \phi(z) \Big\}, \quad \eta(u) := \int_0^u \wp(u')\, du', \end{aligned}$$

for any $z \ge 0$. Next, we compute

$$\begin{aligned} &\partial_t \int_0^\infty |G|\, \phi + \int_0^\infty |G|\, \phi\, \frac{\alpha-1}{x}\, \mathbf{1}_{x \ge x_2} + n_F\, K_0 \int_0^\infty |G|\, \phi \\ &\quad \le K_0 \int_0^\infty \Big\{ \int_0^z |\beta_R^c(z,x)|\, \phi(x)\, dx \Big\}\, |G(z)|\, dz \\ &\quad \le K_0 \int_0^\infty \Big\{ \Phi_1(z) + \Phi_2(z) \Big\}\, |G(z)|\, dz \\ &\quad \le K_0 \Big\{ \frac{4}{\alpha}\, n_F + \eta(x_2/R) + \wp_{\alpha-1} \Big\} \int_0^\infty |G|\, \phi\, dx, \end{aligned}$$

and we take $\alpha > 1$ large enough and next R/x_2 large enough.

We next claim that $U_0 := \mathcal{A} S_{\mathcal{B}_0}$ with $\mathcal{B}_0 := -\partial_x - \mu$, $\mu := n_F\, K_0$, satisfies

$$\|U_0(t) g\|_{L^1_\beta} \le C \left(1 + \frac{1}{t}\right) e^{-\mu t} \|g\|_{-1,1}. \tag{4.13}$$

Starting from the definition

$$\begin{aligned} (U_0(t))(x) &:= e^{-\mu t}\, K_0 \int_x^\infty \kappa_R(y,x)\, \partial_y G(y-t)\, dy \\ &= -e^{-\mu t}\, K_0\, \kappa_R(x,x)\, G(x-t) - e^{-\mu t}\, K_0 \int_x^\infty \partial_y [\kappa_R(y,x)]\, G(y-t)\, dy, \end{aligned}$$

we compute

$$\begin{aligned}\|U_0(t)g\|_{L^1_\beta} &\leq K_0\, e^{-\mu t} \int_0^\infty \chi_R(x)\, \langle x\rangle^\beta\, \wp(1)\, \frac{|G(x-t)|}{x-t+t}\, dx \\ &\quad + K_0\, e^{-\mu t} \int_0^\infty \int_0^y \Big\{ |(\chi_R)'|\, \frac{1}{y} \wp(\frac{x}{y}) + \chi_R(y)\, \frac{x}{y^3}\, |\wp'(\frac{x}{y})| \Big\} \langle x\rangle^\beta\, G(y-t)\, dy \\ &\leq K_0\, e^{-\mu t}\, C_{R,\beta}\, \big(1+\frac{1}{t}\big) \int_0^\infty |G(z)|\, dz,\end{aligned}$$

and that ends the proof of (4.13). We introduce the notations $\mathcal{E}_1 := E$, $\mathcal{E}_0 := \mathcal{E}$ and $\mathcal{E}_{1/2}$ as the 1/2 complex interpolation between the spaces $\mathcal{E}_0$ and $\mathcal{E}_1$. From the preceding estimates, we have for any $a > -\mu$ that

$$\|U_0(t)\|_{\mathcal{E}_j, \mathcal{E}_{j+1/2}}\, C\, t^{-1/2}\, e^{at}.$$

Thanks to (3.24) it is not difficult now to prove that (4.12) holds also in the present case. We conclude again by using the shrinkage of functional space result stated in Theorem 4.6. □

4.4. **Quantified spectral gap for the self-similar fragmentation equation with positive kernel.** We present a second situation where a very accurate and quantitative description of the spectrum is possible.

Proposition 4.7. *Consider the self-similar fragmentation equation and assume that the fragmentation kernel satisfies* (1.35), (1.30) *and* (1.31). *Then, there exists a computable constant $\varepsilon > 0$ such that the spectral gap $s(\Lambda) - a^{**} \geq \varepsilon$ holds.*

Proof of Proposition 4.7. We split the proof into four steps.

Step 1. A priori bounds. We fix $a = a^*/2$ where a^* is given by Proposition 3.10. From Proposition 3.10 and then Theorem 3.3, there exists a constant R_a such that for any eigenfunction f associated to an eigenvalue $\xi \in \Sigma(\Lambda) \cap \Delta_a$ which satisfies the normalization condition

$$\int_0^\infty |f(y)|\, y\, dy = 1 \tag{4.14}$$

there holds

$$\int_0^\infty |f(y)|\, \langle y\rangle^{2+\gamma}\, dy \leq R_a, \quad |\xi| \leq R_a. \tag{4.15}$$

Together with the eigenvalue problem rewritten as

$$\partial_x(x^2 f) = x^2\, [x^\gamma\, f + \xi\, f - \mathcal{F}^+ f],$$

we deduce that $x^2 f \in BV(\mathbb{R}_+) \subset L^\infty(\mathbb{R}_+)$ and then, iterating the argument, that for any $\delta \in (0,1)$ there exists C_δ such that

$$\|f\|_{W^{1,\infty}(\delta, 1/\delta)} \leq C_\delta. \tag{4.16}$$

Step 2. Positivity. From (4.14) and (4.15) we clearly have

$$\int_\delta^{1/\delta} |f(y)|\, y\, dy \geq 1 - 2\, R\, \delta. \tag{4.17}$$

Taking $\delta_1 := 1/(4R)$, we see that there exists at least one point $x_1 \in (2\delta_1, 1/(2\delta_1))$ such that

$$|f(x_1)| \geq \delta_1^2/2. \tag{4.18}$$

Introducing the logarithm function θ defined by $f(y) = |f(y)|\, e^{i\theta(y)}$ which is locally well defined for any $y \in (0,\infty)$ such that $f(y) \neq 0$, we see from (4.18) and (4.16) that there exists an interval $I_1 \subset (\delta_1, 1/\delta_1)$, $x_1 \in I_1$, and a computable real number $\varepsilon_1 := \varepsilon(\delta_1) > 0$ such that

$$|I_1| \geq \varepsilon_1 \quad \text{and} \quad \Re e(f(x)\, e^{-i\theta(x_1)}) \geq \delta_1^2/4 \quad \forall\, x \in I_1, \tag{4.19}$$

and better (since cos and $|f(x)|$ are Lipschitz functions)

$$|I_1| \geq \varepsilon_1 \quad \text{and} \quad |f(x)| \geq \delta_1^2/4, \quad |\theta(x_1) - \theta(x)| \leq \pi/4 \quad \forall\, x \in I_1. \tag{4.20}$$

On the other hand, we know that $\Pi_{\Lambda,0} f = 0$ which yields

$$\int_0^\infty \Re e\Big\{ f(y)\, e^{-i\theta(x_1)} \Big\}\, y\, dy = 0.$$

From (4.15) again, we deduce that

$$\int_\delta^{1/\delta} \Re e\Big\{ f(y)\, e^{-i\theta(x_1)} \Big\}\, y\, dy \leq 2\, R\, \delta.$$

But together with the positivity property (4.19), there exists $\delta_2 < \delta_1$ such that $2R\delta_2 < \kappa_2 := \varepsilon_1\, \delta_1^2/8 > 0$ and for any $\delta \in (0, \delta_2)$

$$\int_\delta^{1/\delta} \Big(\Re e\Big\{ f(y)\, e^{-i\theta(x_1)} \Big\} \Big)_-\, y\, dy \geq \kappa_2.$$

Using the same arguments as above, there exists $x_2 \in (\delta_2, 1/\delta_2)$ such that

$$|f(x_2)|\, \cos(\theta(x_2) - \theta(x_1)) \leq -\kappa_2\, \delta_2^2/2,$$

and then there exist an interval $I_2 \subset (\delta_2, 1/\delta_2)$ and some constants $\varepsilon_2, \kappa_2 > 0$ such that

$$|I_2| \geq \varepsilon_2 \quad \text{and} \quad |f(y)| \geq \kappa_2 \quad \forall\, y \in I_2,$$

as well as

$$\cos[\theta(y) - \theta(x)] \leq 0 \quad \forall\, x \in I_1,\ \forall\, y \in I_2.$$

We may assume without lost of generality that $x_1 > x_2$.

Step 3. By definition of the growth-fragmentation operator Λ, we clearly have

$$\int_0^\infty (\Lambda|f| - (\Lambda f)\, \overline{\text{sign}} f)\, \phi = \int_0^\infty \int_0^y k(y,x)\, |f(y)| (1 - \text{sign} f(y)\, \overline{\text{sign}} f(x))\, \phi(x)\, dx dy.$$

Since

$$\Re e\{1 - \text{sign} f(y)\, \overline{\text{sign}} f(x)\} = 1 - \cos[\theta(y) - \theta(x)] \geq 1 \quad \forall x \in I_1,\ \forall\, y \in I_2,$$

we deduce that

$$\begin{aligned} &\Re e \int_0^\infty (\Lambda|f| - (\Lambda f)\, \overline{\text{sign}} f)\, \phi \\ &\quad \geq \wp_*\, \delta_2^{\gamma-1} \int_{I_2} |f(y)| \int_{I_1} \Big\{ 1 - \cos[\theta(y) - \theta(x)] \Big\}\, x\, dx \Big\}\, dy \\ &\quad \geq \wp_*\, \delta_2^{\gamma-1}\, \delta_1\, \varepsilon_1\, \varepsilon_2\, \kappa_2 =: \eta. \end{aligned}$$

Step 4. Conclusion. For any mass normalized eigenvector $f \in D(\Lambda)$ associated to an eigenvalue $\xi \in \Delta_a \cap \Sigma(\Lambda)\backslash\{0\}$, there holds thanks to step 3

$$\begin{aligned} \Re e\xi \, \langle |f|, \phi\rangle &= \Re e\langle \xi f \, \overline{\mathrm{sign}} f, \phi\rangle \\ &= \Re e\langle \Lambda f \, \overline{\mathrm{sign}} f, \phi\rangle \\ &\leq \langle \Lambda |f|, \phi\rangle - \eta \end{aligned}$$

and then $\Re e\xi \leq -\eta$. As a consequence, $\Delta_\eta \cap \Sigma(\Lambda) = \{0\}$ and we conclude thanks to Theorem 3.1. □

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