Geophys. J. Int. (1992) **111**, 159–169

Spectral analysis of the full gravity tensor

R. Rummel and M. van Gelderen
Delft University of Technology, Faculty of Geodetic Engineering, 2629 JA Delft, Netherlands

Accepted 1992 April 18. Received 1992 April 14; in original form 1991 October 18

SUMMARY

The five independent components Γ_{xz}, Γ_{yz}, Γ_{zz}, $\Gamma_{xx} - \Gamma_{yy}$, and Γ_{xy} of the gravity tensor are measurable by gradiometers. When grouped into $\{\Gamma_{zz}\}$, $\{\Gamma_{xz}, \Gamma_{yz}\}$ and $\{\Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy}\}$, and expanded into an infinite series of pure-spin spherical harmonic tensors, simple eigenvalue connections can be derived between these three sets and the spherical harmonic expansion of the gravity potential. The three eigenvalues are $(n+1)(n+2)$, $-(n+2)\sqrt{n(n+1)}$ and $\sqrt{(n-1)n(n+1)(n+2)}$. This result permits an easy analytical incorporation of all measurable tensor components into a spectral signal and noise analysis of gravity quantities on a sphere. Analogous relations also exist for a 2-D Fourier (flat earth) expansion of these three sets. An additional advantageous feature of the set $\{\Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy}\}$, besides the simple eigenvalue relation, is its invariance with respect to small position uncertainties, e.g., of the trajectory of the satellite or airplane carrying the gradiometer. Hence a complete framework exists in terms of the eigenvectors of all operators connecting the zeroth, first, and second derivatives of the gravitational potential. At the same time the results facilitate the planning of gradiometer missions and their data analysis.

Key words: geodesy, gravity gradiometry, gravity tensor, tensor spherical harmonics.

1 INTRODUCTION

Despite the large amount of gravity material available nowadays the current knowledge of the Earth's gravity field still is far from sufficient. Geopotential models, i.e. sets of spherical harmonic coefficients as derived from the analysis of satellite orbits, cover only the long wavelength part of the field with a spatial resolution of typically 1000 km. Terrestrial gravity measurements are missing for large parts of the Earth's surface, and altimetric data do not cover the polar areas of the oceans. Plans exist to improve this situation during the coming decade by means of satellite and airborne gradiometry. Gravity gradiometry is the measurement of the second derivatives of the gravity potential. A global, detailed picture of the gravity field will improve the understanding of the structure, composition and dynamics of the solid earth and in conjunction with other new methods for probing the Earth's interior, provide a more accurate three-dimensional model, *cf.* Geophysical and Geodetic Requirements (1987), Lambeck (1990), or Solid Earth Science in the 1990s (1991).

The gravity tensor consists of nine components, the nine second derivatives of the gravitational potential V:

$$\mathbf{\Gamma}_{ij} = \frac{\partial^2 V}{\partial x^i\, \partial x^j}, \qquad i, j = 1, 2, 3. \tag{1}$$

In this context the derivatives are

$$\left\{\frac{\partial}{\partial x^i} \,\middle|\, i = 1, 2, 3\right\} \equiv \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right\} = \left\{\frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r \sin\theta}\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial r}\right\},$$

taken with respect to the local spherical, orthonormal triad $\mathbf{e}_i$, where $\mathbf{e}_{i=1} = \mathbf{e}_x$ points north, $\mathbf{e}_{i=2} = \mathbf{e}_y$ points east, and $\mathbf{e}_{i=3} = \mathbf{e}_z$ is directed radially outwards. Since the components Γ_{ij} can be viewed as gradients of the gravity vector components $g_i = \partial V/\partial x^i$, their measurement is commonly referred to as gradiometry. Thus gradiometry is the measurement of Γ_{ij}, either of all nine components, or of some of them, or of a specific linear combination. In case the measurements are taken on the

Earth's surface, the effect of the Earth's rotation has to be taken into account, which means the gravitational potential V has to be replaced by W, the sum of V and the centrifugal potential Z.

Terrestrial gradiometer measurements are very much affected by density variations and topographic features in the immediate vicinity of the observation point. This makes them very suitable for exploration geophysics, *cf.* Jung (1961). However, since the torsion balance, the traditional gradiometric instrument, requires very laborious field and reduction work, and more efficient modern measurement devices do not exist, terrestrial gradiometry is basically not applied anymore. Much is expected, on the other hand, from the development of aerial and satellite gradiometers, see e.g. Paik (1981), Spaceborne Gravity Gradiometers (1983), Balmino *et al.* (1984), and Jekeli (1988), because by gradiometric measurements, taken at a certain altitude above the Earth's surface, the natural gravity attenuation with increasing distance from the Earth's mass distribution is compensated for a great deal. This point shall be illustrated by an example, considering the second radial derivative $\Gamma_{zz} = \partial^2 V/\partial r^2$.

The gravitational potential V is a harmonic function outside the Earth's surface (say, outside a sphere S with radius R, the mean radius of the Earth). Thus it can be expanded into an infinite series of solid spherical harmonics $\varphi_{nm}(P) = (R/r)^{n+1} Y_{nm}(\theta, \lambda)$ with coefficients v_{nm} and $r > R$. $Y_{nm}(\theta, \lambda)$ are the surface spherical harmonics

$$Y_{nm}(\theta, \lambda) = \bar{P}_{n\,|m|}(\cos\theta)\begin{Bmatrix} \cos m\lambda \\ \sin |m|\,\lambda \end{Bmatrix}_{m<0}^{m\geq 0}, \tag{2}$$

and $\bar{P}_{nm}$ the fully normalized associated spherical harmonics. The second radial derivative yields for $r = R$

$$R^2 \frac{\partial^2}{\partial r^2} \varphi_{nm} = R^2 \partial_{rr} \varphi_{nm} = (n+1)(n+2)\varphi_{nm}. \tag{3}$$

Comparison of the original potential function V with Γ_{zz} shows a multiplication of the coefficients v_{nm} by a factor $(n+1)(n+2) \approx n^2$. Since the degree n corresponds approximately to a spatial wavelength $\lambda = 2\pi R/n$ the high sensitivity of Γ_{zz} to local density contrasts is explained. Repeating the same consideration at altitude h $(r = R + h)$ above the Earth's surface, we find

$$r^2 \partial_{rr} \varphi_{nm}(\theta, \lambda, r) = \left(\frac{R}{R+h}\right)^{n+1} (n+1)(n+2)\varphi_{nm}(\theta, \lambda, R). \tag{4}$$

Hence one sees that the attentuation effect $(R/(R+h))^{n+1}$ is, to a certain extent, compensated by the 'differentiation effect' $(n+1)(n+2)$.

This feature as well as the high attainable data rate, both regionally and globally, have led in recent years to considerable activity in the field of spaceborne gradiometry. However, the idea of spaceborne gradiometry has been pursued for more than twenty years (Rummel 1986a). During all these years the fundamental characteristics of spaceborne gradiometry, in terms of gravity field determination, are commonly explained employing spectral analysis as applied to spherical harmonic expansions, analogous to the example shown above, *cf.* Meissl (1971), Glaser & Sherry (1972), Balmino (1974) and Rummel (1975, 1979). If a spectral model of the measurement noise is introduced as well, fairly realistic estimates of the expected signal and noise propagation and of the attainable spatial resolution and accuracy can be derived; we refer to Kaula (1969), Jekeli & Rapp (1980), Rummel (1979), or Rapp (1989).

So far spectral analysis of gradiometry has been confined to the second radial derivative Γ_{zz}. The purpose of this article is to show that it can be applied to all independent components of the gravity tensor. In Section 2 spherical harmonic vectors and second-rank tensors will be discussed. The latter type is used in Section 3 to evaluate the signal and noise propagation of satellite gradiometry. Section 4 contains a short discussion on the effect of orbit errors. In Section 5 spectral analysis of the gravity gradients shall be applied using two-dimensional Fourier expansions. Section 6 contains the conclusions.

2 EXPANSION INTO SPHERICAL HARMONIC VECTORS AND TENSORS

Since the gravitational potential field V is harmonic outside of S and irrotational $(\nabla \times \nabla V = 0)$ it follows for the gravity tensor

$$\sum_{i=1}^{3} \Gamma_{ii} = 0 \qquad \text{(traceless)}, \tag{5a}$$

and

$$\Gamma_{ij} = \Gamma_{ji} \quad \text{and} \quad i \neq j \quad \text{(symmetric)}. \tag{5b}$$

Consequently only five components are independent, e.g. Γ_{xx}, Γ_{yy}, Γ_{xy}, Γ_{yz}, and Γ_{zx}. On $S(0, R)$ the five components are regarded as continuous and continuously differentiable functions. On S the solid spherical harmonics φ_{nm} form a complete set of orthonormal base functions of a separable Hilbert space with $\varphi_{nm} = Y_{nm}$ for $r = R$. This allows V and Γ_{zz} with equation (3) to be written as:

$$V(P)=\frac{\mu}{R}\sum_{n=0}^{\infty}\sum_{m=-n}^{+n} v_{nm}\varphi_{nm}(P)=\frac{\mu}{R}\sum_{n,m} v_{nm}Y_{nm}(\theta_P,\lambda_P), \tag{6}$$

and

$$\Gamma_{zz}(P)=\partial_{rr}V(P)=\frac{\mu}{R^3}\sum_{n,m} v_{nm}(n+1)(n+2)Y_{nm}(\theta_P,\lambda_P), \tag{7}$$

where $\mu = GM$ is the gravitational constant times the mass of the Earth.

Equation (3) can be regarded as the eigenvalue expansion of the linear self-adjoint operator $R^2\partial_{rr}$ with eigenvalues $\lambda_n=(n+1)(n+2)$. On S the self-adjoint operators are isotropic, as explained in Meissl's (1971) discussion of the global covariance function on the sphere. Meissl also extends his theory from scalar to tangent vector functions. Let the surface gradient operator Grad be defined as: $\mathrm{Grad}=r\nabla_s=(-\partial/\partial\theta, \sin^{-1}\theta\,\partial/\partial\lambda)^T$, and the surface divergence as: $\mathrm{Div}=(r\nabla_s\cdot)=(-\partial/\partial\theta, \sin^{-1}\theta\,\partial/\partial\lambda)$. Then we have:

$$\mathrm{Grad}\,Y_{nm}(P)=\sqrt{n(n+1)}\,\mathbf{X}_{nm}(P), \tag{8}$$

with $\mathbf{X}_{nm}$ the orthonormal set of tangent vector spherical harmonics denoted U_{nm} by Meissl (1971). For the adjoint operator we find

$$\mathrm{Div}\,\mathbf{X}_{nm}(P)=-\sqrt{n(n+1)}\,Y_{nm}(P). \tag{9}$$

An immediate consequence is the surface Laplace equation:

$$\mathrm{Div\,Grad}\,Y_{nm}(P)=\mathrm{Lap}\,Y_{nm}(P)=-n(n+1)Y_{nm}(P), \tag{10}$$

with Lap the surface Laplace operator, compare e.g. Heiskanen & Moritz [1967, eq. (1–44)]. By Green's identities it can be shown that the $\mathbf{X}_{nm}(P)$ form a complete, orthonormal system of eigenfunctions on S for tangent, spherical vector fields. Then

$$\begin{aligned}\int_\Omega &\mathbf{X}_{nm}(P)\cdot\mathbf{X}_{k\ell}(P)\,d\Omega_P\\ &=\frac{1}{\sqrt{n(n+1)}}\frac{1}{\sqrt{k(k+1)}}\int_\Omega \mathrm{Grad}\,Y_{nm}(P)\cdot\mathrm{Grad}\,Y_{k\ell}(P)\,d\Omega\\ &=\frac{-1}{\sqrt{n(n+1)}}\frac{1}{\sqrt{k(k+1)}}\int_\Omega Y_{nm}(P)\,\mathrm{Lap}\,Y_{k\ell}(P)\,d\Omega\\ &=4\pi\boldsymbol{\delta}_{nk}\boldsymbol{\delta}_{m\ell},\end{aligned} \tag{11}$$

where Ω is the unit sphere $\Omega(0,1)$ and $d\Omega=\sin\theta\,d\lambda\,d\theta$. Meissl applied the vector spherical harmonics to the expansion of the deflections of the vertical, see also Groten & Moritz (1964).

One may suspect that for the expansion of the second derivatives of V, i.e. of the gravity tensor components, also an orthonormal in this case, a tensorial system exists on S. It is given in Regge & Wheeler (1957) and discussed in more detail in Zerilli (1970). In the case of a second-rank, trace free and symmetric tensor, compare equations (5a, b), the group of the five components is irreducible and belongs to the representation $\mathscr{D}^{(2)}$. We refer to Rose (1957, chapter 17), Joshi (1978, chapter 6.5.3), or Jones (1985, chapter 3.2). For our purpose, three kinds of tensor spherical harmonics are needed, denoted $\mathbf{Z}^{(0)}_{nm}$, $\mathbf{Z}^{(1)}_{nm}$, and $\mathbf{Z}^{(2)}_{nm}$. They suffice to represent the three kinds of tensor components $\{\Gamma_{zz}(=-\Gamma_{xx}-\Gamma_{yy})\}$, $\{\Gamma_{xz},\Gamma_{yz}\}$, and $\{\Gamma_{xx}-\Gamma_{yy}, 2\Gamma_{xy}\}$. In the local spherical triad $\mathbf{e}_i$ they consist of a tangential and a radial part. The tensor spherical harmonics, given here for $r=R$, are defined by the following eigenvalue expansions:

$$R^2\mathscr{Z}^{(0)}\varphi_{nm}=R^2\begin{pmatrix}0&0&0\\0&0&0\\0&0&\dfrac{\partial^2}{\partial z^2}\end{pmatrix}\varphi_{nm}=R^2\begin{pmatrix}0&0&0\\0&0&0\\0&0&\dfrac{\partial^2}{\partial r^2}\end{pmatrix}\varphi_{nm}\equiv(n+1)(n+2)\mathbf{Z}^{(0)}_{nm}, \tag{12a}$$

$$\begin{aligned}R^2\mathscr{Z}^{(1)}\varphi_{nm}&=\frac{R^2}{\sqrt{2}}\begin{pmatrix}0&0&\dfrac{\partial^2}{\partial x\,\partial z}\\0&0&\dfrac{\partial^2}{\partial y\,\partial z}\\ *&*&0\end{pmatrix}\varphi_{nm}=\frac{R}{\sqrt{2}}\begin{pmatrix}0&0&\dfrac{-r\partial^2}{\partial\theta\,\partial r}+\dfrac{\partial}{\partial\theta}\\0&0&\sin^{-1}\theta\left(\dfrac{r\partial^2}{\partial\lambda\,\partial r}-\dfrac{\partial}{\partial\theta}\right)\\ *&*&0\end{pmatrix}\varphi_{nm}\\ &\equiv-(n+2)\sqrt{n(n+1)}\,\mathbf{Z}^{(1)}_{nm},\end{aligned} \tag{12b}$$

$$R^2\mathscr{Z}^{(2)}\varphi_{nm}=\frac{R^2}{\sqrt{2}}\begin{pmatrix}\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2} & -2\frac{\partial^2}{\partial x\,\partial y} & 0\\ * & \frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2} & 0\\ 0 & 0 & 0\end{pmatrix}\varphi_{nm}$$

$$=\frac{1}{\sqrt{2}}\begin{pmatrix}\left(\frac{\partial^2}{\partial\theta^2}-\cot\theta\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}\right) & \frac{2}{\sin\theta}\left(\frac{\partial^2}{\partial\theta\,\partial\lambda}-\cot\theta\frac{\partial}{\partial\lambda}\right) & 0\\ * & -\left(\frac{\partial^2}{\partial\theta^2}-\cot\theta\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}\right) & 0\\ 0 & 0 & 0\end{pmatrix}\varphi_{nm}$$

$$\equiv\sqrt{\frac{(n+2)!}{(n-2)!}}\,\mathbf{Z}^{(2)}_{nm} \tag{12c}$$

(the asterisk indicating symmetry of the component with respect to the main diagonal).

The tensor spherical harmonics $\mathbf{Z}^{(0)}_{nm}$, $\mathbf{Z}^{(1)}_{nm}$, and $\mathbf{Z}^{(2)}_{nm}$ are, apart from the scale factor, essentially identical to the pure-spin tensor harmonics $T^{L0,nm}$, $T^{E1,nm}$, and $T^{E2,nm}$, respectively, of Thorne (1980) or a_{nm}, b_{nm}, and f_{nm} of Zerilli (1970). They are applied in mathematical physics. There they are used for example for the representation of gravitational radiation. However, whereas the latter are expressed as co- and contravariant components with respect to the curvilinear coordinates θ and λ, we prefer to give them as Cartesian components with respect to the local spherical triad $\mathbf{e}_i$. Consequently they provide a more direct connection to the gradiometric observables. With the tensor components indicated by i and j the orthonormality relationship of these tensor spherical harmonics is

$$\int_\Omega\sum_{i,j}[\mathbf{Z}^{(\alpha)}_{nm}]_{ij}[\mathbf{Z}^{(\beta)}_{k\ell}]_{ij}\,d\Omega=4\pi\delta_{nk}\delta_{m\ell}\delta_{\alpha\beta}, \tag{13}$$

where α and $\beta=0$, 1, 2 and $[.]_{ij}$ denote the tensor components.

The operators of equations (12) applied to the gravitational potential (6) give the expansions of the measurable components $\{\Gamma_{zz}\}$, $\{\Gamma_{xz},\Gamma_{yz}\}$ and $\{\Gamma_{xx}-\Gamma_{yy},2\Gamma_{xy}\}$ expressed here as the three second-rank tensors $\mathbf{\Gamma}^{(0)}$, $\mathbf{\Gamma}^{(1)}$ and $\mathbf{\Gamma}^{(2)}$:

$$\mathbf{\Gamma}^{(0)}=\mathscr{Z}^{(0)}V(P)=\frac{\mu}{R}\sum_{n,m}v_{nm}\mathscr{Z}^{(0)}\varphi_{nm}(P)=\frac{\mu}{R}\sum_{n,m}v_{nm}\frac{(n+1)(n+2)}{R^2}\mathbf{Z}^{(0)}_{nm}(P)$$

$$=\frac{\mu}{R^3}\sum_{n,m}z^{(0)}_{nm}\mathbf{Z}^{(0)}_{nm}(P), \tag{14a}$$

$$\mathbf{\Gamma}^{(1)}=\mathscr{Z}^{(1)}V(P)=\frac{\mu}{R}\sum_{n,m}v_{nm}\frac{-(n+2)\sqrt{n(n+1)}}{R^2}\mathbf{Z}^{(1)}_{nm}(P)$$

$$=\frac{\mu}{R^3}\sum_{n,m}z^{(1)}_{nm}\mathbf{Z}^{(1)}_{nm}(P), \tag{14b}$$

$$\mathbf{\Gamma}^{(2)}=\mathscr{Z}^{(2)}V(P)=\frac{\mu}{R}\sum_{n,m}v_{nm}\frac{\sqrt{(n-1)n(n+1)(n+2)}}{R^2}\mathbf{Z}^{(2)}_{nm}(P)$$

$$=\frac{\mu}{R^3}\sum_{n,m}z^{(2)}_{nm}\mathbf{Z}^{(2)}_{nm}(P). \tag{14c}$$

Thus the expansion coefficients of three spherical tensors are related to those of the potential V by the spectral connection (eigenvalues) of Table 1. [Remember: the coefficients of the spherical tensor (or vector) expansions are scalars!] Conversely,

Table 1. Eigenvalues connecting the potential coefficients v_{nm} with those of the irreducible second-rank, spherical tensors containing $\{\Gamma_{zz}\}$, $\{\Gamma_{xz},\Gamma_{yz}\}$, and $\{\Gamma_{xx}-\Gamma_{yy},2\Gamma_{xy}\}$, respectively.

	v_{nm}	
$z^{(0)}_{nm}$	$\lambda^{(0)}_n=(n+1)(n+2)$	(15a)
$z^{(1)}_{nm}$	$\lambda^{(1)}_n=-(n+2)\sqrt{n(n+1)}$	(15b)
$z^{(2)}_{nm}$	$\lambda^{(2)}_n=\sqrt{(n-1)n(n+1)(n+2)}$	(15c)

from the orthogonality relations (13) the components can be determined by

$$v_{nm}\lambda_n^{(\alpha)} = z_{nm}^{(\alpha)} = \left(\frac{\mu}{R^3}\right)^{-1} \int_\Omega \sum_{i,j} [\mathbf{\Gamma}^{(\alpha)}]_{ij}[\mathbf{Z}_{nm}^{(\alpha)}]_{ij}\, d\Omega, \qquad \alpha = 1, 2, 3. \tag{16}$$

Hence the spectral expressions for all five independent tensor components are derived, in complete analogy to the scalar and vector gravity functionals on the sphere. We close this section with three remarks.

(1) The only non-zero component of $\mathbf{\Gamma}^{(0)}$ is Γ_{zz}. Its eigenvalues, equation (15a), are identical to those derived in equation (3). Thus we see that Γ_{zz} can be treated as a scalar function on S as well. Similarly, the combination $\{\Gamma_{xz}, \Gamma_{yz}\}$, contained in $\mathbf{\Gamma}^{(1)}$ could be interpreted as components of a tangent spherical vector field by means of (8) and (9). This has been done by Meissl (1971).

(2) $\mathbf{\Gamma}^{(2)}$ contains the elements $\Gamma_{xx} - \Gamma_{yy}$ and $2\Gamma_{xy}$. These are the well-known observables of a torsion balance, *cf.* Selényi (1953) or Jung (1961), and of the rotating spaceborne gradiometer discussed, e.g. in Spaceborne Gravity Gradiometers (1983) or Rummel (1986b).

(3) If the gravitational potential V is to be determined from the observable tensors $\mathbf{\Gamma}^{(0)}$, $\mathbf{\Gamma}^{(1)}$ or $\mathbf{\Gamma}^{(2)}$ individually, a singularity emerges in the case of $\mathbf{\Gamma}^{(1)}$ for degree $n = 0$, and in the case of $\mathbf{\Gamma}^{(2)}$ for degrees $n = 0$ and $n = 1$, compare Table 1.

3 SPECTRAL ANALYSIS ON THE SPHERE

The expected average signal power of the gravitational potential V, or of the disturbance potential T, i.e. of the remaining part of V after subtracting a known reference field U, is expressed by the dimensionless signal degree variances c_n:

$$c_n = \sum_{m=-n}^{+n} v_{nm}^2. \tag{17a}$$

The expected average size of the individual coefficient is given by the 'degree-order' variance:

$$c_{nm} = \frac{1}{2n+1} c_n = \frac{1}{2n+1} \sum_m v_{nm}^2. \tag{17b}$$

Both variances can be determined either from the coefficients up to the maximum degree of one of the available geopotential models or from a degree variance model, e.g. that of Tscherning & Rapp (1978). From it the degree variances of $\mathbf{\Gamma}^{(0)}$, $\mathbf{\Gamma}^{(1)}$ and $\mathbf{\Gamma}^{(2)}$ are determined by the propagation law of covariances, *cf.* Meissl (1971) or Moritz (1980). With (15) we have

$$c_n(\mathbf{\Gamma}^{(0)}) = (\lambda_n^{(0)})^2 c_n(T) = (n+1)^2(n+2)^2 c_n(T), \tag{18a}$$

$$c_n(\mathbf{\Gamma}^{(1)}) = (\lambda_n^{(1)})^2 c_n(T) = n(n+1)(n+2)^2 c_n(T), \tag{18b}$$

$$c_n(\mathbf{\Gamma}^{(2)}) = (\lambda_n^{(2)})^2 c_n(T) = (n-1)n(n+1)(n+2) c_n(T). \tag{18c}$$

Hence we observe that basically all three types of gradients are related to T by a factor $\lambda_n^2 \approx n^4$. Their spectra are almost identical. On a first inspection this seems somewhat surprising, as we generally have the idea, that the radial component Γ_{zz} has a higher sensitivity to short wavelength features in the gravity spectrum.

The error spectrum is determined by a spherical harmonic expansion of the error covariance function. If we assume for each gradiometer component slightly correlated white noise of constant variance σ_0^2 over S, the degree-order error variance σ_{nm}^2 becomes (*cf.* Heiskanen & Moritz 1967, chapter 7-7):

$$\sigma_{nm}^2 = \frac{S'}{4\pi R^2}, \tag{19}$$

where S' represents the integral (volume) of the covariance function on S. It is the model of a band-limited white noise. For example, for the case of uncorrelated equal area blocks of size $R^2\,\Delta s$ and variance σ_0^2 (see, Jekeli & Rapp 1980) it is $S' = \sigma_0^2 R^2\,\Delta s$ and therefore

$$\sigma_{nm}^2 = \sigma_0^2 \frac{\Delta s}{4\pi}. \tag{20}$$

For spaceborne experiments, several ways are discussed in the literature on how to determine S' from the available samples and their along- and cross-track distribution, *cf.* Migliaccio & Sansò (1989). Let us now assume that each of the five independent gradiometer components can be described independently with the same error model, e.g., unbiased, band-limited white noise of variance $\sigma_0^2 = 10^{-4}\,\mathrm{E}^2\,\mathrm{Hz}^{-1}$ ($1\mathrm{E} = 10^{-9}\,\mathrm{s}^{-2}$). For the component Γ_{xy} two cases shall be distinguished.

Case (1) Γ_{xy} is derived from two independent observations: $\Gamma_{xy} = \frac{1}{2}[(xy) + (yx)]$ [with (xy) and (yx) denoting the measured components]; and

Case (2) only Γ_{xy} is measured.

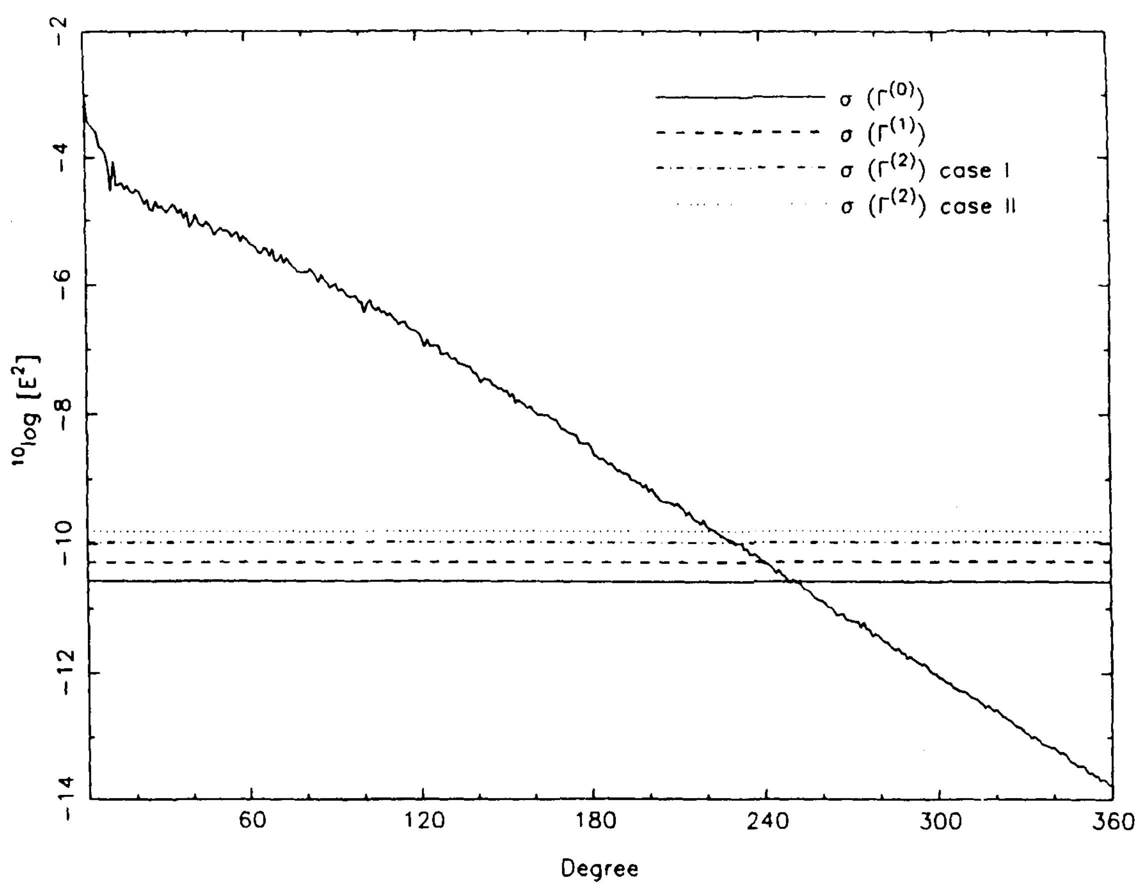

Figure 1. Signal degree-order variances c_{nm} (equation 17b) at 200 km altitude based on the geopotential model OSU89 up to degree and order 360 and noise degree-order variances assuming a standard deviation of 10^{-2} E/$\sqrt{\text{Hz}}$ for each gradient component, see equation (20). The noise spectra refer to Γ_{zz}, $\{\Gamma_{xz}, \Gamma_{yz}\}$ and $\{\Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy}\}$, the latter under the assumption of case (1) and case (2), respectively.

Then straightforward error propagation yields for the ratios of the degree-order variances:

Case (1) $\quad \sigma^2_{nm}(\mathbf{\Gamma}^{(2)}) : \sigma^2_{nm}(\mathbf{\Gamma}^{(1)}) : \sigma^2_{nm}(\mathbf{\Gamma}^{(0)}) = 4:2:1,$ (21a)

Case (2) $\quad \sigma^2_{nm}(\mathbf{\Gamma}^{(2)}) : \sigma^2_{nm}(\mathbf{\Gamma}^{(1)}) : \sigma^2_{nm}(\mathbf{\Gamma}^{(0)}(= 6:2:1.$ (21b)

Hence we see that the error standard deviation of $\mathbf{\Gamma}^{(2)}$ is a factor of 2, or in case (2) even $\sqrt{6}$ higher than that of $\mathbf{\Gamma}^{(0)}$ because of the number of involved tensor components. Hence the superiority of $\{\Gamma_{zz}\}$ above the combinations $\{\Gamma_{xz}, \Gamma_{yz}\}$ or $\{\Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy}\}$ stems not from a higher signal content in the short wavelengths but from the lower noise level and is hardly significant.

In Fig. 1 the signal and noise degree-order spectra are displayed for a satellite altitude of 200 km. The noise spectrum is computed assuming a measurement standard deviation of 10^{-2} E/$\sqrt{\text{Hz}}$ for each component, a mission duration of 6 months and an integrated sample interval of 4 s. The signal degree variances were computed with the OSU89 geopotential model of Rapp & Pavlis (1990). We see the signal spectrum, which is practically the same for $\mathbf{\Gamma}^{(0)}$, $\mathbf{\Gamma}^{(1)}$ and $\mathbf{\Gamma}^{(2)}$, and the white noise spectra. The noise level of $\mathbf{\Gamma}^{(1)}$ is slightly higher than that of $\mathbf{\Gamma}^{(0)}$, and that of $\mathbf{\Gamma}^{(2)}$ (cases 1 and 2) slightly higher than that of $\mathbf{\Gamma}^{(1)}$. As a consequence the attainable resolution, defined by a signal-to-noise ratio of one, is $n = 248$ for $\mathbf{\Gamma}^{(0)}$, $n = 240$ for $\mathbf{\Gamma}^{(1)}$ and $n = 228$ or 223 for $\mathbf{\Gamma}^{(2)}$, respectively. For the same case, the predicted worldwide geoid (and gravity anomaly) precision would be 8 cm (3.1 mGal), 9 cm (3.3 mGal), and 10 or 11 cm (3.6 or 3.7 mGal), respectively. The mutual differences are practically irrelevant. This means that with all three versions, $\mathbf{\Gamma}^{(0)}$, $\mathbf{\Gamma}^{(1)}$ or $\mathbf{\Gamma}^{(2)}$, a worldwide, high resolution ($\approx$100 km) geoid or gravity anomaly field can be determined in a few months time by satellite gradiometry.

Relation (21) has been checked by the following experiment. The gradiometer components Γ_{xx}, Γ_{yy}, Γ_{zz}, Γ_{xy}, Γ_{xz} and Γ_{zx} have been determined from the OSU89 set of potential coefficients on a global $0.25° \times 0.7°$ grid. Then by spherical harmonic analysis the coefficients of each of these components have been computed up to degree and order 240. From them the signal degree variances c_n of each component have been determined. What should be their expected average size? Since according to equation (18) $c_n(\mathbf{\Gamma}^{(0)}) \approx c_n(\mathbf{\Gamma}^{(1)}) \approx c_n(\mathbf{\Gamma}^{(2)}) \approx n^4 c_n(T)$, it can be expected that the signal variance level of the individual

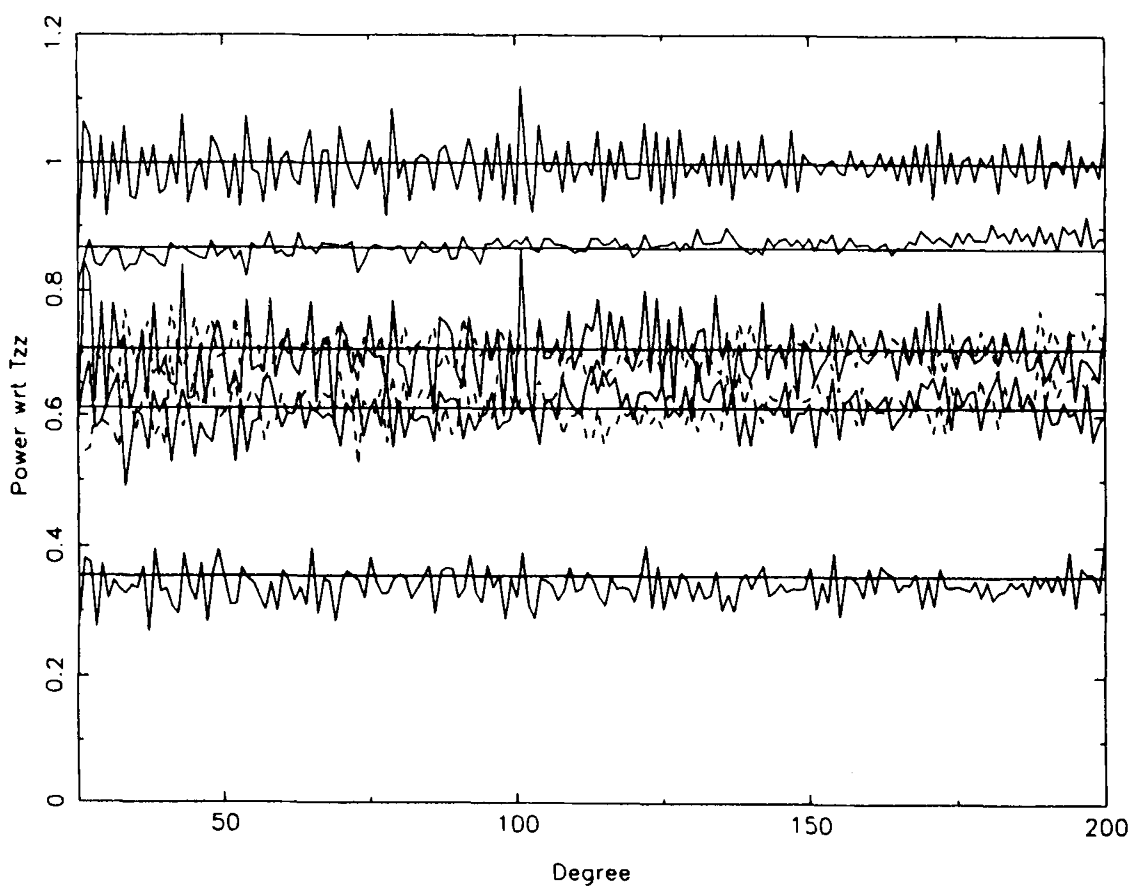

Figure 2. Root mean square (rms) spectra per degree relative to Γ_{zz} of (from top to bottom) $c_n(xz)+c_n(yz)$, $c_n(xx)+c_n(yy)$, $c_n(xz)$, $c_n(yz)$ (dashed line), $c_n(xx)$, $c_n(yy)$ (dashed line), and $c_n(xy)$. The straight lines indicate the theoretically expected values.

component becomes

$$c_n(\Gamma_{xz}) = c_n(\Gamma_{yz}) = \tfrac{1}{2}c_n(\Gamma_{zz}),$$

and

$$c_n(\Gamma_{xy}) = \tfrac{1}{3}c_n(\Gamma_{yy}) = \tfrac{1}{3}c_n(\Gamma_{xx}) = \tfrac{1}{8}c_n(\Gamma_{zz}).$$

This is in essence the inverse reasoning followed for (21a) and (21b), where starting from the same noise level for each gradiometer component that of the composed quantities $\mathbf{\Gamma}^{(0)}$, $\mathbf{\Gamma}^{(1)}$ and $\mathbf{\Gamma}^{(2)}$ was deduced. From the above the following root mean square (rms) average signal ratios per degree follow:

$$\frac{\mathrm{rms}_n\{\Gamma_{zz}\}}{\mathrm{rms}_n\{\Gamma_{xz}\}} = \frac{\mathrm{rms}_n\{\Gamma_{zz}\}}{\mathrm{rms}_n\{\Gamma_{yz}\}} = \frac{1}{\sqrt{2}},$$
$$\frac{\mathrm{rms}_n\{\Gamma_{zz}\}}{\mathrm{rms}_n\{\Gamma_{xx}\}} = \frac{\mathrm{rms}_n\{\Gamma_{zz}\}}{\mathrm{rms}_n\{\Gamma_{yy}\}} = \frac{1}{2\sqrt{2/3}}, \qquad (22)$$
$$\frac{\mathrm{rms}_n\{\Gamma_{zz}\}}{\mathrm{rms}_n\{\Gamma_{xy}\}} = \frac{1}{2\sqrt{2}}.$$

The rms_n spectra relative to Γ_{zz} are displayed in Fig. 2. They confirm equation (22), and consequently (21), see also Rummel, Koop & Schrama (1989).

4 THE EFFECT OF THE ORBIT UNCERTAINTY

In all experiments the observation location is only known approximately. Thus, not only have the coefficients of the gravitational field got to be determined, but the unknown point coordinate corrections (Δx, Δy, Δz) enter as well. In the case of terrestrial observations they are highly irregular and related to the terrain, in satellite applications they describe the difference between a chosen reference and the actual orbit and can be described by orbit dynamics. In Rummel & Colombo

(1985) a linear gradiometric model has been derived in spherical approximation:

$$\begin{pmatrix} \Delta\Gamma_{xx} \\ \Delta\Gamma_{xy} \\ \Delta\Gamma_{xz} \\ \Delta\Gamma_{yy} \\ \Delta\Gamma_{yz} \\ \Delta\Gamma_{zz} \end{pmatrix} = \frac{3\mu}{r^4} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} + \begin{pmatrix} T_{xx} \\ T_{xy} \\ T_{xz} \\ T_{yy} \\ T_{yz} \\ T_{zz} \end{pmatrix} \tag{23}$$

Hereby the anomalous quantities $\Delta\Gamma_{ij}$ represent the difference between a measurable gradiometric component and the corresponding component expressed in a chosen reference gravity field and evaluated at the approximate location. The T_{ij} represent the second derivatives of the unknown disturbance potential. In a least-squares adjustment they are expanded into series of spherical harmonics with unknown coefficients v_{nm}. If from the $\Delta\Gamma_{ij}$ the anomalous tensors $\Delta\mathbf{\Gamma}^{(\alpha)}$ are formed, analogous to (14), we observe that $\Delta\mathbf{\Gamma}^{(2)}$, containing $\Delta\Gamma_{xx} - \Delta\Gamma_{yy}$ and $2\,\Delta\Gamma_{xy}$, becomes invariant with respect to displacements Δx, Δy and Δz. Thus not only does this combination possess simple spectral properties, it also turns out to be, in spherical approximation, insensitive to small displacements, independent of whether they refer to a terrestrial point or an aerial or satellite trajectory.

5 ISOTROPIC EIGENVALUES OF 2-D FFT REPRESENTATION

In this section it shall briefly be shown that analogous isotropic spectral relations, as derived for $\mathbf{\Gamma}^{(\alpha)}$ in terms of spherical harmonics, hold true for 2-D Fourier expansions too. In recent years representation of the local gravity field in terms of 2-D Fourier series received increasing attention. The reason for this is that, on the one hand, better and better geopotential models became available which properly take care of the global, long wavelength part of the gravity field and that, on the other hand, the computational efficiency of FFT makes this approach very attractive. The spectral theory of the gravity field in terms of Fourier expansion is reviewed, for example, by Jordan (1978), Hofmann–Wellenhof & Moritz (1986) or recently by Schwarz, Sideris & Forsberg, (1990). See also Dorman & Lewis (1974).

From the solution of Laplace's equation for the half-plane with $z \geq 0$ (where z is the altitude above the plane representing the surface of the earth) the harmonic base functions become

$$\varphi_{uv}(P) = e^{-wz} e^{-i(ux+vy)}, \tag{24}$$

with the condition that

$$w^2 = u^2 + v^2. \tag{25}$$

The gravitational potential V is now expanded either into the infinite series

$$V(x, y) = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} v_{nm} \varphi_{uv}(x, y), \tag{26}$$

with the understanding that $u = m\dfrac{2\pi}{\lambda_x}$ and $v = n\dfrac{2\pi}{\lambda_y}$, and $\lambda_x \cdot \lambda_y$ is a rectangular area in which V is given and is assumed to be continued periodically in x and y, or as the double integral

$$V(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v(u, v) \varphi_{uv}(x, y)\, du\, dv, \tag{27}$$

for $\lim \lambda_x \to \infty$ and $\lim \lambda_y \to \infty$. The eigenvalues derived from the various differential operators in x, y and z are listed in equation (28), see Table 2.

Applying the tensor operators to φ_{uv} we find:

$$\mathscr{L}^{(0)} \varphi_{uv} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial^2_{zz} \end{pmatrix} \varphi_{uv} \equiv w^2 \mathbf{Z}^{(0)}_{uv}, \tag{29a}$$

$$\mathscr{L}^{(1)} \varphi_{uv} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \partial^2_{xz} \\ 0 & 0 & \partial^2_{yz} \\ * & * & 0 \end{pmatrix} \varphi_{uv} \equiv i w^2 \mathbf{Z}^{(1)}_{uv}, \tag{29b}$$

$$\mathscr{L}^{(2)} \varphi_{uv} = \frac{1}{\sqrt{2}} \begin{pmatrix} \partial^2_{xx} - \partial^2_{yy} & 2\partial^2_{xy} & 0 \\ * & \partial^2_{yy} - \partial^2_{xx} & 0 \\ 0 & 0 & 0 \end{pmatrix} \varphi_{uv} \equiv w^2 \mathbf{Z}^{(2)}_{uv}. \tag{29c}$$

Table 2. Differential operators in x, y and z and corresponding eigenvalues.

operator	eigenvalue	eq.	
∂_x	$-iu$	28.	a
∂_y	$-iv$		b
∂_z	$-w$		c
∂^2_{xx}	$-u^2$		d
∂^2_{xy}	$-uv$		e
∂^2_{xz}	iuw		f
∂^2_{yy}	$-v^2$		g
∂^2_{yz}	ivw		h
∂^2_{zz}	w^2		i

The functions $Z^{(\alpha)}_{uv}$ are orthonormal systems of tensor eigenfunctions:

$$Z^{(0)}_{uv} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \varphi_{uv}, \tag{30a}$$

$$Z^{(1)}_{uv} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \frac{u}{w} \\ 0 & 0 & \frac{v}{w} \\ * & * & 0 \end{pmatrix} \varphi_{uv}, \tag{30b}$$

$$Z^{(2)}_{uv} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{-(u^2 - v^2)}{w^2} & \frac{-2uv}{w^2} & 0 \\ * & \frac{u^2 - v^2}{w^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \varphi_{uv}. \tag{30c}$$

The analogy with the eigenvalue analysis on the sphere is evident, compare with equations (12)–(16). Hence the spectral analysis of signal and noise can be applied along the lines of Section 3.

6 CONCLUSIONS

In recent years plans for gradiometry projects have become more and more concrete. ESA and NASA are jointly planning the Aristoteles mission for the late nineties; for the beginning of the next century a superconducting gravity gradiometer mission (SGGM) is envisaged by NASA. Earlier plans to map the global lunar gravity field by gradiometry are currently under review again. Airborne gradiometry, although technically not mature yet, could soon prove important for measuring details of the gravity field in selected areas. Aristoteles is designed to measure the tensor components Γ_{zz}, Γ_{yz} and Γ_{yy} with a precision of $10^{-2}\,\mathrm{E}/\sqrt{\mathrm{Hz}}$. With this the global gravity field can be determined in six months time with a precision of 2–5 mGal ($1\,\mathrm{mGal} = 10^{-5}\,\mathrm{ms}^{-2}$) in terms of gravity anomalies or 5–10 cm in terms of geoid heights, and with a spatial resolution of 100 km. SGGM would improve these numbers by almost an order of magnitude with a spatial resolution of 50 km.

Although the gradiometer concepts designed so far vary considerably, ranging from one component to full tensor instruments, from inertial to earth pointing orientations, and from fixed to rotating sensors, the error simulations are usually based on the radial component Γ_{zz} only. This component has the highest signal strength and allows very simple error propagation from the gradiometer component to any desired gravity quantity, like geoid heights. With the results of this work the signal power of all measurable gradiometer components can easily be predicted. In terms of signal degree variances the following rule holds: $c_n(\Gamma_{xy}) = \frac{1}{3}c_n(\Gamma_{yy}) = \frac{1}{3}c_n(\Gamma_{xx}) = \frac{1}{4}c_n(\Gamma_{xz}) = \frac{1}{4}c_n(\Gamma_{yz}) = \frac{1}{8}c_n(\Gamma_{zz})$. For example, the signal power of Γ_{xy} is on average only $\frac{1}{8}$ of that of Γ_{zz}. This rule has been confirmed experimentally with one of the available geopotential models.

Furthermore it has been shown that the combinations of gradiometer components $\mathbf{\Gamma}^{(1)} = \{\Gamma_{xz}, \Gamma_{yz}\}$ and $\mathbf{\Gamma}^{(2)} = \{\Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy}\}$ produce the same signal strength as Γ_{zz}. However due to the involvement of two measured components in $\mathbf{\Gamma}^{(1)}$ and three (or four) in $\mathbf{\Gamma}^{(2)}$, in order to reach the same precision and resolution as with Γ_{zz}, their precision must be better by a factor of $\sqrt{2}$ or $\sqrt{6}$ (or 2), respectively. The combination $\mathbf{\Gamma}^{(2)} = \{\Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy}\}$ is of particular interest. It is the combination that, in early days, could be measured with a torsion balance. However modern prototype instruments for airborne and lunar experiments also produce this combination. Error simulation for $\mathbf{\Gamma}^{(1)}$ and $\mathbf{\Gamma}^{(2)}$ turns out to be as simple as that for Γ_{zz}.

Finally, with $\lambda_n^{(1)}$ and the newly found eigenvalue $\lambda_n^{(2)}$, spherical integral formulae can be established, comparable in their structure to the Stokes formula, which convert the measured $\{\Gamma_{xz}, \Gamma_{yz}\}$ or $\{\Gamma_{xx} - \Gamma_{yy}, 2\Gamma_{xy}\}$, given as functions on a sphere (e.g. at satellite altitude), into gravitational potential, geoid height, gravity anomaly or any other desired gravity quantity. This aspect will be elaborated in a later paper. The covariance propagation for the gradient components is discussed by Krarup & Tscherning (1984).

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