# Spectral approach to the communication complexity of multi-party key agreement 

Geoffroy Caillat-Grenier and Andrei Romashchenko

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#### Abstract

In multi-party key agreement protocols it is assumed that the parties are given correlated input data and should agree on a common secret key so that the eavesdropper cannot obtain any information on this key by listening to the communications between the parties. We consider the one-shot setting, when there is no ergodicity assumption on the input data.

It is known that the optimal size of the secret key can be characterized in terms of the mutual information between different combinations of the input data sets, and the optimal key can be produced with the help of the omniscience protocol. However, the optimal communication complexity of this problem remains unknown.

We show that the communication complexity of the omniscience protocol is optimal, at least for some complexity profiles of the input data, in the setting with restricted interaction between parties (the simultaneous messages model). We also provide some upper and lower bounds for communication complexity for other communication problems. Our proof technique combines informationtheoretic inequalities and the spectral method.


Keywords: communication complexity, multiparty secret key agreement, Kolmogorov complexity, expander mixing lemma, information inequalities

## 1 Introduction

In this paper we study a problem of communication complexity motivated by information-theoretic cryptography. We consider multi-party version of the secret key agreement. In this setting, a group of participants is given correlated input data. We assume that some information-theoretic characteristics of this correlation (e.g., the values of the mutual information for various combinations of inputs) are known to the participants of the protocol as well as to the adversary. The aim is to organize a communication over an open channel so that the participants agree on a (preferably large) secret key, while an eavesdropper intercepting the communication gets virtually no information on this key.

This problem was initially studied in the framework of the classical information theory, in terms of Shannon's entropy, starting with the papers [12, 13, 9]. In this setting, the input data of the parties are understood as realizations of correlated probability distributions, and it is usually assumed that these distributions are sequences of i.i.d. random variables. The secrecy of the produced key means that its probability distribution is close the uniform one (conditional on the public data, including the transcript of the communication protocol). Speaking informally, the produced common secret key should look similar to a random output of a symmetric private coin tossed by the parties involved in the protocol and inaccessible to the adversary.

In this paper we study a different approach, which uses algorithmic information theory, [3, 1]. In this setting, the information-theoretic characteristics of the data are defined in terms of Kolmogorov complexity. The secrecy of the produced key means that this key is incompressible, even if we are given all the public data including the transcript of the communication protocol. Practically, this means that the adversary can reveal the value of this secret key (and crack a cryptographic scheme based on this key) only by the brute-force search, see the discussion in 1. This approach follows the general paradigm of building the foundations of cryptography in the framework of algorithmic information theory suggested in 6. We focus on the input data that cannot be produced by correlated i.i.d. sources, and there is no valid ergodicity assumption (so-called "one-shot" paradigm).

In the setting explained above, the optimal size of the secret key in protocols of secret key agreement is fully settled. For two parties, the maximal size of the key is equal to the mutual information between the inputs of the parties; for multi-party protocols the answer is more involved, but the optimal size of the key still can be determined explicitly (as a function of the mutual informations between the input data), see [3. Moreover, for the protocols involving only two parties even the optimal communication complexity is known, see [1]. For $k \geq 3$ parties only an upper bound for communication complexity is known (it is provided by the universal omniscience protocol, see [3).

In this article, we begin the study of the communication complexity of the multi-party version of secret key agreement. For protocols with $k \geq 3$ participants the variety of information parameters of the input data profiles is very large. We restrict ourselves to considering input data (for a triple of parties) with symmetric information profiles (see the values of mutual informations in Fig. 2(b)). We focus on profiles of the input data for which the known communication protocol has a large communication complexity (the number of communicated bits is much larger than the size of the secret key generated by the participants). We prove a somewhat surprising result and show that in this setting the standard omniscience protocol of secret key agreement actually provides the optimal communication complexity among all non-interactive protocols, i.e., among the protocols where the parties broadcast their messages simultaneously (in one round) and then compute the result. We show that, on the other hand, the communication complexity can be improved if we use multi-round protocols. A complete characterization of communication complexity of multi-party secret key agreement with arbitrary tuples of initial data (including non-symmetric information profiles) might be more difficult and requires a more comprehensive research.

We use a technique that combines information-theoretic inequalities and spectral bounds (the expander mixing lemma). An application of a spectral bound per se is not new in communication complexity (see, e.g., the usage of Lindsey's lemma in [15]). Information-theoretic methods are also very common in this area. The combination of these two techniques seems to be less standard.

In a very informal way, our approach can be explained as follows. We assume that Alice, Bob, Charlie are given some (correlated) inputs $x, y$, and $z$ respectively. Using the theoretic-information inequalities we show that the parties can agree on a secret key (of the optimal size) only if the messages sent by Alice contain certain (small but non-negligible) information on other parties' inputs ( $y, z$ ). Then, using spectral bounds (with a version of the expander mixing lemma for bipartite graphs) we show that Alice's message contains certain information on other parties' inputs only if this message is very large (significantly larger than the information on $(y, z)$ contained in this message). Using these considerations we prove a lower bound for communication complexity of secret key agreement. We also provide evidences (see Theorem 1 in Section (4) that this technique of lower bounds can be used in other contexts not related to secret key agreement.

In a communication protocol, the number of bits sent by the parties may depend on the given inputs. Communication complexity of a protocol is usually understood as the worst-case complexity, i.e., the maximum number of communicated bits over all possible valid input data. In our lower bounds, we actually prove that the number of communicated bits must be large for all "typical" inputs (more technically, for all valid tuples of inputs with large enough Kolmogorov complexity). This means that communication complexity of the implied problem is large on average, for a certain class of inputs. The classes of inputs in our constructions are defined by the information profiles (i.e., by the values of the mutual information for all combinations of the inputs) and their combinatorial structure. To make the examples more explicit and intuitive, we use geometric constructions (see the tuples of orthogonal directions in a discrete vector space in Examples 2/3 on p. 5). Our arguments actually apply in more general settings: we only need the inputs to be associated with graphs with a large spectral gap.

## 2 Preliminaries

### 2.1 General notation

For a binary string $x$ we denote its length $|x|$. For a finite set $S$ we denote its cardinality $\# S$.
In what follows we manipulate with equalities and inequalities for Kolmogorov complexity. Since many of them hold up to a logarithmic term, we use the notation $A=^{+} B, A \leq^{+} B$, and $A \geq^{+} B$ for

$$
|A-B|=O(\log n), A \leq B+O(\log n), \text { and } B \leq A+O(\log n)
$$

respectively, where $n$ is clear from the context ( $n$ is usually the length of the strings involved in the inequality).

We denote $\mathbb{F}_{q}$ the field of $q$ elements (usually $q=2^{n}$ ). A $k$-dimensional vector over $\mathbb{F}_{q}$ is a $k$-tuple $\left(x_{1}, \ldots, x_{k}\right) \in \mathbb{F}_{q}^{k}$. We say that two vectors $\left(x_{1}, \ldots, x_{k}\right)$ and $\left(y_{1}, \ldots, y_{k}\right)$ in $\mathbb{F}_{q}^{k}$ are orthogonal to each other if

$$
x_{1} y_{1}+\ldots+x_{k} y_{k}=0
$$

(the addition and multiplication are computed in the field $\mathbb{F}_{q}$ ). A direction in $\mathbb{F}_{q}^{k}$ is an equivalence class of non-zero vectors over $\mathbb{F}_{q}$ that are proportional to each other; we specify a direction as $\left(x_{1}: \ldots: x_{k}\right)$, where $\left(x_{1}, \ldots, x_{k}\right)$ is one of the vectors in this equivalence class. Two directions are orthogonal to each other if every vector in the first one is orthogonal to every vector in the second one.

### 2.2 Kolmogorov Complexity.

Let $M$ be a Turing Machine with two input tapes and one output tape. We say that $p$ is a program that prints a string $x$ given $y$ (a description of $x$ conditional on $y$ ) if $M$ prints $x$ on the pair of inputs $(p, y)$. Kolmogorov complexity of $x$ conditional on $y$ relative to $M$ is defined as

$$
\mathrm{C}_{M}(x \mid y)=\min \{|p|: M(p, y)=x\} .
$$

The invariance theorem (see [17]) claims that there exists an optimal Turing machine $U$ such that for every other Turing machine $V$ there is a number $c_{V}$ such that for all $x$ and $y$

$$
\mathrm{C}_{U}(x \mid y) \leq \mathrm{C}_{V}(x \mid y)+c_{V}
$$

Thus, the algorithmic complexity of $x$ relative to $U$ is minimal up to an additive constant. In the rest of the paper we fix an optimal machine $U$, omit the subscript $U$ and define Kolmogorov complexity of $x$ conditional on $y$ as

$$
\mathrm{C}(x \mid y):=\mathrm{C}_{U}(x \mid y)
$$

Kolmogorov complexity $\mathrm{C}(x)$ of a string $x$ (without a condition) is defined as the Kolmogorov complexity of $x$ conditional on the empty string. We fix an arbitrary computable bijection between binary strings and all finite tuples of binary strings and define Kolmogorov complexity of a tuple $\left\langle x_{1}, \ldots, x_{k}\right\rangle$ as Kolmogorov complexity of the code of this tuple. For brevity we denote this complexity by $\mathrm{C}\left(x_{1}, \ldots, x_{k}\right)$. Similarly, we can fix a bijection between binary strings and elements of finite fields, polynomials over finite fields, directions in vector spaces over finite fields, etc., and talk about Kolmogorov complexities of these objects (implying Kolmogorov complexity of their codes). We use the conventional notation

$$
\mathrm{I}(x: y):=\mathrm{C}(x)+\mathrm{C}(y)-\mathrm{C}(x, y) \text { and } \mathrm{I}(x: y \mid z):=\mathrm{C}(x \mid z)+\mathrm{C}(y \mid z)-\mathrm{C}(x, y \mid z)
$$

(mutual information and conditional mutual information for a pair) and

$$
\mathrm{I}(x: y: z):=\mathrm{C}(x)+\mathrm{C}(y)+\mathrm{C}(z)-\mathrm{C}(x, y)-\mathrm{C}(x, z)-\mathrm{C}(y, z)+\mathrm{C}(x, y, z)
$$

(the triple mutual information). The Kolmogorov-Levin theorem, [16, claims that for all $x, y$

$$
\mathrm{C}(x, y)=^{+} \mathrm{C}(x \mid y)+\mathrm{C}(y) .
$$

Using the Kolmogorov-Levin theorem it is not hard to show that

$$
\mathrm{I}(x: y: z)={ }^{+} \mathrm{I}(x: y)-\mathrm{I}(x: y \mid z)={ }^{+} \mathrm{I}(x: z)-\mathrm{I}(x: z \mid y)={ }^{+} \mathrm{I}(y: z)-\mathrm{I}(y: z \mid x)
$$



Figure 1: Complexity profile for a triple $x, y, z$. On this diagram it is easy to observe several standard equations:

- $\mathrm{C}(x)={ }^{+} \mathrm{C}(x \mid y, z)+\mathrm{I}(x: y \mid z)+\mathrm{I}(x: z \mid y)+\mathrm{I}(x: y: z)$
- $\mathrm{C}(x, y)={ }^{+} \mathrm{C}(x \mid y, z)+\mathrm{C}(y \mid x, z)+\mathrm{I}(x: y \mid z)+\mathrm{I}(x: z \mid y)+\mathrm{I}(y: z \mid x)+\mathrm{I}(x: y: z)$
- $\mathrm{C}(x \mid y)={ }^{+} \mathrm{C}(x \mid y, z)+\mathrm{I}(x: z \mid y)$
- $\mathrm{I}(x: y)={ }^{+} \mathrm{I}(x: y \mid z)+\mathrm{I}(x: y: z)$
- $\mathrm{I}(x: y z)={ }^{+} \mathrm{I}(x: y \mid z)+\mathrm{I}(x: z \mid y)+\mathrm{I}(x: y: z)$
and so on; all these equations are valid up to $O(\log (|x|+|y|+|z|))$.

These relations can be observed on a Venn-like diagram, see Fig. [1]
A string $x$ is said to be (almost) incompressible given $y$ if $C(x \mid y) \geq^{+}|x|$, and $x$ and $y$ are said to be independent, if $\mathrm{I}(x: y)=^{+} 0$. For every $n$, the majority of binary strings of length $n$ are almost incompressible; the vast majority of pairs of strings $x$ and $y$ of length $n$ are independent.

For a pair of strings $(x, y)$ we call by its complexity profile the triple of numbers $(\mathrm{C}(x), \mathrm{C}(y), \mathrm{C}(x, y))$. Due to the Kolmogorov-Levin theorem, the complexity profile of a pair is determined (up to additive error terms $O(\log (|x|+|y|)))$ by the triple of numbers $(\mathrm{C}(x \mid y), \mathrm{C}(y \mid x), \mathrm{I}(x: y))$. Indeed,

$$
\mathrm{C}(x)={ }^{+} \mathrm{C}(x \mid y)+\mathrm{I}(x: y), C(y)={ }^{+} \mathrm{C}(y \mid x)+\mathrm{I}(x: y), C(x, y)={ }^{+} \mathrm{C}(x \mid y)+\mathrm{C}(y \mid x)+\mathrm{I}(x: y)
$$

Similarly, for a triple of strings $(x, y, z)$ we define its complexity profile as the vector with 7 components

$$
(\mathrm{C}(x), \mathrm{C}(y), \mathrm{C}(z), \mathrm{C}(x, y), \mathrm{C}(x, z), \mathrm{C}(y, z), \mathrm{C}(x, y, z))
$$

This profile can be equivalently specified (again, up to additive logarithmic error terms) by the numbers

$$
\mathrm{C}(x \mid y, z), \mathrm{C}(y \mid x, z), \mathrm{C}(z \mid x, y), \mathrm{I}(x: y \mid z), \mathrm{I}(x: z \mid y), \mathrm{I}(y: z \mid x), \mathrm{I}(x: y: z)
$$

see Fig. 1
Example 1. Let $u, v, w$ be three mutually incompressible and independent bit strings of length $n$, i.e., $\mathrm{C}(u, v, w)=^{+} 3 n$. We define $x:=u v$ and $y:=u w$. Then it is easy to see that $\mathrm{C}(x)=^{+} 2 n, \mathrm{C}(y)={ }^{+} 2 n$, and $\mathrm{C}(x, y)=^{+} 3 n$. The complexity profile of $(x, y)$ is shown in the diagram in Fig. 2 (a). In this example, the mutual information $\mathrm{I}(x: y)$ has a clear physical meaning: it is represented by the block $u$ shared by $x$ and $y$.

(a) Complexity profile for Examples 1-2: $\mathrm{C}(x \mid y)={ }^{+} n, \mathrm{C}(y \mid x)={ }^{+} n, \mathrm{I}(x: y)={ }^{+} n$.

(b) Complexity profile for Examples 3: $\mathrm{C}(x \mid y, z)={ }^{+}(k-2) n, \mathrm{I}(x: y)={ }^{+} n$, $\mathrm{I}(x: y z)={ }^{+} 2 n, \mathrm{I}(x: y: z)={ }^{+} 0$.

Figure 2: Diagrams with complexity profiles for Examples 1-3.

Example 2. Let $x=\left(x_{1}: x_{2}: x_{3}\right)$ and $y=\left(y_{1}: y_{2}: y_{3}\right)$ be orthogonal directions in $\mathbb{F}_{2^{n}}^{3}$. To define a direction in the 3 -dimensional space we need to specify 2 coordinates. Hence, $\mathrm{C}(x) \leq+2 n$. Further, in a subspace of co-dimension 1 (directions orthogonal to $y$ ) we need to know only one coordinate to specify $x$, and $\mathrm{C}(x \mid y) \leq n$. Combining these bounds we get $\mathrm{C}(x, y)=^{+} \mathrm{C}(x)+\mathrm{C}(x \mid y) \leq^{+} 3 n$.

The standard counting argument implies that for most pairs of orthogonal directions $(x, y)$ these bounds are tight, i.e., $\mathrm{C}(x))^{+} 2 n, \mathrm{C}(y)={ }^{+} 2 n$, and $\mathrm{C}(x, y)=^{+} 3 n$. Thus, we get another pair of strings with the complexity profile shown in Fig. 2(a). However, this time we cannot "materialize" the mutual information between $x$ and $y$. (This statement that the mutual information of this pair is not materializable can be made more formal, see [10, 11]).
Example 3. Let $k \geq 2$ and $x=\left(x_{1}: x_{2}: \ldots: x_{k+1}\right), y=\left(y_{1}: y_{2}: \ldots: y_{k+1}\right)$, and $z=\left(z_{1}: z_{2}: \ldots: z_{k+1}\right)$ be a triple of pairwise orthogonal directions in $\mathbb{F}_{2^{n}}^{k+1}$. We need to specify $k$ coordinates to define a direction in the $(k+1)$-dimensional space, so we have $\mathrm{C}(x) \leq^{+} k n$. Given $y$ (or $y$ together with $z$ ) we need to specify only $(k-1)$ (respectively, $(k-2)$ ) coordinates to determine $x$. Therefore, $\mathrm{C}(x \mid y) \leq^{+}(k-1) n$ and $\mathrm{C}(x \mid y, z) \leq^{+}(k-2) n$. Similar bounds hold for Kolmogorov complexities of $y$ and $z$.

Again, a simple counting argument implies that for most triples of orthogonal directions $(x, y, z)$ these bounds are tight, up to additive logarithmic terms. The complexity profile of a "typical" triple of pairwise orthogonal directions is shown in Fig. 2(b): the pairwise mutual informations are equal to $n$, and the triple mutual information is zero.

For a survey of the basic properties of Kolmogorov complexity we refer the reader to the introductory chapters in [2] and [4].

### 2.3 Communication Complexity.

We use the conventional notion of a communication protocol for two or three parties (traditionally called Alice, Bob, and Charlie), see for detailed definitions [8]. We discuss deterministic protocols and randomized protocols with a public source of random bits. We usually denote the inputs of Alice, Bob, and Charlie as $x, y$, and $z$ respectively (number-in-hand model). A deterministic communication protocol for inputs $x, y, z \in\{0,1\}^{n}$ returns a result $w=w(x, y, z)$. In a randomized protocol the result depends also on the public source of random bits $r$, and $w=w(x, y, z, r)$. The sequence of messages sent by the parties to each other while following the steps of the protocol is called a transcript $t=t(x, y, z)$ of the communication ( $t=t(x, y, z, r)$ for randomized protocols). Communication complexity of a protocol is the maximal length of its transcript, i.e., $\max _{x, y, z, r}|t(x, y, z, r)|$. In this paper, all theorems are formulated for randomized communication protocols.

In general, a communication protocol may consist of several rounds, when each next message of every party depends on the previously sent messages. In the simultaneously messages model there is no
interaction: all parties send in parallel their messages that depend only on their own input data (and the random bits), and then compute the final result.

A communication protocol computing a function $F(x, y, z)$ returns a correct result if $w(x, y, z, r)=$ $F(x, y, z)$. In a secret key agreement protocol, correctness of the result $w$ means that (i) $w$ is of the required size and (ii) it is almost incompressible even given the transcript of the communication $t$ and the public random bits $r$. For a more detailed discussion of this setting we refer the reader to [3].

If the length of inputs is equal to $n$, we may assume w.l.o.g. that the used string of public random bits $r$ is of length $O(n)$ (using longer sources of random bits may only slightly affect the probability of an erroneous result. For protocols computing a function, $O(\log n)$ bits is enough due to the Newman's theorem, [14]; in secret key agreement protocols we may need $O(n)$ random bits, see [1]). This is important for our setting: we may assume that the terms $O(\log \mathrm{C}(r))$ involved in inequalities for Kolmogorov complexity match in order of magnitude the terms $O(\log (\mathrm{C}(x)+\mathrm{C}(y)+\mathrm{C}(z)))$, where $x, y, z$ are the input data.

We will assume that the communication protocol for $n$-bit inputs can be uniformly computed given $n$. Thus, the protocol as a construction (without input data of the parties) contains in itself negligibly little information, and we cannot "cheat" by embedding in the structure of the protocol any significant information hidden from the eavesdropper.

### 2.4 Spectral technique

Let $G=(L \cup R, E)$ be a bi-regular bipartite graph where each vertex in $L$ has degree $D_{L}$, each vertex in $R$ has degree $D_{R}$, and each edge $e \in E$ connects a vertex from $L$ with a vertex from $R$ (observe that $\left.\# E=\# L \cdot D_{L}=\# R \cdot D_{R}\right)$. The adjacency matrix of such a graph is a zero-one matrix

$$
H=\left(\begin{array}{cc}
0 & J \\
J^{\top} & 0
\end{array}\right)
$$

where $J$ is a matrix of dimension $(\# L) \times(\# R)\left(J_{x y}=1\right.$ if and only if there is an edge between the $x$-th vertex in $L$ and the $y$-th vertex in $R$ ). Let $\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{N}$ be the eigenvalues of $H$, where $N=\# L+\# R$ is the total number of vertices. Since $H$ is symmetric, all $\lambda_{i}$ are real numbers. It is well known that for a bipartite graph the spectrum is symmetric, i.e., $\lambda_{i}=-\lambda_{N-i+1}$ for each $i$, and $\lambda_{1}=-\lambda_{N}=\sqrt{D_{L} D_{R}}$ (see, e.g., [5]). We are interested in graphs such that $\lambda_{2} \ll \lambda_{1}$ (that is, the spectral gap is large). The graphs with a large gap between the first and the second eigenvalues have the property of good mixing, see [7]. We use this property in the form of the following lemma (see [5] for the proof).

Lemma 1 (Expander Mixing Lemma for bipartite graphs). Let $G=(L \cup R, E)$ be a regular bipartite graph where each vertex in $L$ has degree $D_{L}$ and each vertex in $R$ has degree $D_{R}$. Then for each $A \subseteq L$ and $B \subseteq R$ we have

$$
\left|E(A, B)-\frac{D_{L} \cdot \# A \cdot \# B}{\# R}\right| \leq \lambda_{2} \sqrt{\# A \cdot \# B}
$$

where $\lambda_{2}$ is the second largest eigenvalue of the adjacency matrix of $G$ and $E(A, B)$ is the number of edges between $A$ and $B$.

We apply Lemma 1 for the case $\frac{D_{L} \cdot \# A \cdot \# B}{\# R} \geq \lambda_{2} \sqrt{\# A \cdot \# B}$, as shown in the corollary below.
Corollary 1. Let $G=(L \cup R, E)$ be a graph from Lemma 1 with the second eigenvalue $\lambda_{2}$. Then for $A \subseteq L$ and $B \subseteq R$ such that $\# A \cdot \# B \geq\left(\frac{\lambda_{2} \# R}{D_{L}}\right)^{2}$ we have

$$
\begin{equation*}
E(A, B)=O\left(\frac{D_{L} \cdot \# A \cdot \# B}{\# R}\right) \tag{1}
\end{equation*}
$$

We will use the bound for the spectral gap from the following lemma.
Lemma 2. (a) Let $L=R$ be the set of all directions in $\mathbb{F}_{2^{n}}^{k+1}$, and $E$ consists of all pairs of orthogonal directions. In this graph $\# L=\# R=\left(q^{k+1}-1\right) /(q-1)$ and $D_{L}=D_{R}=\left(q^{k}-1\right) /(q-1)$ where $q=2^{n}$, and $\lambda_{2}=\sqrt{2^{(k-1) n}}$.
(b) Let $L$ be the set of all directions $x$ in $\mathbb{F}_{2^{n}}^{k+1}, R$ be the set of orthogonal pairs of directions $(y, z)$ in $\mathbb{F}_{2^{n}}^{k+1}$, and $E$ consists of $(x,(y, z))$ such that all three directions $x, y, z$ are pairwise orthogonal. In this graph

$$
\# L=\frac{q^{k+1}-1}{q-1}, \# R=\frac{\left(q^{k+1}-1\right)\left(q^{k}-1\right)}{(q-1)^{2}}, D_{L}=\frac{\left(q^{k}-1\right)\left(q^{k-1}-1\right)}{(q-1)^{2}}, D_{R}=\frac{q^{k-1}-1}{q-1}
$$

and $\lambda_{2} \leq \sqrt{D_{L}}=O\left(q^{k / 2}\right)$, where $q=2^{n}$.
(This lemma might be known, but for lack of a reference we give a short proof in Appendix.)

## 3 How to say something that the interlocutor already knows

In this section we deal with the following setting. We consider (randomized) communication protocols with two participants, Alice and Bob. We assume that Alice holds an input string $x$ and Bob holds another input string $y$; we assume also that the complexity profile of the pairs $(x, y)$ is known to all parties. The aim of Alice in this protocol is to send to Bob a message that is not completely unpredictable from the point of view of Bob. More precisely, Alice's message must have positive (and non-negligible) mutual information with Bob's input $y$. Such a task may look odd: why should Alice say something that Bob already knows? However, we will see that such an action may be useful and even necessary in a protocol of multi-party secret key agreement.

More formally, in this section we prove Proposition 1 and Proposition 2 In both propositions we use substantially the same argument based on the spectral bound for two different bipartite graphs.

We start with an example when the task of Alice is trivial.
Example 4. Let Alice is given a string $x=u v$ and Bob is given a string $y=u w$ from Example In this setting the task of Alice is very simple. If she wants to send a message having a large mutual information with Bob's $y$, she can send to Bob the first half of her input (i.e., string $u$ ). This string is a part of Bob's input, so the message of Alice has maximal mutual information with $y$.

In Example 4 Alice can send a small message with a high mutual information with $y$ (this message can be a part of string $u$ ), or she can send a message with a low mutual information with $y$ (this message can be a part of string $v$ ). We proceed with a more involved case. We consider a pair $(x, y)$ with the same complexity profile as in Example 4 but with a different combinatorial structure. In this case Alice cannot send a message having a positive mutual information with Bob's data unless this message is long enough.

Proposition 1. Let Alice and Bob are given, respectively, $x$ and $y$ from Example 2. We assume that the pair $(x, y)$ has maximal possible complexity, $\mathrm{C}(x, y)=^{+} 3 n$ (and $\left.\mathrm{C}(x)={ }^{+} 2 n, \mathrm{C}(y)={ }^{+} 2 n, \mathrm{I}(x: y)={ }^{+} n\right)$. We consider a one-round communication protocol where Alice sends to Bob a message $m_{A}=m_{A}(x)$. Then $\mathrm{I}\left(m_{A}: y\right) \leq^{+} \max \left\{0, \mathrm{C}\left(m_{A}\right)-n\right\}$.

This proposition claims that if Alice wants to send a message having $\delta n$ bits of mutual information with $y$, she must send a message of size at least $(1+\delta) n$. This bound is tight (up to $O(\log n)$ ). Indeed, w.l.o.g. we may assume that the direction $x$ is represented by the vector ( $x_{1}, x_{2}, 1$ ). Alice can send to Bob a message that contains completely the first coordinate $x_{1}$ ( $n$ bits of information) and a prefix of length $\delta n$ of the second coordinate $x_{2}$ ( $\delta n$ bits of information). Given $y$ and $x_{1}$, Bob can fully reconstruct the direction $x$ and, in particular, the coordinate $x_{2}$. Therefore, $\mathrm{C}\left(m_{A} \mid y\right) \leq^{+} n$ and $\mathrm{I}\left(m_{A}: y\right) \geq^{+} \delta n$.
Remark 1. In the proof of Proposition 1, to simplify the notation, we ignore the bits $r$ provided by the public source of randomness and handle substantially deterministic protocols. Our argument trivially relativizes given any instance of random bits $r$ independent of $(x, y)$, we only need to add routinely the used random bits to the condition of all terms with Kolmogorov complexity appearing in the proof.

Another formal way to handle the random bits is as follows. In the definition of Kolmogorov complexity, we may supply the universal Turing machine with an oracle tape and define a variant of Kolmogorov complexity literally relativized to on oracle. The proof of the invariance theorem for Kolmogorov complexity (the existence of the optimal decompressor) easily relativizes. Then, we can put on the oracle tape the bit string $r$ produced by the public source of random bits. Our proof of Proposition 1 with the relativized version of $\mathrm{C}(\cdot)$ works in the assumption that the random bits string $r$ provides no information


Figure 3: The profile for the protocol in Proposition 1
on the tuple of inputs $(x, y)$ (technically, we need the assumption $\mathrm{I}(x y: r)={ }^{+} 0$; this means that the plain Kolmogorov complexity of $(x, y)$ and Kolmogorov complexity of $(x, y)$ relativized to the oracle $r$ differ by at most $O(\log n))$. For every pairs $(x, y)$, the condition $\mathrm{I}(x y: r)={ }^{+} 0$ is true for the vast majority of $r$.

We will use a similar abuse of notation in other proofs as well.
Proof. Since Alice computes the message $m_{A}$ given the input data $x$, we have $\mathrm{C}\left(m_{A} \mid x\right)={ }^{+} 0$. We denote $\alpha:=\mathrm{I}\left(m_{A}: x \mid y\right) / n$ and $\beta:=\mathrm{I}\left(m_{A}: x: y\right) / n$ (it is easy to verify that $\left.\mathrm{C}\left(m_{A}\right)=(\alpha+\beta) n\right)$. The complexity profile for the triple $\left(x, y, m_{A}\right)$ is shown in Fig. 3,
Case 1. Assume that $\mathrm{C}\left(m_{A}\right) \leq n-2 \cdot$ const $\cdot \log n$ for some const $>0$ (a constant to be specified later). In this case, to prove the proposition we need to show that $\mathrm{I}\left(m_{A}: y\right)=^{+} 0$. In our notation this is equivalent to $\beta n={ }^{+} 0$. More technically, we are going to show that

$$
\begin{equation*}
\beta n \leq \text { const } \cdot \log n \tag{2}
\end{equation*}
$$

For the sake of contradiction we assume that (2) is false. It is enough to consider the case when $\beta$ is somewhat large but not too large, i.e., just slightly above the threshold (2). Indeed, any communication protocol violating (2) can be converted in a different protocols with the same or a smaller value of $\alpha$ and with $\beta n=$ const $\cdot \log n+O(1)$. To this end, we observe that by discarding a few last bits of Alice's message $m_{A}$ we make the protocol only simpler. By taking shorter and shorter prefixes of $m_{A}$, we gradually reduce complexity of the message itself and in the same time gradually variate the mutual information between the message and $y$. We may replace the initial message $m_{A}$ with the shortest prefix of the initial message that still violates (2). Thus, in what follows, we assume w.l.o.g. that

$$
\text { const } \cdot \log n<\beta n \leq \text { const } \cdot \log n+O(1)
$$

We proceed by defining

$$
A:=\left\{x^{\prime}: \mathrm{C}\left(x^{\prime} \mid m_{A}\right) \leq \mathrm{C}\left(x \mid m_{A}\right)\right\} \text { and } B:=\left\{y^{\prime}: \mathrm{C}\left(y^{\prime} \mid m_{A}\right) \leq \mathrm{C}\left(y \mid m_{A}\right)\right\} .
$$

We use the following standard claim:
Claim. $\quad \# A=2^{\mathrm{C}\left(x \mid m_{A}\right) \pm O(\log n)}=2^{(2-\alpha-\beta) n \pm O(\log n)}$ and $\# B=2^{\mathrm{C}\left(y \mid m_{A}\right) \pm O(\log n)}=2^{(2-\beta) n \pm O(\log n)}$ (see [3, Claim 4.7]).

From the claim we obtain

$$
\# A \cdot \# B=2^{2 n-\alpha n-\beta n+2 n-\beta \pm O(\log n)}=2^{4 n-\alpha n-2 \beta n \pm O(\log n)}=2^{4 n-\mathrm{C}\left(m_{A}\right)-\beta n \pm O(\log n)} .
$$

Since $\mathrm{C}\left(m_{A}\right) \leq n-2 \cdot$ const $\log n$ and $\beta n<$ const $\log n+O(1)$, we can conclude

$$
4 n-\mathrm{C}\left(m_{A}\right)-\beta n \pm O(\log n) \geq 4 n-(n-2 \cdot \text { const } \cdot \log n)-\text { const } \cdot \log n-O(\log n) \geq 3 n
$$

(We should choose the value of const so that const $\cdot \log n$ majorizes the term $O(\log n)$ in the inequality above.)

Thus, $\# A \cdot \# B \geq 2^{3 n}$. We combine Lemma 2(a) and the Expander Mixing Lemma (Corollary (1) and obtain

$$
E(A, B)=O\left(\frac{2^{n} \cdot|A| \cdot|B|}{2^{2 n}}\right)=O\left(\frac{|A| \cdot|B|}{2^{n}}\right) .
$$

Therefore, $\mathrm{C}\left(x, y \mid m_{A}\right) \leq^{+} \log E(A, B) \leq^{+} 3 n-\alpha n-2 \beta n$, and

$$
\mathrm{C}(x, y) \leq^{+} \mathrm{C}\left(m_{A}\right)+\mathrm{C}\left(x, y \mid m_{A}\right) \leq^{+}(\alpha+\beta) n+3 n-\alpha n-2 \beta n=^{+} 3 n-\beta n .
$$

The terms $O(\log n)$ hidden in the notation $\leq^{+}$and $=^{+}$in this inequality do not depend on $\beta$. Thus, by increasing $\beta$, we increase the gap between $(3 k-3) n$ and $\mathrm{C}(x, y)$. On the other hand, $\mathrm{C}(x, y)=^{+} 3 n$. We get a contradiction if $\beta$ is chosen large enough.

Case 2. Now we assume that $\mathrm{C}\left(m_{A}\right)=n+\delta n$ for an arbitrary $\delta$. Denote by $m_{A}^{\prime}$ the prefix of $m_{A}$ of length $(n-$ const $\log n)$ and by $m_{A}^{\prime \prime}$ the suffix of $m_{A}$ of length $(\delta n+\operatorname{const} \log n)$. We know from Case 1 that $\mathrm{I}\left(m_{A}^{\prime}: y\right)=^{+} 0$. It remains to observe that by the chain rule

$$
\mathrm{I}\left(m_{A}: y\right)=^{+} \mathrm{I}\left(m_{A}^{\prime}: y\right)+\mathrm{I}\left(m_{A}^{\prime \prime}: y \mid m_{A}^{\prime}\right)=^{+} \mathrm{I}\left(m_{A}^{\prime \prime}: y \mid m_{A}^{\prime}\right) \leq^{+}\left|m_{A}^{\prime \prime}\right|=^{+} \delta n
$$

and the proposition is proven.
With Lemma 2(b), the proof of Proposition 1 can be adapted to prove a similar statement for the triple from Example 3

Proposition 2. Let Alice be given a direction $x=\left(x_{1}: x_{2}: \ldots: x_{k+1}\right)$ and Bob be given a pair of directions $y=\left(y_{1}: y_{2}: \ldots: y_{k+1}\right)$ and $z=\left(z_{1}: z_{2}: \ldots: z_{k+1}\right)$ in the same $(k+1)$-dimensional space $\mathbb{F}_{2^{n}}^{k+1}$, and it is known that $x, y$, and $z$ are pairwise orthogonal. We assume that the triple $(x, y, z)$ has maximal possible complexity,

$$
\mathrm{C}(x, y, z)=^{+} k n+(k-1) n+(k-2) n=(3 k-3) n
$$

(this implies that $\mathrm{C}(x)={ }^{+} \mathrm{C}(y)={ }^{+} \mathrm{C}(z)={ }^{+} k n, \mathrm{I}(x: y)={ }^{+} \mathrm{I}(x: z)={ }^{+} \mathrm{I}(y: z)={ }^{+} n$, and $\mathrm{I}(x: y z)={ }^{+} \mathrm{I}(y$ : $\left.x z)={ }^{+} \mathrm{I}(z: x y)={ }^{+} 2 n\right)$. We consider one-round communication protocols where Alice sends to Bob a message $m_{A}=m_{A}(x)$. Then $\mathrm{I}\left(m_{A}: y z\right) \leq^{+} \max \left\{0, \mathrm{C}\left(m_{A}\right)-2 n\right\}$.

Proof. The argument is very similar to the proof of Proposition Again, to simplify the notation, we ignore the random bits (see Remark (1). The argument relativizes given any instance of random bits $r$ independent of the input data $(x,\langle y, z\rangle)$ (we would have to put the string of public random bits $r$ to the condition of all terms with Kolmogorov complexity used in the argument).

Since Alice computes the message $m_{A}$ given the input data $x$, we have $\mathrm{C}\left(m_{A} \mid x\right)=^{+} 0$. We denote $\alpha:=\frac{\mathrm{I}\left(m_{A}: x \mid y z\right)}{n}$ and $\beta:=\frac{\mathrm{I}\left(m_{A}: x: y z\right)}{n}$. (Observe that $\mathrm{C}\left(m_{A}\right)=(\alpha+\beta) n$. The complexity profile for the triple $\left(x,\langle y, z\rangle, m_{A}\right)$ is shown in Fig. (4)

Case 1. Assume that $\mathrm{C}\left(m_{A}\right) \leq(k-2) n-2 \cdot$ const $\cdot \log n$ for some const $>0$ (a constant to be specified later). In this case, to prove the proposition we need to show that $\mathrm{I}\left(m_{A}: y z\right)={ }^{+} 0$. In our notation this is equivalent to $\beta n={ }^{+} 0$. More technically, we are going to prove inequality (2), as we did in the proof of Proposition 1

For the sake of contradiction we assume that (2) is false. We may focus on the case when $\beta$ is just slightly above the threshold (2). Indeed, any communication protocol violating (22) can be converted in a different protocol with the same or a smaller value of $\alpha$ and with $\beta n=$ const $\cdot \log n+O(1)$. To this end, we observe that by discarding a few last bits of Alice's message $m_{A}$ we make the protocol only simpler. By taking shorter and shorter prefixes of $m_{A}$, we gradually reduce complexity of the message itself and in the same time gradually variate the mutual information between the message and $y$. We replace the initial message $m_{A}$ with the shortest prefix of the initial message that still violates (2). Thus, in what follows, we assume w.l.o.g. that

```
const}\cdot\operatorname{log}n<\betan\leqconst\cdot\operatorname{log}n+O(1)
```



Figure 4: The profile for the protocol in Proposition 2

We proceed by defining

$$
A:=\left\{x^{\prime}: \mathrm{C}\left(x^{\prime} \mid m_{A}\right) \leq \mathrm{C}\left(x \mid m_{A}\right)\right\} \text { and } B:=\left\{\left\langle y^{\prime}, z^{\prime}\right\rangle: \mathrm{C}\left(\left\langle y^{\prime}, z^{\prime}\right\rangle \mid m_{A}\right) \leq \mathrm{C}\left(\langle y, z\rangle \mid m_{A}\right)\right\}
$$

We use again the standard claim:
Claim. [(see [3, Claim 4.7])]
(i) $\# A=2^{\mathrm{C}\left(x \mid m_{A}\right) \pm O(\log n)}=2^{k n-\alpha n-\beta n \pm O(\log n)}$
(ii) $\# B=2^{\mathrm{C}\left(y, z \mid m_{A}\right) \pm O(\log n)}=2^{(2 k-1) n-\beta n \pm O(\log n)}$

From the claim we obtain

$$
\begin{aligned}
\# A \cdot \# B & =2^{k n-\alpha n-\beta n+(2 k-1) n-\beta \pm O(\log n)}=2^{(3 k-1) n-\alpha n-2 \beta n \pm O(\log n)} \\
& =2^{(3 k-1) n-\mathrm{C}\left(m_{A}\right)-\beta n \pm O(\log n)}
\end{aligned}
$$

Since $\mathrm{C}\left(m_{A}\right) \leq(k-2) n-2 \cdot$ const $\log n$ and $\beta n<$ const $\log n+O(1)$, we can conclude

$$
\begin{aligned}
\# A \cdot \# B & \geq(3 k-1) n-((k-2) n-2 \cdot \text { const } \cdot \log n)-\text { const } \cdot \log n-O(\log n) \\
& \geq(2 k+1) n .
\end{aligned}
$$

(We should choose the value of const so that const $\cdot \log n$ majorizes the term $O(\log n)$ in the inequality above.)

Thus, $\# A \cdot \# B \geq 2^{(2 k+1) n}$. We combine Lemma 2 (b) and Corollary 1 and obtain

$$
E(A, B)=O\left(\frac{2^{(2 k-3) n} \cdot \# A \cdot \# B}{2^{(2 k-1) n}}\right)=O\left(\frac{\# A \cdot \# B}{2^{2 n}}\right)
$$

Therefore, $\mathrm{C}\left(x, y, z \mid m_{A}\right) \leq^{+} \log E(A, B) \leq^{+}(3 k-3) n-\alpha n-2 \beta n$, and

$$
\mathrm{C}(x, y, z) \leq^{+} \mathrm{C}\left(m_{A}\right)+\mathrm{C}\left(x, y, z \mid m_{A}\right) \leq^{+}(\alpha+\beta) n+(3 k-3) n-\alpha n-2 \beta n=^{+}(3 k-3) n-\beta n .
$$

The constants in the $O(\log n)$ terms hidden in the notation $\leq^{+}$and $=^{+}$in the last inequality may depend only on the universal Turing machine and not on the choice of $\beta$. Thus, by choosing $\beta$ larger, we increase the gap between $(3 k-3) n$ and $\mathrm{C}(x, y, z)$.

On the other hand, we have assumed that $\mathrm{C}(x, y, z)=^{+}(3 k-3) n$ (and again, the implied term $O(\log n)$ is independent of $\beta$ ). By making $\beta$ large enough, we get a contradiction.

Case 2. Now we assume that $\mathrm{C}\left(m_{A}\right)=(k-2) n+\delta n$ for an arbitrary $\delta$. Denote by $m_{A}^{\prime}$ the prefix of $m_{A}$ of length $((k-2) n-$ const $\log n)$ and by $m_{A}^{\prime \prime}$ the suffix of $m_{A}$ of length $(\delta n+$ const $\log n)$. We know from Case 1 that $\mathrm{I}\left(m_{A}^{\prime}: y\right)=^{+} 0$. It remains to observe that by the chain rule

$$
\mathrm{I}\left(m_{A}: y\right)=^{+} \mathrm{I}\left(m_{A}^{\prime}: y\right)+\mathrm{I}\left(m_{A}^{\prime \prime}: y \mid m_{A}^{\prime}\right)=^{+} \mathrm{I}\left(m_{A}^{\prime \prime}: y \mid m_{A}^{\prime}\right) \leq^{+}\left|m_{A}^{\prime \prime}\right|=^{+} \delta n
$$

and the proposition is proven.


Figure 5: Alice holding $x$ and Bob holding $y$ send simultaneous messages to Charlie, who computes $z$.

## 4 Simultaneous messages

In this section we use the technical results from the previous section to prove a lower bound for communication complexity of the following problem. Alice and Bob hold, respectively, lines $x$ and $y$ in a plane (intersecting at one point $z$ ). They send to Charlie (in parallel, without interacting with each other) some messages so that Charlie can reconstruct the intersecting point. We argue that the trivial protocol (where Alice and Bob send the full information on their lines) is essentially optimal.

Theorem 1. Let Alice and Bob be given lines in the projective plane over the finite field $\mathbb{F}_{2^{n}}$ (we denote them $x$ and $y$ respectively) intersecting at a point $z$, and Charlie has no input information. Alice and Bob (without a communication with each other) send to Charlie messages $m_{A}$ and $m_{B}$ so that Charlie can find z, see Fig. 5. For every communication protocol for this problem, for some $x, y$ we have

$$
\left|m_{A}\right|+\left|m_{B}\right| \geq^{+} 4 n
$$

which means essentially that in the worst case Alice and Bob must send to Charlie all their data (for a typical pair of lines we have $\left.\mathrm{C}(x, y)={ }^{+} \mathrm{C}(x)+\mathrm{C}(y)={ }^{+} 4 n\right)$.

Proof. In this proof we again abuse the notation and ignore the random bits (see Remark (1). Here, to simplify the notation, we ignore again the random bits and show that the communication complexity is high for a deterministic protocol. The argument trivially relativizes given any instance of random bits $r$ independent of $(x, y)$.

Let $(x, y)$ be a pair of lines in a projective plane over $\mathbb{F}_{2^{n}}$ such that

$$
\mathrm{C}(x, y)=^{+} \mathrm{C}(x)+\mathrm{C}(y)=^{+} 4 n
$$

(this is the case for most pairs of lines in the plane) intersecting at a point $z$. Observe that the graph of possible pairs $(x, z)$ and the graph of possible pairs $(y, z)$ (the configurations (line, point)) is isomorphic to the graph of pairs of orthogonal directions in $\mathbb{F}_{2^{n}}^{3}$ in Example 2 Hence, we can apply Proposition 1 and conclude that

$$
\begin{equation*}
\mathrm{I}\left(m_{A}: z\right) \leq^{+} \max \left\{0, \mathrm{C}\left(m_{A}\right)-n\right\} \text { and } \mathrm{I}\left(m_{B}: z\right) \leq^{+} \max \left\{0, \mathrm{C}\left(m_{B}\right)-n\right\} . \tag{3}
\end{equation*}
$$

In particular, $\mathrm{I}\left(m_{A}: z\right) \geq^{+} n$ and $\mathrm{I}\left(m_{B}: z\right) \geq^{+} n$ only if Kolmogorov complexities of $m_{A}$ and $m_{B}$ are both at least $2 n$.

It is easy to verify that for our construction $\mathrm{I}(x: z)=^{+} n$ and $\mathrm{I}(y: z)=^{+} n$. Since $m_{A}$ and $m_{B}$ are computed from $x$ and $y$ respectively, we conclude that

$$
\begin{equation*}
\mathrm{I}\left(m_{A}: z\right) \leq^{+} n \text { and } \mathrm{I}\left(m_{B}: z\right) \leq^{+} n . \tag{4}
\end{equation*}
$$

We need to show that these two inequalities turn into equalities. Indeed, by construction,

$$
\begin{aligned}
\mathrm{I}(x: y \mid z) & =^{+} \mathrm{C}(x \mid z)+\mathrm{C}(y \mid z)-\mathrm{C}(x, y \mid z) \\
& \leq^{+} n+n-\mathrm{C}(x, y \mid z) \text { [we need } n+O(1) \text { bits to specify a line given a point] } \\
& \leq^{+} 2 n-(\mathrm{C}(x, y)-\mathrm{C}(z)) \\
& \leq^{+} 2 n-4 n+2 n={ }^{+} 0 .[\text { we need } 2 n+O(1) \text { bits to specify a point in the plane] }
\end{aligned}
$$

As $\mathrm{C}\left(m_{A} \mid x\right)={ }^{+} 0$ and $\mathrm{C}\left(m_{B} \mid y\right)={ }^{+} 0$, we have (see Lemma4(ii) in Appendix)

$$
\mathrm{I}\left(m_{A}: m_{B} \mid z\right) \leq^{+} \mathrm{I}(x: y \mid z)=^{+} 0
$$

Therefore, $\mathrm{I}\left(m_{A}: m_{B}: z\right)=^{+} \mathrm{I}\left(m_{A}: m_{B}\right)-\mathrm{I}\left(m_{A}: m_{B} \mid z\right) \geq^{+} 0$, and

$$
\mathrm{I}\left(m_{A} m_{B}: z\right)=^{+} \mathrm{I}\left(m_{A}: z\right)+\mathrm{I}\left(m_{B}: z\right)-\mathrm{I}\left(m_{A}: m_{B}: z\right) \leq^{+} \mathrm{I}\left(m_{A}: z\right)+\mathrm{I}\left(m_{B}: z\right)
$$

On the other hand, since Charlie can compute $z$ from $\left(m_{A}, m_{B}\right)$, we have $\mathrm{I}\left(m_{A} m_{B}: z\right) \geq^{+} 2 n$ and, therefore,

$$
\mathrm{I}\left(m_{A}: z\right)+\mathrm{I}\left(m_{B}: z\right)=^{+} 2 n
$$

Keeping in mind (4), we conclude that $\mathrm{I}\left(m_{A}: z\right)={ }^{+} n$ and $\mathrm{I}\left(m_{B}: z\right)={ }^{+} n$. Due to (3), this is possible only of $\mathrm{C}\left(m_{A}\right) \geq^{+} 2 n$ and $\mathrm{C}\left(m_{B}\right) \geq^{+} 2 n$.

## 5 Secret key agreement with three parties

In this section we prove lower bounds for communication complexity of secret key agreement with three parties. Let us explain our results in some detail. We assume that Alice, Bob, and Charlie are given inputs $x, y, z$ respectively such that

$$
\begin{align*}
& \mathrm{C}(x, y, z){ }^{+} k n+(k-1) n+(k-2) n, \\
& \mathrm{C}(x)=+\mathrm{C}(y))^{+} \mathrm{C}(z)=+k n, \\
& \mathrm{I}(x: y)={ }^{+} \mathrm{I}(x: z)={ }^{+} \mathrm{I}(y: z){ }^{+} n,  \tag{5}\\
& \mathrm{I}(x: y z){ }^{+} \mathrm{I}(y: x z){ }^{+} \mathrm{I}(z: x y)={ }^{+} 2 n
\end{align*}
$$

(the complexity profile of the triple $(x, y, z)$ is like in Example 3. Fig. 2(b), though the combinatorial structure can be arbitrary). This is a symmetric but pretty "generic" complexity profile. By choosing $k$, we control the tradeoff between complexities of $x, y, z$ and the mutual informations between the inputs. We believe that the full understanding of the secret key agreement for inputs satisfying (5) will help to handle eventually inputs with arbitrary complexity profile.

Let Alice, Bob, and Charlie communicate with each other via a public channel. In [3, Theorem 5.5 and Theorem 5.11] it was shown that in this setting the parties can agree on a secret key of size

$$
\frac{1}{2}(\mathrm{I}(x: y \mid z)+\mathrm{I}(x: z \mid y)+\mathrm{I}(y: z \mid x))=^{+} 1.5 n
$$

This size of the key is optimal (up to an additive term $O(\log n)$ ), and a secret key of this size can be obtained in an omniscience protocol. In this protocol, the parties exchange messages so that each of them learns completely the entire triple of inputs $(x, y, z)$. The total number of communicated bits is less than $\mathrm{C}(x, y, z)$, so an eavesdropper can learn only a partial information on the inputs. More specifically, for a triple satisfying (5), in the omniscience protocol the parties send

$$
\begin{equation*}
\frac{1}{2}(\mathrm{C}(y, z \mid x)+\mathrm{C}(x, z \mid y)+\mathrm{C}(x, y \mid z))=^{+}(3 k-4.5) n \tag{6}
\end{equation*}
$$

bits. The gap between $\mathrm{C}(x, y, z)=^{+}(3 k-3) n$ and the amount of the divulged information (6) is used to produce the secret key.

Let us mention that in the omniscience protocol from [3] there is no substantial interaction between the parties: Alice, Bob, and Charlie send in parallel their messages (computed from the input data and the public randomness), and then each of them uses the messages received from two other parties to compute the key.

The omniscience protocol used in [3] provides an upper bound on the communication complexity of secret key agreement. In what follows we prove two lower bounds for the communication of this problem. One of these bound (Theorem 3 below) holds for all ( $x, y, z$ ) satisfying (5) and for all communication protocols. The other one (Theorem(4) applies to ( $x, y, z$ ) from Example 3 and only to one-round protocols with simultaneous messages. Theorem 4 implies that (6) is the optimal communication complexity for inputs satisfying (5) for protocols with simultaneous messages.

Now we proceed with the proof of our bound. We begin with the following technical statement.
Theorem 2. Consider a communication protocol with two parties Alice and Bob where Alice is given $x$, Bob is given y, and everyone (including the eavesdropper) can access some s. If Alice and Bob follow the protocol using a public string of random bits $r$ and agree on a key $w$, on which the eavesdropper gets no information, then $\mathrm{C}(w) \leq^{+} \mathrm{I}(x: y \mid r, s)$. (In the hidden term $O(\log n)$ the value of $n$ is the sum of the lengths of the strings $x, y, w$.)

Remark 2. The condition that the eavesdropper gets no information on the key is understood as follows. Denote by $t$ the transcript of the protocol. Then $\mathrm{I}(w: s, t, r)=^{+} 0$. In other words, the whole public information (the known a priori s, the public randomness, and the transcript $t$ ) contains only negligibly small information on the secret key.

Sketch of proof. Theorem 2 is a relativized version of [3, Theorem 4.2], where $s$ is used as an oracle. More formally, one can follow the argument from [3] substituting $s$ as a supplementary condition in each term of Kolmogorov complexity appearing in the proof. An alternative way is to define a version of Kolmogorov complexity for Turing machines that can access $s$ as on the oracle tape and observe that the proof of Theorem 4.2 in [3] relativizes.

Corollary 2. Consider a communication protocol with three parties where Alice is given x, Bob is given $y$, and Charlie is given $z$. Denote by $m_{C}$ the concatenation of all messages broadcasted by Charlie during the communication. If the parties agree on a secret key $w$ on which the eavesdropper gets no information (even given access to the messages sent by all parties), then $\mathrm{C}(w) \leq{ }^{+} \mathrm{I}\left(x: y \mid r, m_{C}\right)$.

Proof. We apply Theorem 2 substituting $m_{C}$ instead of the public information $s$.
Theorem 3. Let Alice, Bob, and Charlie be given inputs x, y, z satisfying (5). We consider communication protocols (with any number of rounds) where Alice, Bob, and Charlie agree on a secret key $w$ of complexity $\mathrm{C}(w)=^{+} 1.5 n$ (which is optimal due to [3]). Communication complexity of such a protocol is at least $1.5 n-O(\log n)$.

Remark 3. In Theorem 3 we make no assumption about the structure of $(x, y, z)$, so we can use only general information-theoretic considerations. The bound proven in this theorem is rather weak but it applies to all $(x, y, z)$ with the complexity profile shown in Fig. [2(b).

Proof. In this theorem, as usual, we abuse the notation and ignore the random bits (see Remark 11). Our proof of this theorem works with any oracle $r$ that does not change Kolmogorov complexity of the input tuple ( $x, y, z$ ).

We begin the proof with a simple inequality from information theory.
Lemma 3. For all binary strings $x, y, z$ it holds

$$
\mathrm{I}(x: y \mid s)-\mathrm{I}(x: y) \leq^{+} \mathrm{I}(s: x y) .
$$

(See the proof of the lemma in Appendix.)
Denote by $m_{A}, m_{B}, m_{C}$ the concatenation of all messages sent by Alice, Bob, and Charlie respectively. From Corollary 2 we know that the size of the key (in our case $1.5 n$ ) cannot be greater than $\mathrm{I}\left(x: y \mid m_{C}\right)$.

By the theorem assumption we have $\mathrm{I}(x: y)=n$. Therefore, the difference between $\mathrm{I}(x: y)$ and $\mathrm{I}\left(x: y \mid m_{C}\right)$ is at least $0.5 n$. From Lemma 3 it follows that $\mathrm{I}\left(m_{C}: x y\right) \geq^{+} 0.5 n$. This is only possible when $\mathrm{C}\left(m_{C}\right) \geq^{+} 0.5 n$. A similar argument implies $\mathrm{C}\left(m_{A}\right) \geq^{+} 0.5 n$ and $\mathrm{C}\left(m_{B}\right) \geq^{+} 0.5 n$.

Theorem 4. Let Alice be given a direction x, Bob be given a direction y, and Charlie is given a direction $z$ in the $(k+1)$-dimensional space $\mathbb{F}_{2^{n}}^{k+1}$, and it is known that $x, y$, and $z$ are pairwise orthogonal (see Example (3). We consider non-interactive communication protocols where Alice, Bob, and Charlie send messages $m_{A}=m_{A}(x), m_{B}=m_{B}(y)$, and $m_{C}=m_{C}(z)$ respectively and produce a secret key with the optimal (see [3]) complexity $\mathrm{C}(w)=^{+} 1.5 n$. Then

$$
\mathrm{C}\left(m_{A}\right) \geq^{+}(k-1.5) n, \mathrm{C}\left(m_{B}\right) \geq^{+}(k-1.5) n, \mathrm{C}\left(m_{C}\right) \geq^{+}(k-1.5) n
$$

and the communication complexity of the protocol is at least $(3 k-4.5) n-O(\log n)$, which matches the communication complexity of the omniscience protocol (up to an additive logarithmic term).

Remark 4. In Theorem 4 we know the combinatorial structure of the triple ( $x, y, z$ ). This allows us to apply the spectral technique and prove a bound for communication complexity of the protocol that is much stronger than the universal bound from Theorem 3.

Proof. Similarly to the proofs of the previous theorems, we abuse the notation and ignore the random bits (see Remark [1). We begin the argument similarly to Theorem 3. From Corollary 2 we know that the size of the key (in our case $1.5 n$ ) cannot be greater than $\mathrm{I}\left(x: y \mid m_{C}\right)$. By the construction, $\mathrm{I}(x: y)=n$. Therefore, the difference between $\mathrm{I}(x: y)$ and $\mathrm{I}\left(x: y \mid m_{C}\right)$ is at least $0.5 n$. Thus, from Lemma 3 it follows that $\mathrm{I}\left(m_{C}: x y\right) \geq^{+} 0.5 n$.

We apply Proposition 2 and conclude that $\mathrm{I}\left(m_{C}: x y\right) \geq^{+} 0.5 n$ can be true only if

$$
\mathrm{C}\left(m_{C}\right) \geq^{+}(k-2) n+0.5 n
$$

A similar argument applies to $\mathrm{C}\left(m_{A}\right)$ and $\mathrm{C}\left(m_{B}\right)$.
Remark 5. In the statement and in the proof of Theorem 4 we used the typical for the theory of Kolmogorov complexity precision "up to an additive logarithmic term." The same proof, mutatis mutandis, works with any coarser precision. Thus, the statement of the theorem remains true if instead of $O(\log n)$ we take the terms $O(\sqrt{n})$, or $O(f(n))$ for any other $f(n) \geq \log n$.
Remark 6. Using the connection between Kolmogorov complexity and Shannon entropy (see the discussion in (3), one can translate Theorem 4 in the language Shannon's framework. Assume that Alice, Bob, and Charlie are given a randomly chosen triples of orthogonal directions $(x, y, z)$ from $\mathbb{F}_{2^{n}}^{k+1}$ (with the uniform distribution on all such triples). Then, by communicating via a public channel, the parties can agree on a common secret key with Shannon's entropy $1.5 n-o(n)$. Theorem 4 implies that for one-round communication protocols (with simultaneous messages) the total length of the messages sent by Alice, Bob, and Charlie must be at least $(3 k-4.5) n-o(n)$ bits.

In conclusion, we observe that communication complexity in Theorem 4 is not optimal for multi-round protocols.

Theorem 5. In the setting of Theorem 4 there is a multi-round communication protocol (not a simultaneous messages protocol) with communication complexity

$$
(2 k-2.5) n+O(\log n)
$$

where the parties agree on a secret key of the optimal size $1.5 n-O(\log n)$.

Sketch of proof. We adapt the omniscience protocol from 3. In what follows we assume that random hash-functions are chosen with the public source of randomness (e.g., one may assume that random hashing is the multiplication by a randomly chosen matrix).

In the first round, Alice and Bob send messages $m_{A}=m_{A}(x, r)$ and $m_{B}=m_{B}(y, r)$ (random hash-values of $x$ and $y$ ), each of length $(k-1.5) n+O(\log n)$ such that Charlie given $\left(m_{A}, m_{B}, z\right)$ can reconstruct the pair $(x, y)$. Then Charlie sends a message $m_{C}$ that is another random hash-value of $(x, y)$ of length $0.5 n+O(\log n)$.

With a high probability (for a randomly chosen hash-function), the values $m_{B}$ and $m_{C}$ are enough for Alice to reconstruct $y$, and the values $m_{A}$ and $m_{C}$ are enough for Bob to reconstruct $x$. Thus, at the end of communication, with high probability each party knows $(x, y)$. At the same time, the adversary learns from the communication at most $\left|m_{A}\right|+\left|m_{B}\right|+\left|m_{C}\right|=(2 k-2.5) n+O(\log n)$ bits of information.

Now each party applies to ( $x, y$ ) another (independently chosen) random hash function and obtains a hash-value $w=\operatorname{hash}(x, y)$ of length

$$
\mathrm{C}(x, y)-\left|m_{A}\right|-\left|m_{B}\right|-\left|m_{C}\right|-O(\log n)=1.5 n-O(\log n) .
$$

With a high probability the obtained $w$ is incompressible conditional on the data accessible to the eavesdropper (the messages of the parties and the public random bits).

Remark 7. In the proof of Theorem 5, we may define random hashing as random linear mappings, i.e., each hash-values can be computed as the product over $\mathbb{F}_{2}$ of a bit vector by a randomly chosen binary matrix of the appropriate dimension. These matrices can be made publicly known: we can obtain these random matrices from the public source of random bits. Using more sophisticated constructions of hashfunctions, we could reduce the number of used random bits (although this is not necessary to prove Theorem (5).

## 6 Conclusion and open problems

We have settled communication complexity of secret key agreement with simultaneous messages for three parties, for inputs with specific information profiles (shown in Fig. (2). We conjecture that this technique can be adapted to any possible complexity profiles. We may need more involved methods to prove tight lower bounds for multi-round protocols, with interleaving messages and non-trivial interaction between parties.

In this paper we took into consideration only randomized communication protocols with a public source of random bits. We believe that with the help of the techniques from [1], our results can be extended to the setting with private sources of randomness.

Let us recall that Proposition 1 and Proposition 2 from Section 3 were proven with substantially the same argument. These propositions seem to be two special cases of one general phenomenon. It would be interesting to characterize the class of input pairs $(x, y)$ such that the task "given $x$ we need to produce a message with a large mutual information with $y$ " is maximally difficult (i.e., the length of the message must be large), like in Proposition 1 and Proposition 2.

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## A Bounds of the spectral gap

Proof of Lemma圆(a). In this proof we use the notation $q=2^{n}$. To compute $\# L$ and $\# R$ we divide the total number of non-zero vectors in the space by the number of vectors in each equivalence class (the number of non-zero elements in the field). To find $D_{L}$ and $D_{R}$ we do the same calculation in a subspace of co-dimension 1.

Now let us estimate the eigenvalues. Let

$$
H=\left(\begin{array}{cc}
0 & J \\
J^{\top} & 0
\end{array}\right)
$$

be the adjacency matrix of the graph. It is enough to estimate the eigenvalues of $M^{2}$. To this end, we need to estimate the eigenvalues of $J \cdot J^{\top}$, which is the matrix of paths of length 2 in the graph, starting and finishing in $L$. Starting at some $x \in L$, we can go to a $y \in R$ (orthogonal to $x$ ) and then either come back to the same $x$, or to end up in a different $x^{\prime} \in R$. For a fixed $x$, the number of paths

$$
x \rightarrow y \rightarrow x
$$

is equal to $D_{L}=\left(q^{k}-1\right) /(q-1)$ (any $y$ orthogonal to $x$ can serve as the middle point of the path). Further, for $x \neq x^{\prime}$, the number of paths

$$
x \rightarrow y \rightarrow x^{\prime}
$$

is equal to the number of directions $y$ that are orthogonal to the 2-dimensional space spanned on $x$ and $x^{\prime}$. There are $\left(q^{k-1}-1\right) /(q-1)$ such directions. Thus,

$$
J \cdot J^{\top}=\frac{q^{k}-1}{q-1} \cdot\left[\begin{array}{ccccc}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \\
0 & 0 & 0 & \cdots & 1
\end{array}\right]+\frac{q^{k-1}-1}{q-1} \cdot\left[\begin{array}{ccccc}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
1 & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \\
1 & 1 & 1 & \cdots & 0
\end{array}\right]
$$

which rewrites to

$$
J \cdot J^{\top}=\frac{q^{k}-q^{k-1}}{q-1} \cdot\left[\begin{array}{ccccc}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \\
0 & 0 & 0 & \cdots & 1
\end{array}\right]+\frac{q^{k-1}-1}{q-1} \cdot\left[\begin{array}{ccccc}
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \\
1 & 1 & 1 & \cdots & 1
\end{array}\right]
$$

The identity matrix has the eigenvalue 1 of multiplicity $\# L$, and the all-ones matrix has the eigenvalue $\# L=q^{k}$ of multiplicity 1 and the eigenvalue 0 of multiplicity $(\# L-1)$. These matrixes obviously have a common basis of eigenvectors. We conclude that the spectrum of $J \cdot J^{\top}$ consists of the numbers

$$
\lambda_{1}=\frac{q^{k-1}-1}{q-1} \cdot \frac{q^{k+1}-1}{q-1}+\frac{q^{k}-q^{k-1}}{q-1}=D_{L} \cdot D_{R}=\Theta\left(q^{2 k-2}\right)
$$

and

$$
\lambda_{2}=\lambda_{3}=\ldots=\lambda_{\# L}=\frac{q^{k}-q^{k-1}}{q-1}=q^{k-1}
$$

These numbers coincide with the eigenvalues of $H^{2}$, and the absolute values of the eigenvalues of $H$ are the square roots of these numbers.

Proof of Lemma国(b). To calculate $\# L$ we count the number of equivalence classes of non-zero vectors in the space of dimension $(k+1)$, with $q-1$ vectors in each class. Observe that $D_{R}$ is the same number but for the space of co-dimension 2 (we count the directions orthogonal to $y$ and $z$ ).

To calculate $\# R$ we count the number of equivalence classes of non-zero vectors in the space of dimension $(k+1)$ (the number of directions $y$ ) and the number of equivalence classes in the subspace of co-dimension 1 (directions $z$ orthogonal to $y$ ). To find $D_{L}$ we repeat the same computation but with the dimensions decremented by 1 (in the the subspace of vectors orthogonal to $x$ ).

Now we estimate the eigenvalues. We denote again $H=\left(\begin{array}{cc}0 & J \\ J^{\top} & 0\end{array}\right)$ the adjacency matrix of the graph. Similarly to the proof of (a), we estimate the eigenvalues of $J \cdot J^{\top}$, which is the matrix of paths of length 2 in the graph, starting and finishing in $L$. Starting at some $x \in L$, we can go to some $(y, z) \in R$ (orthogonal to $x$ ) and then either come back to the same $x$, or to end up in a different $x^{\prime} \in R$. For a fixed $x$, the number of paths

$$
x \rightarrow(y, z) \rightarrow x
$$

is equal to $D_{L}$ (any $(y, z)$ matching $x$ serves as the middle point of the path). For $x \neq x^{\prime}$, the number of paths

$$
x \rightarrow(y, z) \rightarrow x^{\prime}
$$

is equal to the number of pairs $(y, z)$ that are orthogonal to the 2 -dimensional space spanned on $x$ and $x^{\prime}$. There are $\frac{\left(q^{k-1}-1\right)\left(q^{k-2}-1\right)}{(q-1)^{2}}$ such pairs. Thus,

$$
J \cdot J^{\top}=D_{L} \cdot\left[\begin{array}{ccccc}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \\
0 & 0 & 0 & \cdots & 1
\end{array}\right]+\frac{\left(q^{k-1}-1\right)\left(q^{k-2}-1\right)}{(q-1)^{2}} \cdot\left[\begin{array}{ccccc}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
1 & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \\
1 & 1 & 1 & \cdots & 0
\end{array}\right],
$$

which rewrites to

$$
J \cdot J^{\top}=\left(D_{L}-\frac{\left(q^{k-1}-1\right)\left(q^{k-2}-1\right)}{(q-1)^{2}}\right) \cdot\left[\begin{array}{ccccc}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \\
0 & 0 & 0 & \cdots & 1
\end{array}\right]+\frac{\left(q^{k-1}-1\right)\left(q^{k-2}-1\right)}{(q-1)^{2}} \cdot\left[\begin{array}{ccccc}
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \\
1 & 1 & 1 & \cdots & 1
\end{array}\right]
$$

We know the eigenvalues of the identity and the all-ones matrices. It follows that the spectrum of $J \cdot J^{\top}$ consists of the numbers

$$
\lambda_{1}=D_{L} \cdot D_{R}=\Theta\left(q^{k-2} \cdot q^{k-3} \cdot q^{k}\right) \text { and } \lambda_{2}=\lambda_{3}=\ldots \lambda_{\# L} \leq D_{L}
$$

The absolute values of the eigenvalues of $H$ are the square roots of these numbers.

## B Useful inequalities

Proof of Lemma 3. We need to prove that

$$
\begin{equation*}
\mathrm{I}(x: y \mid s) \leq^{+} \mathrm{I}(x: y)+\mathrm{I}(s: x y) \tag{7}
\end{equation*}
$$

Observe that

$$
\mathrm{I}(x: y)={ }^{+} \mathrm{I}(x: y \mid s)+\mathrm{I}(x: y: s)
$$

and

$$
\mathrm{I}(s: x y)=^{+} \mathrm{I}(s: x)+\mathrm{I}(s: y)-\mathrm{I}(x: y: s)
$$

Therefore, (7) rewrites to

$$
I(x: y \mid s) \leq^{+} \mathrm{I}(x: y \mid s)+\mathrm{I}(x: y: s)+\mathrm{I}(s: x)+\mathrm{I}(s: y)-\mathrm{I}(x: y: s)
$$

This inequality is always true since the terms $\mathrm{I}(s: x)$ and $\mathrm{I}(s: y)$ are non-negative.
Lemma 4. For all $x, y, x^{\prime}, y^{\prime}$
(i) $\mathrm{I}\left(x^{\prime}: y \mid z\right) \leq^{+} \mathrm{I}(x: y \mid z)+\mathrm{C}\left(x^{\prime} \mid x\right)$.
(ii) $\mathrm{I}\left(x^{\prime}: y^{\prime} \mid z\right) \leq^{+} \mathrm{I}(x: y \mid z)+\mathrm{C}\left(x^{\prime} \mid x\right)+\mathrm{C}\left(y^{\prime} \mid y\right)$.

Proof. To prove (i) we observe that by the chain rule for the mutual information

$$
\mathrm{I}\left(x, x^{\prime}: y \mid z\right)={ }^{+} \mathrm{I}(x: y \mid z)+\mathrm{I}\left(x^{\prime}: y \mid x, z\right)={ }^{+} \mathrm{I}\left(x^{\prime}: y \mid z\right)+\mathrm{I}\left(x: y \mid x^{\prime}, z\right)
$$

Therefore,

$$
\begin{aligned}
\mathrm{I}\left(x^{\prime}: y \mid z\right) & =^{+} \mathrm{I}(x: y \mid z)+\mathrm{I}\left(x^{\prime}: y \mid x, z\right)-\mathrm{I}\left(x: y \mid x^{\prime}, z\right) \\
& \leq^{+} \mathrm{I}(x: y \mid z)+\mathrm{I}\left(x^{\prime}: y \mid x, z\right) \\
& \leq^{+} \mathrm{I}(x: y \mid z)+\mathrm{C}\left(x^{\prime} \mid x, z\right) \\
& \leq^{+} \mathrm{I}(x: y \mid z)+\mathrm{C}\left(x^{\prime} \mid x\right) .
\end{aligned}
$$

To prove (ii) we apply the same argument twice (at first we replace $x^{\prime}$ by $x$ and then replace $y^{\prime}$ by $y$ ),

$$
\mathrm{I}\left(x^{\prime}: y^{\prime} \mid z\right) \leq^{+} \mathrm{I}\left(x: y^{\prime} \mid z\right)+\mathrm{C}\left(x^{\prime} \mid x\right) \leq^{+} \mathrm{I}(x: y \mid z)+\mathrm{C}\left(x^{\prime} \mid x\right)+\mathrm{C}\left(y^{\prime} \mid y\right)
$$

