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SPECTRAL APPROXIMATIONS TO THE FRACTIONAL INTEGRAL AND DERIVATIVE

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Abstract

In this paper, the spectral approximations are used to compute the fractional integral and the Caputo derivative. The effective recursive formulae based on the Legendre, Chebyshev and Jacobi polynomials are developed to approximate the fractional integral. And the succinct scheme for approximating the Caputo derivative is also derived. The collocation method is proposed to solve the fractional initial value problems and boundary value problems. Numerical examples are also provided to illustrate the effectiveness of the derived methods.

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Key Words and Phrases: fractional integral, Caputo derivative, spectral approximation, Jacobi polynomials

1. Introduction

Fractional calculus (including the fractional integral and the fractional derivative) has a long history, which is as old as the more familiar integer-order counterparts [14]. Since then, fractional calculus has undergone a rapid development primarily in theoretical mathematics. At present, fractional calculus has been found widely used in many areas of science and engineering [3, 9, 11, 15, 16, 19, 21, 27, 29].

Fractional derivative is more complicated than the classical one, and the calculation of the fractional derivative is also more difficult than that for the typical one. In [4], numerical algorithms for solving the fractional integral and Caputo derivative were considered. Li *et al.* [10] also developed numerical algorithms based on the piecewise polynomial interpolation to approximate the fractional integral and Caputo derivative, and to solve the

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fractional differential equations. An automatic quadrature method based on the Chebyshev polynomials was presented for approximating the Caputo derivative in [25]. Some other computational schemes, such as the L1, L2 and L2C schemes, etc., are also introduced [8, 12, 13, 15, 17, 18, 20, 22, 24, 26, 28].

To increase calculation accuracy, spectral approximations are often chosen. In this paper, we derive effective algorithms to approximate the fractional integral by using the Legendre, Chebyshev and Jacobi polynomials. The numerical algorithm to calculate the Caputo derivative is also derived based on the above computational schemes. Besides, we propose a kind of fractional operational matrix, which can be seen as a generalization of the classical derivative. When the fractional order of the Caputo derivative reduces to an integer, the derived fractional operational matrix reduces to the classical differential matrix. The applications of the constructed algorithms are illustrated to compute the fractional integral, Caputo derivative and the fractional ordinary differential equations. Numerical experiments are displayed to verify the proposed numerical algorithms.

The remainder of this paper is organized as follows. In Section 2, we introduce several definitions of fractional calculus and the Legendre, Chebyshev and Jacobi polynomials. Numerical algorithms for calculating the fractional integral and the Caputo derivative are presented in Sections 3 and 4, respectively. The applications of the methods are illustrated in Section 5. Numerical examples are presented in Section 6, and the conclusion is included in the last section.

2. Definitions and notations

In this section, we introduce the definitions of the fractional calculus. Then we introduce the Legendre, Chebyshev and Jacobi polynomials, which will be used later on.

Definition **2.1**. The fractional integral (or the Riemann–Liouville integral) with order $\alpha > 0$ of the given function $f(t)$ is defined as

$$D_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler's gamma function.

There exist several kinds of fractional derivatives. However, in engineering the Caputo derivative is mostly used, which is introduced below.

DEFINITION **2.2**. The Caputo derivative with order $\alpha > 0$ of the given function $f(t)$ is defined as

$$
\begin{aligned}
{}_C D_{a,t}^{\alpha} f(t) &= D_{a,t}^{-(n-\alpha)} \left[f^{(n)}(t) \right] \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s,
\end{aligned}
\tag{2.2}
$$

where n is a positive integer and $n-1 < \alpha \leq n$.

Next, we introduce the Legendre, Chebyshev and Jacobi polynomials.

The Legendre polynomials $\{L_j(x)\}$, $x \in [-1,1]$ satisfy the three-term recurrence relation

$$
\begin{aligned}
&L_0(x) = 1, \quad L_1(x) = x, \\
&L_{j+1} = \frac{2j+1}{j+1} x L_j(x) - \frac{j}{j+1} L_{j-1}(x), \quad j \geq 1.
\end{aligned}
\tag{2.3}
$$

Their properties which will be used later on are listed as follows

$$
(2j+1) L_j(x) = L'_{j+1}(x) - L'_{j-1}(x), \quad j \geq 1, \tag{2.4}
$$

$$
L_j(\pm 1) = (\pm 1)^j, \quad L'_j(\pm 1) = \frac{1}{2} (\pm 1)^{j-1} j(j+1). \tag{2.5}
$$

The Chebyshev polynomials $\{T_j(x)\}, x \in [-1,1]$ satisfy the three-term recurrence relation

$$
\begin{aligned}
&T_0(x) = 1, \quad T_1(x) = x, \\
&T_{j+1}(x) = 2x T_j(x) - T_{j-1}(x), \quad j \geq 1.
\end{aligned}
\tag{2.6}
$$

Their fundamental properties are presented below

$$
2T_j(x) = \frac{1}{j+1} T'_{j+1}(x) - \frac{1}{j-1} T'_{j-1}(x), \quad j \geq 2, \tag{2.7}
$$

$$
T_j(\pm 1) = (\pm 1)^j, \quad T'_j(\pm 1) = \frac{1}{2} (\pm 1)^{j-1} j^2. \tag{2.8}
$$

The Jacobi polynomials $\{P_j^{a,b}(x)\}$, $a, b > -1, x \in [-1,1]$ are given by the following three-term recurrence relation [23]

$$
\begin{aligned}
&P_0^{a,b}(x) = 1, \quad P_1^{a,b}(x) = \frac{1}{2}(a+b+2)x + \frac{1}{2}(a-b), \\
&P_{j+1}^{a,b}(x) = (A_j^{a,b} x - B_j^{a,b}) P_j^{a,b}(x) - C_j^{a,b} P_{j-1}^{a,b}(x), \quad n \geq 1,
\end{aligned}
\tag{2.9}
$$

where

$$\begin{aligned}
A_j^{a,b} &= \frac{(2j+a+b+1)(2j+a+b+2)}{2(j+1)(j+a+b+1)},\\
B_j^{a,b} &= \frac{(b^2-a^2)(2j+a+b+1)}{2(j+1)(j+a+b+1)(2j+a+b)},\\
C_j^{a,b} &= \frac{(j+a)(j+b)(2j+a+b+2)}{(j+1)(j+a+b+1)(2j+a+b)}.
\end{aligned} \tag{2.10}$$

Here, we list some useful properties of the Jacobi polynomials that will be used in the present paper [23].

$$P_j^{a,b}(1) = \binom{j+a}{j} = \frac{\Gamma(j+a+1)}{j!\Gamma(a+1)}, \quad P_j^{a,b}(-1) = (-1)^j \frac{\Gamma(j+b+1)}{j!\Gamma(b+1)}. \tag{2.11}$$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} P_j^{a,b}(x) = d_{j,m}^{a,b} P_{j-m}^{a+m,b+m}(x), \quad j \geq m, m \in \mathbb{N}, \tag{2.12}$$

where

$$d_{j,m}^{a,b} = \frac{\Gamma(j+m+a+b+1)}{2^m \Gamma(j+a+b+2)}. \tag{2.13}$$

$$P_j^{a,b}(x) = \widehat{A}_j^{a,b} \frac{\mathrm{d}}{\mathrm{d}x} P_{j-1}^{a,b}(x) + \widehat{B}_j^{a,b} \frac{\mathrm{d}}{\mathrm{d}x} P_j^{a,b}(x) + \widehat{C}_j^{a,b} \frac{\mathrm{d}}{\mathrm{d}x} P_{j+1}^{a,b}(x), \quad j \geq 1, \tag{2.14}$$

in which

$$\begin{aligned}
\widehat{A}_j^{a,b} &= \frac{-2(j+a)(j+b)}{(j+a+b)(2j+a+b)(2j+a+b+1)},\\
\widehat{B}_j^{a,b} &= \frac{2(a-b)}{(2j+a+b)(2j+a+b+2)},\\
\widehat{C}_j^{a,b} &= \frac{2(j+a+b+1)}{(2j+a+b+1)(2j+a+b+2)}.
\end{aligned} \tag{2.15}$$

If $j=1$, $\widehat{A}_1^{a,b}$ in (2.15) is set to be zero.

Remark **2.1**. The case $a=b=0$ in (2.9) yields the Legendre polynomials ($P_j^{0,0}(x) = L_j(x)$). If $a=b=-\frac{1}{2}$, then $P_j^{-\frac{1}{2},-\frac{1}{2}}(x) = \frac{\Gamma(j+1/2)}{j!\sqrt{\pi}} T_j(x)$, $j \geq 0$.

3. Approximation to the fractional integral

In the section, we develop algorithms to approximate the fractional integral of a given function. Among three kinds of polynomials, the Legendre polynomials are commonly used. We mainly study the Legendre approximation to the fractional integral in details in this section. The Chebyshev and Jacobi approximations are almost the same as that of the Legendre approximation.

Let $u(x)$ be a function defined on the interval $[-1,1]$, and N be a positive integer. Denote by

$$p_N(x) = \sum_{j=0}^{N} \tilde{l}_j \, L_j(x), \tag{3.1}$$

where $p_N(x)$ is an approximation of $u(x)$, and $\tilde{l}_j$ are the coefficients determined by $u(x)$. If $p_N(x)$ is an orthogonal projection of $u(x)$, then $\tilde{l}_j$ can be determined by the orthogonality of $\{L_j(x)\}$. In this paper, we assume that $p_N(x)$ is the interpolation of $u(x)$. If $p_N(x)$ is the interpolation of $u(x)$ on the Legendre–Gauss–Lobatto points $\{x_k\}_{k=0}^{N}$, then $\tilde{l}_j$ can be determined by

$$\tilde{l}_j = \frac{1}{\gamma_j} \sum_{k=0}^{N} u(x_k) L_j(x_k) \omega_k, \tag{3.2}$$

where $\gamma_j = \frac{2}{2j+1}$ for $0 \le j \le N-1$, $\gamma_N = \frac{2}{N}$, and $\{\omega_k\}_{k=0}^{N}$ are the corresponding quadrature weights [23].

Therefore, for any $\alpha > 0$, the fractional integral $D_{-1,x}^{-\alpha} u(x)$ can be approximated by

$$\begin{aligned} D_{-1,x}^{-\alpha} u(x) \approx D_{-1,x}^{-\alpha} p_N(x) &= \frac{1}{\Gamma(\alpha)} \int_{-1}^{x} (x-s)^{\alpha-1} p_N(s) \, \mathrm{d}s \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{N} \tilde{l}_j \int_{-1}^{x} (x-s)^{\alpha-1} L_j(s) \, \mathrm{d}s \\ &= \sum_{j=0}^{N} \tilde{l}_j \, \widehat{L}_j^{\alpha}(x), \end{aligned} \tag{3.3}$$

where $\widehat{L}_j^{\alpha}(x) = \frac{1}{\Gamma(\alpha)} \int_{-1}^{x} (x-s)^{\alpha-1} L_j(s) \, \mathrm{d}s$. Next, we give an effective recurrence formula to calculate $\widehat{L}_j^{\alpha}(x)$.

We can easily get $\widehat{L}_0^{\alpha}(x) = \frac{(x+1)^{\alpha}}{\Gamma(\alpha+1)}$ and $\widehat{L}_1^{\alpha}(x) = \frac{x(x+1)^{\alpha}}{\Gamma(\alpha+1)} - \frac{\alpha(x+1)^{\alpha+1}}{\Gamma(\alpha+2)}$ from (2.3). For $j \geq 1$, by using (2.3), we have

$$\begin{aligned}
\widehat{L}_{j+1}^{\alpha}(x) =& \frac{1}{\Gamma(\alpha)} \int_{-1}^{x} (x-s)^{\alpha-1} L_{j+1}(s)\,\mathrm{d}s \\
=& \frac{1}{\Gamma(\alpha)} \frac{1}{j+1} \int_{-1}^{x} (x-s)^{\alpha-1} \Big[(2j+1)sL_j(s) - jL_{j-1}(s)\Big]\,\mathrm{d}s \\
=& \frac{1}{j+1}\Big\{(2j+1)x\widehat{L}_j^{\alpha}(x) - j\widehat{L}_{j-1}^{\alpha}(x) \\
& - \frac{2j+1}{\Gamma(\alpha)} \int_{-1}^{x} (x-s)^{\alpha} L_j(s)\,\mathrm{d}s\Big\}.
\end{aligned} \tag{3.4}$$

Noticing that $(2j+1)L_j(x) = L'_{j+1}(x) - L'_{j-1}(x)$ for $j \geq 1$ (see Eq. (2.4)), we have

$$\begin{aligned}
\widehat{L}_{j+1}^{\alpha}(x) =& \frac{1}{j+1}\Big\{(2j+1)x\widehat{L}_j^{\alpha}(x) - j\widehat{L}_{j-1}^{\alpha}(x) \\
& - \frac{1}{\Gamma(\alpha)} \int_{-1}^{x} (x-s)^{\alpha} (L'_{j+1}(s) - L'_{j-1}(s))\,\mathrm{d}s\Big\} \\
=& \frac{1}{j+1}\Big\{(2j+1)x\widehat{L}_j^{\alpha}(x) - j\widehat{L}_{j-1}^{\alpha}(x) \\
& - \frac{1}{\Gamma(\alpha)} \Big[(x-s)^{\alpha}(L_{j+1}(s) - L_{j-1}(s))\Big]_{-1}^{x} \\
& - \alpha(\widehat{L}_{j+1}^{\alpha}(x) - \widehat{L}_{j-1}^{\alpha}(x))\Big\} \\
=& \frac{1}{j+1}\Big\{(2j+1)x\widehat{L}_j^{\alpha}(x) - j\widehat{L}_{j-1}^{\alpha}(x) - \alpha(\widehat{L}_{j+1}^{\alpha}(x) - \widehat{L}_{j-1}^{\alpha}(x))\Big\}.
\end{aligned} \tag{3.5}$$

Hence, for $j \geq 1$, we get the following recurrence relation

$$\widehat{L}_{j+1}^{\alpha}(x) = \frac{1}{j+1+\alpha}\Big\{(2j+1)x\widehat{L}_j^{\alpha}(x) - (j-\alpha)\widehat{L}_{j-1}^{\alpha}(x)\Big\}. \tag{3.6}$$

So, $\widehat{L}_j^{\alpha}(x)$ can be calculated by the following formula

$$\begin{cases}
\widehat{L}_0^{\alpha}(x) = \dfrac{(x+1)^{\alpha}}{\Gamma(\alpha+1)}, \\
\widehat{L}_1^{\alpha}(x) = \dfrac{x(x+1)^{\alpha}}{\Gamma(\alpha+1)} - \dfrac{\alpha(x+1)^{\alpha+1}}{\Gamma(\alpha+2)}, \\
\widehat{L}_{j+1}^{\alpha}(x) = \dfrac{1}{j+1+\alpha}\Big\{(2j+1)x\widehat{L}_j^{\alpha}(x) - (j-\alpha)\widehat{L}_{j-1}^{\alpha}(x)\Big\}, j \geq 1.
\end{cases} \tag{3.7}$$

Therefore, $D_{-1,x}^{-\alpha}u(x)$ is approximated by

$$D_{-1,x}^{-\alpha}\,u(x)\approx D_{-1,x}^{-\alpha}\,p_N(x)=\sum_{j=0}^{N}\tilde{l}_j\,\widehat{L}_j^{\alpha}(x),$$

where $\widehat{L}_j^{\alpha}(x)$ is defined by (3.7), and $\tilde{l}_j$ is defined by (3.2).

Similarly, we can get the similar results for the Chebyshev polynomials and the Jacobi polynomials.

For the Chebyshev polynomials, let $\widehat{T}_j^{\alpha}(x)=\frac{1}{\Gamma(\alpha)}\int_{-1}^{x}(x-s)^{\alpha-1}T_j(s)\,\mathrm{d}s$, by many calculations, one can get

$$\begin{cases}\widehat{T}_0^{\alpha}(x)=\dfrac{(x+1)^{\alpha}}{\Gamma(\alpha+1)},\\ \widehat{T}_1^{\alpha}(x)=\dfrac{x(x+1)^{\alpha}}{\Gamma(\alpha+1)}-\dfrac{\alpha(x+1)^{\alpha+1}}{\Gamma(\alpha+2)},\\ \widehat{T}_2^{\alpha}(x)=\dfrac{4x}{2+\alpha}\widehat{T}_1^{\alpha}(x)-\dfrac{2}{2+\alpha}\widehat{T}_0^{\alpha}(x)+\dfrac{\alpha(x+1)^{\alpha}}{(2+\alpha)\Gamma(\alpha+1)},\\ \widehat{T}_{j+1}^{\alpha}(x)=\dfrac{2(j+1)x}{j+1+\alpha}\widehat{T}_j^{\alpha}(x)-\dfrac{(j+1)(j-1-\alpha)}{(j+1+\alpha)(j-1)}\widehat{T}_{j-1}^{\alpha}(x)\\ \qquad+\dfrac{2(-1)^j\alpha(x+1)^{\alpha}}{\Gamma(\alpha+1)(j+1+\alpha)(j-1)},\quad j\geq 2.\end{cases}\tag{3.8}$$

If $u(x)\approx p_N(x)=\sum_{j=0}^{N}\tilde{t}_jT_j(x)$, then ${}_CD_{-1,x}^{-\alpha}u(x)$ can be approximated by

$$D_{-1,x}^{-\alpha}\,u(x)\approx D_{-1,x}^{-\alpha}\,p_N(x)=\sum_{j=0}^{N}\tilde{t}_j\,\widehat{T}_j^{\alpha}(x),$$

where $\tilde{t}_j$ can be similarly determined, see [23] for details.

For the Jacobi polynomials, we denote by $\widehat{P}_j^{a,b,\alpha}(x)=\frac{1}{\Gamma(\alpha)}\int_{-1}^{x}(x-s)^{\alpha-1}P_j^{a,b}(s)\,\mathrm{d}s$. Using the recurrence formulae (2.9)–(2.10), the properties

(2.11) and (2.14), and tedious calculations, we have

$$\begin{cases} \widehat{P}_0^{a,b,\alpha}(x) = \dfrac{(x+1)^\alpha}{\Gamma(\alpha+1)}, \\ \widehat{P}_1^{a,b,\alpha}(x) = \dfrac{a+b+2}{2}\left(\dfrac{x(x+1)^\alpha}{\Gamma(\alpha+1)} - \dfrac{\alpha(x+1)^{\alpha+1}}{\Gamma(\alpha+2)}\right) + \dfrac{a-b}{2}\widehat{P}_0^{a,b,\alpha}(x), \\ \widehat{P}_{j+1}^{a,b,\alpha}(x) = \dfrac{A_j^{a,b}x - B_j^{a,b} - \alpha A_j^{a,b}\widehat{B}_j^{a,b}}{1+\alpha A_j^{a,b}\widehat{C}_j^{a,b}}\widehat{P}_j^{a,b,\alpha}(x) \\ \quad - \dfrac{C_j^{a,b} + \alpha A_j^{a,b}\widehat{A}_j^{a,b}}{1+\alpha A_j^{a,b}\widehat{C}_j^{a,b}}\widehat{P}_{j-1}^{a,b,\alpha}(x) \\ \quad + \dfrac{\alpha A_j^{a,b}\left(\widehat{A}_j^{a,b}P_{j-1}^{a,b}(-1) + \widehat{B}_j^{a,b}P_j^{a,b}(-1) + \widehat{C}_j^{a,b}P_{j+1}^{a,b}(-1)\right)}{\Gamma(\alpha+1)\left(1+\alpha A_j^{a,b}\widehat{C}_j^{a,b}\right)}(x+1)^\alpha, \\ \quad j \geq 1. \end{cases} \tag{3.9}$$

If $u(x) \approx p_N(x) = \sum_{j=0}^N \tilde{p}_j^{a,b} P_j^{a,b}(x)$, then $D_{-1,x}^{-\alpha}u(x)$ can be approximated by

$$D_{-1,x}^{-\alpha}\, u(x) \approx D_{-1,x}^{-\alpha}\, p_N(x) = \sum_{j=0}^N \tilde{p}_j^{a,b}\, \widehat{P}_j^{a,b,\alpha}(x),$$

where $\tilde{p}_j^{a,b}$ can be determined almost similarly as $\tilde{l}_j$ in (3.2).

REMARK **3.1**. If $a = b = 0$, then $\widehat{P}_j^{0,0,\alpha}(x) = \widehat{L}_j^\alpha(x)$. If $a = b = -\frac{1}{2}$, then $\widehat{P}_j^{-\frac{1}{2},-\frac{1}{2},\alpha}(x) = \frac{\Gamma(j+1/2)}{j!\sqrt{\pi}}\widehat{T}_j^\alpha(x)$.

4. Approximation to the Caputo derivative

This section deals with the numerical approximation of the Caputo derivative of a given function $u(x), x \in [-1,1]$. The algorithm is based on the numerical approximation of the fractional integral derived in the previous section. Among the Legendre, Chebyshev and Jacobi polynomials, the Jacobi polynomials are the most general. Then in the present section, we study the Jacobi approximation to the Caputo derivative.

Suppose that $p_N(x)$ is the approximate polynomial of $u(x)$, which can be expressed as

$$p_N(x) = \sum_{j=0}^N \tilde{p}_j^{a,b} P_j^{a,b}(x), \quad x \in [-1,1]. \tag{4.1}$$

Let $n-1<\alpha<n$, n is a positive integer, and the αth Caputo derivative of $p_N(x)$ reads as

$$\begin{aligned}
{}_CD^{\alpha}_{-1,x}p_N(x) &= \frac{1}{\Gamma(n-\alpha)}\int_{-1}^{x}(x-s)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}s^n}p_N(s)\,\mathrm{d}s\\
&= \frac{1}{\Gamma(n-\alpha)}\sum_{j=0}^{N}\tilde{p}^{a,b}_j\int_{-1}^{x}(x-s)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}s^n}P^{a,b}_j(s)\,\mathrm{d}s\\
&= \sum_{j=n}^{N}\tilde{p}^{a,b}_j d^{a,b}_{j,n}\left(\frac{1}{\Gamma(n-\alpha)}\int_{-1}^{x}(x-s)^{n-\alpha-1}P^{a+n,b+n}_j(s)\,\mathrm{d}s\right)\\
&= \sum_{j=n}^{N}\tilde{p}^{a,b}_j\, d^{a,b}_{j,n}\,\widehat{P}^{a+n,b+n,n-\alpha}_j(x),
\end{aligned}$$

where Eq. (2.12) is used. $d^{a,b}_{j,n}$ is defined by (2.13), and $\widehat{P}^{a+n,b+n,n-\alpha}_j(x)$ is defined by (3.9) with a,b and α being replaced by $a+n,b+n$ and $n-\alpha$, respectively.

Denote by

$$D^{a,b,\alpha,m}_j=\frac{1}{\Gamma(n-\alpha)}\int_{-1}^{x}(x-s)^{n-\alpha-1}\frac{\mathrm{d}^m}{\mathrm{d}s^m}P^{a,b}_j(s)\,\mathrm{d}s,\quad 0\le m\le n,$$

one has

$$D^{a,b,\alpha,n}_j(x)=d^{a,b}_{j,n}\widehat{P}^{a+n,b+n,n-\alpha}_j(x), \tag{4.2}$$

where $D^{a,b,\alpha,n}_j(x)=0$ for $0\le j\le n-1$, $d^{a,b}_{j,n}$ is given by (2.13), and $\widehat{P}^{a+n,b+n,n-\alpha}_j$ is defined by (3.9) with parameters $a+n,b+n,n-\alpha$ being replaced with a,b,α, respectively.

Therefore, the αth Caputo derivative ${}_CD^{\alpha}_{-1,x}u(x)$ can be approximated by

$${}_CD^{\alpha}_{-1,x}u(x)\approx{}_CD^{\alpha}_{-1,x}p_N(x)=\sum_{j=n}^{N}\tilde{p}^{a,b}_jD^{a,b,\alpha,n}_j(x),$$

where $D^{a,b,\alpha,n}_j(x)$ is given by (4.2) and $\tilde{p}^{a,b}_j$ can be similarly obtained as $\tilde{l}_j$ in (3.2).

5. Applications of the algorithms

Let $u(x)$ be a real-valued function defined on the interval $[x_a,x_b]$, and $\hat{x}_i\,(i=0,1,...,N)$ be the collocation points on the reference interval $[-1,1]$, then $x_i=\frac{1}{2}[(x_b-x_a)\hat{x}_i+x_a+x_b]$ are the corresponding collocation points on $[x_a,x_b]$. Similar to the previous section, we use the more general polynomials, i.e., the Jacobi polynomials, to illustrate our numerical methods.

For a positive integer N, suppose that $u(x)$ can be approximated by the following polynomial

$$u(x) \approx p_N(x) = \sum_{j=0}^{N} \tilde{p}_j^{a,b} P_j^{a,b}(\hat{x}), \quad \hat{x} = \frac{2x - x_a - x_b}{x_b - x_a} \in [-1,1], \tag{5.1}$$

Therefore, for any $\alpha > 0$, we have

$$\begin{aligned} D_{x_a,x}^{-\alpha} p_N(x) =& \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{N} \tilde{p}_j^{a,b}\, D_{x_a,x}^{-\alpha} P_j^{a,b}(\hat{x}) \\ =& \left(\frac{x_b - x_a}{2}\right)^{\alpha} \sum_{j=0}^{N} \tilde{p}_j^{a,b}\, \widehat{P}_j^{a,b,\alpha}(\hat{x}), \end{aligned} \tag{5.2}$$

in which

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} D_{x_a,x}^{-\alpha} P_j^{a,b}(\hat{x}) =& \frac{1}{\Gamma(\alpha)} \int_{x_a}^{x} (x-s)^{\alpha-1} P_j^{a,b}(\hat{s})\, \mathrm{d}s \\ =& \left(\frac{x_b - x_a}{2}\right)^{\alpha} \frac{1}{\Gamma(\alpha)} \int_{-1}^{\hat{x}} (\hat{x}-\hat{s})^{\alpha-1} P_j^{a,b}(\hat{s})\, \mathrm{d}\hat{s} \\ =& \left(\frac{x_b - x_a}{2}\right)^{\alpha} \widehat{P}_j^{a,b,\alpha}(\hat{x}). \end{aligned} \tag{5.3}$$

Here $s = \frac{1}{2}[(x_b - x_a)\hat{s} + x_a + x_b]$ and $x = \frac{1}{2}[(x_b - x_a)\hat{x} + x_a + x_b]$, $\hat{s}$, $\hat{x} \in [-1,1]$ are used. $\widehat{P}_j^{a,b,\alpha}(\hat{x})$ is defined by (3.9).

So, the fractional integral of $u(x)$ at any point $x \in [x_a, x_b]$ can be approximated by (5.2). If we choose $x = x_i$ $(i = 0,1,...,N)$, then we have

$$\begin{pmatrix} D_{x_a,x_0}^{-\alpha} u(x_0) \\ D_{x_a,x_1}^{-\alpha} u(x_1) \\ \vdots \\ D_{x_a,x_N}^{-\alpha} u(x_N) \end{pmatrix} \approx \begin{pmatrix} D_{x_a,x_0}^{-\alpha} p_N(x_0) \\ D_{x_a,x_1}^{-\alpha} p_N(x_1) \\ \vdots \\ D_{x_a,x_N}^{-\alpha} p_N(x_N) \end{pmatrix} = {}_C D_{x_a,x_b}^{(a,b,-\alpha)} \tilde{\mathbf{p}}^{a,b} \tag{5.4}$$

where $\tilde{\mathbf{p}}^{a,b} = (\tilde{p}_0^{a,b}, \tilde{p}_1^{a,b}, ..., \tilde{p}_N^{a,b})^T$, and

$$\left[D_{x_a,x_b}^{(a,b,-\alpha)}\right]_{i,j} = \left(\frac{x_b - x_a}{2}\right)^{\alpha} \widehat{P}_j^{a,b,\alpha}(\hat{x}_i), \quad i,j = 0,...,N.$$

Similarly, for $n-1 < \alpha < n \in \mathbb{N}$, we can also get the following formula for the Caputo derivative

$$\begin{pmatrix} D_{x_a,x_0}^{\alpha} u(x_0) \\ D_{x_a,x_1}^{\alpha} u(x_1) \\ \vdots \\ D_{x_a,x_N}^{\alpha} u(x_N) \end{pmatrix} \approx \begin{pmatrix} D_{x_a,x_0}^{\alpha} p_N(x_0) \\ D_{x_a,x_1}^{\alpha} p_N(x_1) \\ \vdots \\ D_{x_a,x_N}^{\alpha} p_N(x_N) \end{pmatrix} = D_{x_a,x_b}^{(a,b,\alpha)} \tilde{\mathbf{p}}^{a,b} \tag{5.5}$$

where

$$\left[{}_CD^{(a,b,\alpha)}_{x_a,x_b}\right]_{i,j} = \left(\frac{x_b - x_a}{2}\right)^{-\alpha} D^{a,b,\alpha,n}_j(\hat{x}_i), \quad i,j = 0,...,N, \tag{5.6}$$

and $D^{a,b,\alpha,n}_j$ is defined by (4.2).

REMARK **5.1**. For a given positive integer N, the matrix ${}_CD^{(a,b,\alpha)}_{x_a,x_b}$ can be calculated effectively by $O(N^2)$ arithmetic operations. The similar work can be found in [20], where the operational matrix $\mathbf{D}^{(\alpha)}$ based on the explicit form of the Legendre polynomials was obtained, which takes $O(N^3)$ arithmetic operations. The similar matrix $\mathbf{D}^{(\alpha)}$ based on the Chebyshev polynomials can be found in [5, 6], where the computational complexity is $O(N^3)$.

REMARK **5.2**. If $p_N(x)$ (see (5.1)) is the Legendre-Gauss-Lobatto interpolation of $u(x)$, $u \in H^r(I), I = [x_a, x_b]$, then

$$\|u - p_N\|_{L_\infty(I)} \leq CN^{3/4-r}\|u\|_{H^r(I)},\ r \in \mathbb{N},$$

where C is a positive constant (see [2]). Therefore, we can get the following the error bounds

$$\|D^{-\alpha}_{x_a,x}(u - p_N)\|_{L_\infty(I)} \leq CN^{3/4-r}\|u\|_{H^r(I)}, \quad \alpha > 0, r \geq 1,$$

and

$$\|{}_CD^{\alpha}_{x_a,x}(u-p_N)\|_{L_\infty(I)} \leq CN^{3/4+2n-r}\|u\|_{H^r(I)},\ n-1 < \alpha < n, n \in \mathbb{N}, r \geq 2n.$$

The above estimates will be verified by the numerical experiments in the following section.

Next we consider the eigenvalues of the Caputo fractional operator. Let $x_a = -1, x_b = 1$, and $x_i (i = 0, 1, ..., N)$ be the Legendre–Gauss–Lobatto points. We get the matrix ${}_CD^{(0,0,\alpha)}_{-1,1} \in R^{(N+1)\times(N+1)}$ from Eq. (5.6). ${}_CD^{(0,0,\alpha)}_{-1,1}$ is actually a kind of the operational differential matrix corresponding to the Caputo derivative. If $\alpha = n$, i.e., a positive integer, then ${}_CD^{(0,0,\alpha)}_{-1,1}$ reduces to the classical differential matrix $D^{(n)}$.

In the following, we numerically study the spectral radius of the Caputo derivative operator. Consider the following model problem

$$\begin{cases} {}_CD^{\alpha}_{-1,x}u(x) = \lambda u(x), \quad x \in [-1, 1], \quad 0 < \alpha \leq 2, \\ u(-1) = u_{-1}, \quad u(1) = u_1, \end{cases} \tag{5.7}$$

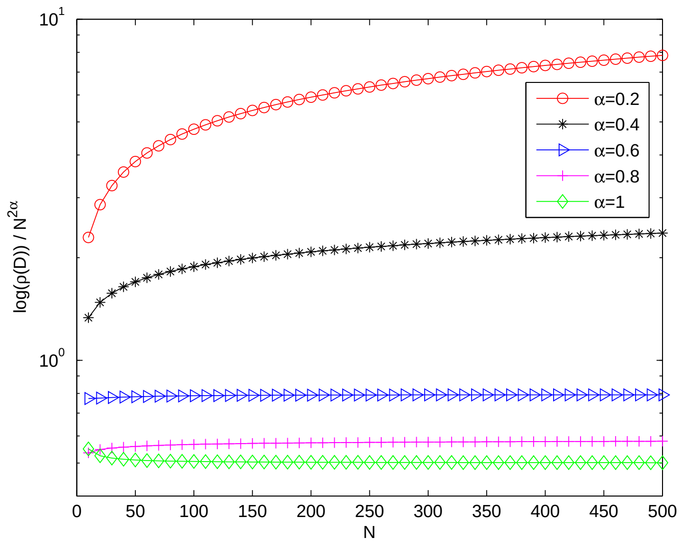

FIGURE 1. The spectral radius associated with the Caputo fractional operator for $0 < \alpha \leq 1$.

where besides $u(-1) = u_{-1}$, the condition $u(1) = u_1$ is necessary for $1 < \alpha \leq 2$.

For $0 < \alpha \leq 1$, let $p_N(x)$ (see (5.1)) be the interpolation of $u(x)$ on the Legendre–Gauss–Lobatto points $\{x_i\}$. Letting $p_N(x)$ satisfy the equation (5.7) at $x = x_i$ $(i = 1, 2, ..., N)$, and combining the initial condition $u(-1) = u_{-1}$, we have

$$D\tilde{\mathbf{p}}^{a,b} = \lambda \tilde{\mathbf{p}}^{a,b}, \tag{5.8}$$

where $[D]_{i,j} = [{}_C D_{-1,1}^{(0,0,\alpha)}]_{i,j} (i = 1, 2, ..., N; j = 0, 1, ..., N)$, and $[D]_{0,j} = (-1)^j (j = 0, 1, ..., N)$. It is easy to see that λ is just the eigenvalue of the matrix D. We know that, if $\alpha = 1$, the spectral radius $\rho(D)$ of D satisfies the following relation

$$\rho(D) \leq C_0 N^2, \quad C_0 > 0.$$

For $1 < \alpha \leq 2$, we can also get almost the same relation as (5.8), except that the last row of D in (5.8) is replaced by $[D]_{N,j} = 1\,(j = 0, 1, ..., N)$. We plot the spectral radius of D for different α and N in Figures 1 and 2. From Figures 1 and 2, we can see that the spectral radius $\rho(D)$ of D is bounded by

$$\rho(D) \leq C_0 N^{2\alpha}, \quad 0 < \alpha \leq 2, C_0 > 0.$$

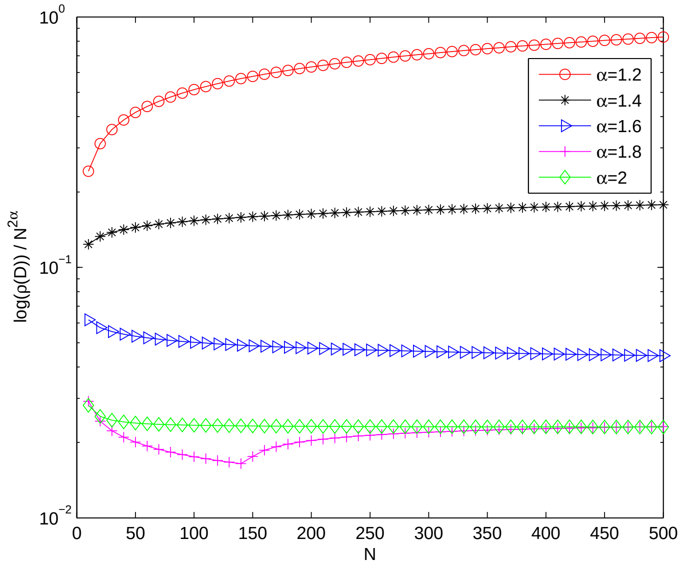

FIGURE 2. The spectral radius associated with the Caputo fractional operator for $1 < \alpha \leq 2$.

Next, we simply illustrate how to use our method to solve the fractional differential equations. Consider the fractional equation in the following form

$$\begin{cases} A(x)u'(x) + B(x)\,{}_C D_{0,x}^{\alpha} u(x) + C(x)u(x) = f(x), \quad x \in (0, L], \\ u(0) = u_0, \end{cases} \tag{5.9}$$

where $0 < \alpha < 1$, $A(x), B(x)$ and $C(x)$ are real-valued functions.

Suppose that $p_N(x)$ (see Eq. (5.1) and set $x_a = 0, x_b = L$) is the approximate solution of $u(x)$, $x_i\,(i = 0, 1, ..., N)$ are the collocation points on the interval $[0, L]$ satisfying $p_N(x_i) = u(x_i)\,(i = 0, 1, ..., N)$ and

$$A(x_i)p_N'(x_i) + B(x_i)\,{}_C D_{0,x_i}^{\alpha} p_N(x_i) + C(x_i)u(x_i) = f(x_i), \quad i = 1, 2, ..., N.$$

Noting $p_N(x_0) = u_0$, we get the following algebraic equation

$$M\tilde{\mathbf{p}}^{a,b} = F, \tag{5.10}$$

where $F=(u_0, f(x_1), f(x_2), ..., f(x_N))^T$, and

$$\begin{cases} [M]_{0,j} = P_j^{a,b}(-1), \\ [M]_{i,j} = A(x_i)\left[D_{0,L}^{(a,b,1)}\right]_{i,j} + B(x_i)\left[{}_C D_{0,L}^{(a,b,\alpha)}\right]_{i,j} + C(x_i)\left[D_{0,L}^{(a,b,0)}\right]_{i,j}, \\ \qquad i=1,...,N,\ j=0,1,...,N. \end{cases} \tag{5.11}$$

For the following fractional initial value problem

$$\begin{cases} A(x)u^{(m)}(x) + B(x)\,{}_C D_{0,x}^{\alpha} u(x) + C(x)u(x) = f(x), \quad x \in (0, L], \\ u(0) = u_0, \quad u'(0) = u_0', \end{cases} \tag{5.12}$$

where $1 < \alpha < 2$, $m = 1$ or 2. We can also get the algebraic equation of the form like (5.10), where $F = (u_0, f(x_1), f(x_2), ..., f(x_{N-1}), u_0')^T$, and M satisfies

$$\begin{cases} [M]_{0,j} = P_j^{a,b}(-1), \quad [M]_{N,j} = \dfrac{2}{L}\dfrac{\mathrm{d}P_j^{a,b}(-1)}{\mathrm{d}x}, \\ [M]_{i,j} = A(x_i)\left[D_{0,L}^{(a,b,m)}\right]_{i,j} + B(x_i)\left[{}_C D_{0,L}^{(a,b,\alpha)}\right]_{i,j} + C(x_i)\left[D_{0,L}^{(a,b,0)}\right]_{i,j}, \\ \qquad i=1,...,N-1,\ j=0,1,...,N. \end{cases} \tag{5.13}$$

If the condition $u'(0) = u_0'$ in (5.12) is replaced by $u(L) = u_L$, then the original fractional initial value problem is reduced to a fractional boundary value problem. We can still get the same form of algebraic equation (5.10), where $F = (u_0, f(x_1), f(x_2), ..., f(x_{N-1}), u_L)^T$, and M is defined the same as (5.13) except that $[M]_{N,j} = \frac{2}{L}\frac{\mathrm{d}P_j^{a,b}(-1)}{\mathrm{d}x}$ is replaced by $[M]_{N,j} = P_j^{a,b}(1)$.

6. Numerical examples

This section provides the numerical examples to verify the methods obtained in the preceding sections. The first two examples are used to test the efficiency and accuracy of the formulae (3.9) and (4.2).

EXAMPLE **6.1**. Let $u(x) = x^{\mu}, x \in [0, L] = [0, 1]$. Now we calculate the numerical solutions of the fractional integral $D_{0,x}^{-\alpha}u(x)$ and the Caputo derivative ${}_C D_{0,x}^{\alpha}u(x)$, $\alpha > 0$.

The analytical forms of the fractional integral and the Caputo derivative of $u(x)$ are given by

$$D_{0,x}^{-\alpha} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} x^{\mu+\alpha}, \quad \mu > -1,$$

TABLE 1. The absolute errors for Example **6.1** with $a = b = 0$ and $\mu = 3.5$.

N	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 1.2$	$\alpha = 1.5$	$\alpha = 1.8$
10	4.57e-08	3.57e-08	1.78e-08	5.18e-09	1.67e-09	6.04e-10
20	2.89e-10	1.52e-10	5.37e-11	9.88e-12	2.54e-12	1.31e-12
40	1.82e-12	6.36e-13	1.52e-13	1.74e-14	3.68e-15	2.77e-15
80	1.12e-14	2.59e-15	4.11e-16	1.67e-16	1.67e-16	1.18e-16
10	2.32e-07	1.82e-06	8.40e-06	1.61e-04	3.14e-04	3.18e-04
20	2.49e-09	2.90e-08	1.99e-07	6.63e-06	1.92e-05	2.67e-05
40	2.70e-11	4.73e-10	4.88e-09	2.81e-07	1.22e-06	2.55e-06
80	2.88e-13	7.62e-12	1.19e-10	1.18e-08	7.77e-08	2.44e-07

and

$$ {}_C D_{0,x}^{\alpha} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)} x^{\mu-\alpha}, \quad \mu > -1. $$

Suppose that $p_N(x)$ is the interpolation of $u(x)$ on the Jacobi–Gauss–Lobatto points x_j $(j = 0, 1, ..., N)$ on the interval $[0, L]$, and p_N is expressed by

$$ p_N(x) = \sum_{j=0}^{N} \tilde{p}_j^{a,b} P_j^{a,b}\Big(\frac{2x}{L} - 1\Big), \tag{6.1} $$

where $\tilde{p}_j^{a,b}$ can be easily calculated like (3.2).

We first set $a = b = 0$ and choose different N and α for our computation, the results are shown in Table 1. The first four rows of Table 1 show the absolute maximum errors at the Legendre–Gauss–Lobatto points by using the formula (5.4) (or (3.9)) for the fractional integral. The last four rows of Table 1 give the absolute maximum errors at the Legendre–Gauss–Lobatto points by using the formula (5.5) for the Caputo derivative. Obviously, Table 1 displays the spectral accuracy of the derived method for the approximation of the fractional integral and Caputo derivative.

Now we set $a = b = -1/2$ to test our algorithms. The numerical results are shown in Table 2. The first four rows and the last four rows give the maximum errors for numerical solutions of the fractional integral and the Caputo derivative of $u(x)$ at the Chebyshev–Gauss–Lobatto points. We can see that the spectral accuracy is achieved.

EXAMPLE **6.2**. Let $u(x) = \sin x, x \in [0, L]$, and we calculate the fractional integral $D_{0,x}^{-\alpha} u(x)$ and ${}_C D_{0,x}^{\alpha} u(x)$, $\alpha > 0$.

TABLE 2. The absolute errors for Example **6.1** with $a = b = -1/2$ and $\mu = 3.5$.

N	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 1.2$	$\alpha = 1.5$	$\alpha = 1.8$
10	5.49e-08	4.59e-08	2.54e-08	8.33e-09	3.09e-09	1.67e-09
20	3.08e-10	1.96e-10	7.62e-11	1.70e-11	4.97e-12	2.93e-12
40	1.81e-12	7.79e-13	2.14e-13	3.23e-14	8.09e-15	5.61e-15
80	1.06e-14	3.05e-15	5.66e-16	3.33e-16	1.80e-16	1.73e-16
10	2.23e-07	1.48e-06	7.44e-06	1.40e-04	3.23e-04	4.55e-04
20	2.12e-09	2.11e-08	1.62e-07	5.37e-06	1.86e-05	3.95e-05
40	2.15e-11	3.22e-10	3.77e-09	2.17e-07	1.14e-06	3.66e-06
80	2.20e-13	5.00e-12	8.89e-11	8.92e-09	7.10e-08	3.45e-07

The exact expressions of $D_{0,x}^{-\alpha}\sin x$ and ${}_CD_{0,x}^{\alpha}\sin x$ are given by

$$D_{0,x}^{-\alpha}\sin x = x^{\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{\Gamma(\alpha+2k+1)}, \quad \alpha > 0,$$

and

$$\begin{cases} {}_CD_{0,x}^{\alpha}\sin(x) = x^{1-\alpha}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{\Gamma(2k+2-\alpha)}, & 0 < \alpha < 1, \\ {}_CD_{0,x}^{\alpha}\sin(x) = x^{3-\alpha}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^{k+1} x^{2k}}{\Gamma(2k+4-\alpha)} & 1 < \alpha < 2. \end{cases}$$

We use formula (6.1) as that used in Example **6.1**. The numerical experiments are displayed in Tables 3 and 4. The first three rows and the last three rows of Table 3 display the absolute maximum errors of the numerical solutions of the fractional integral and the Caputo derivative of $u(x)$ by using the methods (5.4) and (5.5) with $a = b = 0$, respectively. Table 4 gives the corresponding errors of the case $a = b = -1/2$. We can see that the satisfactory results are obtained.

Next, we use our method to solve the fractional differential equations.

EXAMPLE **6.3**. Consider the following Baglay–Torvik equation [16, 6]

$$u''(x) + {}_CD_{0,x}^{1.5}u(x) + u(x) = f(x), \quad x \in (0,1], \tag{6.2}$$

with the initial conditions

$$u(0) = 0, \quad u'(0) = w. \tag{6.3}$$

Choosing appropriate $f(x)$ such that (6.2) has the exact solution $u(x) = \sin wx$.

TABLE 3. The absolute errors for Example **6.2** with $a = b = 0$.

N	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 1.2$	$\alpha = 1.5$	$\alpha = 1.8$
4	4.46e-06	5.40e-06	3.88e-06	1.71e-06	6.94e-07	2.90e-07
8	4.79e-12	4.72e-12	2.73e-12	8.94e-13	2.88e-13	6.96e-14
16	6.66e-16	2.22e-16	2.22e-16	1.67e-16	1.67e-16	8.33e-17
4	1.05e-05	6.17e-05	2.31e-04	1.02e-03	2.18e-03	5.66e-03
8	1.48e-11	1.08e-10	5.96e-10	4.14e-09	1.47e-08	5.33e-08
16	3.22e-15	1.91e-14	9.29e-14	6.39e-13	2.45e-12	9.03e-12

TABLE 4. The absolute errors for Example **6.2** with $a = b = -1/2$.

N	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 1.2$	$\alpha = 1.5$	$\alpha = 1.8$
4	6.08e-06	7.91e-06	6.26e-06	3.41e-06	1.87e-06	1.27e-06
8	6.58e-12	6.63e-12	3.96e-12	1.38e-12	4.97e-13	1.36e-13
16	3.33e-16	3.33e-16	2.22e-16	5.55e-17	2.22e-16	5.55e-17
4	1.34e-05	5.66e-05	1.95e-04	9.93e-04	2.05e-03	5.40e-03
8	1.98e-11	1.04e-10	3.92e-10	3.34e-09	1.15e-08	4.34e-08
16	4.44e-16	1.22e-15	7.55e-15	4.40e-14	2.32e-13	9.82e-13

We first set $a = b = 0$, and the results are shown in Table 5, which shows the maximum absolute errors of the method (5.13) and the shifted Chebyshev tau (SCT) method developed in [6] for the same parameters. From this example, we can see that our method gives the more accurate results than the SCT method used in [6]. Table 6 gives the maximum absolute errors for (6.2) with the boundary value conditions, say, the condition $u'(0) = w$ is replaced by $u(1) = \sin w$. From Tables 5–6, we find that the spectral accuracy is attained.

Tables 7 and 8 show the maximum absolute errors of the method (5.13) with $a = b = -1/2$. Table 7 displays the errors of the numerical solutions of (6.2) with the initial conditions (6.3), and Table 8 gives the errors of the numerical solutions of (6.2) with the boundary value conditions as mentioned above. Both two cases in this example give the better results than the SCT method reported in [6].

This equation is often used to test the numerical algorithms. In the following we consider the homogeneous Baglay–Torvik model.

EXAMPLE **6.4**. We consider the homogeneous Baglay–Torvik equation with order $\alpha \in (1, 2)$ [1, 16, 7]

$$u''(x) + {}_CD^{\alpha}_{0,t}u(x) + u(x) = 0,\ x \in (0, L],\ u(0) = 1,\ u'(0) = 0. \qquad (6.4)$$

TABLE 5. The absolute errors for Example **6.3** with different N and $a = b = 0$.

N	w	Method (5.13)	SCT [6]	w	Method (5.13)	SCT [6]
4	1	2.42e-04	3.4e-04	4π	1.25e+01	3.9e-00
8	1	7.40e-10	4.3e-07	4π	1.38e+00	4.7e-01
16	1	3.33e-16	1.8e-08	4π	8.55e-05	3.5e-05
32	1	4.44e-16	7.1e-10	4π	5.10e-13	1.4e-06
48	1	3.33e-16	9.9e-11	4π	6.81e-13	1.9e-07
64	1	4.44e-16	2.4e-11	4π	1.90e-13	4.8e-08

TABLE 6. The absolute errors for Example **6.3** with boundary value conditions and $a = b = 0$.

N	w	error	w	error	w	error
4	1	2.39e-05	2π	1.51e-01	4π	3.47e+00
8	1	7.53e-11	2π	9.20e-04	4π	1.62e-01
16	1	1.11e-16	2π	1.07e-10	4π	8.51e-06
32	1	3.33e-16	2π	1.78e-15	4π	1.13e-12
48	1	2.22e-16	2π	1.47e-15	4π	2.89e-13
64	1	2.22e-16	2π	1.44e-15	4π	1.22e-13

TABLE 7. The absolute errors for Example **6.3** with initial conditions and $a = b = -1/2$.

N	w	error	w	error	w	error
4	1	1.41e-04	2π	4.59e-01	4π	1.36e+01
8	1	2.89e-10	2π	3.40e-03	4π	5.28e-01
16	1	5.55e-16	2π	2.96e-10	4π	2.33e-05
32	1	4.44e-16	2π	2.00e-15	4π	3.32e-13
48	1	4.44e-16	2π	5.33e-15	4π	3.70e-13
64	1	5.55e-16	2π	3.87e-15	4π	2.85e-13

We apply the method (5.13) with $a = b = 0$ to solve the equation (6.4), where the collocation points are chosen as the Legendre–Gauss–Lobatto points. We set $L = 50, N = 256$ for different α in Figures 3 and 4. When $\alpha = 1.25$, our result coincides with the results reported in [7], see Figure 3. Figure 4 is also consistent with the numerical result in [7].

At last, we use scheme (5.13) to solve an oscillation model.

TABLE 8. The absolute errors for Example **6.3** with boundary value conditions and $a = b = -1/2$.

N	w	error	w	error	w	error
4	1	4.43e-05	2π	3.94e-01	4π	6.51e+00
8	1	6.66e-11	2π	8.71e-04	4π	1.97e-01
16	1	5.55e-16	2π	5.04e-11	4π	4.19e-06
32	1	5.55e-16	2π	1.61e-15	4π	3.21e-13
48	1	5.55e-16	2π	8.88e-16	4π	4.45e-13
64	1	5.55e-16	2π	1.39e-15	4π	3.53e-13

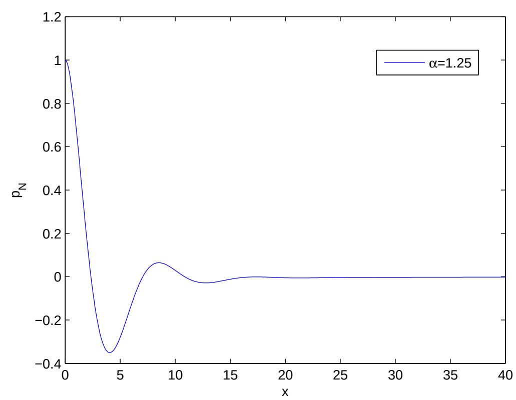

FIGURE 3. Numerical result for Example **6.4** with $N = 256, L = 40$.

EXAMPLE **6.5**. Consider the following fractional oscillation equation [7]

$$ {}_{C}D_{0,t}^{\alpha}u(x) + u(x) = f(x), \quad 1 < \alpha < 2,\ x \in (0, L], \tag{6.5}$$

with the initial conditions

$$u(0) = 1, \quad u'(0) = 0.$$

In this example, we still use the method (5.13) with $a = b = 0$ to solve (6.5), where the Legendre–Gauss–Lobatto collocation points are used again. We still let $N = 256, L = 50$ for our computation, and set $\alpha = 1.3, 1.5, 1.8, 1.95$ as in [7]. The numerical result is displayed in Figure 5, which coincides with the result in [7].

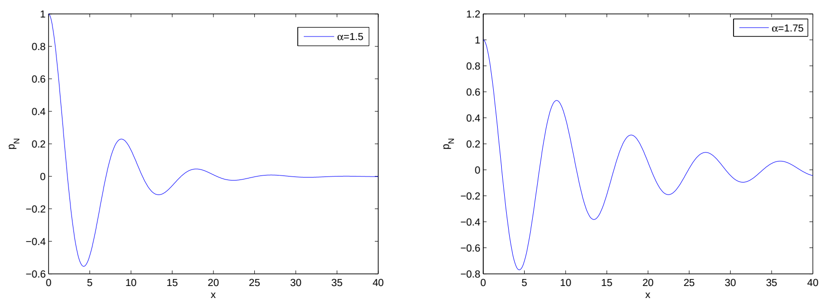

FIGURE 4. Numerical results for Example **6.4** with $N = 256, L = 40$ and $\alpha = 1.5, 1.75$.

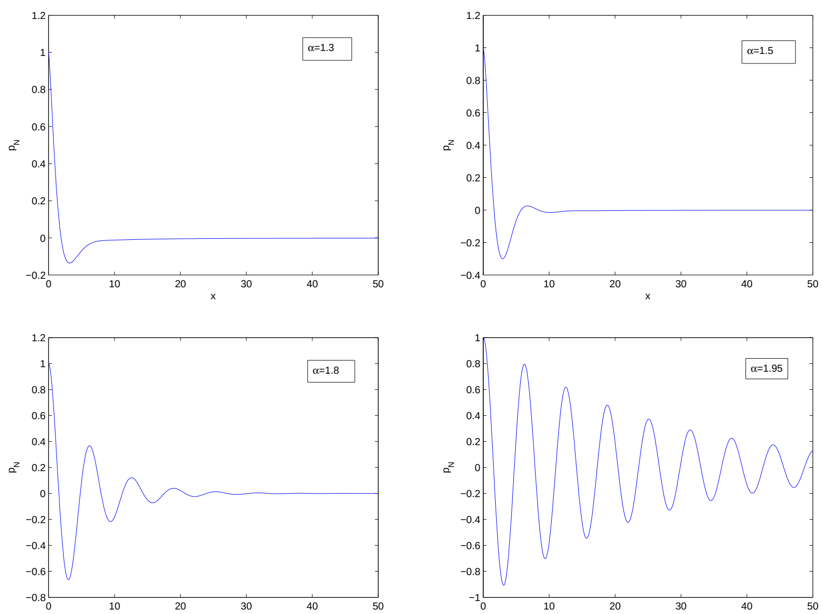

FIGURE 5. Numerical results for Example **6.5** with $N = 256, L = 50$.

7. Conclusion

In this paper, based on the Legendre, Chebyshev and Jacobi polynomials we develop the effective numerical algorithms to compute the fractional integral and Caputo derivative. The operational differential matrix based on the Jacobi–Gauss–Lobatto points is also obtained, which can be seen as a generalization of the classical differential matrix. The computational cost for deriving the operational differential matrix is $O(N^2)$, which is much less than that in [5, 6].

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