

## SPECTRAL AVERAGING, PERTURBATION OF SINGULAR SPECTRA, AND LOCALIZATION

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ABSTRACT. A spectral averaging theorem is proved for one-parameter families of self-adjoint operators using the method of differential inequalities. This theorem is used to establish the absolute continuity of the averaged spectral measure with respect to Lebesgue measure. This is an important step in controlling the singular continuous spectrum of the family for almost all values of the parameter. The main application is to the problem of localization for certain families of random Schrödinger operators. Localization for a family of random Schrödinger operators is established employing these results and a multi-scale analysis.

### 1. INTRODUCTION AND MAIN RESULTS

Spectral averaging techniques play an important role in controlling the singular continuous spectrum of families of self-adjoint operators. Such methods have been used in the theory of random Schrödinger operators (cf. [CL]) and in some approaches to quantum stability of time-dependent models [H2]. In the application to random Schrödinger operators, spectral averaging can also be used to derive a Wegner lemma and to prove the Lipschitz continuity of the integrated density of states (cf. [KS]). The method of Kotani and Simon [KS] requires some analyticity of the potential in the random variables. We present here a technique applicable to more general families of potentials depending only on differentiability on some parameter. The main tool is the method of differential inequalities. We apply these results to prove exponential localization at low energies for random Schrödinger operators of the form

$$(1.1) \quad H_\omega = -\Delta + \sum_{j \in \mathbb{Z}^d} v(\lambda_j(\omega))(x - j),$$

on  $L^2(\mathbb{R}^d)$ ,  $d \geq 1$ , where  $v \geq 0$  is a compactly supported single-site potential specified in section 4, and  $\{\lambda_j(\omega)\}$  are independent, identically distributed random variables. This type of model, as well as some of the models treated in [CH], have no lattice analogs, and cannot be treated by previously known methods.

Our main technical result is the following.

**Theorem 1.1.** *Let  $H_\lambda$ ,  $\lambda \in \Gamma \equiv (\lambda_0, \lambda_1)$  be a  $C^2$ -family of self-adjoint operators such that  $D(H_\lambda) = D_0 \subset \mathcal{H} \forall \lambda \in \Gamma$ , and such that  $R_\lambda(z) \equiv (H_\lambda - z)^{-1}$  is twice*

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strongly differentiable in  $\lambda \forall z, \text{Im } z \neq 0$ . Assume that  $\exists$  finite positive constants  $C_j, j = 0, 1$ , and a positive bounded self-adjoint operator  $B$  such that, on  $D_0$ ,

$$(D1) \quad \dot{H}_\lambda \equiv \frac{dH_\lambda}{d\lambda} \geq C_0 B^2;$$

$$(D2) \quad |\ddot{H}_\lambda| \equiv \left| \frac{d^2 H_\lambda}{d\lambda^2} \right| \leq C_1 \dot{H}_\lambda.$$

Then  $\forall E \in \mathbb{R}$  and  $\forall$  real, positive  $g \in C_0^2(\Gamma)$ , there exists a finite positive constant  $C$  depending only on  $\|g^{(j)}\|_1, j = 0, 1, 2$ , s.t.  $\forall \phi \in \mathcal{H}$ ,

$$(1.2) \quad \sup_{\delta > 0} \left| \int_\Gamma g(\lambda) \langle \phi, B(H_\lambda - E - i\delta)^{-1} B\phi \rangle \right| \leq C \|\phi\|^2.$$

Let us present some consequences of Theorem 1.1. The first corollary will be used in the proof of Wegner’s lemma for the model (1.1). In the case that  $\Gamma = \mathbb{R}$ , one can take  $g_t(\lambda) \equiv (1 + t\lambda^2)^{-1}$  in (1.2) and improve the next corollary to  $h \in L^\infty(\mathbb{R})$ , as in [CH].

**Corollary 1.2.** *Under the hypotheses of Theorem 1.1, for any Borel set  $J \in \mathbb{R}$  and for all real, non-negative  $h \in C_0^2(\Gamma)$ , there exists a finite, positive constant  $C$ , depending only on  $\|h^{(j)}\|_1, j = 0, 1, 2$ , such that*

$$(1.3) \quad \left\| \int_\Gamma h(\lambda) B E_\lambda(J) B d\lambda \right\| \leq C |J|,$$

where  $E_\lambda(\cdot)$  is the spectral family for  $H_\lambda$  and  $|J|$  is the Lebesgue measure of  $J$ .

The next corollary is a version of the so-called “Kotani’s trick”.

**Corollary 1.3.** *In addition to the hypotheses of Theorem 1.1, assume that  $\text{Ran } B$  is cyclic for  $H_\lambda \forall \lambda \in \Gamma$  in the sense that  $\{f(H_\lambda)B\phi, f \in L^\infty(\mathbb{R}), \phi \in \mathcal{H}\}$  is dense in  $\mathcal{H}$ . Then for any Borel set  $J \subset \mathbb{R}$  with  $|J| = 0$ , one has  $E_\lambda(J) = 0$  a.e.  $\lambda \in \Gamma$ .*

Remarks 1.4.

1. Theorem 1.1 can be proved under the following related set of hypotheses, which we will use in section 4.

$$(D1)^* \quad -\dot{H}_\lambda \equiv -\frac{dH_\lambda}{d\lambda} \geq C_0 B^2;$$

$$(D2)^* \quad |\ddot{H}_\lambda| \equiv \left| \frac{d^2 H_\lambda}{d\lambda^2} \right| \leq -C_1 \dot{H}_\lambda.$$

The proof proceeds as in section 2 provided one defines  $R(\lambda, \varepsilon, \delta)$  in (2.1) by

$$R(\lambda, \varepsilon, \delta) \equiv (H_\lambda - E + i\delta - i\varepsilon \dot{H}_\lambda)^{-1}.$$

2. Assumption (D1) of Theorem 1.1 can easily be changed to the following more general, but local, assumption:

$$(D1)' \quad \begin{aligned} \dot{H}_\lambda &= a(H_\lambda - E) + D(E), \text{ for some constant } a > 0, \\ \text{and } D(E) &\geq C_0 B^2, \text{ for all } E \in I, I \in \mathbb{R}. \end{aligned}$$

Then the conclusions of Theorem 1.1 and Corollaries 1.2–1.3 still hold locally in  $I$ . Using these results and the techniques of [CH], one can recover the results of Klopp [Kl] concerning Wegner estimates for negative energies and not necessarily positive potentials.

3. For many of the models for which the above corollaries apply, it can be shown that singular continuous spectrum is generic in the topological sense (cf. [RJMS]). This is not, of course, incompatible with our results but shows that in general one cannot expect that they hold *for every*  $\lambda \in \Gamma$ .

In a related paper [CHST], we use a differential inequality to prove results similar to these applicable to multiplicative perturbations of the Laplacian. Such perturbations can be used to describe the propagation of electromagnetic and acoustic waves in random media.

We present the proof of Theorem 1.1 and Corollaries 1.2–1.3 in section 2. In section 3, we apply Corollary 1.3 to obtain a result on perturbation of singular spectrum. This theorem is similar to a result of [H1] and of [CH]. It is a continuous version, for relatively compact perturbations, of Simon and Wolff's [SW] development of rank one perturbations of self-adjoint operators on lattices. We discuss localization for the breather model (1.1) in section 4. We prove the estimates necessary to apply the methods developed in [CH]. In particular, we establish a Wegner estimate (using Corollary 1.2), prove the continuity of the integrated density of states, and obtain initial exponential decay estimates on the finite-volume Green's function. Finally, in section 5, we make some additional remarks. We present a related theorem about absolutely continuous spectrum and indicate the relation between our work and Howland's positive commutator method [H1].

## 2. PROOF OF THEOREM 1.1 AND COROLLARIES 1.2–1.3

*Proof of Theorem 1.1.* We define for  $\varepsilon > 0$  and  $0 < \delta < 1$ ,

$$(2.1) \quad R(\lambda, \varepsilon, \delta) \equiv (H_\lambda - E + i\delta + i\varepsilon\dot{H}_\lambda)^{-1},$$

and set

$$(2.2) \quad K(\lambda, \varepsilon, \delta) \equiv BR(\lambda, \varepsilon, \delta)B.$$

We first derive an *a priori* estimate on  $K$ . By the Cauchy-Schwarz inequality, one has  $\forall \phi \in \mathcal{H}$ ,  $\|\phi\| = 1$ ,

$$(2.3) \quad \begin{aligned} \|K(\lambda, \varepsilon, \delta)\phi\| &\geq -\operatorname{Im}\langle \phi, K(\lambda, \varepsilon, \delta)\phi \rangle \\ &\geq \varepsilon C_0 \|K(\lambda, \varepsilon, \delta)\phi\|^2. \end{aligned}$$

The last inequality follows from assumption D1 and the fact that

$$-\operatorname{Im}\langle \phi, K(\lambda, \varepsilon, \delta)\phi \rangle = \langle \phi, BR(\lambda, \varepsilon, \delta)^*(\delta + \varepsilon\dot{H}_\lambda)(R(\lambda, \varepsilon, \delta)B\phi) \rangle.$$

Let  $g \in C_0^2(\Gamma)$  and define

$$(2.4) \quad F(\varepsilon, \delta) \equiv \int_\Gamma g(\lambda) \langle \phi, K(\lambda, \varepsilon, \delta)\phi \rangle d\lambda.$$

Inequality (2.3) implies the bound

$$(2.5) \quad |F(\varepsilon, \delta)| \leq (C_0\varepsilon)^{-1} \|g\|_1.$$

We also need a related result for  $K(\lambda, \varepsilon, \delta)^*$ . Note that

$$K(\lambda, \varepsilon, \delta)^* = K(\lambda, -\varepsilon, -\delta),$$

so that in place of (2.3), we have

$$(2.6) \quad \begin{aligned} \|K(\lambda, \varepsilon, \delta)^*\phi\| &\geq \operatorname{Im}\langle \phi, K(\lambda, -\varepsilon, -\delta)\phi \rangle \\ &\geq C_0 \|K(\lambda, \varepsilon, \delta)^*\phi\|^2. \end{aligned}$$

Under the differentiability assumption on  $H_\lambda$ , one has

$$\begin{aligned}
 i \frac{dF}{d\varepsilon}(\varepsilon, \delta) &= \int g(\lambda) \langle \phi, BR(\lambda, \varepsilon, \delta) \dot{H}_\lambda R(\lambda, \varepsilon, \delta) B\phi \rangle d\lambda \\
 (2.7) \qquad &= - \int g(\lambda) \frac{d}{d\lambda} \langle \phi, K(\lambda, \varepsilon, \delta) \phi \rangle \\
 &\quad - i\varepsilon \int g(\lambda) \langle \phi, BR(\lambda, \varepsilon, \delta) \ddot{H}_\lambda R(\lambda, \varepsilon, \delta) B\phi \rangle.
 \end{aligned}$$

To handle the second term, we write

$$\ddot{H}_\lambda = |\ddot{H}_\lambda|^{1/2} U |\ddot{H}_\lambda|^{1/2},$$

where  $U$  is a partial isometry commuting with the self-adjoint operator  $\ddot{H}_\lambda$ . The second matrix element in (2.7) can be written as

$$\begin{aligned}
 (2.8) \qquad |\langle \phi, BR \ddot{H}_\lambda R B\phi \rangle| &= |\langle |\ddot{H}_\lambda|^{1/2} R^* B\phi, U |\ddot{H}_\lambda|^{1/2} R B\phi \rangle| \\
 &\leq \frac{1}{2} \{ \langle \phi, BR |\ddot{H}_\lambda| R^* B\phi \rangle + \langle \phi, BR^* |\ddot{H}_\lambda| R B\phi \rangle \}.
 \end{aligned}$$

Let us consider the first term on the right in (2.8). In light of condition (D2), we have

$$(2.9) \qquad \langle \phi, BR |\ddot{H}_\lambda| R^* B\phi \rangle \leq C_1 \langle \phi, BR \dot{H}_\lambda R^* B\phi \rangle.$$

Note that from (2.1)–(2.2) and the fact that  $\delta > 0$ ,

$$(2.10) \qquad \langle \phi, BR(\varepsilon \dot{H}_\lambda) R^* B\phi \rangle \leq -\text{Im} \langle \phi, K\phi \rangle.$$

Inequalities (2.9) and (2.10) imply that the first term on the right side of (2.8) is bounded above by

$$(2.11) \qquad -(c_1/\varepsilon) \text{Im} \langle \phi, K\phi \rangle.$$

Since a similar estimate holds for the second term on the right in (2.8), we obtain by the positivity of  $g$ ,

$$(2.12) \qquad \int g(\lambda) |\langle \phi, BR \ddot{H}_\lambda R B\phi \rangle| d\lambda \leq C_1 \varepsilon^{-1} |\text{Im} F(\varepsilon, \delta)|.$$

By this result and integration by parts on the first term on the right in (2.7), we obtain

$$(2.13) \qquad \left| \frac{dF}{d\varepsilon}(\varepsilon, \delta) \right| \leq \left| \int_\Gamma g'(\lambda) \langle \phi, K\phi \rangle d\lambda \right| + C_1 |\text{Im} F(\varepsilon, \delta)|.$$

Estimates (2.3) and (2.5) imply the bound

$$(2.14) \qquad \left| \frac{dF}{d\varepsilon}(\varepsilon, \delta) \right| \leq (\varepsilon C_0)^{-1} (\|g'\|_1 + C_1 \|g\|_1) \equiv C_2 \varepsilon^{-1}.$$

Integrating this differential inequality yields an improved estimate for  $F$ ,

$$(2.15) \qquad |F(\varepsilon, \delta)| \leq C_3 |\log \varepsilon| + |F(1, \delta)|,$$

where  $C_3$  is independent of  $\delta$  and  $|F(1, \delta)|$  is uniformly bounded in  $\delta$ . Now, we consider a function  $\tilde{F}(\varepsilon, \delta)$  defined by

$$(2.16) \qquad \tilde{F}(\varepsilon, \delta) = \int g'(\lambda) \langle \phi, K(\lambda, \varepsilon, \delta) \phi \rangle d\lambda.$$

As in (2.5),  $\tilde{F}$  satisfies

$$(2.17) \quad |\tilde{F}(\varepsilon, \delta)| \leq (C_0\varepsilon)^{-1}\|g'\|_1.$$

We repeat the arguments (2.8)–(2.11). Since  $g'$  is not necessarily positive, we replace (2.12) by

$$(2.18) \quad \left| \int g'(\lambda)\langle\phi, BR\ddot{H}_\lambda RB\phi\rangle \right| \leq C_1\varepsilon^{-1} \int |g'(\lambda)|\{\|K^*\phi\| + \|K\phi\|\} d\lambda \\ \leq \|g'\|_1(C_1/C_0)\varepsilon^{-2}.$$

In place of (2.14), we obtain,

$$(2.19) \quad \left| \frac{d\tilde{F}}{d\varepsilon}(\varepsilon, \delta) \right| \leq (\varepsilon C_0)^{-1}(C_1\|g'\|_1 + \|g''\|_1),$$

leading to

$$(2.20) \quad |\tilde{F}(\varepsilon, \delta)| \leq C_4|\log \varepsilon| + |\tilde{F}(1, \delta)|.$$

We now return to (2.13) and obtain from (2.15) and (2.20)

$$(2.21) \quad \left| \frac{dF}{d\varepsilon}(\varepsilon, \delta) \right| \leq C_5|\log \varepsilon| + C_6,$$

where  $C_5$  is independent of  $\delta$  and depends on  $\|g^{(p)}\|_1$ ,  $p = 0, 1, 2$ , and  $C_6$  depends on  $|F(1, \delta)|$  and  $|\tilde{F}(1, \delta)|$ , which are bounded, independent of  $\delta$ , by (2.5) and (2.17). Consequently, we obtain by integrating (2.21),

$$(2.22) \quad |F(\varepsilon, \delta)| \leq C,$$

where  $C$  depends on  $\|g^{(p)}\|_1$ ,  $p = 0, 1, 2$ , and is independent of  $\delta$  and uniform in  $\varepsilon$ ,  $0 < \varepsilon < 1$ . The proof of the theorem now follows from the fact that  $R(\lambda, \varepsilon, \delta)$  converges weakly to  $R(\lambda, \delta)$  as  $\varepsilon \rightarrow 0$  provided  $\delta > 0$ , and the dominated convergence theorem since

$$\left| \int_{\Gamma} g(\lambda)\langle\phi, BR(\lambda, \varepsilon, \delta)B\phi\rangle d\lambda \right| \leq C,$$

by (2.22). □

*Proof of Corollary 1.2.* As in [CH], Stone's formula gives, for any  $\phi \in \mathcal{H}$ ,

$$(2.23) \quad \langle\phi, BE_\lambda(J)B\phi\rangle \leq \frac{1}{\pi} \lim_{\delta \downarrow 0} \operatorname{Im} \int_J dE\langle\phi, B(H_\lambda - E + i\delta)^{-1}B\phi\rangle.$$

One checks that for  $z \equiv E - i\delta$ , with  $E \in \mathbb{R}$  and  $\delta \equiv \operatorname{Im} z > 0$ ,

$$(2.24) \quad B(H_\lambda - z)^{-1}B = n\text{-}\lim_{\varepsilon \downarrow 0} K(\lambda, \varepsilon, \delta).$$

Consequently, it follows from (2.23)–(2.24), Theorem 1.1, and Fubini's Theorem, that

$$(2.25) \quad \left| \int_{\Gamma} h(\lambda)\langle\phi, BE_\lambda(J)B\phi\rangle d\lambda \right| \\ \leq \frac{1}{\pi} \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \left| \int_{\Gamma} d\lambda h(\lambda) \int_J dE\langle\phi, K(\lambda, \varepsilon, \delta)\phi\rangle \right| \\ \leq C|J| \|\phi\|^2,$$

where the constant  $C$  depends on  $\|g^{(p)}\|$ , for  $p = 0, 1, 2$ , as above. Since the bounded operator  $\int_{\Gamma} h(\lambda)BE_{\lambda}(J)B$  is self-adjoint, the result follows from (2.25).  $\square$

*Proof of Corollary 1.3.* For any non-negative  $h \in C_0^{\infty}(\Gamma)$ , it follows from Theorem 1.1 and Corollary 1.2 that  $\int_{\Gamma} h(\lambda)\langle \phi, BE_{\lambda}(J)B\phi \rangle = 0 \forall \phi \in \mathcal{H}$ . Consequently, the non-negative function  $\langle \phi, BE_{\lambda}(J)B\phi \rangle = 0$  for a.e.  $\lambda \in \Gamma$ . Since  $\text{Ran } B$  is assumed to be  $H$ -cyclic, we obtain that  $\langle \psi, E_{\lambda}(J)\psi \rangle = 0 \forall \lambda \in \Gamma_{\psi} \subset \Gamma$  and  $|\Gamma_{\psi}| = |\Gamma|$ . Now let  $\{\psi_n\}$  be a complete orthonormal basis for  $\mathcal{H}$  and set  $\Gamma_{\infty} \equiv \bigcap_n \Gamma_{\psi_n}$ . Then  $|\Gamma_{\infty}| = |\Gamma|$  and  $\langle \psi, E_{\lambda}(J)\psi \rangle = 0, \forall \lambda \in \Gamma_{\infty}$  and  $\forall \psi$  in a dense set in  $\mathcal{H}$ . By standard arguments, this can be extended to all  $\psi \in \mathcal{H}$ . Since  $E_{\lambda}$  is a projection, this shows  $E_{\lambda}(J) = 0$  a.e.  $\lambda \in \Gamma$ .  $\square$

### 3. PERTURBATION OF SINGULAR SPECTRA

Theorem 1.1 allows us to generalize Theorem 3.2 of [CH] to families of potentials  $V(\lambda)$  which are not necessarily of the form  $\lambda V$ . We consider a family of Schrödinger operators  $H_{\lambda} \equiv H_0 + V(\lambda), \lambda \in \Gamma = (\lambda_0, \lambda_1)$ , a finite interval, with a common domain of self-adjointness  $D(H_0) \subset \mathcal{H}$ , a separable Hilbert space. The potential  $V(\lambda)$  is assumed to be a symmetric operator, which can be factorized as  $V(\lambda) = C_{\lambda}D_{\lambda}^*$ , for bounded operators  $D_{\lambda}$  and  $C_{\lambda}$ . Let us assume  $V(\lambda') = 0$  for exactly one  $\lambda' \in \Gamma$ . We change parameters so that  $\lambda' = 0$ . We assume that the map  $\lambda \in \Gamma \rightarrow R_{\lambda}(z) \equiv (H_{\lambda} - z)^{-1}, \text{Im } z \neq 0$ , is continuous. We consider a fixed interval  $I \subset \mathbb{R}$  and the following assumptions:

$$(P1) \quad \Gamma_{\lambda}(z) \equiv D_{\lambda}^*R_0(z)C_{\lambda} \text{ is compact } \forall \lambda \in \Gamma \text{ and } \text{Im } z \neq 0.$$

$$(P2) \quad \exists I_0 \subset I, |I_0| = |I|, \text{ such that } \forall E \in I_0 \text{ and } \forall \lambda \in \Gamma,$$

$$\sup_{\varepsilon \neq 0} \|R_0(E + i\varepsilon)X_{\lambda}\| < \infty, \text{ for } X_{\lambda} = C_{\lambda} \text{ and } D_{\lambda}.$$

Let  $\tilde{H}_{\lambda}$  denote the restriction of  $H_{\lambda}$  to the subspace

$$\tilde{\mathcal{H}} \equiv [f(H_{\lambda})C_{\lambda}\phi, f \in L^{\infty}(\mathbb{R}), \phi \in \mathcal{H}]^{cl},$$

where  $cl$  denotes closure.

**Theorem 3.1.** *Assume (P1)–(P2) and the hypotheses of Theorem 1.1. For a.e.  $\lambda \in \Gamma, \sigma(\tilde{H}_{\lambda}) \cap I$  is pure point with finitely degenerate eigenvalues.*

*Proof.* The proof is basically the same as in [CH]. Note that  $I_0$  is independent of  $\lambda$ . The analog of the Aronszajn-Donoghue formula for this case is

$$(1 + \Gamma_{\lambda}(z))D_{\lambda}^*R_{\lambda}(z)C_{\lambda} = \Gamma_{\lambda}(z),$$

for  $\text{Im } z \neq 0$ . As in [CH], one concludes that there is no absolutely continuous spectrum in  $I$  and that the singular continuous spectrum of  $H_{\lambda}$  for  $\lambda \neq \lambda'$  lies in  $I \setminus I_0$ , and hence has Lebesgue measure zero. Since the hypotheses of Theorem 1.1 are satisfied by the family  $H_{\lambda}$ , we conclude by Corollary 1.3 that  $\sigma_{sc}(H_{\lambda}) \cap I = \emptyset$  a.e.  $\lambda \in \Gamma$ . Thus,  $\sigma(H_{\lambda}) \cap I$  is pure point for a.e.  $\lambda \in \Gamma$  and the eigenvalues are finitely degenerate.  $\square$

4. LOCALIZATION FOR THE BREATHER MODEL

We apply the methods developed in the last two sections to prove localization for the family of random Hamiltonians on  $L^2(\mathbb{R}^d)$  of the form

$$(4.1) \quad H_\omega = -\Delta + \sum_{i \in \mathbb{Z}^d} v(\lambda_i(\omega)(x - i)) = -\Delta + V_\omega,$$

where the “breather” potential  $V_\omega$  is described as follows. The random variables  $\{\lambda_i(\omega)\}$  are independent, identically distributed (iid) random variables with a common distribution function  $g \in C_0^2([M_0, M_1])$  for  $0 < M_0 < 1 < M_1 < 3$ . Let  $\Lambda_l(x)$  denote the cube of side  $l$  centered at  $x \in \mathbb{R}^d$ . The compactly supported, single-site potential  $v \geq 0$  belongs to  $C^2(\mathbb{R}^d)$  with  $v|_{\Lambda_1(0)} \geq E_0 > 0$  and  $\text{supp } v \subset \Lambda_{3/2}(0)$ . The function  $v$  is *repulsive* in that it satisfies

$$(A1) \quad -x \cdot \nabla v(x) \geq 0.$$

We also need to control the second derivatives of  $v$ . Let  $\text{Hess}[v]$  denote the Hessian of  $v$  and  $(\cdot, \cdot)$  the Euclidean inner product on  $\mathbb{R}^d$ . We require

$$(A2) \quad \exists c_0 > 0 \text{ s.t. } \left| \frac{(x, \text{Hess}[v]x)}{x \cdot \nabla v} \right| \leq c_0 < \infty.$$

We note that suitable smooth truncations of functions of the form  $e^{-\|x\|^2}$  and  $(1 + \|x\|^2)^{-k}$  satisfy the assumptions (A1)–(A2).

We need a condition on the distribution function  $g$  in order to locate the a.s.  $\sigma_{\text{ess}}(H_\omega)$ .

For some  $a, 1 < a < M_1, \exists \kappa_0 > 0$  s.t.

$$(A3) \quad \int_a^{M_1} g(\lambda) d\lambda \geq \kappa_0 > 0.$$

Let  $\langle v \rangle$  denote the mean  $\int v(x) dx$ .

**Proposition 4.1.** *Assume (A1)–(A3). The family  $H_\omega$  defined in (4.1) is self-adjoint on  $H^2(\mathbb{R}^d)$  and  $\mathbb{Z}^d$ -ergodic. The deterministic spectrum  $\Sigma \subset [\Sigma_0, \infty)$ , with  $\Sigma_0 \equiv \inf \sigma_{\text{ess}}(H_\omega)$  a.s., and  $0 \leq \Sigma_0 \leq a^{-d} \langle v \rangle$ , for  $a, 1 < a < M_1$ , as in (A3).*

*Proof.* Since  $H_\omega$  in (4.1) is  $\mathbb{Z}^d$ -ergodic and measurable,  $\sigma(H_\omega) = \sigma_{\text{ess}}(H_\omega)$  is deterministic (cf. [CL]). We use Persson’s formula (cf. [CFKS]) to estimate  $\Sigma_0 \equiv \inf \Sigma$ , where  $\Sigma = \sigma(H_\omega) = \sigma_{\text{ess}}(H_\omega)$  a.e. This formula is

$$(4.2) \quad \Sigma_0 = \sup_{\substack{K \subset \mathbb{R}^d \\ \text{compact}}} \left\{ \inf_{\substack{\phi \in C_0^\infty(\mathbb{R}^d \setminus K) \\ \|\phi\|=1}} \langle \phi, H_\omega \phi \rangle \right\}.$$

For any  $L > 0$ , and any  $a$ , with  $1 < a < M_1$  satisfying (A3), define the event  $A_n^L$  by

$$A_n^L \equiv \{ \omega \in \Omega \mid \exists \Lambda_{L/2} \subset \Lambda_{nL}(0) \setminus \Lambda_{(n-1)L}(0) \text{ s.t. } \text{for } i \in \Lambda_{L/2} \cap \mathbb{Z}^d, \\ \text{supp } v(\lambda_i(\omega)(x - i)) \subset \Lambda_{3/2a}(i) \}.$$

The independence of the random variables allows us to compute

$$\mathbb{P}(A_n^L) \geq \left[ \int_a^{M_1} g(\lambda) d\lambda \right]^{(L/2)^d} > 0.$$

The events  $A_n^L$  are independent and as  $\mathbb{P}(A_n^L)$  is independent of  $n$ ,  $\sum_n \mathbb{P}(A_n^L) = \infty$ . By the second Borel-Cantelli lemma,  $A_n^L$  occur infinitely often with probability one. By choosing a sequence of non-negative test functions supported in these  $\Lambda_{L/2}$ -boxes, we obtain from (4.2)

$$\Sigma_0 \leq \left(\frac{L}{2}\right)^{-d} \int_{\Lambda_{L/2}} V_\omega(x) d^d x + c_1 L^{-2},$$

where the  $L^{-2}$ -term comes from estimating the gradients. By a simple calculation, the integral on the right is bounded above by  $a^{-d} \langle v \rangle (L/2)^d$ , where  $\langle v \rangle = \int v(x) dx$ . By now considering a sequence of increasing lengths  $L$ , we obtain the result.  $\square$

Let us note the following important consequence of Proposition 4.1. Since the probability of any isolated real number being an eigenvalue of  $H_\omega$  is zero, there must be a non-empty interval of essential spectrum beginning at  $\Sigma_0$  in the deterministic spectrum. We will prove that there is a.s. only dense pure point spectrum in this interval near  $\Sigma_0$ . Let us consider energies in the interval  $I_0 \equiv [\Sigma_0, E_0]$ , where  $E_0$  is defined by  $v|\Lambda_1(0) \geq E_0 > 0$ . Since  $\Sigma_0 < \langle v \rangle / a^d$ , and  $\langle v \rangle \geq E_0$ , we see that  $\Sigma_0 \rightarrow E_0$  as  $M_1 \rightarrow 1^+$ . Indeed, if  $M_1 = 1$ , the potential is everywhere greater than  $E_0$ . We will prove a.s. localization for the breather model (4.1) in the interval  $[\Sigma_0, E_0]$  for  $M_1$  sufficiently close to 1. In this way, if  $\delta_0 \equiv (M_1 - 1)^{-1}$  measures the disorder of the system, large  $\delta_0$  implies decay of the Green's function at energies in  $I_0$ . We note that by taking  $M_0$  small, we can replace  $E_0$  by an increasing function  $E_0(M_0)$  and obtain localization in a larger interval  $[\Sigma_0, E_0(M_0)]$ , where  $\lim_{M_0 \rightarrow 0} E_0(M_0) \equiv \max v(x)$ . (We won't present these details here.) We also note that we are limited to energies in the range  $[\Sigma_0, \max v(x)]$  due to the quantum tunneling estimates.

**Theorem 4.2.** *Consider the breather model (4.1) with  $v \geq 0$  as specified there and satisfying (A1)–(A3). For all  $\delta_0 \equiv (M_1 - 1)^{-1}$  sufficiently large, there exists a non-empty interval  $I_0 \equiv [\Sigma_0, E_0]$ , where  $\Sigma_0 \equiv \inf \sigma(H_\omega)$  a.s. and  $v|\Lambda_1(0) \geq E_0$ , such that  $\sigma(H_\omega) \cap I_0$  is pure point a.s. with exponentially decaying eigenfunctions.*

Our proof of Theorem 4.2 follows [CH] so we only point out the necessary modifications of the arguments presented there for the breather model (4.1).

We first show that model (4.1) satisfies the hypotheses of Theorem 1.1. We define  $H_\lambda$  by freezing all the variables  $\{\lambda_i(\omega)\}_{i \neq 0}$ , writing  $\lambda \equiv \lambda_0(\omega)$ , for simplicity, and setting

$$H_\lambda = \{-\Delta + \sum_{i \neq 0} v(\lambda_i(\omega)(x - i))\} + v(\lambda x) = H_0 + V_\lambda(x),$$

where

$$V_\lambda(x) = v(\lambda x),$$

and  $\lambda \in [M_0, M_1]$ . Under the assumptions on  $v$ ,  $H_\lambda$  is twice strongly differentiable on  $D(H_0)$ . The derivatives are easily computed to be

$$\frac{dH_\lambda}{d\lambda} = x \cdot \nabla v(\lambda x),$$

and,

$$\frac{d^2 H_\lambda}{d\lambda^2} = (x, \text{Hess}[v](\lambda x)x).$$



Condition (A2) implies that  $\dot{H}_\lambda$  is non-positive, so we use the second version of the hypotheses of Theorem 1.1 given in Remarks 1.4. From the above calculations and condition (A2), we see that

$$|\ddot{H}_\lambda| \leq -(c_0/M_0)\dot{H}_\lambda,$$

which shows that (D2)\* is satisfied with  $C_1 \equiv (c_0/M_0)$ . As for condition (D1)\*, the repulsive condition (A1) and the assumptions on the support of the random variables and the support of  $v$  imply that  $\exists c_0, 0 < c_0 < \infty$ , s.t.

$$c_0 \chi_{A(1/8,1/4)}(x) \leq -x \cdot \nabla v(\lambda x),$$

where  $A_{(1/8,1/4)} \equiv \Lambda_{1/4}(0) \setminus \Lambda_{1/8}(0)$ , and  $\chi_K$  is the characteristic function for the set  $K \subset \mathbb{R}^d$ . The operator  $B$  of hypothesis (D1)\* can be taken to be  $\chi_{A(1/8,1/4)}$ . This shows that Theorem 1.1 can be applied to the model (4.1) and consequently the Corollaries 1.2–1.3 and Theorem 3.1 hold for  $H_\lambda$ . We note that by Proposition A2.2 of [CH], the subspace  $\tilde{\mathcal{H}}$  of Theorem 3.1 is the entire Hilbert space  $\mathcal{H} = L^2(\mathbb{R}^d)$ .

To compute the proof of localization, Theorem 4.2, we must verify conditions [H1]( $\gamma_0, l_0$ ) and [W] of [CH]. The Wegner lemma necessary for this model follows easily from Theorem 1.1 and the argument of [CH].

**Theorem 4.3.** *Assume (A1)–(A3). Let  $H_\omega$  be as in (1.1). For a cube  $\Lambda \subset \mathbb{R}^d$ , let  $H_\Lambda$  denote  $H_\omega$  restricted to  $\Lambda$  with Dirichlet boundary conditions and the random variables in  $\mathbb{R}^d \setminus \Lambda$  frozen. For any interval  $[0, \kappa]$ ,  $\exists$  a finite constant  $C_W > 0$ , depending on  $\|g^{(p)}\|_1$  for  $p = 0, 1, 2$ , s.t. if  $\eta < 1$  and  $I_\eta \equiv [E_0 - \eta, E_0 + \eta] \subset [0, \kappa]$ , then*

$$\mathbb{P}\{\text{dist}(\sigma(H_\Lambda), E_0) < \eta\} \leq C_W \eta |\Lambda|,$$

provided  $|\Lambda|$  is large enough.

As for [H1]( $\gamma_0, l_0$ ), we have the following lemma. Let  $E_0 = V|\partial\Lambda_1(0) > 0$ , as above. We will prove that for any  $\varepsilon > 0$ , the resolvent of  $H_{\Lambda_{l_0}}$  decays exponentially, for suitable  $l_0$  and disorder, at energies in  $[\Sigma_0, E_0 - \varepsilon]$ .

**Lemma 4.4.** *Assume that the distribution function  $g$  and single-site potential  $v$  satisfy the conditions listed after (4.1) and (A3). Given  $\xi > 2d$ . For any  $l_0$  sufficiently large,  $\exists M_1 > 1$  (i.e., for sufficiently large disorder  $\delta_0 = (M_1 - 1)^{-1}$ ) such that*

$$(4.3) \quad \mathbb{P}\{V_\omega | \Lambda_{l_0} > E_0\} \geq 1 - l_0^{-\xi}.$$

*Proof.* The probability is bounded below by  $P_{l_0} \equiv \mathbb{P}\{\lambda_i(\omega) \geq 1 \forall i \in \mathbb{Z}^d \cap \Lambda_{l_0}\}$ . By independence,

$$P_{l_0} = \left[ 1 - \int_1^{M_1} g(\lambda) d\lambda \right]^{l_0^d} \equiv \eta(M_1)^{l_0^d}.$$

We require that  $P_{l_0} \geq 1 - l_0^{-\xi}$  for suitable  $M_1$ . This follows from the monotone increasing behavior  $P_{l_0}(M_1)$  as  $M_1 \rightarrow 1^+$  ( $\delta_0 \rightarrow \infty$ ),  $P_{l_0}(1) = 1$ , and the inequality

$$0 \leq -\log \eta(M_1) \leq \frac{1}{l_0^d} \log \left( \frac{l_0^\xi}{l_0^\xi - 1} \right). \quad \square$$

Recall that  $\Sigma_0 < \langle v \rangle a^{-d}$  and that  $\frac{1}{M_1} < \frac{1}{a} < 1$ . This allows a rough estimate of the size of the energy interval:  $|I_0| \sim \delta_0^{-d} = (M_1 - 1)^d$ . Condition (4.3) insures that we can use the quantum tunneling estimates for the resolvent of  $H_\Lambda$  for energies in  $I_0$ . In exactly the same way as in section 5 of [CH], we obtain the following estimate. For  $\xi > 2d$  fixed,  $l_0$  large, and  $I_0$  as determined by Lemma 4.4, if  $E \in I_0$ , then

$$\|W(\chi_{\Lambda_{l_0}})(H_{\Lambda_{l_0}} - E - i\varepsilon)^{-1}\chi_{\Lambda_{l_0/3}}\| \leq e^{-\delta(E_0 - E)^{1/2}l_0},$$

for some  $0 < \delta < 1$ , with probability larger than  $1 - l_0^{-\xi}$ . This verifies [H1]( $\gamma_0, l_0$ ) of the model (4.1). The remaining parts of the proof of Theorem 4.2 follow as in [CH].

Another corollary of Theorem 4.3 concerns the Lipschitz continuity of the integrated density of states (cf. [CL] for a proof of the existence of the IDS).

**Corollary 4.5.** *The integrated density of states for the breather model (4.1), under the hypotheses of Theorem 4.3, is Lipschitz continuous.*

## 5. REMARKS

**A. Absolutely Continuous Spectrum.** In certain situations, the family of operators  $H_\lambda$ ,  $\lambda \in \Gamma$ , may have absolutely continuous spectrum. In general, this does not follow from the assumptions of Theorem 1.1. Random Schrödinger operators provide examples of families with an interval of dense pure point spectrum. In these cases, however, it follows from Corollaries 1.2–1.3 that the singular part of the spectral measures for different  $\lambda$ 's must be mutually singular. We note a situation in which this is not the case.

**Proposition 5.1.** *Let  $H$  and  $A$  be self-adjoint operators such that the family  $H_\lambda \equiv e^{-iA\lambda} H e^{iA\lambda}$ ,  $\lambda \in \Gamma$  with  $0 \in \Gamma$ , satisfies the hypotheses of Theorem 1.1. Furthermore, assume  $\text{Ran } B$  (in condition (D1)) is cyclic for  $H_\lambda$ . Then  $H$  (and, consequently  $H_\lambda$ ,  $\lambda \in \Gamma$ ) is purely absolutely continuous.*

*Proof.* This proposition is a corollary of Theorem 1.1. By the above remark, the singular parts of the spectrum of  $H_\lambda$ ,  $\lambda \in \Gamma$ , must be empty for they are both mutually singular and unitarily equivalent.  $\square$

To explain this result more clearly, let us note that the assumptions of Theorem 1.1 read here:

- (a)  $i[H, A] \geq C_0 B^2$ ,
- (b)  $|[A, [A, H]]| \leq iC_1 [H, A]$ ,

for positive constants  $C_0$  and  $C_1$ .

This can be compared to the Kato-Putnam Theorem [RS]. This theorem states that if there is a bounded self-adjoint operator  $A$  such that (a) holds and if  $\text{Ran } B$  is cyclic for  $H$ , then  $H$  is purely absolutely continuous. In our situation, there is no boundedness assumption on  $A$ , but instead we require the extra condition (b). Notice that this type of restriction on the second commutator also appears in Mourre's positive commutator theorem (cf. [CFKS]).

**B. Howland's Method of Positive Commutators.** Howland [H1] introduced an alternate proof of Kotani's trick based on the Kato-Putnam Theorem (cf. [RS])

concerning positive commutators. The relationship between this and our Theorem 1.1 is as follows. Let  $H_\lambda$  be a twice continuously differentiable family of self-adjoint operators on a Hilbert space  $\mathcal{H}$ , with  $\lambda \in \mathcal{M}$  a finite measure space with measure  $\mu$ . Consider the constant fiber direct sum Hilbert space

$$(5.1) \quad \tilde{\mathcal{H}} \equiv L^2(\mathcal{M}, \mathcal{H}; \mu) = \int_{\mathcal{M}}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda),$$

where  $\mathcal{H}_\lambda \cong \mathcal{H} \forall \lambda \in \mathcal{M}$ . The family  $\{H_\lambda\}_{\lambda \in \mathcal{M}}$  lifts to a self-adjoint operator  $\mathbb{H} \equiv \int_{\mathcal{M}}^{\oplus} H_\lambda d\mu(\lambda)$ , defined for suitable  $\phi \in \tilde{\mathcal{H}}$ , by

$$(5.2) \quad (\mathbb{H}\phi)(\lambda) = H_\lambda \phi(\lambda).$$

The importance of the absolute continuity of  $\mathbb{H}$  is shown in the following theorem of Howland [H1].

**Theorem 5.2.** *Let  $\mathbb{H}$  as defined in (5.2) be spectrally absolutely continuous. Assume  $\exists$  a fixed set  $S \subset \mathbb{R}$ ,  $|S| = 0$ , s.t.  $S$  supports the singular continuous part of  $H_\lambda$  a.e.  $\lambda \in \mathcal{M}$ . Then  $H_\lambda$  has no singular continuous part for  $\mu$  a.e.  $\lambda \in \mathcal{M}$ .*

We note that the hypotheses of Theorem 1.1 and Corollary 1.3 show that  $\mathbb{H}$ , constructed as in (5.1)–(5.2) from  $H_\lambda$  satisfying the hypotheses of Theorem 1.1, is absolutely continuous. Instead of the differential inequalities used here, Howland used the positive commutator method to prove the absolute continuity of  $\mathbb{H}$ .

To see how this occurs, we sketch the idea. Let  $\mathbb{A}$  denote the skew-adjoint operator  $-d/d\lambda$ , i.e.  $\mathbb{A}$  acts on suitable  $\psi \in \tilde{\mathcal{H}}$  by

$$(5.3) \quad (\mathbb{A}\psi)(\lambda) = -\frac{d\psi(\lambda)}{d\lambda}.$$

The commutator between  $\mathbb{H}$  and  $\mathbb{A}$  can be computed from (5.2)–(5.3) to obtain

$$(5.4) \quad [\mathbb{H}, \mathbb{A}] = \int_{\mathcal{M}}^{\oplus} \dot{H}_\lambda d\mu(\lambda).$$

Our condition (D1) guarantees the strict positivity of this commutator. We define  $\mathbb{B}$  on  $\mathcal{H}$  by

$$(5.5) \quad (\mathbb{B}\phi)(\lambda) = B\phi(\lambda).$$

Then by (D1) and (5.4),

$$(5.6) \quad [\mathbb{H}, \mathbb{A}] \geq c_0 \mathbb{B}^2.$$

Hence, our condition (D1) is a positive commutator condition. If it were possible to replace  $\mathbb{A}$  in (5.3) by a bounded operator such that (5.6) still holds, the Kato-Putnam Theorem would guarantee the absolute continuity of  $\mathbb{H}$ . Unfortunately, the boundedness of  $\mathbb{A}$  in the Kato-Putnam Theorem is essential. For linear perturbations of the form

$$H_\lambda = H_0 + \lambda V,$$

Howland shows how to replace  $\mathbb{A}$  in (5.3) by a bounded operator on  $\tilde{\mathcal{H}}$ , thus proving the absolute continuity of  $\mathcal{H}$  for certain non-negative  $V$ . It is not clear how to replace  $\mathbb{A}$  by a bounded (or even relatively  $\mathbb{H}$ -bounded) operator in general.

**C. Other Comments.** The strategy developed here has other applications. For example, it is related to the Carey-Pincus Theorem on the perturbation of operators with singular spectrum. This theorem states that an operator with purely singular spectrum can be changed into one with only pure point spectrum under a perturbation by a positive trace-class operator of arbitrarily small trace-norm (in fact, by an operator of rank  $m$  if the spectrum has multiplicity  $m$  [H3]). The methods of this paper also have applications to the proof of quantum stability for many time-dependent models.

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#### REFERENCES

- [CFKS] H. Cycon, R. Froese, W. Kirsch and B. Simon, *Schrödinger operators with applications to quantum mechanics and global geometry*, Springer Texts and Monographs in Physics, Springer-Verlag, New York, 1987. MR **88g**:35003
- [CH] J. M. Combes and P. D. Hislop, *Localization for some continuous random Hamiltonians in  $d$ -dimensions*, J. Funct. Anal. **124** (1994), 149–180. MR **95g**:82047
- [CHST] J. M. Combes, P. D. Hislop, C. A. Shubin, and A. Tip, in preparation.
- [CL] R. Carmona and J. Lacroix, *Spectral theory of random Schrödinger operators*, Birkhäuser, Boston, 1990. MR **92k**:47143
- [H1] J. Howland, *Perturbation theory of dense point spectra*, J. Funct. Anal. **74** (1987), 52–80. MR **89b**:47023
- [H2] ———, *Quantum stability*, Schrödinger Operators (E. Balslev, ed.), Lecture Notes in Physics No. 403, Springer Verlag, 1992, pp. 100–122. MR **94b**:81026
- [H3] ———, *On a theorem of Carey and Pincus*, J. Math. Anal. Appl. **145** (1990), 562–565. MR **91f**:47020
- [KI] F. Klopp, *Localisation for some continuous random Schrödinger operators*, Comm. Math. Phys. **167** (1995), 553–569. MR **95m**:82080
- [KS] S. Kotani and B. Simon, *Localization in general one-dimensional systems II*, Comm. Math. Phys. **112** (1987), 103–120. MR **89d**:81034
- [RS] M. Reed and B. Simon, *Methods of modern mathematical physics*, Vol. IV, Analysis of operators, Academic Press, New York, 1978. MR **58**:12429c
- [RJMS] R. del Rio, S. Jitomirskaya, N. Makarov, and B. Simon, *Singular continuous spectrum is generic*, Bull. Amer. Math. Soc. (N.S.) **31** (1994), 208–212. MR **95a**:47015
- [SW] B. Simon and T. Wolff, *Singular continuous spectrum under rank one perturbation and localization for random Hamiltonians*, Comm. Pure Appl. Math. **39** (1986), 75–90. MR **87k**:47032

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