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SPECTRAL BOUNDS FOR MATRIX POLYNOMIALS
WITH UNITARY COEFFICIENTS[*]

THOMAS R. CAMERON[†]

Abstract. It is well known that the eigenvalues of any unitary matrix lie on the unit circle. The purpose of this paper is to prove that the eigenvalues of any matrix polynomial, with unitary coefficients, lie inside the annulus $A_{\frac{1}{2},2}(0) := \left\{ z \in \mathbb{C} \mid \frac{1}{2} < |z| < 2 \right\}$. The foundations of this result rely on an operator version of Rouché's theorem and the intermediate value theorem.

Key words. Matrix polynomial, Polynomial eigenvalue problem, Unitary matrix.

AMS subject classifications. 65L15, 34L15.

1. Introduction. Developing bounds for the eigenvalues of matrix polynomials is a beautiful problem full of applications. For example, information about the location of eigenvalues is valuable for computing them by an iterative method [1, 2]. When computing the pseudospectra of matrix polynomials, one must obtain a particular region that contains the eigenvalues of interest. The spectrum bounds help determine such a region. Our main result is specific to providing lower and upper bounds on the eigenvalues of matrix polynomials whose coefficients are unitary. However, much of our development can be applied to any matrix polynomial and has been used in iterative methods for finding eigenvalues of matrix polynomials [1].

An $n \times n$ matrix polynomial of degree m is a mapping $P : \mathbb{C} \to \mathbb{C}^{n \times n}$ defined by

$$P(z) = \sum_{i=0}^{m} A_i z^i,$$

where $A_i \in \mathbb{C}^{n \times n}$ and $A_m \neq 0$, the zero matrix. For a given matrix polynomial $P(z)$, we say that $\lambda \in \mathbb{C}$ is an eigenvalue, if $\det P(\lambda) = 0$. We denote the set of all eigenvalues of $P(z)$ by $\sigma(P)$, and call this set the spectrum of $P(z)$. Moreover, we say that $x \in \mathbb{C}^n \setminus \{0\}$ is an eigenvector of $P(z)$ corresponding to λ, if $P(\lambda)x = 0$.

It is well known that the eigenvalues of a linear matrix polynomial, with unitary coefficients, lie on the unit circle. However, once we consider matrix polynomials of

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degree greater than 1, this result no longer holds.

EXAMPLE 1.1. Consider the quadratic matrix polynomial

$$P(z) = z^2 I + z I - I,$$

whose coefficients are unitary. The eigenvalues of $P(z)$ satisfy the equation $z^2 + z - 1 = 0$, and therefore, $\sigma(P) = \left\{ \frac{-1 \pm \sqrt{5}}{2} \right\}$.

Let $P(z)$ be an $n \times n$ matrix polynomial of degree m with unitary coefficients. Given Example 1.1, we know that if $m > 1$, then the eigenvalues of $P(z)$ are not guaranteed to lie on the unit circle. However, we will show that the eigenvalues lie inside the annulus $A_{\frac{1}{2},2}(0)$. To prove this result we introduce an operator version of Rouché's theorem in Section 2, and use it to find bounds on the eigenvalues of matrix polynomials. Then, in Section 3, we use these bounds and the intermediate value theorem to prove our main result.

2. Rouché and Pellet theorems for matrix polynomials. Throughout this section H is a separable Hilbert space, G is an open connected subset of $\mathbb{C}$ called a region, and $\mathcal{L}(H)$ is the space of bounded linear operators from H to H. We begin with a definition which will be used in the succeeding theorems.

DEFINITION 2.1. A bounded linear operator $A \in \mathcal{L}(X, Y)$, where X and Y are complex Banach spaces, is called a *Fredholm operator* if its range ImA is closed and the numbers

$$n(A) = \dim \mathrm{Ker} A, \; d(A) = \dim \; (Y/\mathrm{Im} A)$$

are finite. Note that if X and Y are finite dimensional, then A is always Fredholm.

THEOREM 2.2. *Let $W, S : G \to \mathcal{L}(H)$ be analytic operator functions. Assume that W is normal with respect to the simple closed curve $\gamma \subseteq G$. If $\left\| W(z)^{-1} S(z) \right\| < 1$ for all $z \in \gamma$, then $W + S$ is also normal with respect to γ. Moreover, $W + S$ and W have the same number of eigenvalues inside γ, counting multiplicities.*

Proof. See p. 205 of [5]. □

We note that Theorem 2.2 holds for any Cauchy contour in G such that its interior is a subset of G, but we only need concern ourselves with simple closed curves. We say that W is normal with respect to the curve $\gamma \subseteq G$, if $W(z)$ is invertible for all $z \in \gamma$ and W is Fredholm on the interior of γ. Moreover, the norm used can be any norm induced by a norm on H, and it is easy to see that this result holds for analytic matrix valued functions. We state and prove this now as a reference for the remainder of this paper.

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THEOREM 2.3. *Let $A, B : G \to \mathbb{C}^{n \times n}$ be analytic matrix-valued functions. Assume that $A(z)$ is invertible on the simple closed curve $\gamma \subseteq G$. If $\left\| A(z)^{-1}B(z) \right\| < 1$ for all $z \in \gamma$, then $A+B$ and A have the same number of eigenvalues inside γ, counting multiplicities.*

Proof. Given $z \in G$, $A(z)$ and $B(z)$ are bounded linear operators from $\mathbb{C}^n$ to $\mathbb{C}^n$, where $\mathbb{C}^n$ is a separable Hilbert space. Moreover, $A(z)$ is a Fredholm operator, since $\mathbb{C}^n$ is of finite dimension. Therefore, A is normal with respect to the simple closed curve $\gamma \subseteq G$, and the result follows from Theorem 2.2. ☐

When $n = 1$ Theorem 2.3 is equivalent to the classical Rouché theorem (see [12, p.391]), which has many applications. For example, Rouché's theorem has been used to prove the fundamental theorem of algebra and provide bounds on the roots of a complex scalar polynomial. With regards to the latter, A.L. Cauchy provided upper and lower bounds in [3]. A more general approach was taken up by M.A. Pellet in [11], and has been revised by M. Marden in [9].

Now that we have a Rouché type theorem for analytic matrix valued functions we can prove a generalization of Pellet and Cauchy bounds for the eigenvalues of matrix polynomials. For our purposes we only need the Cauchy bounds, that is the upper and lower bounds on the spectrum of a matrix polynomial. However, we can easily prove a generalization of both Cauchy and Pellet bounds at the same time. We choose to do so, since both Cauchy and Pellet bounds have been used in numerical methods for computing the eigenvalues of a matrix polynomial [1, 2].

The generalization of Pellet and Cauchy bounds for matrix polynomials can be found in [10] and [2]. For our purposes, we present a slightly different statement of these results in Theorem 2.4. The detailed proof of this result was influenced by J.L. Walsh in [14]

THEOREM 2.4. *Let $P(z)$ be a $n \times n$ matrix polynomial of degree $m \geq 2$, where $A_0 \neq 0$. For each $k \in \{0, 1, \dots, m\}$, such that A_k is nonsingular, consider the equation*

$$(2.1) \qquad \left\| A_k^{-1} \right\|^{-1} r^k = \sum_{i \neq k} \|A_i\| r^i,$$

where r is a real positive number and $\|\cdot\|$ is any induced matrix norm.

1. *If $k = 0$, then there exists one real positive solution r, and $P(z)$ has no eigenvalues of moduli less than r.*
2. *If $0 < k < m$, then there are either no real positive solutions or two real positive solutions $r_1 \leq r_2$. In the latter case, $P(z)$ has no eigenvalues in the annulus $A_{r_1, r_2}(0)$.*
3. *If $k = m$, then there exists one real positive solution R, and $P(z)$ has no*

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eigenvalues of moduli greater than R.

Proof. For each $k \in \{0, 1, \ldots, m\}$, such that A_k is nonsingular, we define $F :$ $\mathbb{R} \to \mathbb{R}$ by

$$F(x) = \|A_0\| + \cdots + \|A_{k-1}\| \, x^{k-1} - \left\|A_k^{-1}\right\|^{-1} x^k + \cdots + \|A_m\| \, x^m,$$

and $A(z) = A_k z^k$ and $B(z) = \sum_{i \neq k} A_i z^i$.

Let $k = 0$, then $F(x)$ has one sign change and by Descartes' rule of signs $F(x)$ has one real positive root r. Since $F(0) < 0$ and $F(r) = 0$, it follows that $F(x) < 0$ for all $0 \leq x < r$ and $\|B(z)\| < \left\|A^{-1}(z)\right\|^{-1}$ for all $|z| < r$. For any $\varepsilon > 0$, define $\gamma = (r - \varepsilon)e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Then $\left\|A(z)^{-1}B(z)\right\| < 1$ for all $z \in \gamma$. Since this holds for all $\varepsilon > 0$, by Theorem 2.3, $P(z)$ has no eigenvalues of moduli less than r.

Let $0 < k < m$, then $F(x)$ has either no real positive roots or two real positive roots. Suppose that $r_1 < r_2$ are two real positive roots of $F(x)$, then $F(x) < 0$ for all $r_1 < x < r_2$ and $\|B(z)\| < \left\|A^{-1}(z)\right\|^{-1}$ for all $r_1 < |z| < r_2$. For $\varepsilon > 0$, define $\gamma_1 = (r_1 + \varepsilon)e^{i\theta}$ and $\gamma_2 = (r_2 - \varepsilon)e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Then $\left\|A(z)^{-1}B(z)\right\| < 1$ for all $z \in \gamma_1$ and all $z \in \gamma_2$, and for all $\varepsilon > 0$. So, Theorem 2.3 implies that $P(z)$ has kn eigenvalues in $B_{r_2}(0)$ and $\overline{B_{r_1}}(0)$. Since $\overline{B_{r_1}}(0) \subseteq B_{r_2}(0)$, it follows that $P(z)$ has no eigenvalues inside the annulus $A_{r_1,r_2}(0)$.

Let $k = m$, then $F(x)$ has one real positive root R. Moreover, $F(x) < 0$ for all $x > R$ and $\|B(z)\| < \left\|A^{-1}(z)\right\|^{-1}$ for all $|z| > R$. For $\varepsilon > 0$, define $\gamma = (R + \varepsilon)e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Then, $\left\|A(z)^{-1}B(z)\right\| < 1$ for all $z \in \gamma$ and for all $\varepsilon > 0$. Therefore, by Theorem 2.3, $P(z)$ has mn eigenvalues inside $\overline{B_R}(0)$. Since $\det P(z)$ is a polynomial of degree mn, it follows that $P(z)$ has no eigenvalues of moduli greater than R. $\square$

3. The main result. Now that we have introduced an operator version of Rouché's theorem and used it to prove a generalized version of Pellet and Cauchy bounds for the spectrum of matrix polynomials, we can prove our main result. We begin with several definitions that lay the foundation for this result.

DEFINITION 3.1. Let

$$G(n, \mathbb{C}) = \left\{A \in \mathbb{C}^{n \times n} \mid A^* A = I\right\}$$

denote the group under matrix multiplication of all $n \times n$ unitary matrices. We define the set of all $n \times n$ matrix polynomials with unitary coefficients by

$$U(n) = \left\{P(z) = \sum_{i=0}^{m} A_i z^i \mid A_i \in G(n, \mathbb{C}) \text{ for } i = 0, 1, \ldots, m, \text{ where } m \in \mathbb{N}\right\}.$$

We then define the family of all matrix polynomials with unitary coefficients by

$$\mathcal{U} = \bigcup_{n \in \mathbb{N}} U(n).$$

It is interesting that although $G(n, \mathbb{C})$ is a group, $U(n)$ fails to be a group. Indeed, $U(n)$ is closed under matrix polynomial multiplication and $I \in U(n)$. However, not every element of $U(n)$ has an inverse which is also an element of $U(n)$. For this reason, $U(n)$ is a semigroup with the identity element.

Now, corresponding to each $P(z) \in \mathcal{U}$ there is a spectrum $\sigma(P)$, which is the set of all eigenvalues of $P(z)$. Let $\sigma_\mathcal{U} = \{\sigma(P) \mid P(z) \in \mathcal{U}\}$, and $|\sigma_\mathcal{U}| = \{|\sigma(P)| \mid P(z) \in \mathcal{U}\}$, where $|\sigma(P)|$ denotes the set of moduli of the eigenvalues of $P(z)$. The following result provides upper and lower bounds on $|\sigma_\mathcal{U}|$.

THEOREM 3.2. *Let $P(z) \in \mathcal{U}$. Then for any $\lambda \in \mathbb{C}$, such that λ is an eigenvalue of $P(z)$, it follows that $\frac{1}{2} < |\lambda| < 2$.*

Proof. Define $u : \mathbb{R} \to \mathbb{R}$ by

$$u(x) = x^m - x^{m-1} - \cdots - 1.$$

Since the spectral norm of a unitary matrix is 1, it follows from Theorem 2.4 that the one real positive root of $u(x)$ is an upper bound on the moduli of the eigenvalues of $P(z)$. Note that $2^m > 2^{m-1} + \cdots + 2^0$ for all positive integers m. Therefore, $u(2) > 0$ and $u(1) < 0$. Since u is a continuous function, the intermediate value theorem states that there exists an $R \in (1, 2)$ such that $u(R) = 0$. The moduli of any eigenvalue of $P(z)$ is bounded above by R and therefore bounded above by 2.

Define $l : \mathbb{R} \to \mathbb{R}$ by

$$l(x) = x^m + \cdots + x - 1.$$

Again, by Theorem 2.4, the one real positive root of $l(x)$ is a lower bound on the modulie of the eigenvalues of $P(z)$. Note that $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = 1$; therefore, $l(\frac{1}{2}) < 0$ and $l(1) > 0$. By the intermediate value theorem, there exists $r \in \left(\frac{1}{2}, 1\right)$ such that $l(r) = 0$. The moduli of any eigenvalue of $P(z)$ is bounded below by r and therefore bounded below by $\frac{1}{2}$. $\square$

Theorem 3.2 provides upper and lower bounds on the set $|\sigma_\mathcal{U}|$. We conclude by proving that these bounds are in fact optimal; that is, they are the least upper bound and greatest lower bound of the set $|\sigma_\mathcal{U}|$.

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THEOREM 3.3. $\sup |\sigma_{\mathcal{U}}| = 2$ *and* $\inf |\sigma_{\mathcal{U}}| = \frac{1}{2}$.

Proof. Given Theorem 3.2, we just have to show that there is no number less than 2 or greater than $\frac{1}{2}$ which serves as an upper or lower bound to $|\sigma_{\mathcal{U}}|$, respectively.

Consider a real number r, such that $\frac{1}{2} < r < 1$. Then $\sum_{i=1}^{\infty} r^i > 1$, and it follows that there exists an $m \in \mathbb{N}$ such that $\sum_{i=1}^{m} r^i > 1$. Now define $P(z) = -I + \sum_{i=1}^{m} z^i I$, which is clearly an element of $\mathcal{U}$. Then $P(z)$ has a real positive eigenvalue that is also a root of the polynomial $l(x) = x^m + \cdots + x - 1$. Since $l(r) > 0$ and $l(\frac{1}{2}) < 0$, it follows from the intermediate value theorem that there exists a $\lambda \in \left(\frac{1}{2}, r\right)$ such that $l(\lambda) = 0$. Therefore, r cannot be a lower bound of $|\sigma_{\mathcal{U}}|$.

Define the matrix polynomial $P(z) = z^m I - \sum_{i=0}^{m-1} z^i I$, which is clearly an element of $\mathcal{U}$ for any $m \in \mathbb{N}$. Moreover, $P(z)$ has a real positive eigenvalue which is also a root of the polynomial $u(x) = x^m - \left(x^{m-1} + \cdots + 1\right)$. If $R \in (1, 2)$ is an upper bound of $|\sigma_{\mathcal{U}}|$, then $R^m \geq R^{m-1} + \cdots + 1$. For, if $R^m < R^{m-1} + \cdots + 1$, then there exists a $\lambda \in (R, 2)$ such that $u(\lambda) = 0$. We note that

$$R^{m-1} + \cdots + 1 = \frac{1 - R^m}{1 - R}.$$

Therefore, if $R \in (1, 2)$ is an upper bound of $|\sigma_{\mathcal{U}}|$, then $R^m \geq \frac{1-R^m}{1-R}$. Since $(1 - R) < 0$, it follows that $R^m (2 - R) \leq 1$ and we have

$$(2 - R) \leq \frac{1}{R^m}.$$

Since $P(z) \in \mathcal{U}$ for any $m \in \mathbb{N}$, the above inequality must hold for all $m \in \mathbb{N}$ in order for R to be an upper bound of $|\sigma_{\mathcal{U}}|$. Since this is not possible, it follows that $R \geq 2$. $\square$

REFERENCES

[1] D.A. Bini and V. Noferini. Solving polynomial eigenvalue problems by means of the Ehrlich-Aberth method. Preprint, arXiv:1207.6292 [math.NA], 2012.
[2] D.A. Bini, V. Noferini, and M. Sharify. Locating the eigenvalues of matrix polynomials. Preprint, arXiv:1206.3632 [math.NA], 2012.
[3] A.L. Cauchy. *Exercices de mathématique.* Oeuvres (2), 9:122, 1829.
[4] I. Gohberg and E.I. Sigal. *An operator generalization of the logarithmic residue theorem and Rouché's theorem. Mat. Sb. (N.S.)*, 84(126):607–629, 1971 (in Russian); English translation in *Math. USSR-Sb.*, 13:603–625, 1971.
[5] I. Gohberg, S. Goldberg, and M.A. Kaashoek. *Classes of Linear Operators*, Vol. 1. *Operator Theory: Adv. Appl.*, Birkhauser Verlag, Basel, 49, 1990.
[6] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials.* SIAM, Philadelphia, PA, 2009.
[7] N.J. Higham and F. Tisseur. Bounds for eigenvalues of matrix polynomials. *Linear Algebra Appl.*, 358:5–22, 2003.

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[8] B.W. Levinger. A generalization of Pellet's theorem concerning the zeros of a polynomial. *Proc. Amer. Math. Soc.*, 18(5):767–774, 1967.

[9] M. Marden. A refinement of Pellet's theorem. *Bull. Amer. Math. Soc.*, 54:550–557, 1948.

[10] A. Melman. Generalization and variations of Pellet's theorem for matrix polynomials. Preprint, arXiv:1210.0172 [math.NA], 2013.

[11] M.A. Pellet. Sur un mode de séparation des racines des équations et la formule de Lagrange. *Bull. Sci. Mat.*, 5:393–395, 1881.

[12] R. Remmert. *Theory of Complex Functions*. Springer-Verlag, New York, 1991.

[13] E. Rouché. Mémoire sur la série de Lagrange. *J. École Polytech.*, 22:217–218, 1862.

[14] J.L. Walsh. On Pellet's theorem concerning the roots of a polynomial. *Ann. of Math.*, 26(1/2):59–64, 1924.