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SPECTRAL CONVERGENCE OF RIEMANNIAN MANIFOLDS

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Abstract. We introduce a spectral distance on the set of compact Riemannian manifolds, making use of their heat kernels, and show some basic properties of the distance on a class of compact Riemannian manifolds with diameters uniformly bounded from above and Ricci curvatures uniformly bounded from below.

CONTENTS

Introduction. Gromov [10] introduced a distance on the set of compact Riemannian manifolds and established, among other things, a precompactness theorem on a class of compact Riemannian manifolds with a uniform upper bound for the diameters and a uniform lower bound for the Ricci curvatures. Various works of interest have since appeared around the Gromov-Hausdorff distance. For example, Fukaya [9] gave a notion of measured Hausdorff topology on the set of compact Riemannian manifolds and discussed the eigenvalue problem for the Laplace operators. The main result in [9] concerns the convergence of the spectra with respect to the topology in the set of compact Riemannian manifolds with a uniform bound for the diameters and a uniform bound for the sectional curvatures in their absolute values. It was improved later in [12].

On the other hand, from the point of view of spectral geometry, Bérard, Besson and Gallot [3] defined a family of ditances on the set of compact Riemannian manifolds and proved a precompactness theorem under the assumption similar to the one used by Gromov (see also Muto [15]).

In this paper, motivated by these results, we shall introduce a new uniform topology

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on the set of compact Riemannian manifolds, making use of the heat kernels, and then discuss some basic properties of the topology in relations with the Gromov-Hausdorff distance under certain geometric conditions.

Now we shall explain briefly the main results of this paper. For a compact Riemannian manifold M, we denote by $p_M(t, x, y)$ the heat kernel of M with respect to the normalized Riemannian measure μ_M^{can} (i.e., the Riemannian measure divided by the volume of M). Given two compact Riemannian manifolds M and N, not necessarily continuous maps $f: M \to N$ and $h: N \to M$ are said to be *ε-spectral approximations* between M and N if they respectively satisfy

$$e^{-(t+1/t)}|p_M(t, x, y) - p_N(t, f(x), f(y))| < \varepsilon \quad \text{for all} \quad t > 0 \quad \text{and} \quad x, y \in M$$

and

$$e^{-(t+1/t)}|p_M(t, h(a), h(b)) - p_N(t, a, b)| < \varepsilon \quad \text{for all} \quad t > 0 \quad \text{and} \quad a, b \in N .$$

The spectral distance between M and N is by definition the lower bound of the numbers $\varepsilon > 0$ such that they admit ε-spectral approximations. The spectral distance gives a uniform structure on the set of compact Riemannian manifolds. Given an integer $n > 1$, and constants $D > 0$ and $\kappa > 0$, we write $\mathcal{M}(n, D, \kappa)$ for the set of compact Riemannian manifolds of dimension n with diameter $\mathrm{diam}(M) \leq D$ and Ricci curvature $\geq -(n-1)\kappa^2$. Then we can show the following results:

(i) On $\mathcal{M}(n, D, \kappa)$, the topology given by the spectral distance is finer than that of the Gromov-Hausdorff distance (cf. Theorem 3.5).

(ii) $\mathcal{M}(n, D, \kappa)$ is precompact with respect to the spectral distance (cf. Theorem 3.6). Moreover, a boundary element of $\mathcal{M}(n, D, \kappa)$ in its completion with respect to the spectral distance can be regarded as a triad (X, μ, p) of a compact length space X, a Radon measure μ of unit total mass on X, and a positive Lipschitz function p on $(0, \infty) \times X \times X$ which is the heat kernel of a C_0-semigroup on $L^2(X, \mu)$ (cf. Theorem 3.8).

(iii) The continuity of eigenvalues and the convergence of eigenfunctions in a certain sense hold in $\mathcal{M}(n, D, \kappa)$ with respect to the spectral distance (cf. Theorem 4.5).

To prove these results, we use basically several estimates on the heat kernels obtained by some authors (cf. §2).

As indicated later in Example 1 of §1, from the nature of the problem considered here, we shall in fact investigate *Riemannian manifolds endowed with weight functions and the associated operators* rather than Riemannian manifolds and the Laplace operators.

1. Spectral distance. In this section we shall introduce a uniform topology on the set of equivalence classes of compact Riemannian manifolds endowed with weight functions.

1.1. To begin with, we recall the definiton of the Hausdorff distance on the set of isometry classes of metric spaces introduced by Gromov [10]. Given two metric spaces X and Y, a distance δ on the disjoint union $X \sqcup Y$ is said to be admissible if its restriction to X and Y are equal to the original distances d_X and d_Y in X and Y, respectively. The Gromov-Hausdorff distance $\mathrm{HD}(X, Y)$ is by definition the lower bound $\inf \mathrm{H}^{\delta}(X, Y)$, where δ runs over all admissible distances on $X \sqcup Y$ and $\mathrm{H}^{\delta}(X, Y)$ stands for the Hausdorff distance in $(X \sqcup Y, \delta)$, namely, the lower bound of the numbers $\varepsilon > 0$ such that $\delta(x, Y) < \varepsilon$ and $\delta(y, X) < \varepsilon$ for all $x \in X$ and $y \in Y$. The Gromov-Hausdorff distance HD enjoys all the properties of a distance when it is restricted to the set of compact metric space. Observe that if $\mathrm{HD}(X, Y) < \varepsilon$, then there exists a map $f: X \to Y$ such that (i) the 2ε-neighborhood of $f(X)$ covers Y; and (ii) $|d_X(x, y) - d_Y(f(x), f(y))| < 2\varepsilon$ for all $x, y \in X$, and also there is a map $h: Y \to X$ satisfying (i) and (ii). Indeed, we take a distance δ on $X \sqcup Y$ such that $\mathrm{H}^{\delta}(X, Y) < \varepsilon$ and then choose maps $f: X \to Y$ and $h: Y \to X$ in such a way that $\delta(x, f(x)) < \varepsilon$ and $\delta(a, h(a)) < \varepsilon$ for all $x \in X$ and $a \in Y$. Not necessarily continuous maps $f: X \to Y$ and $h: Y \to X$ satisfying the above properties (i) and (ii) are called *2ε-Hausdorff approximations* between X and Y. Let us denote by $\mathrm{HD}'(X, Y)$ the lower bound of the numbers $\varepsilon > 0$ for which there exist ε-Hausdorff approximations $f: X \to Y$ and $h: Y \to X$. Then we have

$$\frac{1}{2} \mathrm{HD}'(X, Y) \leq \mathrm{HD}(X, Y) \leq 2\mathrm{HD}'(X, Y).$$

For this reason, $\mathrm{HD}'(X, Y)$ induces the same uniform topology in the set of isometry classes of compact metric spaces as the Gromov-Hausdorff distance.

1.2. Let M be a complete Riemannian manifold without boundary and w a positive smooth function on M. Throughout this paper, manifolds are always assumed to be connected. We consider an elliptic differential operator $\mathscr{L}_w$ of second order defined by

$$\mathscr{L}_w = -\Delta_M - \nabla \log w,$$

where Δ_M stands for the Laplace operator of M acting on functions, namely, $\Delta_M \psi = \mathrm{trace} \nabla d\psi$. This operator $\mathscr{L}_w$ is associated with the quadratic form $\mathscr{D}(\phi, \psi) = \int_M \langle \nabla\phi, \nabla\psi \rangle w \, d\mathrm{vol}_M$ on the space of smooth functions with compact support, where $d\mathrm{vol}_M$ denotes the Riemannian measure of M. In what follows, we write simply μ_w for the (Radon) meaure $w \, d\mathrm{vol}_M$, and we denote by $p_w(t, x, y)$ the heat kernel of the operator $\mathscr{L}_w$ in $L^2(M, \mu_w)$. When M is compact, by the Sturm-Liouville decomposition, we have the eigenfunction expansion of the kernel:

$$p_w(t, x, y) = \sum_{\nu=0}^{\infty} e^{-\lambda_\nu t} u_\nu(x) u_\nu(y).$$

Here $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty$ are the eigenvalues of $\mathscr{L}_w$ and $\{u_\nu\}$ is a complete orthonormal system of $L^2(M, \mu_w)$ consisting of eigenfunctions with u_ν having eigenvalue λ_ν.

Throughout this paper, two triads (M, μ_v, p_v) and (N, μ_w, p_w) are said to be equivalent and will be identified if there is a map $f: M \to N$ which preserve the heat kernels, namely,

$$p_v(t, x, y)=p_w(t, f(x), f(y))$$

for all $t>0$ and $x, y\in M$. In fact, since

$$\lim_{t\to 0} 4t\log p_v(t, x, y)=-d_M(x, y)^2$$

for all $x, y\in M$ and for every triad (M, μ_v, p_v) by a theorem due to Varadhan [20] and Cheng, Li and Yau [7], we see that $\dim M=\dim N$ and the above map $f: M\to N$ is a distance preserving map from M onto N, and hence an isometry between M and N. Moreover for a continuous function ψ on N,

$$\begin{aligned}\int_M \psi(f(x))d\mu_v(x) &= \lim_{t\to 0}\int_M\int_N p_w(t, f(x), b)\psi(b)d\mu_w(b)d\mu_v(x)\\ &= \lim_{t\to 0}\int_M\int_M p_w(t, f(x), f(y))\psi(f(y))d\mu_v(x)d\mu_w(f(y))\\ &= \lim_{t\to 0}\int_M\int_M p_v(t, x, y)\psi(f(y))d\mu_v(x)d\mu_w(f(y))\\ &= \int_M \psi(f(y))d\mu_w(f(y)).\end{aligned}$$

This shows that f preserves the measures, $f_*\mu_v=\mu_w$.

Given two triads $\tau_1=(M, \mu_v, p_v)$ and $\tau_2=(N, \mu_w, p_w)$, not necessarily continuous maps $f: M\to N$ and $h: N\to M$ are said to be *ε-spectral approximations* between τ_1 and τ_2 if they satisfy

$$e^{-(t+1/t)}|p_v(t, x, y)-p_w(t, f(x), f(y))|<\varepsilon$$
$$e^{-(t+1/t)}|p_v(t, h(a), h(b))-p_w(t, a, b)|<\varepsilon$$

for all $t>0$, $x, y\in M$ and $a, b\in N$. The *spectral distance* $\mathrm{SD}(\tau_1, \tau_2)$ between τ_1 and τ_2 is defined to be the lower bound of the numbers $\varepsilon>0$ so that they admit ε-spectral approximations. Here we understand $\mathrm{SD}(\tau_1, \tau_2)=\infty$ if there are no such maps. Let us denote by $\mathcal{M}_{w,c}$ the set of the equivalence classes of all triads (M, μ_v, p_v) with M compact, then obviously $\mathrm{SD}(\tau_1, \tau_2)$ is finite for all τ_1 and $\tau_2\in\mathcal{M}_{w,c}$. Moreover if $\mathrm{SD}(\tau_1, \tau_2)=0$, then $\tau_1=\tau_2$ in $\mathcal{M}_{w,c}$. Indeed, by the definition of the spectral distance SD, we have a sequence of $\varepsilon(i)$-spectral approximations $f_i: M\to N$ and $h_i: N\to M$ between $\tau_1=(M, \mu_v, p_v)$ and $\tau_2=(N, \mu_w, p_w)$ with $\varepsilon(i)$ converging to zero as i tends to infinity. Taking a subsequence if necessarily, we may assume that these maps $f_i: M\to N$ and $h_i: N\to M$ respectively converge to maps $f: M\to N$ and $h: N\to M$ which preserve the heat kernels. Thus $\mathcal{M}_{w,c}$ equipped with the spectral distance SD becomes a metric space.

In the above definition of ε-spectral approximations, we multiply the function

$e^{-(t+1/t)}$ by the differences of the heat kernels for convenience, to take the asymptotic behavior of the heat kernels as $t \to 0$ into account.

1.3. Before concluding this section, we shall exhibit some examples. In the rest of the section, for a compact Riemannian manifold M, we denote simply by μ_M^{can} the normalized Riemannian measure of M, namely, the Riemannian measure $d\mathrm{vol}_M$ of M divided by its volume $\mathrm{vol}(M)$, and also by p_M the heat kernel of the Laplace operator of M in $L^2(M, \mu_M^{\mathrm{can}})$.

EXAMPLE 1. Let M be a compact Riemannian manifold of dimension n and w a positive smooth function with $\mu_w(M)=1$. Take another compact Riemannian manifold N of dimension k, and denote by M_ε the warped product of M and N with warping function $\varepsilon w^{1/k}$ $(\varepsilon>0)$. Letting ε tend to zero, we see that $(M_\varepsilon, \mu_{M_\varepsilon}^{\mathrm{can}}, p_{M_\varepsilon})$ converges to (M, μ_w, p_w) with respect to the spectral distance. Indeed, the canonical projection $\pi: M_\varepsilon \to M$ and any section $\sigma: M \to M_\varepsilon$ give $\delta(\varepsilon)$-spectral approximations with $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0$. We remark that the push-forward measure $\pi_*\mu_{M_\varepsilon}^{\mathrm{can}}$ coincides with μ_w for any ε.

EXAMPLE 2. Let $\pi: E \to T$ be a Riemannian covering over a compact flat Riemannian manifold T and let Γ be the covering transformation group. Suppose we have an isomertic action of Γ on a compact Riemannian manifold S, namely, we have a homomorphism $\alpha: \Gamma \to \mathrm{Isom}(S)$. Then we obtain a family of Riemannian manifolds which consists of the quotient manifolds $M_\varepsilon=(\varepsilon E)\times_\Gamma S$ $(\varepsilon>0)$ of the Riemannian products $(\varepsilon E)\times S$ with respect to the diagonal action of Γ. Observe that as ε tends to zero, the triad $(M_\varepsilon, \mu_{M_\varepsilon}^{\mathrm{can}}, p_{M_\varepsilon})$ converges to $(S/K, \mu, p)$ with respect to the spectral distance. Here K denotes the closure of the subgroup $\alpha(\Gamma)$ in $\mathrm{Isom}(S)$, μ is the push-forward measure $\rho_*\mu_S^{\mathrm{can}}$ of μ_S^{can} under the canonical projection $\rho: S \to S/K$, and the pull-back ρ^*p of p is given by

$$\rho^*p(t,x,y)=\sum_{\nu=0}^{\infty} e^{-\lambda_\nu t}u_\nu(x)u_\nu(y)\,,$$

where $\{u_\nu\}_{\nu=0}^{\infty}$ is a complete orthonormal system of eigenfunctions of S in the subspace $L^2(S/K, \rho_*\mu_S^{\mathrm{can}})$ of $L^2(S, \mu_S^{\mathrm{can}})$ which consists of K-invariant square integrable functions, and λ_ν stands for the corresponding ν-th eigenvalue of S in $L^2(S/K, \rho_*\mu_S^{\mathrm{can}})$. We note that at a regular point a of the quotient space S/K, the density of the measure μ with respect to the m-dimensional Hausdorff measure of S/K $(m=\dim S-\dim\rho^{-1}(a))$ is equal to the ratio of the volume of the submanifold $\rho^{-1}(a)$ to that of S.

This is a typical example of Riemannian manifolds collapsing to a lower-dimensional space while keeping their curvatures and diameters bounded. According to Fukaya [9], when such a family, say $\{M_i\}$, converges to a metric space with a Radon measure with respect to the measured Hausdorff topology in his sense, the eigenvalues and eigenfunctions of M_i converge in a certain sense. In fact, the triad $(M_i, \mu_{M_i}^{\mathrm{can}}, p_{M_i})$

converges with respect to the spectral distance SD.

EXAMPLE 3. We consider an example in [21] of Riemannian manifolds collapsing to a lower-dimensional space while keeping their curvatures bounded below and their diameters bounded above. Let G be a compact Lie group of positive dimension acting on a compact Riemannian manifold M effectively. Take a G-invariant metric g on M, and consider a G-action ϕ on $G\times M$ defined by $\phi_a(b,x)=(ab,a(x))$. Let $\pi\colon G\times M\to M$, $\pi(a,x)=a^{-1}(x)$, be the projection along the G-orbits, and ω the connection on the principal bundle $(G\times M,\pi,M)$ such that $0\times T_xM$ $(x\in M)$ are the horizontal spaces. We define a family of metrics g_ε on $G\times M$ by

$$g_\varepsilon(\xi,\xi')=g(d\pi(\xi),d\pi(\xi'))+\varepsilon^2\langle\omega(\xi),\omega(\xi')\rangle\,,$$

where $\langle\ ,\ \rangle$ denotes a bi-invariant metric of G. Let ψ be another G-action on $G\times M$ defined by $\psi_a(b,x)=(ba^{-1},x)$. Since g_ε is ψ-invariant, g_ε induces a metric g_ε^* on M, and we write M_ε for (M,g_ε^*). Then M_ε converges to the quotient metric space M/G with respect to HD as ε tends to zero. Moreover if we denote by $w_\varepsilon(x)$ $(x\in M)$ the volume of $G\times\{x\}$ with respect to g_ε and set $w_\varepsilon^*(x)=w_\varepsilon(x)/\int_{M_\varepsilon}w_\varepsilon d\mathrm{vol}_{M_\varepsilon}$, then we see that as ε tends to zero, the triad $(M_\varepsilon,\mu_{w_\varepsilon^*},p_{w_\varepsilon^*})$ converges to $(M/G,\mu,p)$ with respect to the spectral distance. Here $(M/G,\mu,p)$ is given in a manner similar to Example 2.

The sequences of these examples converge to lower-dimensional spaces. See for instance, [1], [2], [9], [11], [16] and the references therein for different kinds of examples and related topics.

2. **Bounds for heat kernels.** Throughout this section, M is assumed to be an n-dimensional complete Riemannian manifold. Let w be a positive smooth function on M and μ the Radon measure $wd\mathrm{vol}_M$ as in Section 1. The purpose of this section is to give an analog of the Bishop-Gromov inequality and then derive some geometric estimates for the heat kernel $p_w(t,x,y)$ of $\mathscr{L}_w$. Our arguments are based on the methods which have been established when w is constant. We refer the reader to Bérard, Besson and Gallot [3], Chavel [5], Davies [7], Li and Yau [13], [14], Sturm [19] and the references therein. The results of the present section provide the basic ingredients in proving the main theorems in this paper.

2.1. First of all, we introduce a symmetric tensor associated with a given positive function w. For an integer $k>0$, we define a symmetric tensor $R_{w,k}$ by

$$R_{w,k}=\mathrm{Ric}_M-\frac{k}{w^{1/k}}Ddw^{1/k}$$

$$=\mathrm{Ric}_M-\frac{1}{k}d\log w\otimes d\log w-\mathring{D}d\log w\,,$$

where Ric_M stands for the Ricci tensor of M. For $k=0$, we set $R_{w,0}=\mathrm{Ric}_M$. In this case,

w is always assumed to be a constant. In the case $k=1$, the tensor $R_{w,1}$ was introduced in Setti [17], [18], where upper and lower estimates for the first nonzero eigenvalue of $\mathscr{L}_w$, for M compact, and also those for the heat kernel of $\mathscr{L}_w$ were given in terms of a lower bound of $R_{w,1}$. In what follows, we shall explain some implications of a lower bound for $R_{w,k}$ and then derive some results which will be needed later.

Let us fix any compact Riemannian manifold N of dimension k and consider the warped product $M_{w,k}=M\times_{w^{1/k}}N$ of M and N with warping function $w^{1/k}$. We denote by π the natural projection of $M_{w,k}$ onto M. Then the Ricci tensor $\mathrm{Ric}_{M_{w,k}}$ of $M_{w,k}$ restricted to the horizontal subspaces coincides with the pull-back of the tensor $R_{w,k}$ by π. Namely we see that

$$\mathrm{Ric}_{M_{w,k}}(X, Y)=R_{w,k}(d\pi(X), d\pi(Y)) \tag{2.1}$$

for all horizontal vectors X and Y of $M_{w,k}$. We notice that for any smooth function ψ on an open set of M,

(2.2)
(i) the gradient of $\psi\circ\pi$ is the horizontal lift of the gradient of ψ ;
(ii) $\Delta_{M_{w,k}}(\psi\circ\pi)=\{\Delta_M\psi+\nabla\log w\cdot\psi\}\circ\pi=-(\mathscr{L}_w\psi)\circ\pi$.

Let r be the distance to a point x in M and r^* the distance to the fiber $\pi^{-1}(x)=\{x\}\times N$ in $M_{w,k}$. Clearly $r^*=r\circ\pi$. Suppose that

$$R_{w,k}\geq-(n+k-1)\kappa^2 \tag{2.3}$$

on M, where $n=\dim M$ and κ is a positive constant. Then the standard comparison argument together with (2.1) yields the following estimate:

$$\Delta_{M_{w,k}}r^*<(n+k-1)\kappa\frac{\cosh\kappa r^*}{\sinh\kappa r^*} \tag{2.4}$$

on $M_{w,k}$, or equivalently,

$$-\mathscr{L}_w r<(n+k-1)\kappa\frac{\cosh\kappa r}{\sinh\kappa r} \tag{2.4'}$$

on M. Because r and r^* are only Lipschitz functions on the cut loci, respectively, (2.4) and (2.4′) should be understood in a generalized sense. In fact, an argument due to Calabi allows us to assume without loss of generality that they are smooth when applying the comparison theorem or the maximum principle (cf. [5]). Now let us put for convenience

$$V_{n+k,\kappa}(t)=\int_0^t(\kappa^{-1}\sinh\kappa s)^{n+k-1}ds\,.$$

Then the proof of the Bishop-Gromov inequality combined with the inequality (2.4) shows that the ratio of the volume of $\pi^{-1}(B_x(t))$ in $M_{w,k}$ to $V_{n+k,\kappa}(t)$ is monotone decreasing in $t>0$, where $B_x(t)$ stands for the metric ball of M around a point x with radius t. Since the ratio of the volume of $\pi^{-1}(B_x(t))$ in $M_{w,k}$ to the volume of N is equal to the mass $\mu_w(B_x(t))$ of the ball with respect to the measure μ_w, we obtain the following:

PROPOSITION 2.1. *Let M be a complete Riemannian manifold of dimension n and w a positive smooth function on M satisfying* (2.3) *for an integer $k\geq 0$ and a constant $\kappa>0$. Then*

$$\frac{\mu_w(B_x(r))}{\mu_w(B_x(R))}>\frac{V_{n+k,\kappa}(r)}{V_{n+k,\kappa}(R)}$$

for all $x\in M$ and $0<r<R$.

Let M, w, k and κ be as above. Then it follows immediately from the proposition that given $R>0$ and $\varepsilon>0$, there exists a positive integer ν depending only on n, k, κ, R and ε such that the ball $B_x(R)$ around a point x of M with radius R contains at most ν disjoint balls of radius ε. Thus by virtue of Gromov's precompactness theorem [10, Chap. 5], we have the following:

COROLLARY 2.2. *Given constants n, k and κ as above and given $D>0$, the set of (isometry classes of) all compact Riemannian n-manifolds M, each of which admits a positive smooth function w satisfying* (2.3), *and the diameter of which is bounded from above by D, is precompact with respect to the Gromov-Hausdorff distance* HD.

2.2. We now derive some bounds for the heat kernel $p_w(t, x, y)$ of $\mathscr{L}_w$ under the assumption (2.3). We begin with the following:

PROPOSITION 2.3. *Let M and w be as in Proposition* 2.1. *Let $u(t, x)$ be a positive solution on* $(0, \infty)\times M$ *of the equation*

$$\left(\frac{\partial}{\partial t}+\mathscr{L}_w\right)u(t, x)=0\,.$$

Then (i) *for any $\alpha>1$,*

$$\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}\leq\frac{(n+k)\alpha^2}{2}\left\{\frac{1}{t}+\frac{(n+k-1)\kappa^2}{2(\alpha-1)}\right\};$$

(ii) *for all $t>0$, $s>0$, $\alpha>1$ and $x, y\in M$,*

$$u(t, x)\leq u(t+s, y)\left(\frac{t+s}{t}\right)^{(n+k)\alpha/2}\exp\left(\frac{\alpha d_M(x, y)^2}{4s}+\frac{(n+k)(n+k-1)\kappa^2 s}{4(\alpha-1)}\right).$$

PROOF. On account of (2.1) and (2.2), this proposition follows in the same methods as those for [14, Theorems 1.3 and 2.2]. q.e.d.

Secondly, we shall give upper bounds for the heat kernel.

PROPOSITION 2.4. *Let M and w be as in Proposition* 2.1. *Then the heat kernel $p_w(t, x, y)$ satisfies*

$$p_w(t, x, y) \leq$$
$$C_{n+k,\kappa}(\varepsilon)\mu_w(B_x(\sqrt{t}))^{-1/2}\mu_w(B_y(\sqrt{t}))^{-1/2}\exp\left(-(1-\varepsilon)\frac{d_M(x, y)^2}{4t}+(\varepsilon-\lambda_0)t\right)$$

for all $t>0$, $\varepsilon>0$ and $x, y \in M$, where $C_{n+k,\kappa}(\varepsilon)$ is a positive constant depending only on $n+k$, κ and ε in such a way that $C_{n+k,\kappa}(\varepsilon)$ diverges as ε tends to zero, and λ_0 is the bottom of the spectrum of the operator $\mathscr{L}_w$ on $L^2(M, \mu_w)$. Moreover if M is compact and the diameter is bounded above by a positive constant D, then

$$p_w(t, x, y) \leq C_{n+k,\kappa}(\varepsilon)\mu_w(M)^{-1}\frac{V_{n+k,\kappa}(D)}{V_{n+k,\kappa}(\sqrt{t})}\exp\left(-(1-\varepsilon)\frac{d_M(x, y)^2}{4t}+\varepsilon t\right)$$

for all $t \in (0, D^2]$, $\varepsilon>0$ and $x, y \in M$.

PROOF. The same arguments as in [7] combined with (2.1) and (2.2) give the first estimate, which implies the second, because of Proposition 2.1. q.e.d.

As for a lower bound of the heat kernel, we have:

PROPOSITION 2.5. *Let M and w be as in Proposition* 2.1. *Then the heat kernel $p_w(t, x, y)$ satisfies*

$$p_w(t, x, y) \geq$$
$$C'_{n+k,\kappa}(\varepsilon)\mu_w(B_x(\sqrt{t}))^{-1/2}\mu_w(B_y(\sqrt{t}))^{-1/2}\exp\left(-(1+\varepsilon)\frac{d_M(x, y)^2}{4t}-(\varepsilon+C'')t\right)$$

for all $t>0$, $\varepsilon>0$ and $x, y \in M$, where $C'_{n+k,\kappa}(\varepsilon)$ is a positive constant depending only on $n+k$, κ and ε in such a way that $C'_{n+k,\kappa}(\varepsilon)$ converges to zero as ε tends to zero, and $C''=(n+k-1)^2\kappa^2/4$.

PROOF. This follows by the same arguments as in [19] together with (2.1) and (2.2). q.e.d.

We notice that the Poincaré inequality holds for a Riemannian manifold endowed with a measure. Namely we have the following:

PROPOSITION 2.6. *Let M and w be as in Proposition* 2.1 *and let f be a smooth function on M. Then*

$$\int_{B_x(r)} |f-f_{x,r,w}|^2 d\mu_w \leq C^{1+r\kappa}r^2\int_{B_x(r)} |\nabla f|^2 d\mu_w,$$

for all $x \in M$ and $r>0$, where C is a positive constant depending only on $n+k$ and

$$f_{x,r,w}=\frac{1}{\mu_w(B_x(r))}\int_{B_x(r)} f d\mu_w .$$

PROOF. This can be derived by the same arguments as in [4] together with (2.1) and (2.2). q.e.d.

2.3. We have considered so far a complete Riemannian manifold M with a weight function w. In what follows, we shall discuss the case where M is compact and derive several estimates for the heat kernel p_w and also for the eigenvalues and eigenfunctions of $\mathscr{L}_w$. Let $\{\lambda_\nu : 0=\lambda_0<\lambda_1\leq\lambda_2\leq\cdots\nearrow\infty\}$ be the eigenvalues of $\mathscr{L}_w$ and $\{u_\nu\}$ a complete orthonormal system of $L^2(M, \mu_w)$ consisting of eigenfunctions with u_ν having eigenvalue λ_ν. Noting (2.1) and (2.2), and then applying the same methods as in Li and Yau [13] and also Cheng [6], we are able to verify the following:

PROPOSITION 2.7. *In the above notation and under the assumption* (2.3), *the following assertions hold*:

(i) *An eigenfunction u of $\mathscr{L}_w$ with eigenvalue λ satisfies*

$$|\nabla u|^2\leq 4\left\{\frac{(n+k-1)\beta\lambda}{\beta-1}+(n+k-1)^2\kappa^2\right\}(\beta\sup|u|-u)^2$$

for any $\beta>1$.

(ii) *The first nonzero eigenvalue λ_1 enjoys*

$$\lambda_1\geq\frac{\exp[-1-\sqrt{1+4(n+k-1)^2\kappa^2\operatorname{diam}(M)^2}]}{2(n+k-1)\operatorname{diam}(M)^2} .$$

(iii) *The ν-th eigenvalue λ_ν satisfies*

$$\lambda_\nu\leq C\kappa^2+C'\frac{\nu^2}{\operatorname{diam}(M)^2}$$

for some positive constants C and C' depending only on $n+k$.

Let us now assume, in addition to (2.3), that the diameter $\operatorname{diam}(M)$ of M satisfies

$$\operatorname{diam}(M)\leq D \tag{2.5}$$

for a constant $D>0$, and further the measure μ has unit total mass

$$\mu_w(M)=1 . \tag{2.6}$$

Then the same arguments as in [3: Theorem 3] yield the following estimates:

$$\sum_{\nu=1}^{\infty}\lambda_\nu^\alpha e^{-\lambda_\nu t}|u_\nu(x)||u_\nu(y)|\leq C_1(\alpha)t^{-\alpha-(n+k)/2} \tag{2.7}$$

for any $\alpha\geq 0$ and for all $t>0$ and $x, y\in M$, where $C_1(\alpha)$ is a positive constant depending only on $n+k$, D, κ and α;

$$\lambda_\nu \geq C_2 \nu^{2/(n+k)}\,; \tag{2.8}$$

$$u_\nu^2 \leq C_3\left(\kappa^2 + \frac{\nu^2}{\operatorname{diam}(M)^2}\right)^{(n+k)/2}, \tag{2.9}$$

where C_2 and C_3 are some positive constants depending only on $n+k$, D and κ. In particular, letting $\alpha = 1$ in (2.7), we have

$$\left|\frac{\partial}{\partial t} p_w(t, x, y)\right| \leq C_4 t^{-(2+n+k)/2} \qquad (C_4 = C_1(1)) \tag{2.10}$$

for all $t > 0$ and $x, y \in M$, and hence by Proposition 2.3 (i), we obtain

$$|\nabla_x p_w(t, x, y)| \leq C_5(\kappa^2 + t^{-(1+n+k)/2}) \tag{2.11}$$

for all $t > 0$ and $x, y \in M$, where $C_5 > 0$ is a constant depending only on $n+k$, D and κ. Hence it follows from (2.10) and (2.11) that

$$\begin{aligned} |p_w(s, x, y) - p_w(t, x', y')| \leq{}& C_5(\kappa^2 + t^{-(n+k+1)/2})(d_M(x, x') + d_M(y, y')) \\ &+ \frac{2C_4}{n+k}\,|t^{-(n+k)/2} - s^{-(n+k)/2}|\,. \end{aligned} \tag{2.12}$$

Now rescaling what we have obtained in (2.7), (2.8), (2.9) and (2.12), we can deduce the following:

PROPOSITION 2.8. *Let M be a compact Riemannian manifold of dimension n and w a positive smooth function on M satisfying* (2.3), (2.5) *and* (2.6) *for some $k \geq 0$, $\kappa > 0$ and $D > 0$. Let $p_w(t, x, y)$ be the heat kernel of $\mathscr{L}_w$, $\{\lambda_\nu\}_{\nu=0}^\infty$ the eigenvalues of $\mathscr{L}_w$ written in increasing order and repeated according to multiplicity, and $\{u_\nu\}$ a complete orthonormal system in $L^2(M, \mu_w)$ of eigenfunctions with u_ν having eigenvalue λ_ν. Then*

(i) *for any $\alpha \geq 0$ and for all $t > 0$ and $x, y \in M$,*

$$\sum_{\nu=1}^\infty \lambda_\nu^\alpha e^{-\lambda_\nu t} |u_\nu(x)|\,|u_\nu(y)| \leq C(\alpha) d^{n+k} t^{-\alpha-(n+k)/2},$$

where $C(\alpha)$ is a positive constant depending only on α, $n+k$ and κD;

(ii) *for all $s, t > 0$ and $x, x', y, y' \in M$,*

$$\begin{aligned} &|p_w(s, x, y) - p_w(t, x', y')| \\ &\quad \leq C'd\left(d^2\kappa^2 + \left(\frac{d}{\sqrt{s}}\right)^{n+k+1} + \left(\frac{d}{\sqrt{t}}\right)^{n+k+1}\right)(d_M(x, x') + d_M(y, y')) \\ &\qquad + C''\left|\left(\frac{d}{\sqrt{s}}\right)^{n+k} - \left(\frac{d}{\sqrt{t}}\right)^{n+k}\right|; \end{aligned}$$

(iii) *for all* $\nu \geq 1$,

$$\lambda_\nu \geq C^{(3)} d^{-2} \nu^{2/(n+k)} ;$$

(iv) *for all* $\nu \geq 1$ *and* $x \in M$,

$$u_\nu(x)^2 \leq C^{(4)} (d^2 \kappa^2 + \nu^2)^{(n+k)/2} .$$

Here we put $d = \operatorname{diam}(M)$, *and* C', C'', $C^{(3)}$ *and* $C^{(4)}$ *are all positive constants depending only on* $n+k$ *and* κD.

3. A precompactness theorem. One of Gromov's theorems asserts that the set $\mathscr{I}$ of isometry classes of compact metric spaces of length is complete with respect to the Gromov-Hausdorff distance HD (cf. [10, Chap. 5]). In particular, the completion of the set $\mathscr{M}_c$ of isometry classes of compact Riemannian manifolds endowed with HD can be realized as a subspace of $(\mathscr{I}, \mathrm{HD})$. In this section, we shall restrict our attention to certain subspaces of $\mathscr{M}_{w,c}$, and carry out the completion of them. More precisely, given integers $n > 1$, $k \geq 0$, and constants $D > 0$, $\kappa > 0$, let $\mathscr{M}_w^*(n, k, D, \kappa)$ be the subspace of $\mathscr{M}_{w,c}$ consisting of triads (M, μ_w, p_w) such that the dimension of M is equal to n, and the conditions (2.3), (2.5) and (2.6) are respectively satisfied by these constants. Here the weight function w is assumed to be equal to $1/\mathrm{vol}(M)$ in the case $k = 0$. First of all, we shall show that the projection ρ of $\mathscr{M}_{w,c}$ onto $\mathscr{M}_c$ which sends (M, μ_w, p_w) to M is uniformly continuous on the subspace $\mathscr{M}_w^*(n, k, D, \kappa)$ (cf. Theorem 3.5). Hence the restriction of the projection ρ to this subspace extends uniquely to a uniformly continuous map $\bar{\rho}$ from the completion of the subspace into $\mathscr{I}$. Then Gromov's precompactness theorem says, as mentioned in Corollary 2.2, that the image is precompact. Actually we shall show that the subspace $\mathscr{M}_w^*(n, k, D, \kappa)$ itself is precompact (cf. Theorem 3.6). Finally, we shall be concerned with the boundary elements of $\mathscr{M}_w^*(n, k, D, \kappa)$ in its completion (cf. Theorem 3.8).

3.1. Let us begin with the following:

LEMMA 3.1. *Let* (M, μ_w, p_w) *be an element of* $\mathscr{M}_w^*(n, k, D, \kappa)$. *Then*

$$|4t \log p_w(t, x, y) + d_M(x, y)^2| \leq \varepsilon_1(t)$$

for all $x, y \in M$ *and* $t \in (0, D^2]$, *where* $\varepsilon_1(t)$ *is a positive continuous function on* $(0, D^2]$ *depending only on* $n+k$, D *and* κ *which converges to zero as* t *tends to zero.*

PROOF. By Proposition 2.4, we first have

$$p_w(t, x, y) \leq C_{n+k,\kappa}(\varepsilon) \frac{V_{n+k,\kappa}(D)}{V_{n+k,\kappa}(\sqrt{t})} \exp\left(-(1-\varepsilon) \frac{d_M(x, y)^2}{4t} + \varepsilon t \right)$$

for all $x, y \in M$, $t \in (0, D^2]$ and $\varepsilon > 0$. Let us here choose a positive continuous function $\varepsilon(t)$ on $(0, D^2]$ so that $\varepsilon(t)$ converges to zero as t tends to zero and

$$C_{n+k,\kappa}(\varepsilon(t)) \leq \exp t^{-1/2}$$

for $t \in (0, D^2]$. Then we obtain

$$\begin{aligned} &4t \log p_w(t, x, y) + d_M(x, y)^2 \\ &\quad \leq 4t^{1/2} + \varepsilon(t) d_M(x, y)^2 + 4\varepsilon(t) t^2 + 4t\{\log V_{n+k,\kappa}(D) - \log V_{n+k,\kappa}(\sqrt{t})\} . \end{aligned}$$

Secondly, applying Proposition 2.5, we can deduce that

$$4t \log p_w(t, x, y) + d_M(x, y)^2 \geq -4t^{1/2} - \varepsilon'(t) d_M(x, y)^2 - (4\varepsilon'(t) + (n+k-1)^2\kappa^2) t^2 ,$$

for some positive continuous function $\varepsilon'(t)$ on $(0, \infty)$ chosen in such a way that $\varepsilon'(t)$ converges to zero as t tends to zero and

$$C'_{n+k,\kappa}(\varepsilon'(t)) \geq \exp(-t^{-1/2}) ,$$

where $C'_{n+k,\kappa}(\varepsilon)$ is as in Proposition 2.5. The above two inequalities show the lemma.

q.e.d.

LEMMA 3.2. *Let (M, μ_v, p_v) and (N, μ_w, p_w) be two triads of $\mathcal{M}^*_w(n, k, D, \kappa)$. Suppose there are a mapping $f: A \to N$ of a subset A of M into N and a positive number r satisfying*

$$e^{-(t+1/t)} | p_v(t, x, y) - p_w(t, f(x), f(y)) | \leq r$$

for all x, y of A and $t > 0$. Then

$$| d_M(x, y) - d_N(f(x), f(y)) | \leq \varepsilon_2(r)$$

for all x, y of A, where $\varepsilon_2(r)$ is a continuous increasing function on $(0, \infty)$ with $\lim_{t\to 0} \varepsilon_2(t) = 0$, which depends only on $n+k$, D and κ.

PROOF. We first notice that

$$\begin{aligned} (3.1) \qquad &| \log p_w(t, f(x), f(y)) - \log p_v(t, x, y) | \\ &\quad \leq C_1 \exp\left(\frac{C_2}{t} + C_3 t\right) | p_w(t, f(x), f(y)) - p_v(t, x, y) | \end{aligned}$$

for all $x, y \in A$ and $t \in (0, \infty)$, because

$$\min\{p_w(t, f(x), f(y)), p_v(t, x, y)\} \geq \frac{1}{C_1} \exp\left(-\frac{C_2}{t} - C_3 t\right)$$

by Proposition 2.5, where C_1, C_2 and C_3 are positive constants depending only on $n+k$, D and κ. In view of the identity

$$\begin{aligned} &d_M(x, y)^2 - d_N(f(x), f(y))^2 \\ &\quad = \{d_M(x, y)^2 + 4t \log p_v(t, x, y)\} - \{d_N(f(x), f(y))^2 + 4t \log p_w(t, f(x), f(y))\} \\ &\qquad + 4t\{\log p_w(t, f(x), f(y)) - \log p_v(t, x, y)\} , \end{aligned}$$

it follows from Lemma 3.1, Proposition 2.1, and (3.1) that

$$(3.2)\qquad |d_M(x, y)^2 - d_N(f(x), f(y))^2| \leq 8\varepsilon_1(t) + 4C_1 rt \exp\left(\frac{C_2+1}{t} + (C_3+1)t\right)$$

for all $x, y \in A$ and $t \in (0, D^2]$. Let us choose a continuous function $t(r)$ with $t(0)=0$ and a positive constant C_4 depending only on $n+k$, D and κ in such a way that $t(r) \leq D^2$ and

$$4C_1 t(r) \exp\left(\frac{C_2+1}{t(r)} + C_3 t(r)\right) \leq r^{-1/2} + C_4 .$$

Then substituting $t(r)$ into t on the right-hand side of (3.2), and defining $\varepsilon_2(r)$ by

$$\varepsilon_2(r)^2 = 8\varepsilon_1(t(r)) + r^{1/2} + C_4 r ,$$

we obtain

$$|d_M(x, y)^2 - d_N(f(x), f(y))^2| \leq \varepsilon_2(r)^2$$

for all $x, y \in A$, which implies

$$|d_M(x, y) - d_N(f(x), f(y))| \leq \varepsilon_2(r) .$$

This shows the lemma. q.e.d.

LEMMA 3.3. *Let (M, μ_v, p_v) and (N, μ_w, p_w) be elements of $\mathcal{M}^*_w(n, k, D, \kappa)$ and let A be a δ-dense subset of M. Suppose there is a map $f: A \to N$ from A into N such that*

(i) *$f(A)$ is δ'-dense in N for some $\delta' > 0$;*

(ii) $e^{-(t+1/t)}|p_v(t, x, y) - p_w(t, f(x), f(y))| < r$

for some $r > 0$, and for all $t > 0$ and $x, y \in A$. Then there are r'-spectral approximations $\bar{f}: M \to N$ and $\bar{h}: N \to M$ between (M, μ_v, p_v) and (N, μ_w, p_w)) satisfying

$$d_M(x, \bar{h} \circ \bar{f}(x)) < \delta + \varepsilon_2(r) , \qquad d_N(a, \bar{f} \circ \bar{h}(a)) < \delta'$$

for all $x \in M$ and $a \in N$, where $r' = r + C_5(\delta + \delta')$, $\varepsilon_2(r)$ is as in Lemma 3.2, and C_5 is a positive constant depending only on $n+k$, D and κ.

Here a subset A of a metric space X is said to be δ-dense if $d_X(x, A) < \delta$ for any $x \in X$.

PROOF OF LEMMA 3.3. From the second assumption (ii) in this lemma and Lemma 3.2, we can deduce that

$$|d_M(x, y) - d_N(f(x), f(y))| \leq \varepsilon_2(r)$$

for all $x, y \in A$. Since A is δ-dense in M, let us take a map $\zeta: M \to A$ of M into A such that $d_M(\zeta(x), x) \leq \delta$ for all $x \in M$, and extend the map f to a map $\bar{f}: M \to N$ by setting $\bar{f} = f \circ \zeta$. Then it follows from Proposition 2.8 (ii) that

$$e^{-(t+1/t)}|p_v(t, x, y) - p_w(t, \bar{f}(x), \bar{f}(y))| < r + C_6 \delta$$

for all $t>0$ and $x, y \in M$, and for some constant $C_6>0$ depending only on $n+k$, D and κ. Now we choose a map $h: f(A) \to M$ from the image of f into M in such a way that $f \circ h(a) = a$ for all $a \in f(A)$, and further define a map $\bar{h}$: $N \to M$ by $\bar{h} = h \circ \eta$, where η is a map from N into $f(A)$ satisfying $d_N(\eta(a), a) \leq \delta'$ for all $a \in N$. Then it is easy to see that these maps $\bar{f}$: $M \to N$ and $\bar{h}$: $N \to M$ give r'-spectral approximations. q.e.d.

The following lemma is an easy consequence of the theorem by Gromov mentioned at the beginning of this section and so we omit the proof.

LEMMA 3.4. *Let K be a relatively compact subset of $(\mathscr{I}, \mathrm{HD})$. Given $\varepsilon>0$, there exists $\delta>0$ such that for any pair of elements $X, Y \in K$, and for every pair of maps $f: X \to Y$ and h: $Y \to X$, the image of f and that of h are respectively ε-dense in Y and X, provided that they satisfy respectively*

$$|d_X(x, y) - d_Y(f(x), f(y))| < \delta$$

for all $x, y \in X$, and

$$|d_X(h(a), h(b)) - d_Y(a, b)| < \delta$$

for all $a, b \in Y$.

Now the following is clear from the above Lemmas 3.1–3.4.

THEOREM 3.5. *The natural projection of $(\mathscr{M}_{w,c}, \mathrm{SD})$ onto $(\mathscr{M}_c, \mathrm{HD})$ which sends (M, μ_w, p_w) to M is continuous uniformly on the subspace $\mathscr{M}^*_w(n, k, D, \kappa)$ for given integers $n>1$, $k \geq 0$, constants $D>0$ and $\kappa>0$.*

3.2. We are now in a position to prove the following:

THEOREM 3.6. *For given integers $n>1$, $k \geq 0$, constants $D>0$ and $\kappa>0$, the subspace $\mathscr{M}^*_w(n, k, D, \kappa)$ of the uniform space $(\mathscr{M}_{w,c}, \mathrm{SD})$ is precompact.*

PROOF. Let $\{(M_i, \mu_{w_i}, p_{w_i})\}_{i=1,2,\ldots}$ be a sequence in $\mathscr{M}^*_w(n, k, D, \kappa)$. We would like to show that it contains an SD-Cauchy subsequence. Taking Corollary 2.2 into account, we may assume that M_i converges to a compact metric space X with respect to the Gromov-Hausdorff distance, namely, there are $r(i)$-Hausdorff approximations f_i: $M_i \to X$ and h_i: $X \to M_i$ between M_i and X with $r(i) \to 0$. Let us consider a sequence $\{q_i\}$ of functions on $(0, \infty) \times X \times X$ defined by $q_i(t, a, b) = p_{w_i}(t, h_i(a), h_i(b))$. Observe first from Proposition 2.8 (ii) that

$$|q_i(s, a, b) - q_i(t, a', b')| \leq C_7 d_i \left(d_i^2 \kappa^2 + \left(\frac{d_i}{\sqrt{s}} \right)^{n+k+1} + \left(\frac{d_i}{\sqrt{t}} \right)^{n+k+1} \right)$$
$$\times \{d_X(a, a') + d_X(b, b') + 2r(i)\} + C_7 \left| \left(\frac{d_i}{\sqrt{s}} \right)^{n+k} - \left(\frac{d_i}{\sqrt{t}} \right)^{n+k} \right|$$

for all $s, t>0$ and $a, b, a', b' \in X$, where d_i stands for the diameter of M_i and C_7 is a positive constant depending only on $n+k$ and κD. We now prove that the sequence $\{q_i\}$ contains a subsequence which converges to a positive Lipschitz function $p(t, a, b)$ on $(0, \infty)\times X\times X$. For this, we choose a family of finite subsets A_ν $(\nu=1, 2, \dots)$ of X such that A_ν is $(1/\nu)$-dense in X. Then $\{q_i\}$ is uniformly bounded and equicontinuous on $[1/\nu, \nu]\times A_\nu\times A_\nu$ for each ν. Hence we have a subsequence $\{i_\nu\}$ such that q_{i_ν} converges to a function on $[1/\nu, \nu]\times A_\nu\times A_\nu$ as i_ν tends to infinity. Here we may assume that $\{i_{\nu+1}\}\subset\{i_\nu\}$. Then by the diagonal argument, we can assert that there exists a subsequence $\{q_j\}$ of $\{q_i\}$ which converges to a function p on $(0, \infty)\times A\times A$ as j tends to infinity, where we set $A=\bigcup_{\nu=1}^{\infty} A_\nu$. This function p clearly satisfies

$$(3.3)\qquad |p(s, a, b)-q_i(t, a', b')|\le C_7 d\left(d^2\kappa^2+\left(\frac{d}{\sqrt{s}}\right)^{n+k+1}+\left(\frac{d}{\sqrt{t}}\right)^{n+k+1}\right)$$
$$\times\{d_X(a, a')+d_X(b, b')\}+C_7\left|\left(\frac{d}{\sqrt{s}}\right)^{n+k}-\left(\frac{d}{\sqrt{t}}\right)^{n+k}\right|$$

for all $s, t>0$ and $a, b, a', b' \in A$, where we set $d=\mathrm{diam}(X)$. Since A is dense in X, the function p extends uniquely to a Lipschitz function on $(0, \infty)\times X\times X$, which will be denoted by the same letter p. It is easy to see that $q_j(t, a, b)$ converges to $p(t, a, b)$ for every $(t, a, b)\in(0, \infty)\times X\times X$. Hence p satisfies (3.3) for all $s, t>0$ and $a, b, a', b' \in X$; in particular

$$(3.4)\qquad |p(t, a, b)-1|\le C_7 d^2\left(d^2\kappa^2+\left(\frac{d}{\sqrt{t}}\right)^{n+k+1}\right)+C_7\left(\frac{d}{\sqrt{t}}\right)^{n+k}$$

for all $t>0$ and $a, b\in X$. We claim that

$$(3.5)\qquad e^{-(t+1/t)}|q_j(t, a, b)-p(t, a, b)|\le r'(j)\,,$$

for all $(t, a, b)\in(0, \infty)\times X\times X$, where $\{r'(j)\}$ is a sequence of positive numbers which goes to zero as j tends to infinity. Indeed, suppose to the contrary that there exist a positive constant ρ, a sequence $\{t_j\}$ of positive numbers, and families of points $\{a_j\}$ and $\{b_j\}$ of X for which the following inequality holds:

$$e^{-(t_j+1/t_j)}|q_j(t_j, a_j, b_j)-p(t_j, a_j, b_j)|\ge\rho>0\,.$$

Then by (3.4), we may assume that t_j converges to a number $\tau>0$, and further both a_j and b_j converge respectively to some points a and b of X, since X is compact. Hence we see that the left-hand side of the above inequality tends to zero as j goes to infinity. This is absurd. Thus our claim (3.5) is verified.

Now it is not hard to see that $\{(M_j, \mu_{w_j}, p_{w_j})\}$ is an SD-Cauchy sequence. Indeed, for any (j, k) and for every $(t, x, y)\in(0, \infty)\times M_j\times M_j$, we have

$$
\begin{aligned}
&e^{-(t+1/t)}|p_{w_j}(t,x,y)-p_{w_k}(t,h_k\circ f_j(x),h_k\circ f_j(y))|\\
&\quad\le e^{-(t+1/t)}|p_{w_j}(t,x,y)-p_{w_j}(t,h_j\circ f_j(x),h_j\circ f_j(y))|\\
&\qquad+e^{-(t+1/t)}|p_{w_j}(t,h_j\circ f_j(x),h_j\circ f_j(y))-p(t,f_j(x),f_j(y))|\\
&\qquad+e^{-(t+1/t)}|p_{w_j}(t,f_j(x),f_j(y))-p_{w_k}(t,h_k\circ f_j(x),h_k\circ f_j(y))|\\
&\quad\le C_7e^{-(t+1/t)}d_j\left(d_j^2\kappa^2+2\left(\frac{d_j}{\sqrt{t}}\right)^{n+k}\right)\\
&\qquad\times\{d_{M_j}(x,h_j\circ f_j(x))+d_{M_j}(y,h_j\circ f_j(y))\}+r'(j)+r'(k)\\
&\quad\le C_8r(j)+r'(j)+r'(k)\,,
\end{aligned}
$$

where we have applied Proposition 2.8 (ii) and put

$$C_8=C_7\max_{t>0}e^{-(t+1/t)}D\left(D^2\kappa^2+2\left(\frac{D}{\sqrt{t}}\right)^{n+k}\right).$$

This implies that $h_k\circ f_j\colon M_j\to M_k$ and $h_j\circ f_k\colon M_k\to M_j$ are $r(j,k)$-spectral approximations with $r(j,k)$ converging to zero as $j,k\to\infty$. Thus the proof of Theorem 3.6 is completed. q.e.d.

3.3. Let $\{(M_i,\mu_{w_i},p_{w_i})\}_{i=1,2,\dots}$ be an SD-Cauchy sequence in $\mathscr{M}_w^*(n,k,D,\kappa)$. In the rest of this section, we shall describe the limit element in the completion. By virtue of Theorem 3.5, we see that M_i converges, as i tends to infinity, to a compact metric space X in the topology of the Gromov-Hausdorff distance. Moreover, it turns out from the argument in the proof of Theorem 3.6 that there exist a Lipschitz function $p(t,a,b)$ on $(0,\infty)\times X\times X$ and $r(i)$-Hausdorff approximations $f_i\colon M_i\to X$ and $h_i\colon X\to M_i$, with $r(i)$ converging to zero as $i\to\infty$, which enjoy

$$
\begin{aligned}
&e^{-(t+1/t)}|p_{w_i}(t,x,y)-p(t,f_i(x),f_i(y))|\le r(i)\\
&e^{-(t+1/t)}|p_{w_i}(t,h_i(a),h_i(b))-p(t,a,b)|\le r(i)
\end{aligned}\tag{3.6}
$$

for all $t>0$, $x,y\in M_i$, and $a,b\in X$;

$$d_{M_i}(x,h_i\circ f_i(x))\le r(i)\,,\qquad d_X(a,f_i\circ h_i(a))\le r(i)\tag{3.7}$$

for all $x\in M_i$ and $a\in X$. In particular, in addition to (3.3) and (3.4), p satisfies

$$p(t,a,b)\le C_{n+k,\kappa}(\varepsilon)\frac{V_{n+k,\kappa}(D)}{V_{n+k,\kappa}(\sqrt{t})}\exp\left(-(1-\varepsilon)\frac{d_X(a,b)^2}{4t}+\varepsilon t\right)\tag{3.8}$$

for all $t\in(0,D^2]$, $\varepsilon>0$ and $a,b\in X$, where $C_{n+k,\kappa}(\varepsilon)$ is as in Proposition 2.4.

Now without loss of generality, we may assume that the above maps f_i and h_i are all Borel measurable. As a result, we have a family of the push-forward measures $f_{i*}\mu_{w_i}$ on X. Each $f_{i*}\mu_{w_i}$ has unit total mass. We claim the following:

LEMMA 3.7. *There is a measure μ on X such that as i tends to infinity, (M_i, μ_{w_i}) converges to (X, μ) with respect to the measured Hausdorff topology in the sense of* [9], *namely*

(3.9) $$f_{i*}\mu_{w_i} \to \mu \quad \text{in the weak* topology}.$$

The proof of this lemma will be postponed until the end of this section. Let us continue the arguments. We observe that the limit measure μ also has unit total mass, and further it satisfies

$$\frac{\mu(B_a(r))}{\mu(B_a(R))} \geq \frac{V_{n+k,\kappa}(r)}{V_{n+k,\kappa}(R)}$$

for all r, R with $0 < r \leq R$, and $a \in X$, because of Proposition 2.1 and the fact that

$$\mu(B_a(r)) = \lim_{i\to\infty} \mu_{w_i}(f_i^{-1}(B_a(r))) = \lim_{i\to\infty} \mu_{w_i}(B_{h_i(a)}(r)).$$

As a result, $\mu(\Omega) > 0$ for any open set Ω in X, which implies in particular that the space $C(X)$ of continuous functions on X is dense in $L^2(X, \mu)$. Moreover we see that p has the same bounds as in Propositions 2.4 and 2.5, namely,

$$p(t, a, b) \leq C_{n+k,\kappa}(\varepsilon)\mu(B_a(\sqrt{t}))^{-1/2}\mu(B_b(\sqrt{t}))^{-1/2}\exp\left(-(1-\varepsilon)\frac{d_X(a, b)^2}{4t} + \varepsilon t\right);$$

$$p(t, a, b) \geq C'_{n+k,\kappa}(\varepsilon)\mu(B_a(\sqrt{t}))^{-1/2}\mu(B_b(\sqrt{t}))^{-1/2}\exp\left(-(1+\varepsilon)\frac{d_X(a, b)^2}{4t} - (\varepsilon + C'')t\right)$$

for any $\varepsilon > 0$ and for all $t > 0$ and $a, b \in X$, where $C'' = (n+k+1)^2\kappa^2/4$.

Now we can deduce from (3.3), (3.4), (3.6), and (3.7) that

(3.10) $$\int_X p(t, a, b)d\mu(b) = 1,$$

$$\int_X p(t, a, c)p(s, c, b)d\mu(c) = p(t+s, a, b).$$

Thus if we set

$$T_t(\psi)(a) = \int_X p(t, a, b)\psi(b)d\mu(b)$$

for $\psi \in L^2(X, \mu)$, then $\{T_t\}$ is a symmetric Markov semigroup on $L^2(X, \mu)$. Moreover, we can show that for any continuous function ψ on X, $T_t(\psi)$ converges to ψ uniformly on X as $t \to 0$:

(3.11) $$\lim_{t\to 0} \|T_t(\psi) - \psi\|_\infty = 0.$$

Indeed, given $\varepsilon>0$, we first take a positive number δ such that $|\psi(a)-\psi(b)|\leq\varepsilon$ if $d_X(a, b)\leq\delta$. Secondly, by virtue of (3.8), we can choose a positive number T in such a way that $p(t, a, b)\leq\varepsilon$ for any $t\leq T$ and all $a, b\in X$ with $d_X(a, b)\geq\delta$. Therefore we see that for every $a\in X$,

$$\begin{aligned}|T_t(\psi)(a)-\psi(a)|&=|T_t(\psi-\psi(a))|\\&\leq\int_{B_a(\delta)}p(t, a, b)|\psi(b)-\psi(a)|\,d\mu(b)+\int_{X\setminus B_a(\delta)}p(t, a, b)|\psi(b)-\psi(a)|\,d\mu(b)\\&\leq\varepsilon(1+2\|\psi\|_\infty)\,.\end{aligned}$$

Thus (3.11) is verified. As a consequence, we see that $\{T_t\}$ is a strongly continuous semigroup with kernel p on $L^2(X, \mu)$. Let us denote by $\mathscr{L}_p$ the infinitesimal generator of $\{T_t\}$. Then $\mathscr{L}_p$ has the eigenvalues $\{\lambda_\nu\}_{\nu=0}^\infty$ written in increasing order and repeated according to multiplicity. Let $\Phi=\{u_\nu\}$ be a complete orthonormal system in $L^2(X, \mu)$, which consists of eigenfunctions of $\mathscr{L}_p$ with u_ν having eigenvalue λ_ν. Then the kernel p has the eigenfunction expansion

$$p(t, a, b)=\sum_{\nu=0}^{\infty}e^{-\lambda_\nu t}u_\nu(a)u_\nu(b)\,.$$

Thus we are allowed to use the notion of r-spectral approximations between two elements in the completion of the uniform space $\mathscr{M}_w^*(n, k, D, \kappa)$ for given constants n, k, D and κ as before.

Summing up what we have observed so far, we have the following:

THEOREM 3.8. *The limit element of a sequence $\{\tau_i=(M_i, \mu_{w_i}, p_{w_i})\}$ in the uniform space $\mathscr{M}_w^*(n, k, D, \kappa)$ can be regarded as a triad $\tau=(X, \mu, p)$ which consists of a compact metric space X of length, a Radon measure μ of unit total mass on X, and a positive Lipschitz function $p(t, a, b)$ on $(0, \infty)\times X\times X$ such that*

(i) *X is the limit of $\{M_i\}$ with respect to the Gromov-Hausdorff distance;*

(ii) *μ satisfies*

$$\frac{\mu(B_a(r))}{\mu(B_a(R))}\geq\frac{V_{n+k,\kappa}(r)}{V_{n+k,\kappa}(R)}$$

for $a\in X$, $0<r\leq R$;

(iii) *p is the heat kernel of a strongly continuous semigroup on $L^2(X, \mu)$ enjoying*

$$p(t, a, b)\leq C_{n+k,\kappa}(\varepsilon)\mu(B_a(\sqrt{t}))^{-1/2}\mu(B_b(\sqrt{t}))^{-1/2}\exp\left(-(1-\varepsilon)\frac{d_X(a, b)^2}{4t}+\varepsilon t\right);$$

$$p(t, a, b)\geq C'_{n+k,\kappa}(\varepsilon)\mu(B_a(\sqrt{t}))^{-1/2}\mu(B_b(\sqrt{t}))^{-1/2}\exp\left(-(1+\varepsilon)\frac{d_X(a, b)^2}{4t}-(\varepsilon+C'')t\right)$$

for any $\varepsilon>0$ *and for all* $t>0$ *and* $a, b\in X$, *where* $C''=(n+k+1)^2\kappa^2/4$. *In addition, p satisfies*

$$|p(s, a, b)-p(t, a', b')|\leq C_7 d\left(d^2\kappa^2+\left(\frac{d}{\sqrt{s}}\right)^{n+k+1}+\left(\frac{d}{\sqrt{t}}\right)^{n+k+1}\right)$$
$$\times\{d_X(a, a')+d_X(b, b')\}+C_7\left|\left(\frac{d}{\sqrt{s}}\right)^{n+k}-\left(\frac{d}{\sqrt{t}}\right)^{n+k}\right|$$

for all $s, t>0$ *and* $a, b, a', b'\in X$, *where we set* $d=\mathrm{diam}(X)$. *Furthermore, there exist Borel measurable* $r(i)$*-spectral approximations* $f_i\colon M_i\to X$ *and* $h_i\colon X\to M_i$ *between* τ_i *and* τ *satisfying* (3.7) *and* (3.9).

PROOF OF LEMMA 3.7. Suppose there exist two subsequences, say $\{f_{j*}\mu_j\}$ and $\{f_{k*}\mu_k\}$, of $\{f_{i*}\mu_i\}$, which converges respectively to measures μ and μ' in the weak* topology. Then for any $\psi\in C(X)$, we have

$$\begin{aligned}\int_X \psi(a)d\mu(a)&=\lim_{t\to 0}\iint_{X\times X} p(t, a, b)\psi(b)d\mu'(b)d\mu(a)\\&=\lim_{t\to 0}\iint_{X\times X} p(t, a, b)d\mu(a)\psi(b)d\mu'(b)\\&=\int_X \psi(b)d\mu'(b)\,.\end{aligned}$$

This shows that $\mu=\mu'$. q.e.d.

4. Convergence of eigenvalues and eigenfunctions. Bérard, Besson and Gallot [3] defined a family of spectral distances on the set of compact Riemannian manifolds by embedding them into the same Hilbert space, the space of real-valued, square integrable series. The embedding is built by means of the heat kernels of the manifolds, and it is proved that the set $\mathcal{M}_w^*(n, 0, D, \kappa)$ is precompact with respect to each of the spectral distances in their sense. Their distances are, however, different from ours. For instance, as we have seen in Theorem 3.5, the spectral distance SD in our sense is closely related to the Gromov-Hausdorff distance. Moreover taking the Sturm-Liouville decomposition of the heat kernel into account, we may consider a point of a compact Riemannian manifold endowed with a measure as *a curve* in the Hilbert space. This is our point of view.

In this section we shall first define a distance on the set of equivalence classes of elements (M, μ_w, p_w, Φ) where $(M, \mu_w, p_w)\in\mathcal{M}_{w,c}$ with $\mu_w(M)=1$ and $\Phi=\{u_\nu\}_{\nu=0,1,\dots}$ is a complete orthonormal system in $L^2(M, \mu_w)$ consisting of the eigenfunctions of $\mathcal{L}_w$, and discuss its properties in relations with the spectral distance SD and the Gromov-Hausdorff distance (cf. Theorems 4.1 and 4.2). Secondly, we shall show that when a sequence $\{\tau_i=(M_i, \mu_{w_i}, p_{w_i})\}$ in $\mathcal{M}_w^*(n, k, D, \kappa)$ converges to an element $\tau=(X, \mu, p)$, the ν-th eigenvalue of τ_i tends to that of τ and moreover the eigenfunctions of τ_i also converge to those of τ in a certain sense (cf. Theorem 4.6).

4.1. Let us begin with defining two Hilbert spaces l_2 and h_1 by

$$l_2=\left\{(a_\nu)_{\nu=1,2,\ldots}:\sum_{\nu=1}^{\infty}a_\nu^2<+\infty\right\}$$

$$h_1=\left\{(a_\nu)_{\nu=1,2,\ldots}:\sum_{\nu=1}^{\infty}(1+\nu^2)a_\nu^2<+\infty\right\}.$$

We remark that the embedding $h_1\to l_2$ is a compact operator. Let us consider the space $C_\infty([0,\infty),l_2)$ of continuous curves $\gamma:[0,\infty)\to l_2$ such that the l_2-norm $|\gamma(t)|_{l_2}$ of $\gamma(t)$ tends to zero as $t\to\infty$. This space is endowed with the distance

$$d_\infty(\gamma,\sigma):=\sup_{t>0}|\gamma(t)-\sigma(t)|_{l_2}.$$

For any subset A of $C_\infty([0,\infty),l_2)$ and a positive constant r, $\mathcal{N}_r(A)$ stands for the r-neighborhood of A, namely, $\mathcal{N}_r(A):=\{\gamma\in C_\infty([0,\infty),l_2):d_\infty(A,\gamma)<r\}$, and the Hausdorff distance δ_H on the set of bounded closed subsets of the metric space $C_\infty([0,\infty),l_2)$ is defined by

$$\delta_H(A,B)=\inf\{r>0:A\subset\mathcal{N}_r(B),\ B\subset\mathcal{N}_r(A)\}.$$

Given a positive constant C and a nonnegative continuous function $\eta(t)$ $(t\geq 0)$ which tends to zero at $t\to\infty$, if we set

$$K(C,\eta):=\{\gamma\in C_\infty([0,\infty),l_2):|\gamma(t)|_{h_1}\leq\eta(t)\quad\text{for all}\quad t\geq 0,$$
$$|\gamma(t)-\gamma(s)|_{l_2}\leq C|t-s|\quad\text{for all}\quad t,s\geq 0\},$$

then it is easy to see that $K(C,\eta)$ equipped with the distance d_∞ becomes a compact metric space. Here we recall a well-known fact that the set of closed subsets of a compact metric space is compact with respect to the Hausdorff distance (cf. Federer [8, p. 183]).

4.2. Let M be a compact Riemannian manifold of dimension n and w a positive smooth function on M which gives a positive Radon measure $\mu_w=w\,d\mathrm{vol}_M$ of unit mass: $\mu_w(M)=1$. Using the eigenvalues and the eigenfunctions of the operator $\mathcal{L}_w$, we embed M into the metric space $C_\infty([0,\infty),l_2)$ as follows: Let $0=\lambda_0<\lambda_1\leq\lambda_2\leq\cdots$ be the eigenvalues of $\mathcal{L}_w$ and $\Phi=\{u_\nu\}$ a complete orthonormal system of eigenfunctions of $\mathcal{L}_w$ in $L^2(M,\mu_w)$ with u_ν having eigenvalue λ_ν. For a point x of M, we define an element $F_\Phi[x]$ of $C_\infty([0,\infty),l_2)$ by

$$F_\Phi[x](t)=(e^{-(t+1/t)/2}e^{-\lambda_\nu t/2}u_\nu(x))_{\nu=1,2,\ldots}.$$

Then it turns out that the map F_Φ of M into $C_\infty([0,\infty),l_2)$ given by $x\mapsto F_\Phi[x]$ gives rise to a continuous imbedding of M. Indeed, the injectivity of the map follows from the fact that the eigenfunctions of $\mathcal{L}_w$ separate the points of M. We observe that for all $x,y\in M$ and $t>0$,

$$\langle F_\Phi[x](t), F_\Phi[y](t)\rangle_{l_2} = e^{-(t+1/t)}(p_w(t,x,y)-1)$$

$$d_\infty(F_\Phi[x], F_\Phi[y])^2 = \sup_{t>0} e^{-(t+1/t)}(p_w(t,x,x)+p_w(t,y,y)-2p_w(t,x,y)),$$

where $p_w(t, x, y)$ stands as before for the heat kernel of the operator $\mathscr{L}_w$ on $L^2(M, \mu_w)$. If we set

$$\Theta_{M,w}(x,y) = \left\{\sup_{t>0} e^{-(t+1/t)}(p_w(t,x,x)+p_w(t,y,y)-2p_w(t,x,y))\right\}^{1/2}$$

for $x, y \in M$, then we have a distance $\Theta_{M,w}$ on M which induces the same topology of M. This distance will play an important role when we investigate a class of compact Riemannian manifolds endowed with measures such that no uniform lower bound for the modified Ricci tensors $R_{w,k}$ exists. This topic will be discussed elsewhere.

Let us now define a pseudo-distance on the set of elements $(M, \mu_w, p_w, \Phi = \{u_\nu\})$ as above by

$$\mathrm{SD}^*(\alpha, \beta) = \delta_H(F_\Phi[M], F_\Psi[N])$$

for $\alpha = (M, \mu_v, p_v, \Phi = \{u_\nu\})$ and $\beta = (N, \mu_w, p_w, \Psi = \{v_\nu\})$. Notice that

$$\mathrm{SD}^*(\alpha, \beta) < r$$

if and only if there exist not necessarily continuous maps $f: M \to N$ and $h: N \to M$ such that

$$e^{-(t+1/t)} \sum_{\nu=1}^{\infty} |e^{-\lambda_\nu t/2} u_\nu(x) - e^{-\rho_\nu t/2} v_\nu(f(x))|^2 < r^2 \tag{4.1}$$

for all $t > 0$ and $x \in M$;

$$e^{-(t+1/t)} \sum_{\nu=1}^{\infty} |e^{-\lambda_\nu t/2} u_\nu(h(a)) - e^{-\rho_\nu t/2} v_\nu(a)|^2 < r^2 \tag{4.2}$$

for all $t > 0$ and $a \in N$. Here $\{\lambda_\nu : 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots\}$ and $\{\rho_\nu : 0 = \rho_0 < \rho_1 \le \rho_2 \le \cdots\}$ are respectively the eigenvalues of $\mathscr{L}_v$ and $\mathscr{L}_w$. In particular, $\mathrm{SD}^*(\alpha, \beta) = 0$ if and only if there is an isometry $f: M \to N$ between M and N so that $f^*w = v$ and $f^*v_\nu = u_\nu$ for all $\nu = 1, 2, 3, \ldots$. In what follows, we identify such elements and denote by $\mathscr{FM}_{w,c}$, the set of equivalence classes of elements $(M, \mu_w, p_w, \Phi = \{u_\nu\})$ equipped with the distance SD^*. Let $\alpha = (M, \mu_v, p_v, \Phi = \{u_\nu\})$ and $\beta = (N, \mu_w, p_w, \Psi = \{v_\nu\})$ be two elements of $\mathscr{FM}_{w,c}$ such that $\mathrm{SD}^*(\alpha, \beta) < r$, and then take maps $f: M \to N$ and $h: N \to M$ satisfying (4.1) and (4.2), respectively. Let $p_v(t, x, y)$ (resp. $p_w(t, a, b)$) be the heat kernel of $\mathscr{L}_v$ on M (resp. $\mathscr{L}_w$ on N). Since

$$e^{-(t+1/t)}\{p_v(t,x,y) - p_w(t,f(x),f(y))\} = \langle F_\Phi[x](t) - F_\Psi[f(x)](t), F_\Phi[y](t)\rangle_{l_2} + \langle F_\Psi[f(x)](t), F_\Phi[y](t) - F_\Psi[f(y)](t)\rangle_{l_2},$$

it follows that for all $x, y \in M$ and $t>0$,

$$e^{-(t+1/t)}|p_v(t, x, y)-p_w(t, f(x), f(y))| \leq \sigma r, \tag{4.3}$$

where $\sigma=\sup\{|F_\Phi[x](t)|_{l_2}+|F_\Psi[a](t)|_{l_2} : x\in M, a\in N, t>0\}$. In the same way, we have

$$e^{-(t+1/t)}|p_v(t, h(a), h(b))-p_w(t, a, b)| \leq \sigma r \tag{4.4}$$

for all $a, b \in N$ and $t>0$.

4.3. Given positive integers $n>1$, $k\geq 0$ and positive constants D, κ, we set

$$\mathscr{F}\mathscr{M}_w^*(n, k, D, \kappa)=\{(M, \mu_w, p_w, \Phi)\in \mathscr{F}\mathscr{M}_{w,c} : (M, \mu_w, p_w)\in \mathscr{M}_w^*(n, k, D, \kappa)\} .$$

Then it follows from Proposition 2.8 (i) and (iii) that

$$F_\Phi[M]\subset K(C, \eta)$$

for any $(M, \mu_w, p_w, \Phi)\in \mathscr{F}\mathscr{M}_w^*(n, k, D, \kappa)$, where C and η depend only on $n+k$, D and κ. Thus we have the following:

THEOREM 4.1. *The metric space $\mathscr{F}\mathscr{M}_w^*(n, k, D, \kappa)$ with the distance* SD* *is precompact.*

Now if we denote by π (resp. π') the natural projection from $\mathscr{F}\mathscr{M}_{w,c}$ onto $\mathscr{M}_{w,c}$ (resp. $\mathscr{I}$) which sends (M, μ_w, p_w, Φ) to (M, μ_w, p_w) (resp. M), then by (4.3), (4.4) and Lemma 3.2, we have the following:

THEOREM 4.2. *Given constants n, k, D and κ as before, both of the projections π and π' are uniformly continuous on the space $\mathscr{F}\mathscr{M}_w^*(n, k, D, \kappa)$. To be precise, given two elements $\alpha=(M, \mu_v, p_v, \Phi=\{u_\nu\})$ and $\beta=(N, \mu_w, p_w, \Psi=\{v_\nu\})$ of $\mathscr{F}\mathscr{M}_w^*(n, k, D, \kappa)$, there exist $\delta(\mathrm{SD}^*(\alpha, \beta))$-spectral approximations $f: M\to N$ and $h: N\to M$ between $\pi(\alpha)$ and $\pi(\beta)$, which are also $\delta(\mathrm{SD}^*(\alpha, \beta))$-Hausdorff approximations between $\pi'(\alpha)$ and $\pi'(\beta)$, such that*

$$e^{-(t+1/t)}\sum_{\nu=1}^{\infty}|e^{-\lambda_\nu t/2}u_\nu(x)-e^{-\rho_\nu t/2}v_\nu(f(x))|^2<r^2$$

for all $t>0$ and $x\in M$;

$$e^{-(t+1/t)}\sum_{\nu=1}^{\infty}|e^{-\lambda_\nu t/2}u_\nu(h(a))-e^{-\rho_\nu t/2}v_\nu(a)|^2<r^2$$

for all $t>0$ and $a\in N$. Hence we put $r=2\mathrm{SD}^*(\alpha, \beta)$, λ_ν and ρ_ν are the ν-th eigenvalue of α and β, respectively, and $\delta(t)$ is a monotone increasing continuous function depending only on $n+k$, D and κ with $\delta(0)=0$.

By this theorem, the projection π (resp. π') extends uniquely to a continuous map $\bar{\pi}$ (resp. $\bar{\pi}'$) from the completion of $\mathscr{F}\mathscr{M}_w^*(n, k, D, \kappa)$ onto that of $\mathscr{M}_w^*(n, k, D, \kappa)$ (resp. that of $\pi'(\mathscr{F}\mathscr{M}_w^*(n, k, D, \kappa))=\rho(\mathscr{M}_w^*(n, k, D, \kappa))$ in $\mathscr{I}$). Therefore Theorem 3.6 and also

Corollary 2.2 can be derived from Theorems 4.1 and 4.2 without referring to Gromov's precompactness theorem.

4.4 We shall now prove the uniform continuity of the eigenvalues with respect to SD.

LEMMA 4.3. *Let n, k, D and κ be as before, and let D' be a positive constant less than D. Then given a positive integer ν, there exists a monotone increasing continuous function $\varepsilon_\nu(t)$ with $\varepsilon_\nu(0)=0$, depending only on $n+k$, D, κ, D' and ν, such that for all $\alpha=(M, \mu_v, p_v, \Phi)$, $\beta=(N, \mu_w, p_w, \Psi)\in\mathscr{FM}_w^*(n, k, D, \kappa)$ with* $\mathrm{diam}(M)\geq D'$ *and* $\mathrm{diam}(N)\geq D'$,

$$|\lambda_\nu-\sigma_\nu|\leq\varepsilon_\nu(\mathrm{SD}^*(\alpha, \beta)),$$

where λ_ν and σ_ν are the ν-th eigenvalues of $\mathscr{L}_v$ and $\mathscr{L}_w$, respectively.

PROOF. Let u_ν and v_ν be the ν-th eigenfunctions of $\mathscr{L}_v$ and $\mathscr{L}_w$ having the eigenvalues λ_ν and σ_ν, respectively, and let $f: M\to N$ and $h: N\to M$ be as in Theorem 4.2. Then we have

$$e^{-(t+1/t)}|e^{-\lambda_\nu t/2}u_\nu(x)-e^{-\sigma_\nu t/2}v_\nu(f(x))|^2<r^2$$

for all $t>0$ and $x\in M$;

$$e^{-(t+1/t)}|e^{-\lambda_\nu t/2}u_\nu(h(a))-e^{-\sigma_\nu t/2}v_\nu(a)|^2<r^2$$

for all $t>0$ and $a\in N$. Here we put $r=2\mathrm{SD}^*(\alpha, \beta)$. Since $\int_M u_\nu^2\, d\mu_w=1$, there is a point x_ν of M such that $|u_\nu(x_\nu)|=1$. Therefore we see that

$$e^{(\sigma_\nu-\lambda_\nu)t/2}\leq re^{(t+1/t+\sigma_\nu t)/2}+|v_\nu(f(x_\nu))|.$$

Recall that

$$\sigma_\nu\leq C_\nu, \qquad |v_\nu(f(x_\nu))|\leq C'_\nu$$

for some positive constants C_ν and C'_ν depending only on $n+k$, D, κ, D' and ν, because of Proposition 2.7 (iii) and Proposition 2.8 (iv), respectively. Let us set $\xi_\nu(t)=\exp(t+1/t+C_\nu t)$ for simplicity and denote by A_ν the positive number where ξ_ν takes the minimum value m_ν, and in addition, let us take a continuous function $T_\nu(s)$ in such a way that $\xi_\nu(T_\nu(s))=1/s$ for $s\leq 1/m_\nu$ and $T_\nu(s)=A_\nu$ for $s\geq 1/m_\nu$. Then we have

$$\sigma_\nu-\lambda_\nu\leq\begin{cases}\dfrac{2}{T_\nu(r)}\log(1+C'_\nu) & \text{if } r\leq\dfrac{1}{m_\nu}\\[2ex] \dfrac{2}{T_\nu(r)}\log(m_\nu r+C'_\nu) & \text{if } r\geq\dfrac{1}{m_\nu}.\end{cases}$$

In exactly the same way, we get this bound for $\lambda_\nu-\sigma_\nu$. q.e.d.

4.5. At this stage, we shall study the boundary of $\mathscr{FM}_w^*(n, k, D, \kappa)$ in its completion $(\mathcal{F}\mathcal{M}_w^*(n,k,D,\kappa))^-$. Let $\{\alpha_i=(M_i,\mu_{w_i},p_{w_i},\Phi_i=\{u_v^i\})\}$ be an SD*-Cauchy sequence in $\mathcal{F}\mathcal{M}_w^*(n,k,D,\kappa)$. Then $\{\pi(\alpha_i)=(M_i,\mu_{w_i},p_{w_i})\}$ is an SD-Cauchy sequence in $\mathcal{M}_w^*(n,k,D,\kappa)$. Indeed, if we set $r(i,j)=\mathrm{SD}^*(\alpha_i,\alpha_j)$ and $r'(i,j)=\delta(r(i,j))$, then by Theorem 4.2, we have $r'(i,j)$-spectral approximations $f_{ij}\colon M_i\to M_j$ and $h_{ij}\colon M_j\to M_i$ between $\pi(\alpha_i)$ and $\pi(\alpha_j)$, with $r'(i,j)$ tending to zero as $i,j\to\infty$, which enjoy (4.1) and (4.2), namely,

$$e^{-(t+1/t)}\sum_{v=1}^{\infty}|e^{-\lambda_v^i t/2}u_v^i(x)-e^{-\lambda_v^j t/2}u_v^j(f_{ij}(x))|^2<r(i,j)^2$$

for all $t>0$ and $x\in M_i$;

$$e^{-(t+1/t)}\sum_{v=1}^{\infty}|e^{-\lambda_v^i t/2}u_v^i(h_{ij}(a))-e^{-\lambda_v^j t/2}u_v^j(a)|^2<r(i,j)^2 \tag{4.5}$$

for all $t>0$ and $a\in M_j$, where $\{\lambda_v^i: 0=\lambda_0^i<\lambda_1^i\le\lambda_2^i\le\cdots\}$ are the eigenvalues of $\mathcal{L}_{w_i}$. Suppose that M_i does not converge to a point, that is, the diameters of M_i are uniformly bounded below. Let (X,μ,p) be the limit of the sequence $\{\pi(\alpha_i)=(M_i,\mu_{w_i},p_{w_i})\}$ in $\mathcal{M}_w^*(n,k,D,\kappa)$. Then there exist $r'(i)$-spectral approximations $f_i\colon M_i\to X$ and $h_i\colon X\to M_i$ between $\pi(\alpha_i)$ and (X,μ,p), with $r'(i)$ tending to zero as $i\to\infty$, which satisfy

$$d_{M_i}(h_{ij}\circ h_j(a),h_i(a))\le 2r'(i,j) \tag{4.6}$$

for all $a\in X$. Observe that

$$\begin{aligned}
&e^{-(t+1/t)}\sum_{v=1}^{\infty}|e^{-\lambda_v^i t/2}u_v^i(h_i(a))-e^{-\lambda_v^j t/2}u_v^j(h_j(a))|^2\\
&\quad\le 2e^{-(t+1/t)}\sum_{v=1}^{\infty}|e^{-\lambda_v^i t/2}u_v^i(h_i(a))-e^{-\lambda_v^i t/2}u_v^i(h_{ij}\circ h_j(a))|^2\\
&\quad\quad+2e^{-(t+1/t)}\sum_{v=1}^{\infty}|e^{-\lambda_v^i t/2}u_v^i(h_{ij}\circ h_j(a))-e^{-\lambda_v^j t/2}u_v^j(h_j(a))|^2 .
\end{aligned}$$

The second term on the right-hand side in this inequality is bounded from above by $2r(i,j)^2$, because of (4.5). On the other hand, if we put $x=h_i(a)$ and $y=h_{ij}\circ h_j(a)$ for simplicity, the first term is equal to

$$2e^{-(t+1/t)}\{p_{w_i}(t,x,x)+p_{w_i}(t,y,y)-2p_{w_i}(t,x,y)\} .$$

Hence by Proposition 2.8 (ii) and (4.6), we see that the first term is less than $Cr'(i,j)$, where C is a positive constant depending only on $n+k$, D and κ. Thus we obtain

$$e^{-(t+1/t)}\sum_{v=1}^{\infty}|e^{-\lambda_v^i t/2}u_v^i(h_i(\alpha))-e^{-\lambda_v^j t/2}u_v^j(h_j(a))|^2\le r''(i,j)\,, \tag{4.7}$$

for all $t>0$ and $a\in X$, where $r''(i,j)$ converges to zero as $i,j\to\infty$. In particular, this implies that for any fixed integer $v>0$, (4.8) $$e^{-(t+1/t)}|e^{-\lambda_\nu^i t/2}u_\nu^i(h_i(a))-e^{-\lambda_\nu^j t/2}u_\nu^j(h_j(a))|^2 \le r''(i,j)\,,$$

for all $t>0$ and $a \in X$. Since $\{\lambda_\nu^i\}_{i=1,2,3,\dots}$ for each ν is a Cauchy sequence by Lemma 4.3, we have the limit, say λ_ν. Moreover it follows from (4.8) that $\{u_\nu^i \circ h_i\}$ is also a Cauchy sequence in $L^\infty(X)$. Let us denote by u_ν the function to which $u_\nu^i \circ h_i$ converges as i tends to infinity. Then it is not hard to see that u_ν is a Lipschitz continuous function on X and further it is an eigenfunction of $\mathscr{L}_p$ with the eigenvalue λ_ν. Here $\mathscr{L}_p$ stands for the infinitesimal generator of the C_0-semigroup $\{T_t\}$ in $L^2(X, \mu)$ with kernel $p(t, a, b)$. Thus by putting $u_0=1$, we obtain a complete orthonormal system $\Phi=\{u_\nu\}_{\nu=0,1,2,\dots}$ in $L^2(X,\mu)$ and also we have an embedding $F_\Phi\colon X \to C_\infty([0,\infty), l_2)$ in exactly the same manner as in the case of Riemannian manifolds. Clearly the image $F_\Phi[X]$ coincides with the Hausdorff limit of $F_{\Phi_i}[M_i]$ in $C_\infty([0,\infty), l_2)$ as i tends to infinity. In other words, $\alpha_i=(M_i, \mu_{w_i}, p_{w_i}, \Phi_i)$ converges to (X, μ, p, Φ) with respect to the distance SD* as i tends to infinity. In particular, letting j go to infinity in (4.7), we get

$$e^{-(t+1/t)}\sum_{\nu=1}^{\infty}|e^{-\lambda_\nu^i t/2}u_\nu^i(h_i(a))-e^{-\lambda_\nu t/2}u_\nu(a)|^2 \le r''(i)\,,$$

for all $t>0$ and $a \in X$, where $r''(i)$ converges to zero as i tends to infinity. In addition, the same argument as above shows that

$$e^{-(t+1/t)}\sum_{\nu=1}^{\infty}|e^{-\lambda_\nu^i t/2}u_\nu^i(x)-e^{-\lambda_\nu t/2}u_\nu(f_i(x))|^2 \le r''(i)\,,$$

for all $t>0$ and $x \in M_i$. Thus we have:

THEOREM 4.4. *Given constants n, k, D and κ as before, the completion $(\mathscr{F}\mathscr{M}_w^*(n,k,D,\kappa))^-$ of $\mathscr{F}\mathscr{M}_w^*(n,k,D,\kappa)$ consists of the elements $(X,\mu,p,\Phi=\{u_\nu\})$, where (X,μ,p) belongs to the completion $(\mathscr{M}_w^*(n,k,D,\kappa))^-$ of $\mathscr{M}_w^*(n,k,D,\kappa)$ and $\Phi=\{u_\nu\}$ is a complete orthonormal system of eigenfunctions of $\mathscr{L}_p$ with u_ν having the ν-th eigenvalue.*

4.6. We are now in a position to state the main theorem in this section. For two elements σ, τ of $(\mathscr{M}_w^*(n,k,D,\kappa))^-$, we set

$$\Gamma(\sigma,\tau)=\max\{\mathrm{SD}^*(\alpha,\bar\pi^{-1}(\tau)) : \alpha \in \bar\pi^{-1}(\sigma)\}\,,$$

$$\Theta(\sigma,\tau)=\min\{\mathrm{SD}^*(\alpha,\beta) : \alpha \in \bar\pi^{-1}(\sigma), \beta \in \bar\pi^{-1}(\tau)\}\,.$$

Then given a sequence $\{\sigma_i\}$ and an element τ in $(\mathscr{M}_w^*(n,k,D,\kappa))^-$, we can assert that the following three conditions are mutually equivalent:

(i) $\lim_{i\to\infty}\mathrm{SD}(\sigma_i,\tau)=0$, namely σ_i converges to τ in $(\mathscr{M}_w^*(n,k,D,\kappa))^-$;

(ii) $\lim_{i\to\infty}\Gamma(\sigma_i,\tau)=0$;

(iii) $\lim_{i\to\infty}\Theta(\sigma_i,\tau)=0$.

In particular, for an element $\tau=(X,\mu,p) \in (\mathscr{M}_w^*(n,k,D,\kappa))^-$, if we denote by $\lambda_\nu(\tau)$ the ν-th eigenvalue of the operator $\mathscr{L}_p$, then it follows from Lemma 4.3 that

$$\lim_{i\to\infty} \lambda_\nu(\sigma_i)=\lambda_\nu(\tau)\,,$$

provided a sequence $\{\sigma_i\}$ converges to τ. Here we understand $\lambda_\nu(\tau)=\infty$ for $\nu>0$, when X is a point. In this trivial case, we set evidently $p=1$. We notice that if every eigenvalue of τ is simple, then the following condition (iv) is also equivalent to the above ones (i)–(iii):

(iv) $\lim_{i\to\infty} \Gamma(\tau, \sigma_i)=0$.

Thus we obtain the following:

THEOREM 4.5. *Given integers $n>1$, $k\geq 0$, positive constants D and κ, the following assertions hold*:

(i) *The ν-th eigenvalue λ_ν for each ν, which is regarded as a function on the uniform space $\mathcal{M}_w^*(n, k, D, \kappa)$, extends continuously to the completion $(\mathcal{M}_w^*(n, k, D, \kappa))^-$.*

(ii) *Suppose a sequence $\{(M_i, \mu_{w_i}, p_{w_i})\}$ in $\mathcal{M}_w^*(n, k, D, \kappa)$ converges to an element $\tau=(X, \mu, p)$. Then for any complete orthonormal system $\Phi_i=\{u_\nu^i\}_{\nu=1,2,\ldots}$ in $L^2(M_i, \mu_{w_i})$ which consists of eigenfunctions u_ν^i with u_ν^i having the ν-th eigenvalue, there exist such a system $\Psi_i=\{v_\nu^i\}$ in $L^2(X, \mu)$, and $r(i)$-spectral approximations $f_i\colon M_i\to X$ and $h_i\colon X\to M_i$ satisfying*

$$e^{-(t+1/t)}\sum_{\nu=1}^{\infty} |e^{-\lambda_\nu^i t/2}u_\nu^i(h_i(a))-e^{-\lambda_\nu t/2}v_\nu^i(a)|^2\leq r(i)\,,$$

for all $t>0$ and $a\in X$;

$$e^{-(t+1/t)}\sum_{\nu=1}^{\infty} |e^{-\lambda_\nu^i t/2}u_\nu^i(x)-e^{-\lambda_\nu t/2}v_\nu^i(f_i(x))|^2\leq r(i)\,,$$

for all $t>0$ and $x\in M_i$. Here $r(i)$ does not depend on the choice of Φ_i and tends to zero as $i\to\infty$.

5. Spectral convergence and resolvents. Let $\tau=(M, \mu_w, p_w)$ be a triad of $\mathcal{M}_{w,c}$. For a positive number σ, the inverse $R_{\tau,\sigma}$ of the operator $\mathcal{L}_w+\sigma I$ in $L^2(M, \mu_w)$ has the kernel $g_{\tau,\sigma}$, called the Green function, which is given by

$$g_{\tau,\sigma}(x, y)=\int_0^\infty e^{-\sigma t}p_w(t, x, y)dt\,.$$

This holds for an element τ of the completion of the uniform space $\mathcal{M}_w^*(n, k, D, \kappa)$ for given constants n, k, D and κ as before. In this section, we describe some conditons for the convergence of a sequence in $\mathcal{M}_w^*(n, k, D, \kappa)$ in terms of the resolvents.

5.1. Let $\{\tau_i=(M_i, \mu_{w_i}, p_{w_i})\}_{i=1,2,\ldots}$ be a sequence of $\mathcal{M}_w^*(n, k, D, \kappa)$ such that M_i converges to a compact metric space X with respect to the Gromov-Hausdorff distance as i tends to infinity. Let $f_i\colon M_i\to X$ and $h_i\colon X\to M_i$ be $r(i)$-Hausdorff approximations between M_i and X with $r(i)$ converging to zero as i tends to infinity. Without loss of generality, we may assume that both maps f_i and h_i are Borel measurable. Fix a positive number σ. Then we have a sequence of linear maps $R_i^*: C(X)\to B(X)$ of the space of continuous functions on X into that of bounded Borel measurable functions defined by

$$R_i^*(\psi)(a)=h_i^*\circ R_{\tau,\sigma}\circ f_i^*(\psi)(a)=\int_{M_i} g_{\tau_i,\sigma}(h_i(a), y)\psi(f_i(y))d\mu_{w_i}(y)$$

for $\psi\in C(X)$ and $a\in X$. In addition, we have a sequence of the push-forward measures $f_{i*}\mu_i$ on X. The main result of this section is stated as follows:

THEOREM 5.1. *Let $\tau_i=(M_i, \mu_{w_i}, p_{w_i})$, X, f_i, h_i, R_i^* and $f_{i*}\mu_{w_i}$ be as above. Then the conditions below are mutually equivalent:*

(i) *The maps f_i and h_i are $r'(i)$-spectral approximations between τ_i and an element $\tau_\infty=(X, \mu, p)$ with $\lim_{i\to\infty} r'(i)=0$.*

(ii) *The measure $f_{i*}\mu_{w_i}$ converges as i tends to infinity in the weak* topology and, furthermore, $\lim_{i\to\infty} R_i^*(\psi)(a)$ exists for any $\psi\in C(X)$ and $a\in X$.*

(iii) *The measure $f_{i*}\mu_{w_i}$ converges as i tends to infinity to a measure μ in the weak* topology and further $\lim_{i\to\infty}\int_X R_i^*(\psi)\phi d\mu$ exists for any $\psi\in C(X)$ and every $\phi\in B(X)$.*

When one (and hence all) of these conditions is satisfied, for any $\psi\in C(X)$ and $a\in X$,

$$R_{\tau_\infty,\sigma}(\psi)(a)=\lim_{i\to\infty} R_i^*(\psi)(a)\ .$$

5.2. We prove two lemmas for the proof of this theorem. The first is stated as follows:

LEMMA 5.2. *Let $\tau=(M, \mu_w, p_w)$ be an element of $\mathcal{M}_w^*(n, k, D, \kappa)$ and $g_{\tau,\sigma}$ the Green function of $(\mathcal{L}_w+\sigma I)^{-1}$. Then for any $\varepsilon>0$ and $\delta>0$, there is a constant $\gamma>0$ depending only on $n+k$, D, κ, ε and δ such that*

$$|g_{\tau,\sigma}(x, y)-g_{\tau,\sigma}(x, z)|\le\varepsilon$$

if $d_M(x, y)\ge\delta$, $d_M(x, z)\ge\delta$ and $d_M(y, z)\le\gamma$.

PROOF. Observe first from Proposition 2.4 ($\varepsilon=1/2$) that if $d_M(x, y)\ge\delta$,

$$\int_0^T e^{-\sigma t}p_w(t, x, y)dt\le C_1\int_0^T e^{-\sigma t-\delta^2/8t+t/2}\frac{V_{n+k,\kappa}(D)}{V_{n+k,\kappa}(\sqrt{t})}dt \tag{5.1}$$

for any $T: 0<T\le D^2$, where C_1 is a positive constant depending only on $n+k$, D and κ. Secondly by Proposition 2.8 (ii), we have

$$\left|\int_T^\infty e^{-\sigma t}\{p_w(t, x, y)-p_w(t, x, z)\}dt\right|\le C_2\int_T^\infty t^{-(n+k+1)/2}e^{-\sigma t}dtd_M(y, z)$$

for all $T>0$ and $x, y, z\in M$, where $C_2>0$ depends only on $n+k$, D and κ. Hence it follows that if $d_M(x, y)\ge\delta$ and $d_M(x, z)\ge\delta$,

$$
\begin{aligned}
&|g_{\tau,\sigma}(x, y)-g_{\tau,\sigma}(x, z)| \\
&\quad\leq 2C_1 \int_0^T e^{-\sigma t-\delta^2/8t+t/2} \frac{V_{n+k,\kappa}(D)}{V_{n+k,\kappa}(\sqrt{t})}\, dt+C_2 \int_T^\infty t^{-(n+k+1)/2} e^{-\sigma t} dt\, d_M(y, z)
\end{aligned}
$$

for any $T\in(0, D^2]$. Hence taking T appropriately, we see that the assertion of the lemma holds. q.e.d.

LEMMA 5.3. *In the same notation as in Theorem* 5.1, *suppose the first condition* (i) *holds. Let* $\varepsilon>0$ *and* $\delta>0$ *be given constants. Then for sufficiently large i, one has*

$$|g_{\tau_i,\sigma}(h_i(a), h_i(b))-g_{\tau_\infty,\sigma}(a, b)|\leq\varepsilon$$

if $d_X(a, b)\geq\delta$;

$$|g_{\tau_i,\sigma}(x, y)-g_{\tau_\infty,\sigma}(f_i(x), f_i(y))|\leq\varepsilon$$

if $d_{M_i}(x, y)\geq\delta$.

PROOF. Let us consider indices i so large that $2r(i)\leq\delta$. Then for all $a, b\in X$ with $d_X(a, b)\geq\delta$, we have

$$d_{M_i}(h_i(a), h_i(b))\geq d_X(a, b)-r(i)\geq\frac{1}{2}\delta\,,$$

and hence by (5.1)

$$\left|\int_0^T e^{-\sigma t}\{p_{w_i}(t, h_i(a), h_i(b))-p(t, a, b)\}dt\right|\leq 2C_1 \int_0^T e^{-\sigma t-\delta^2/8t+t/2} \frac{V_{n+k,\kappa}(D)}{V_{n+k,\kappa}(\sqrt{t})}\, dt$$

for any $T: 0<T\leq D^2$. On the other hand, since h_i is an $r'(i)$-spectral approximation, we obtain

$$\left|\int_T^\infty e^{-\sigma t}\{p_{w_i}(t, h_i(a), h_i(b))-p(t, a, b)\}dt\right|\leq r'(i)\int_T^\infty e^{-\sigma t+t+1/t} dt$$

for all $T>0$. Thus we can deduce that for all $a, b\in X$ with $d_X(a, b)\geq\delta$ and for any $T\in(0, D^2]$,

$$
\begin{aligned}
&|g_{\tau_i,\sigma}(h_i(a), h_i(b))-g_{\tau_\infty,\sigma}(a, b)| \\
&\quad\leq 2C_1 \int_0^T e^{-\sigma t-\delta^2/8t+t/2} \frac{V_{n+k,\kappa}(D)}{V_{n+k,\kappa}(\sqrt{t})}\, dt+r'(i)\int_T^\infty e^{-\sigma t+t+1/t} dt\,.
\end{aligned}
$$

This shows the first assertion of the lemma. In a similar way, we can verify the second one. q.e.d.

5.3. We are now in a position to give a proof for Theorem 5.1.

PROOF OF THEOREM 5.1. (i) $\Rightarrow$ (ii) We have already shown that $f_{i*}\mu_i$ converges to a measure μ in the weak* topology (cf. Lemma 3.7). Let ψ be a continuous function on X. For any $\varepsilon>0$, take $\delta>0$ so small that

$$|\psi(a)-\psi(b)|\leq\sigma\varepsilon$$

if $d_X(a, b)\leq\delta$. Fix a point $a\in X$ and set $\psi_a=\psi-\psi(a)$. For simplicity, we write B for the metric ball around a with radius δ and put $B^c=X\setminus B$, $B_i=f_i^{-1}(B)$ and $B_i^c=M_i\setminus B_i$. Then

$$(5.2)\quad \begin{aligned} &|R_i^*(\psi)(a)-R_{\tau_\infty,\sigma}(\psi)(a)|=|R_i^*(\psi_a)(a)-R_{\tau_\infty,\sigma}(\psi_a)(a)|\\ &\leq\left|\int_{B_i} g_{\tau_i,\sigma}(h_i(a), y)\psi_a(f_i(y))d\mu_{w_i}(y)\right|+\left|\int_B g_{\tau,\sigma}(a, b)\psi_a(b)d\mu(b)\right|\\ &+\left|\int_{B_i^c}\{g_{\tau_i,\sigma}(h_i(a), y)-g_{\tau_i,\sigma}(h_i(a), h_i\circ f_i(y))\}\psi_a(f_i(y))d\mu_{w_i}(y)\right|\\ &+\left|\int_{B^c}\{g_{\tau_i,\sigma}(h_i(a), h_i(b))-g_{\tau_\infty,\sigma}(a, b)\}\psi_a(b)df_{i*}\mu_{w_i}(b)\right|\\ &+\left|\int_{B^c} g_{\tau_\infty,\sigma}(a, b)\psi_a(b)df_{i*}\mu_{w_i}(b)-g_{\tau_\infty,\sigma}(a, b)\psi_a(b)d\mu(b)\right|.\end{aligned}$$

The first two terms on the right-hand side of this inequality are both bounded by ε, since $|\psi_a|\leq\sigma\varepsilon$ on B. As for the third term, we observe that

$$d_{M_i}(y, h_i\circ f_i(y))\leq r(i)$$

for all $y\in M_i$, and further notice that if $y\in B_i^c$, then

$$\begin{aligned} &d_{M_i}(h_i(a), h_i\circ f_i(y))\geq d_X(f_i(y), a)-r(i)\geq\delta-r(i);\\ &d_{M_i}(h_i(a), y)\geq d_X(f_i(y), f_i\circ h_i(a))-r(i)\\ &\qquad\geq d_X(f_i(y), a)-d_X(a, f_i\circ h_i(a))-r(i)\geq\delta-2r(i).\end{aligned}$$

Hence taking sufficiently large i so that $\delta>4r(i)$, and applying Lemma 5.2 to the third term on the right-hand side of the inequality (5.2), we see that this is bounded by ε for large i. Moreover by virtue of Lemma 5.3, we deduce that the fourth term there is also bounded by ε for large i. Finally, since the push-forward measure $f_{i*}\mu_i$ converges weakly to the measure μ as $i\to\infty$, it follows that the last term there converges to zero as $i\to\infty$. Thus we have shown that

$$|R_i^*(\psi)(a)-R_{\tau,\sigma}(\psi)(a)|\leq 5\varepsilon$$

for large i.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) is also clear from the arguments in the proof of Theorem 3.6 and that of the first assertion (i) $\Rightarrow$ (ii).

6. Further discussions. Let M be a compact Riemannian manifold of dimension n. Given smooth functions $w>0$ and V on M, we consider the equation

$$\left(\frac{\partial}{\partial t}+\mathscr{L}_{w,V}\right)u(t,x)=0\,,$$

where we set

$$\mathscr{L}_{w,V}f=-\Delta_M f-\nabla\log w\cdot f+Vf\,.$$

Let us denote by $p_{w,V}(t,x,y)$ the fundamental solution of the above equation. We have restricted our attention so far to the case $V=0$. However, it is possible to introduce the spectral distance SD on the set of equivalence classes of triads $(M, w, p_{w,V})$ and carry out the discussions similar to what we have done in the previous sections, although some obvious changes should be made.

In fact, given integers $n>1$, $k\geq 0$ and positive constants D, κ and η, we write $\mathscr{M}^*_{w,V}(n,k,D,\kappa,\eta)$ for the uniform space of elements $(M, w, p_{w,V})$ as above such that the dimension of M is equal to n, (2.3) the tensor $R_{w,k}\geq-(n+k-1)\kappa^2$, (2.5) the $\operatorname{diam}(M)\leq D$, (2.6) the measure μ_w has unit total mass $\mu_w(M)=1$, and moreover V satisfies

$$|V|_\infty\leq\eta^2\,. \tag{6.1}$$

Furthermore, let $\mathscr{F}\mathscr{M}^*_{w,V}(n,k,D,\kappa,\eta)$ be the metric space with the distance SD* which consists of elements $(M, w, p_{w,V}, \Phi=\{u_\nu\})$, where $(M, w, p_{w,V})\in\mathscr{M}^*_{w,V}(n,k,D,\kappa,\eta)$ and $\Phi=\{u_\nu\}_{\nu=0,1,2,\ldots}$ is as before a complete orthonormal system in $L^2(M,\mu_w)$ of eigenfunctions u_ν of $\mathscr{L}_{w,V}$ with u_ν having the ν-th eigenvalue λ_ν. We remark that u_ν is uniformly Hölder continuous, namely,

$$|u_\nu(x)-u_\nu(y)|\leq C(\nu)d_M(x,y)^\alpha$$

for some $\alpha\in(0,1)$ depending only on the given constants n, k, D, κ and η, and a positive constant $C(\nu)$ depending only on ν and the given constants in such a way that $C(\nu)\leq O(\nu^2)$. This can be verified by the standard elliptic regularity theory together with the Poincaré inequality described in Proposition 2.6. Then results similar to those in Sections 3–5 can be shown to be true for these spaces $\mathscr{M}^*_{w,V}(n,k,D,\kappa,\eta)$ and $\mathscr{F}\mathscr{M}^*_{w,V}(n,k,D,\kappa,\eta)$. Obviously the constants and the functions there must depend also on the given constant η, and the equality in (3.10), for example, should be read as

$$e^{-\eta t}\leq\int_X p(t,a,b)d\mu(b)\leq e^{+\eta t}\,.$$

In addition, the components of the embedding F_Φ: $M\to C_\infty([0,\infty), l_2)$ in 4.2 should be begun by

$$\zeta_0(t)e^{-\lambda_0 t/2}u_0(x)\,,$$

where $\zeta_0(t)=\exp(-t-1/t-\eta t)/2$.

Let us conclude this section with a direct application of the generalization mentioned above.

THEOREM 6.1. *Let M be a compact Riemannian manifold of dimension n and let $w>0$ and V be smooth functions on M. Suppose* (2.3), (2.5) *and* (6.1) *are satisfied for some positive constants k, κ, D and η, respectively. Then there is a constant $C=C(n+k, D, \kappa, \eta)$ depending only on the quantities in the parenthesis such that*

$$\lambda_1-\lambda_0\geq\frac{C}{\operatorname{diam}(M)^2},$$

where λ_i $(i=0, 1)$ are the first two eigenvalues of the operator $\mathscr{L}_{w,V}$.

PROOF. By rescaling the metric of M and multiplying the weight function w by a constant if necessarily, we may assume that the diameter of M is equal to one and the measure μ_w has unit total mass. Moreover, replacing the given V by $V-\min V$, we may assume that $V\geq 0$. Then the assertion follows by contradiction. Indeed, suppose to the contrary that there exists a sequence $\{\tau_j=(M_j, w_j, p_{w_j,V_j})\}$ in $\mathscr{M}^*_{w,V}(n, k, D, \kappa, \eta)$ with $\operatorname{diam}(M_j)=1$ such that the gap between the first two eigenvalues of $\mathscr{L}_{w_j,V_j}$ tends to zero as $j\to\infty$. Then by virtue of an analog to Theorem 3.8, we may assume that this sequence converges to an element $\tau=(X, \mu, p)$ in the completion of $\mathscr{M}^*_{w,V}(n, k, D, \kappa, \eta)$. This leads to a contradiction, because we have an analog to Theorem 4.5, and the first eigenvalue of τ is simple.

REMARK. In the present paper, we have focused on compact Riemannian manifolds without boundary. However it is possible to discuss a class of complete (pointed) Riemannian manifolds including noncompact manifolds or manifolds with boundary. This topic will be taken up elsewhere.

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