Spectral density of fluctuations of a double-well Duffing oscillator driven by white noise

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(Received 26 May 1987; revised manuscript received 22 July 1987)

The power spectrum of the archetypal fluctuating bistable system, the underdamped double-well Duffing oscillator, is investigated both experimentally, with use of an electronic circuit, and theoretically. The experiment confirms previous analytic results for the structure of the spectrum, including the existence of three distinct peaks within a certain parameter range. The theory is extended to describe analytically the shape of the peak due to intrawell fluctuations for arbitrary noise intensities as well as certain other features of the spectrum. Good quantitative agreement of theory and experiment is demonstrated.

I. INTRODUCTION

The bistable potential has long been a basic element in the study of dynamical systems in contact with fluctuating heat baths. Several reviews[1-4] serve to introduce the general topical area and to explore the wide variety of physical properties of such systems. An early preview of the very large role to be played by the bistable potential in fluctuating nonequilibrium systems and subsequent discussions of the physical applications has been given by Landauer.[5] More recent studies on white-noise forcing have emphasized the Fokker-Planck approach in order to investigate the roles played by additive as well as multiplicative noise,[6] and to obtain the stationary, multidimensional statistical densities[7] or escape rates.[4,8] All these studies have drawn substantially on the original treatise of Stratonovich;[9] escape rates may also be obtained by use of some other quite general methods.[10] More recently, powerful numerical methods for solving multidimensional Fokker-Planck systems have been set forth in Risken's monograph,[11] and a multitude of physical applications resulting from parametric noise are to be found in the work of Horsthemke and Lefever.[12] Current emphasis is on systems driven by colored noise,[13] a body of work which once again owes much of its inspiration to Stratonovich,[14] in the sense that many analytic, approximate solutions of higher-dimensionality systems result from expansions of one kind or another about a white-noise (one-dimensional) theory.[15] Although it is the theoretical challenges posed by colored noise that account in large measure for the high level of current interest in such systems, the power spectrum or correlation function of a nonlinear system driven simply by white noise raises equally challenging problems. In the present paper, we focus our attention on the Langevin equation which describes the full (i.e., nonadiabatically reduced) bistable oscillator.

Though it is possible to obtain an exact theory for the correlation function (and hence for the power spectrum through the Wiener-Khintchine theorem) in the limit of zero damping or zero nonlinearity (for a review, see Ref. 16) the usual difficulties have plagued efforts to obtain solutions for arbitrary damping and strength of the nonlinearity. Even in the case of weak damping and nonlinearity, as follows from the explicit expressions obtained for this case,[17] the spectrum turns out to be complicated because of "competition" between damping and nonlinearity and it is their *ratio* that determines the shape of the spectrum. A rather large amount of literature exists commencing with early efforts to obtain the correlation functions, for arbitrary damping and nonlinearity with ordinary continued fractions[18,19] or statistical linearization[20] techniques, and continuing with more recent discussions[21,22] (see also reviews[16,23]). The problem becomes much more complicated in the case of a double-well oscillator. This is due, in particular, to the characteristic "slowing down" of the motion in the vicinity of the top of the potential barrier with weak damping, so that even weak noise causes relatively strong fluctuations. The spectrum of an underdamped bistable system displays some distinctive features, as we shall see.

The theoretical approach used here, based on the Langevin equation, is particularly convenient for comparison with the results of analogue simulation, since in the latter case it is always the Langevin equation itself which is directly simulated. As we show below with direct measurements made on an analogue simulator of

the noise-driven, double-well Duffing oscillator, three distinct peaks appear in the power spectra for appropriate values of the damping and noise intensity. Other groups have also done analogue simulations,[18,23,24] but the power spectra for the system in question have not previously been measured. Analytically, the power spectrum (excluding the zero-frequency peak) was first described by Onodera[25] in the limit of zero damping. The structure of the spectrum for the case of weak damping, with account taken of all three peaks, and a discussion of the mechanisms giving rise to this structure, were given in Ref. 26. More recently developed matrix continued fraction and associated numerical techniques have also revealed this structure.[27]

This paper is organized as follows. In Sec. II we outline and review the theory and discuss in physical terms the effect on the shape of the power spectrum of two dimensionless parameters related to the damping and the noise strength. In Sec. III an explicit expression for the shape of the peak due to intrawell trajectories is obtained. It is valid over a broad range of noise intensities. The ranges where different parameters strongly influence the peak are set forth. In Sec. IV the shape of the peak near zero frequency is discussed. Section V describes the analogue simulator, the measurements made on it, and compares the results to the theory. Finally, Sec. VI summarizes the results and draws conclusions.

II. REVIEW OF THE THEORY AND DISCUSSION OF THE SHAPE OF THE POWER SPECTRUM

By changing the equation describing Brownian motion of the Duffing oscillator with a bistable potential,

$$\ddot{q}-\omega_0^2 q+\gamma q^3+2\Gamma\dot{q}=f(t)\ , \qquad \langle f(t)f(t')\rangle=2B\Gamma\delta(t-t')\ , \tag{1}$$

to the dimensionless variables "time" $\omega_0 t$ and "coordinate" $(\gamma^{1/2}/\omega_0)q$, one can easily see that the dynamics of the oscillator as a whole and the shape of the spectral density of fluctuations of coordinates

$$Q(\omega)=\frac{1}{\pi}\,\mathrm{Re}\int_0^\infty dt\,\exp(i\omega t)\langle q(t)q(0)\rangle \tag{2}$$

are determined[17,26] by two dimensionless parameters, Γ/ω_0 and β, where

$$\beta=\frac{\gamma B}{2\omega_0^4}\equiv\frac{B}{8\Delta U},\quad \Delta U=\tfrac{1}{4}\omega_0^4/\gamma\ . \tag{3}$$

The former, Γ/ω_0, is seen from (1) to characterize friction in the system, while the latter, β, characterizes the relative intensity of a random force $f(t)$ which acts on the oscillator. The physical meaning of β is most evident in the case when both $f(t)$ and the friction force $-2\Gamma\dot{q}$ in (1) result from coupling of the oscillator to a thermostat. In that case $B=2kT$ and $4\beta=kT/\Delta U$ [the quantity ΔU in (3) is the height of the potential barrier between the equilibrium states of the oscillator as shown in Fig. 1].

For the underdamped oscillator,

$$\Gamma/\omega_0\ll 1\ , \tag{4}$$

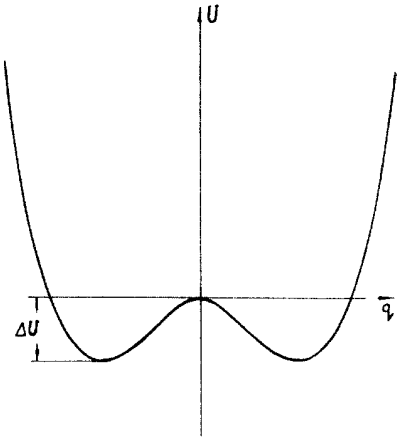

FIG. 1. Potential energy $U(q)$ of the double-well Duffing oscillator.

the function $Q(\omega)$ can have[26] either two or *three distinct peaks* depending on the value of β. One of them is located at the frequency $\omega=0$. At low noise intensities, $4\beta\ll 1$, the shape of this peak near the maximum is formed by fluctuational transitions between the stable states. As a result of such a transition, the oscillator coordinate q changes from a value $q\approx q_i$, corresponding to that stable state i ($i=1,2$) occupied prior to transition to that corresponding to the other state, $q\approx -q_i$, where $q_i=(-1)^i q_0$, and $q_0=\omega_0\gamma^{-1/2}$. Respectively, the intensity of the peak is $\sim q_0^2$, while the halfwidth is of the order of the transition probability and thus is exponentially small; $\sim\Gamma\exp(-1/4\beta)$, as shown previously.[16,26]

At $4\beta\ll 1$, the oscillator mainly performs small-amplitude vibrations about one of the stable equilibrium positions q_i. The contribution of this motion to the function $Q(\omega)$ is the peak at the frequency $\omega_{\max}^{(1)}$, which is close to the eigenfrequency of these vibrations in the limit of zero amplitude,

$$\omega_{\max}^{(1)}\approx\omega_0\sqrt{2}\ . \tag{5}$$

With rising β an increasingly important role in the formation of $Q(\omega)$ is played by vibrations with trajectories above the barrier of the oscillator potential $U(q)$ (the overbarrier vibrations). They give rise to the peak of $Q(\omega)$ at the frequency $\omega_{\max}^{(2)}$, which at comparatively small β is given by[26]

$$\omega_{\max}^{(2)}=\pi\omega_0/\ln(16\beta^{-1})\ \text{ for }\ \ln(16\beta^{-1})\gg 1\ . \tag{6}$$

The emergence of this peak is connected with the slowing down of the oscillator motion near the top of the barrier. This causes the eigenfrequency $\omega(E)$ to tend to zero rapidly as the oscillator energy E, which is given by

$$E=\tfrac{1}{2}p^2+U(q),\quad U(q)=-\tfrac{1}{2}\omega_0^2 q^2+\tfrac{1}{4}\gamma q^4\ , \tag{7}$$

approaches zero, so that $\omega(E)\propto\ln^{-1}(\Delta U/|E|)$ for $|E|/\Delta U\ll 1$. As a consequence, the peak turns out to be relatively narrow for small β. Estimates based on the analytical results[26] and on numerical calculations[25,26] show that for $\beta\gtrsim 0.06$, the ratio of the height of this peak at $\omega_{\max}^{(2)}$ to that of the peak caused by the vibrations

about equilibrium positions at $\omega_{\max}^{(1)}$ is $\gtrsim 0.2$ (at sufficiently small damping), and increases as $\exp(-1/4\beta)$ with increasing β. We note that the emergence of all the peaks stated above is a general property of the spectral density of fluctuations in underdamped bistable systems.[26]

In the region of the minimum between the peaks at $\omega_{\max}^{(1)}$ and $\omega_{\max}^{(2)}$, the spectrum $Q(\omega)$ is formed by the motion near the top of the potential hump with both positive and negative energies. Generalizing our previous approach[26] to include the motion with negative energies one can show that, for $\ln(16/\beta) \gg 1$, the position of the minimum is given by

$$\omega_{\min}^{(1)} \approx \pi\omega_0[\ln(16/\beta) - \ln\ln(16/\beta)]^{-1} . \tag{8}$$

With further increases in β, the intensity of the peak due to overbarrier vibrations at $\omega_{\max}^{(2)}$ increases, while the peak due to vibrations about equilibrium positions at $\omega_{\max}^{(1)}$ becomes smeared and ultimately coalesces with the former. For $\beta > 0.35$ it cannot, in practice, be singled out in the spectrum. At $\beta \gg 1$ the maximum of the unified peak is located at $\omega_{\max}^{(2)} \approx 1.2\beta^{1/4}\omega_0$.[17(b),26]

The explicit analytical expression for the spectral density $Q(\omega)$ may be obtained neglecting dissipation[16,25-27]. In this approximation Eq. (1) reduces to an equation which describes free vibrations of the oscillator,

$$\ddot{q} - \omega_0^2 q + \gamma q^3 = 0 . \tag{1'}$$

To find $Q(\omega)$ one can multiply the solution $q(t)$ of (1') by $q(0)$, average the product over the initial values of the coordinate $q(0)$ and momentum $p(0)$ with the weight w_{st},

$$w_{st} \equiv w_{st}(E) = Z^{-1}\exp(-2E/B),$$
$$Z = \int dq\, dp \exp(-2E/B) , \tag{9}$$

and then obtain from (2) the Fourier transform.[16] The resulting expression for $Q(\omega)$ can then be analyzed in the limiting cases treated earlier.[26]

For $4\beta \gg 1 \gg \Gamma/\omega_0$ this "dissipationless" approximation describes $Q(\omega)$ well over the whole range of the broad peak at nonzero frequency. For $4\beta < 1$ and $\Gamma/\omega_0 \ll 1$ the range of its applicability becomes more limited in frequency. However, it may be used to describe semiquantitatively the middle peak of $Q(\omega)$, when it is distinct, and also to describe the substantial part of the peak due to intrawell vibrations for $\beta \gg \Gamma/\omega_0$. In the range $\beta \lesssim \Gamma/\omega_0$ the dissipation qualitatively influences the shape of the latter peak. It determines also the shape of the zero-frequency peak. The dependence of $Q(\omega)$ on Γ is nonanalytic in the corresponding frequence ranges.

III. THE SHAPE OF THE PEAK DUE TO INTRAWELL VIBRATIONS

In the limit of a vanishingly small "reduced" noise intensity β, the oscillator vibrations about the stable equilibrium positions are practically linear. In the absence of dissipation they give rise to the δ-function-shaped peak of $Q(\omega)$ at their eigenfrequency $\omega_0\sqrt{2}$. Dissipation results in the vibration frequency uncertainty $\sim\Gamma$ and therefore causes the broadening of the peak. For

$$2\beta \ll \Gamma/\omega_0 \ll 1 , \tag{10}$$

this broadening mechanism is the dominant one, and near the maximum, i.e., when $|\omega - \omega_0\sqrt{2}| \ll \omega_0$, the shape of the peak of $Q(\omega)$ is Lorentzian with a halfwidth Γ.

As the noise intensity increases, another broadening mechanism plays an increasingly important role, namely, modulational broadening.[17,23] It is connected with the nonlinearity of the now larger amplitude vibrations about the stable states. This nonlinearity causes a dependence of the vibration eigenfrequency $\omega(E)$ on the oscillator energy E. As a result of this dependence energy straggling, which is due to a random force, gives rise to frequency straggling and, therefore, to broadening of the peak of $Q(\omega)$. The characteristic modulational broadening $\delta\omega$ at small β is

$$\delta\omega \sim B\,|[d\omega(E)/dE]_{E_{\min}}| \sim \beta\omega_0, \quad E_{\min} = -\Delta U \equiv -\frac{\omega_0^4}{4\gamma}, \tag{11}$$

where B determines the characteristic width of the energy distribution [cf. (9)] and where $E_{\min}$ is the minimum oscillator energy. For a single-well quantum oscillator the emergence of modulational broadening was shown when considering localized vibrations in solids.[28] The dynamics of such vibrations has much in common with the Brownian motion of an underdamped nonlinear oscillator.[16]

In the parameter range where $2\beta \sim \Gamma/\omega_0$ both broadening mechanisms make contributions of the same order of magnitude, that is $\delta\omega \sim \Gamma$, and the shape of the spectral peak under consideration is determined by the competition between them. The peak may be described here using previous results:[17(a)] with the aid of a certain special method, the spectrum of fluctuations was found for a single-well nonlinear oscillator at arbitrary $\delta\omega/\Gamma$, neglecting the corrections $\sim\Gamma/\omega_0$ and $\sim 4\beta$. The approximations made in Ref. 17(a) correspond, in essence, to account being taken only of the linear terms in the expansion of the vibration eigenfrequency $\omega(E)$ in $E - E_{\min}$ as well as of only the term $\propto (E - E_{\min})^{1/2}$ in the expansion of the vibration amplitude.

For nonlinear vibrations about the stable states of the oscillator under consideration, where, to first order in $E - E_{\min}$,[29]

$$\omega(E) \approx \omega_0\sqrt{2}(1 - \tfrac{3}{16}\epsilon), \quad \epsilon = (E - E_{\min})/\Delta U, \quad \epsilon \ll 1 , \tag{12}$$

one obtains, by generalizing slightly the previous results,[17] that at small but finite damping

$$Q(\omega)\approx\tilde{Q}(\Omega),\quad \Omega=\omega-\omega_0\sqrt{2},\quad |\Omega|\ll\omega_0\ ,$$
$$\tilde{Q}(\Omega)=C_1\mathrm{Re}\int_0^\infty dt\,\exp(i\Omega t)\tilde{Q}(t)\ ,$$
$$\tilde{Q}(t)=\exp(\Gamma t)[\cosh(at)+(\Gamma/a)(1-2i\alpha)\sinh(at)]^{-2}\ , \tag{13a}$$
$$a=\Gamma(1-4i\alpha)^{1/2},\quad \alpha=\tfrac{3}{8}\sqrt{2}\beta\frac{\omega_0}{\Gamma}$$
$$C_1\equiv C_1(\Omega)=C_1'C_1''(\Omega),\quad C_1'=\frac{\sqrt{2}}{8}\frac{B^2}{\omega_0^3}Z^{-1}\exp\left[\frac{1}{4\beta}\right]$$
$$C_1''(\Omega)=\left[1+\frac{\Omega}{2\sqrt{2}\omega_0}\right]^{-2}.$$

We note that to lowest order in β,

$$Z=\pi\sqrt{2}B\omega_0^{-1}\exp(1/4\beta),\quad C_1'=\frac{1}{4\pi}q_0^2\beta;\quad q_0=\omega_0\gamma^{-1/2}\ . \tag{13b}$$

Just this value of C_1' was used, in essence, in Ref. 17(a) instead of C_1. C_1' coincides with C_1 near the maximum of peak ($|\Omega|\ll\omega_0$) with the accuracy to $\beta,\Gamma/\omega_0$ adopted in Ref. 17(a). The additional approximating factor $C_1''(\Omega)$ is introduced in (13a) in such a way that it is close to unity at $|\Omega|\ll\omega_0$ and describes here correctly the addition $\sim|\Omega|/\omega_0$ for $\beta\gg\Gamma/\omega_0$ [within the framework of the model adopted in deriving (13a), in particular, when (12) is fulfilled]. At the same time the factor $C_1''(\Omega)$ makes it possible to use (13a) when describing the wings relatively far from the peak for small $\beta\ll1$.

It is seen from Eq. (13a) that the shape of the peak depends on the single parameter α which is proportional to a ratio of two small quantities, β and Γ/ω_0. The function $\tilde{Q}(\Omega)/C_1$, for different α, was analyzed in detail in Ref. 17(a) [for convenience the sign of the parameter α in (13a) is opposite to that in Ref. 17(a)]. At $4\alpha\ll1$ the shape of peak of $\tilde{Q}(\Omega)$ is close to a Lorentzian, and the maximum lies at

$$\omega_{\max}^{(1)}=\omega_0\sqrt{2}(1-\tfrac{3}{2}\beta),\quad 2\beta\ll\Gamma/\omega_0\ . \tag{14}$$

With rising α the deviation of the shape of the peak from a Lorentzian becomes more and more substantial, the asymmetry of the peak becomes appreciable, and at $4\alpha\gg1$, near the maximum,

$$\tilde{Q}(\Omega)\approx\tilde{Q}^{(0)}(\Omega)\ ,$$
$$\tilde{Q}^{(0)}(\Omega)=\pi C_1\frac{|\Omega|}{\Omega_0^2}\exp\left[-\frac{|\Omega|}{\Omega_0}\right]\theta(-\Omega)\ , \tag{15}$$
$$\Omega_0=\frac{3}{2\sqrt{2}}\beta\omega_0,\quad |\Omega|\gg\Gamma|\alpha|^{1/2},\quad 2\beta\gg\Gamma/\omega_0\ ,$$

where $\theta(x)$ is the step function. The expression (15) does not contain the friction coefficient Γ. It coincides with the expression for $Q(\omega)$ given by the dissipationless approximation in the range $\omega\approx\omega_0\sqrt{2}$. This may be easily understood if one notes that the energy of the vibrations with the eigenfrequency $\omega(E)=\omega\approx\omega_0\sqrt{2}$ [only resonant vibrations, including the overtones, contribute to $Q(\omega)$ in the dissipationless approximation] is $E-E_{\min}\approx-\frac{1}{2}B\Omega/\Omega_0$ according to (12) and (15), and that the oscillator distribution in energy is of the form (9). The maximum of the function (15) lies at

$$\omega_{\max}^{(1)}\approx\omega_0\sqrt{2}(1-\tfrac{3}{4}\beta),\quad 2\beta\gg\Gamma/\omega_0\ \ (\beta\lesssim0.03)\ . \tag{14'}$$

Near the maximum, $\Omega\approx-\Omega_0$, the dissipation-induced corrections to $\tilde{Q}^{(0)}(\Omega)$ are $\sim\alpha^{-1}\ll1$.[17(a)]

The function $\tilde{Q}^{(0)}(\Omega)$ is seen from (15) to have a singular point at $\Omega=0$, that is, when $\omega=\omega_0\sqrt{2}$. The derivative $d\tilde{Q}^{(0)}/d\omega$ is discontinuous here. This point corresponds to the end point in the spectrum of intrawell vibration eigenfrequencies. The presence of a singular point in the eigenfrequency spectrum results in a nonanalytic dependence of $Q(\omega)$ on Γ in the respective frequency range,

$$\tilde{Q}(\Omega)\sim C_1\Omega_0^{-1}\alpha^{-1/2},\quad |\Omega|\lesssim\alpha^{1/2}\Gamma\ , \tag{16}$$

that is, $\tilde{Q}(\Omega)\propto\Gamma^{1/2}$.

In the wings of the peak where $|\Omega|\gg\Gamma,\Omega_0\ln(1+\alpha)$, the function $\tilde{Q}(\Omega)$ is also determined by dissipation. According to (13a),

$$\tilde{Q}(\Omega)\approx8\omega_0^2C_1'\Gamma(\omega^2-2\omega_0^2)^{-2},\quad |\Omega|\gg\Gamma,\Omega_0\ln(1+\alpha)\ . \tag{17}$$

The spectrum (17) coincides with the spectrum of a harmonic oscillator with an eigenfrequency $\omega_0\sqrt{2}$ and a friction coefficient Γ. Weak nonlinearity of the vibrations about the stable states requires corrections of the order of β in the range $|\Omega|\sim\omega_0$.

For $\beta\lesssim\Gamma/\omega_0$ an inaccuracy of Eq. (13a) is due to the neglect of both the corrections $\sim\Gamma/\omega_0$ and those of the order of β, with the result that it weakly depends on frequency. At $\beta\gg\Gamma/\omega_0$ the main source of inaccuracy is the neglect of the terms of higher order in $\epsilon=(E-E_{\min})/\Delta U$ in (12) and in the expression for the vibration amplitude. For such values of $\beta\omega_0/\Gamma$ the inaccuracy grows as ω moves toward the low-frequency wing of $Q(\omega)$, since for a given $\omega(|\Omega|\sim\Omega_0)$ the spectrum is formed mainly by vibrations with $\omega(E)=\omega$ and, therefore, an increase in $|\Omega|$ causes an increase in the value of ϵ for resonant vibrations [$\epsilon\approx-4\beta\Omega/\Omega_0$ according to (12)]. It follows from the explicit expressions for the vibration parameters that if one wants the expression (15) to differ from the exact result of the dissipationless approximation by $\lesssim20\%$ up to frequencies Ω for which $\tilde{Q}^{(0)}(\Omega)\gtrsim0.2\tilde{Q}^{(0)}_{\max}(\Omega)$, the values of β should be limited to $\beta\lesssim0.03$ (then $\epsilon\lesssim\frac{1}{2}$). This limitation is adopted in (14'). It follows from these results that it is practicable to obtain an expression for the spectrum $Q(\omega)$ in the range of the peak, located at $\omega\approx\omega_0\sqrt{2}$, which includes the effects of dissipation, on the one hand, and on the other hand, is not limited to the range of very small β; this is one of the aims of the present paper. The relevant expression is shown below to be of the form

$$Q(\omega)\approx Q^{(0)}(\omega)+\tilde{Q}(\Omega)-\tilde{Q}^{(0)}(\Omega)\ , \tag{18}$$

where $Q^{(0)}(\omega)$ is the result of the dissipationless approxi-

mation[25-27] while $\tilde{Q}(\Omega)$ and $\tilde{Q}^{(0)}(\Omega)$ are given by (13a) and (15).

For small $(\omega_0\sqrt{2}-\omega)/\omega_0\sqrt{2}$ the main term in $Q^{(0)}(\omega)$ goes over into $\tilde{Q}^{(0)}(\Omega)$,

$$|Q^{(0)}(\omega)-Q_{\text{la}}^{(0)}(\omega)-\tilde{Q}^{(0)}(\Omega)|/\tilde{Q}^{(0)}_{\max}\sim\left[\frac{\omega_0\sqrt{2}-\omega}{\omega_0\sqrt{2}}\right]^2,$$

$$\frac{\omega_0\sqrt{2}-\omega}{\omega_0\sqrt{2}}\ll 1\ . \tag{19}$$

The term $Q_{\text{la}}^{(0)}(\omega)$ in (19) is due to vibrations with large amplitude, and is thus exponentially small for small β; we note that the contribution to $Q(\omega)$ made by large-amplitude vibrations at $\omega>\omega_0$ and $4\beta\gtrsim 1$ is well described by $Q^{(0)}(\omega)$, since the frequency straggling for the actual vibrations $\delta\omega\gtrsim\omega_0\gg\Gamma$.[26] It is evident from (19) that in the range $\beta\lesssim\Gamma/\omega_0$, where $\tilde{Q}(\Omega)$ approximates $Q(\omega)$ with an accuracy of Γ/ω_0, the inaccuracy of Eq. (18) is also $\sim\Gamma/\omega_0\ll 1$. The latter is the case as well for $\beta\gg\Gamma/\omega_0$.

At $\beta\gg\Gamma/\omega_0$ it is convenient to show the validity and to estimate accuracy of Eq. (18) by analyzing the Fourier-transformed Einstein-Fokker-Planck equation for the probability density of a transition from one point in the phase space to another. In the frequency range $|\Omega|\gg\Gamma\alpha^{1/2}$ this equation may be solved by perturbation theory in α^{-1}, with the zeroth-order approximation corresponding to the dissipationless solution.[26] At $-\Omega\sim\Omega_0$, i.e., near the maximum of the spectral peak under consideration, this gives Eq. (18) for $Q(\omega)$ within an accuracy $\delta Q(\omega)/Q(\omega)\sim\Gamma/\omega_0\ll\beta$, if $\beta\ll 1$. [Note that the dissipation-induced correction is itself $\tilde{Q}(\Omega)-\tilde{Q}^{(0)}(\Omega)$, and its ratio to $Q^{(0)}(\omega)$ is approximately $\Gamma/\Omega_0\gg\Gamma/\omega_0$.]

Equation (18) also well describes the spectrum in the range $|\Omega|\lesssim\Gamma\alpha^{1/2}$ where the peak goes over into the high-frequency wing and $Q(\omega)$ is nonanalytic in Γ. The inaccuracy of (18), $\delta Q(\omega)/Q(\omega)$, is $\sim\beta/\sqrt{\alpha}\ll\beta$ here, while $\delta Q(\omega)/Q(\omega^{(1)}_{\max})\sim\Gamma/\omega_0$. This estimate can be obtained rigorously by making use of a method for solution of the Einstein-Fokker-Planck equation which some of us have suggested previously,[17(b)] but it follows also from qualitative considerations. Indeed, to obtain the above estimate it suffices to show that $Q(\omega)$ at $|\Omega|\lesssim\Gamma\alpha^{1/2}$ is determined mainly by vibrations with dimensionless energies $\epsilon\sim\beta/\sqrt{\alpha}$ [with the exception of the contribution $Q_{\text{la}}^{(0)}(\omega)$ made by large-amplitude vibrations, which is, however, taken into account in $Q^{(0)}(\omega)$ in (18)]. Then the inaccuracy of the equation $Q(\omega)-Q_{\text{la}}^{(0)}(\omega)=\tilde{Q}(\Omega)$ is $\sim\beta/\sqrt{\alpha}$, since just those corrections $\sim\epsilon$ and $\sim\Gamma/\omega_0\ll\beta/\sqrt{\alpha}$ were neglected in obtaining the expression (13a) for $\tilde{Q}(\Omega)$. This is also true of the inaccuracy of Eq. (18) as a whole, since $|Q^{(0)}(\omega)-Q_{\text{la}}^{(0)}(\omega)-\tilde{Q}^{(0)}(\Omega)|\sim(\beta^2/\sqrt{\alpha})Q(\omega)$ for respective frequencies delineated in (19) and (16).

To determine the characteristic value of ϵ, we note that in the model of Ref. 26 the drift (v_E,v_ϕ) and diffusion (D_E,D_ϕ) coefficients for the vibration energy E and the "slow" part of phase $\phi-\omega_0\sqrt{2}t$ are $v_E=-2\Gamma\Delta U(\epsilon+2\beta)$, $v_\phi=-(3\sqrt{2}/16)\omega_0\epsilon$, $D_E=8\Gamma(\Delta U)^2\epsilon\beta$, and $D_\phi=2\Gamma\beta/\epsilon$. The characteristic value t_c of the time, within which the spectrum $Q(\omega)$ is formed at $|\Omega|\lesssim\Gamma\alpha^{1/2}$, and the characteristic value $\epsilon=\epsilon_c$, are determined obviously by the condition $|\omega-\omega_0\sqrt{2}-v_\phi|t_c\sim(D_\phi t_c)^{1/2}\sim 1$. This gives $t_c\sim(\Gamma\alpha^{1/2})^{-1}$, $\epsilon_c\sim\beta/\sqrt{\alpha}\sim(\beta\Gamma/\omega_0)^{1/2}$ (for such values of ϵ_c, t_c, the change of the vibration phase due to drift and diffusion in energy is ~ 1 as well). Thus $\epsilon_c\sim\beta/\sqrt{\alpha}$ indeed, and therefore the inaccuracy of (18) is $\sim\beta/\sqrt{\alpha}\sim(\beta\Gamma/\omega_0)^{1/2}$.

In the range of the wings relatively far from the peak, and at sufficiently small β where $Q(\omega)$ is determined by dissipative processes and (17) holds, the relative inaccuracy of (18) is $\sim\beta$, but $Q(\omega)$ itself is very small here. At higher β this inaccuracy decreases in the range $\omega_0\sqrt{2}>\omega>\omega_0$ due to a sharp increase of the contributions from processes unrelated to dissipation. As a result of this, the halfwidth of the peak increases as well. In particular, for β as small as 0.07 the halfwidth turns out to be of the order of ω_0.

In the range $\omega<\omega_0$ an addend Q_{fl}, which is nonanalytic in Γ,

$$Q_{\text{fl}}\sim\frac{\omega_0}{\gamma}\left[\frac{\Gamma}{\beta\omega_0}\right]^{1/2}\exp\left[-\frac{1}{4\beta}\right], \tag{20}$$

should be taken into account in (18) at not too small β. It is caused by motion within a narrow energy band near the top of the potential barrier, $|E|\lesssim E_c$, where $E_c=(\beta\Gamma/\omega_0)^{1/2}\Delta U$. This motion is essentially random and aperiodic. The term (20) gives the main correction to the result of the dissipationless approximation in the range $\omega_0>\omega\gg\Gamma$, if

$$\frac{\Gamma}{\omega_0}\gg\beta^{-3}\exp\left[-\frac{1}{2\beta}\right]. \tag{21}$$

The aperiodicity of motion with energies $|E|\lesssim E_c$ causes the middle (in frequency) peak of $Q(\omega)$ to be somewhat different in shape from that of $Q^{(0)}(\omega)$. The difference is most prominent in the frequency ranges $\omega\lesssim\omega_c$ and $\omega^{(1)}_{\min}\lesssim\omega\lesssim 2\omega_c$, with $\omega_c=\omega(E_c)\approx\pi\omega_0/\ln(64\,\Delta U/E_c)$, where $Q^{(0)}(\omega)$ is formed just by vibrations whose energies E [determined from the equation $\omega(E)=\omega$] are $\lesssim E_c$ in absolute value. In particular, the smearing of frequencies of the respective vibrations causes $Q(\omega)$ to be smaller than $Q^{(0)}(\omega)$ in the vicinity of ω_c and $2\omega_c$.

IV. THE WING OF THE ZERO-FREQUENCY PEAK

The form of $Q(\omega)$ at very small frequencies is determined by dissipation and turns out to be quite complicated. At $\omega=0$ the function $Q(\omega)$ was mentioned to have a peak. At small β this peak near its maximum is described by the exponentially narrow[16,26,27(a)] distribution $Q_t(\omega)$, given by

$$Q(\omega)\approx Q_t(\omega),\quad \omega\ll\Gamma,\quad \exp(-1/4\beta)\ll 1\ ,$$

$$Q_t(\omega)=\frac{1}{\pi}q_0^2\Gamma'/(\omega^2+\Gamma'^2)\ , \tag{22}$$

$$\Gamma'=\lambda_0\Gamma\beta^{-1}\exp(-1/4\beta),\quad \lambda_0\sim 1\ .$$

The halfwidth of $Q_t(\omega)$, Γ', is determined by the probability of interwell transitions, which is the fundamental problem originally discussed by Kramers.[30] The value of λ_0 for the double-well Duffing oscillator has been calculated by Risken and Voigtlaender.[27(b)]

In the frequency range $\omega \sim \Gamma \gg \Gamma'$, the function $Q_t(\omega)$ is exponentially small. The spectral density $Q(\omega)$ in this range is formed by vibrations about the stable states. In order to calculate $Q(\omega)$ it is convenient to express the oscillator coordinate for the vibrations within the jth well ($j=1,2$) in terms of the energy E and phase ϕ,

$$q_j = \sum_{n=0}^{\infty} q_j^{(n)}(E)\cos(n\phi) \ . \tag{23}$$

(The explicit expressions for $q_j^{(n)}$ are given in Ref. 26.) The terms with $n \neq 0$ in this expansion oscillate in time as $\exp[\pm in\omega(E)t]$, and therefore on substitution of (23) into the expression (2) for $Q(\omega)$, they make only small, nonresonant contributions to $Q(\omega)$ at frequencies $\omega \ll \omega(E)$. The main contribution is due to the term with $n=0$, i.e., to fluctuations of the value of coordinate averaged over the vibration period $2\pi\omega^{-1}(E)$,

$$\begin{aligned} &Q(\omega) \approx \bar{Q}(\omega), \quad \Gamma' \ll \omega \sim \Gamma \ll \omega_0, \quad \exp(-1/4\beta) \ll 1, \\ &\bar{Q}(\omega) = \sum_{j=1,2} \bar{Q}_j(\omega) \ , \\ &\bar{Q}_j(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} dt e^{i\omega t}\langle \bar{q}_j(t)\bar{q}_j(0)\rangle \ , \\ &\bar{q}_j(t) \equiv q_j^{(0)}(E(t)) \ . \end{aligned} \tag{24}$$

According to Ref. 26

$$q_j^{(0)}(E) = (-1)^j q_0 \omega(E)/\omega_0\sqrt{2}, \quad q_0 = \omega_0\gamma^{-1/2} \ . \tag{25}$$

It is evident from (24) that the distribution $\bar{Q}(\omega)$ is worked out in terms of the oscillator energy E, and thus of $q_j^{(0)}(E)$, varying in time due to dissipation and fluctuations. At small β, when small $E - E_{\min}$ are essential, the dependence of $q_j^{(0)}$ on $E - E_{\min}$ is seen from (25) and (12) to be linear, and

$$\bar{Q}(\omega) = \frac{9}{16\pi} q_0^2 \beta^2 \frac{2\Gamma}{4\Gamma^2 + \omega^2}, \quad \Gamma' \ll \omega \ll \omega_0 \ . \tag{26}$$

We have taken into account here that for small β the nonlinearity of vibrations about stable states is small, and for such vibrations the energy averaged over the random force realizations, $\langle E - E_{\min}\rangle_f$, varies in time as[31]

$$\langle E(t) - E_{\min}\rangle_f \approx [E(0) - E_{\min} - B/2]\exp(-2\Gamma t) + B/2 \ .$$

The distribution $\bar{Q}(\omega)$ is seen from (26) to have the form of a Lorentzian peak with a halfwidth 2Γ. The intensity of the peak is proportional to the small quantity β^2. It should be noted that the emergence of a peak of similar type for a single-well quantum oscillator with a weakly asymmetric[32] (and weakly anharmonic) potential was demonstrated, in essence, when analyzing the low-frequency absorption of electromagnetic waves by localized vibrations in solids.[33]

For sufficiently small β when the ranges of applicability of Eqs. (13) and (26) overlap, i.e., when the wings of the zero-frequency peak and the peak at $\omega \approx \omega_0\sqrt{2}$ overlap, the position $\omega_{\min}^{(2)}$ of the lowest frequency minimum of $Q(\omega)$, as follows from (17) and (26), is given by

$$\omega_{\min}^{(2)} = \left(\frac{9\beta}{4}\right)^{1/4} \omega_0 \ . \tag{27}$$

Equation (27) is satisfied provided only the small-amplitude vibrations about the stable states contribute to $Q(\omega)$ at $\omega \approx \omega_{\min}^{(2)}$, while the contribution made by vibrations with energies $E - E_{\min} \sim \Delta U$ is negligible. Taking into account the explicit expression for the latter obtained in Ref. 26 one obtains a criterion of validity of (27) in the form

$$\begin{aligned} \frac{\Gamma}{\omega_0} \gg M\beta^{-3}\exp\left[-\frac{1}{4\beta} - \pi\left(\tfrac{9}{4}\beta\right)^{-1/4}\right], \\ \beta^{-4}\exp\left[-\frac{1}{2\beta}\right], \end{aligned} \tag{28}$$

where these inequalities correspond to $|\partial\tilde{Q}(\Omega)/\partial\omega| \gg |\partial Q^{(0)}(\omega)/\partial\omega|$ at $\omega = \omega_{\min}^{(2)}$ and $\bar{Q}(\omega_{\min}^{(2)}) \gg Q_{\mathrm{fl}}$ with $M \sim 10^2$. It should be noted that, as a consequence of the sharp increase in the intensity of fluctuations when the oscillator energy approaches the local potential maximum, the approximation (26) is valid only at rather small β.

Thus, we have analyzed the features of the spectral density of fluctuations $Q(\omega)$ for an underdamped double-well Duffing oscillator and pointed out the physical mechanisms causing these features. The analytic expressions for $Q(\omega)$ in a wide parameter range have been given as well. For particular parameter values the function $Q(\omega)$ may also be calculated in another way, by solving numerically the Einstein-Fokker-Planck equation. Such a solution for $\beta = 0.05$ and 0.25 and several Γ/ω_0 was obtained in a recent paper.[27(a)]

V. ANALOGUE SIMULATION EXPERIMENTS

In order to establish whether or not the interesting phenomena discussed in the preceding sections occur in actuality in a real physical system, we have studied the effect of pseudowhite external noise on an electronic circuit model of (1) with $\omega_0 = 1$. The circuit in question was essentially the same as one described previously[34] in connection with colored noise experiments. For the present work, however, the damping constant Γ was made as small as possible consistent with the avoidance of instabilities or self oscillation. The correlation time τ_N of the noise applied to the circuit was made very much shorter than the time constants τ_I of the two integrators: for the measurements to be reported $\tau_N = 10$ μs, $\tau_I = 0.33$ ms. The noise was therefore perceived by the circuit as white and of intensity

$$B = (\tau_N/\Gamma\tau_I)\langle V_N^2\rangle \ , \tag{29}$$

where $\langle V_N^2\rangle$ was the mean-square noise voltage applied to the circuit. The magnitude of Γ was too small to be measured directly, being to a significant extent determined by nonidealities of the analogue components comprising the circuit, but it was inferred from measure-

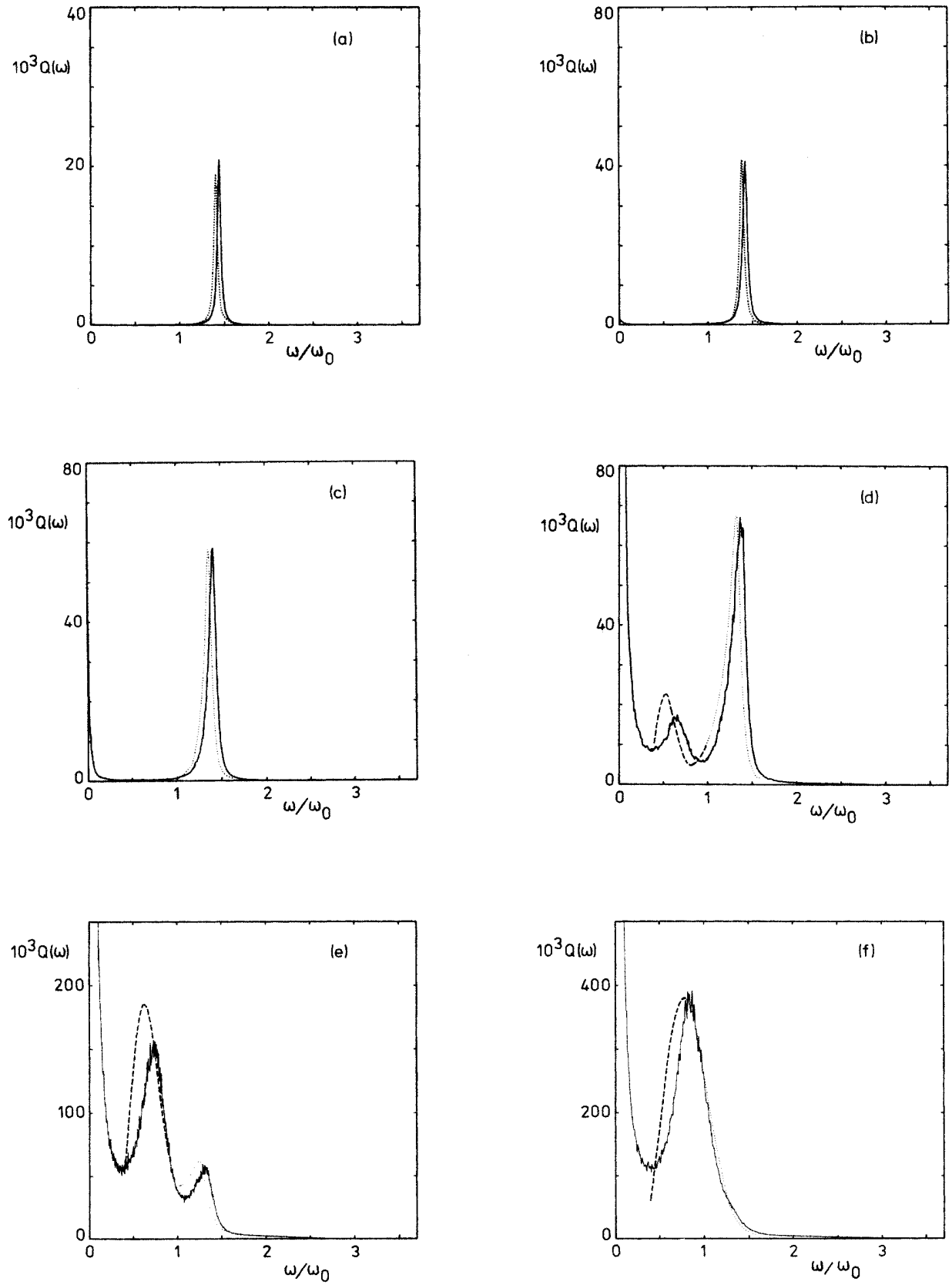

FIG. 2. Power spectral densities $Q(\omega)$ measured for the electronic circuit model of a Duffing oscillator driven by a random force (solid curves) compared with theoretical predictions (dashed curves) for (a) $\beta=0.0055$; (b) 0.0142; (c) 0.0291; (d) 0.060; (e) 0.149; (f) 0.369. Those sections of the theoretical curves that have been calculated in the nondissipative approximation are shown dashed; where explicit account has been taken of dissipative effects, the curves are dotted.

ments of $\langle \dot{q}^2 \rangle = B$ for known values of $\langle V_N^2 \rangle$. It was found by this method that $\Gamma = 0.015 \pm 5\%$, a value which turned out to be consistent with that obtained from the halfwidth of the intrawell oscillation peak for small β (see below).

The $q(t)$ output of the circuit was analyzed by means of a Nicolet 1080 data processor. The signal was discretized into blocks of 1024 samples, each of which was digitized with 12-bit (binary digit) precision. A standard fast-Fourier-transform (FFT) technique[35] was applied in turn to each block to compute $Q(\omega)$. This operation was repeated typically 500–1000 times, the resultant $Q(\omega)$ spectra being summation averaged in order to enhance their statistical quality. It should be emphasized that these procedures in no way affected the operation of the circuit and that the role of the digital data processor was purely instrumental and/or analytical.

A set of experimental $Q(\omega)$ spectra measured in this way for a range of values of β is shown in Fig. 2. It may be noted immediately that the results are in excellent qualitative agreement with the discussion given above and that, for the range $0.06 \lesssim \beta \lesssim 0.2$, $Q(\omega)$ has the three distinct peaks predicted by the theory.[26]

To facilitate a more detailed comparison of theory and experiment there are also plotted in Fig. 2(a)–2(f) the theoretical curves obtained for the appropriate values of β and Γ/ω_0 by using the explicit expressions (18), (13), and (26) (the parts of curves calculated within the dissipationless approximation are shown dashed). The theoretical and experimental curves are evidently close to each other over a wide frequency range. The general form of the peaks of $Q(\omega)$, including their width, asymmetry, and height ratio (for the peaks at frequencies $\omega_{\max}^{(1)}$ and $\omega_{\max}^{(2)}$), as well as the evolution of their overall shape and the shift of maxima with varying β in the experiment are the same as in theory. The positions of the high-frequency and middle maxima $\omega_{\max}^{(1,2)}$ as well as the positions of the minimum between them $\omega_{\min}^{(1)}$ in the experimental and theoretical curves are also close (cf. Fig. 3). It can be seen, however, that the theoretical values of $\omega_{\max}^{(1,2)}, \omega_{\min}^{(1)}$ are systematically smaller than the experimental ones.

At $\beta = 0.0055$ the peak in the experimental spectrum is seen from Fig. 2(a) to be practically Lorentzian. This makes it convenient to use that data for an alternative determination of the friction coefficient Γ. In the theory the peak shape at small β is described by the function $\tilde{Q}(\Omega)$ in (13a). The simple algebraic expression for $\tilde{Q}(\Omega)$ when the peak shape is close to being Lorentzian, i.e., when the parameter $\alpha = (3/4\sqrt{2})\beta\omega_0/\Gamma$ is small, has been obtained previously.[17(a)] It follows from this expression that the halfwidth of the peak is $\approx \Gamma(1 + \frac{52}{15}\alpha^2)$. By comparing the latter expression with the halfwidth of the experimental curve, assumed symmetrical, we obtain $\Gamma/\omega_0 \approx 0.022$ and $\alpha \approx 0.13$ for $\beta = 0.0055$. It follows from (14) that for such α the maximum of $Q(\omega)$ is shifted from $\omega_0\sqrt{2}$ to lower frequencies by $\approx (3/\sqrt{2})\beta\omega_0 \approx 0.012\omega_0$. At the same time the maximum of the experimental curve is displaced from $\omega_0\sqrt{2}$ to higher frequencies by $\approx 0.025\omega_0$. Because the frequency $\omega_0\sqrt{2}$ is the largest eigenfrequency of intrawell vibrations of the Duffing oscillator, the displacement of that peak to frequencies exceeding $\omega_0\sqrt{2}$ at small β must necessarily be connected with small systematic differences between the real system investigated experimentally, and the idealized Duffing oscillator model considered by the theory.

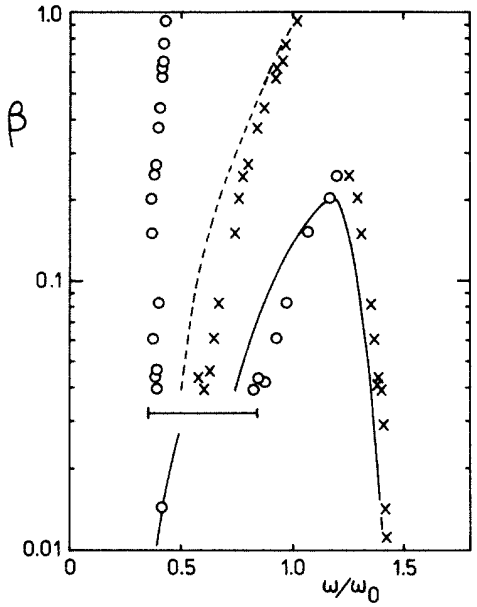

FIG. 3. Frequencies of the extrema in $Q(\omega)$ plotted as functions of the noise intensity β. The positions of maxima $\omega_{\max}^{(1)}$ and $\omega_{\max}^{(2)}$ measured for the analogue electronic circuit are shown as crosses; those of the minimum between them, $\omega_{\min}^{(1)}$, and of the minimum between the zero-frequency peak and $\omega_{\max}^{(2)}$, are shown by open circles. The curves are theoretical predictions, with calculations in the nondissipative approximation shown dashed, and those that include dissipative effects shown as solid curves. The short section of solid curve in the lower left of the figure refers to the range of small β where only a single minimum is to be expected; the gap separating it from the other curves and data represents a parameter range where three adjacent extrema are so broad that useful measurements of their positions become impossible, as indicated by the single wide bar.

As β increases, intrawell vibrations with higher energies and correspondingly smaller eigenfrequencies contribute to $Q(\omega)$ more and more, and the peak under consideration is shifted to smaller frequencies. At the same time the nonlinearity of the vibrations influences more and more the shape of the peak. Since this influence depends on the ratio of the two small parameters, β and Γ/ω_0, even for β as small as 0.0142 [Fig. 2(b)] the peak turns out to be somewhat asymmetric in both the theoretical and experimental curves; the curves are parallel, and their halfwidths exceed the $\beta = 0$ value by a factor of ~ 1.4; the theoretical curve is obtained from (13a) where $\alpha \approx 0.35$.

The asymmetry and the halfwidth of the peak under consideration increase rapidly with increasing β [cf. Figs. 2(b)–2(e)]. The theoretical curves in the region of

the peak in Figs. 2(c)–2(e) are calculated from (18), which, as it was shown above, is valid in a wide range of β and ω [for the values of β in Fig. 2(a) and 2(b), Eqs. (18) and (13) coincide to within $\sim 2\%$ inside the peak]. It is clear that (18) describes very well the experimental curves (provided that account is taken of the small systematic shift in frequency mentioned above). The experiment can therefore be taken to confirm the conclusion[17(a)] that the peak shape arises as the result of a competition between dissipative and nondissipative broadening mechanisms. We note that for a single-well nonlinear underdamped oscillator the change of shape of the spectral density peak, and the shift in its position, as α is varied was observed in an earlier analogue experiment by Fronzoni *et al.*;[23] however, a detailed comparison with theory was not carried out in that paper.

For the values of β in Figs. 2(a)–2(d) the halfwidth Γ' (22) of the zero-frequency peak of $Q(\omega)$ is so small that most of the peak could not in practice be plotted. For $\beta=0.0055$ the peak as a whole is δ shaped within the scale chosen in Fig. 2(a). The wing of the peak at $\beta=0.0142$ [Fig. 2(b)] is formed mainly by small-amplitude nonlinear vibrations about the stable states and is described by (26). For the higher values of β investigated experimentally the additions to (26) caused by the oscillator motion with relatively high energies can be shown to be essential. Within the range of its applicability (26) clearly fits the experimental data very well as shown, for example, in Fig. 2(b).

The most distinctive feature of the spectra shown in Figs. 2(d) and 2(e) is the presence of three separate peaks of $Q(\omega)$. It is seen from these figures, that although the shapes and heights of the middle peaks in the theoretical and experimental curves are similar, the disagreement between theory and experiment in the region of the middle peak is much more prominent than in the case of the high-frequency peak; furthermore, it does not simply correspond to the shift in frequency. This arises in large measure from the neglect of dissipation in calculating $Q(\omega)$ in the vicinity of the middle peak [cf. the discussion which follows (20) and (21) in Sec. III]. In particular, for the experimental values of Γ/ω_0 and β, the frequency ω_c turns out to be close to the position (6) of the maximum of $Q^{(0)}(\omega)$; it follows, therefore, from the arguments of Sec. III, that the maximum of $Q(\omega)$ shifts to higher frequencies and that the height of the peak is smaller than in the dissipationless approximation. An appreciable influence of dissipation on the middle peak at $4\beta \ll 1$ is seen also from the results of Voigtlaender and Risken[27(a)] for $\beta=0.05$.

An important additional source of disagreement between theory and experiment in the range of the middle peak of $Q(\omega)$ probably lies in the small systematic differences between the real system investigated experimentally and the idealized model (1). The coordinate q and momentum p of the oscillator performing the over-barrier vibrations, which form the peak, vary over particularly wide intervals exceeding $2q_0\sqrt{2}$ and $\omega_0 q_0\sqrt{2}$, respectively. Therefore, to describe the peak shape quantitatively, the model should be close to the real system throughout this very large parameter range. This applies to the potential shape, the friction force, and the distribution of the random force $f(t)$ (in particular, in a real system the friction coefficient and the random force may depend on dynamical variables). The sensitivity of the middle peak of $Q(\omega)$ to parameters of a system is especially high when $4\beta \ll 1$, since the peak arises here as the result of a competition between large addends in the exponent[26] [one of them reflecting the distribution of large fluctuations of $f(t)$]. We note also a strong dependence of the peak height on the noise-intensity parameter β when $4\beta \ll 1$, namely that $\partial \ln Q(\omega^{(2)}_{\max})/\partial \ln \beta \approx (4\beta)^{-1} \gg 1$.

The loci of the extrema of $Q(\omega)$ as functions of β were investigated experimentally by local parabolic interpolation of spectra such as those of Fig. 2. The results of this procedure are shown by the data points of Fig. 3, where the circles represent minima and the crosses represent maxima. The lines in the figure represent theoretical values based on the discussion of Secs. II–IV. These data correspond to the positions $\omega^{(1)}_{\max}$ and $\omega^{(2)}_{\max}$ of the maxima of the right-hand side and middle peaks of $Q(\omega)$, as well as to the values of the position $\omega^{(1)}_{\min}$ of the minimum between these peaks [at $\beta \lesssim 0.06$ the values of $\omega^{(1,2)}_{\max}$ and $\omega^{(1)}_{\min}$ resulting from numerical calculations based on (18) are well described by simple analytic expressions (14), (14') (6), and (8)]. Also plotted in Fig. 3 are the experimental data for the position $\omega^{(2)}_{\min}$ of the left minimum, which lies between the zero-frequency peak of $Q(\omega)$ and the middle one, and theoretical and experimental data for the position $\omega_{\min}$ of the minimum of $Q(\omega)$ within the range of β for which (at the experimental value of Γ/ω_0) the middle peak is not manifested.

The theoretical values of $\omega_{\min}$ are obtained from (27). It should be noted that in the region $\beta \approx 0.03$–0.04, where the middle peak of $Q(\omega)$ becomes distinct for a given Γ/ω_0, the values of $\omega_{\min}$ and $\omega^{(2)}_{\max}$ given by (27) and (6), respectively, are very close to one another. Therefore, it seems when looking at Fig. 3 that, in a narrow range of β, the frequency $\omega_{\min}$ "splits" three ways into $\omega^{(2)}_{\min}$, $\omega^{(2)}_{\max}$, and $\omega^{(1)}_{\min}$. In reality, since $\omega^{(2)}_{\max}$ as given by (6) is slightly smaller than $\omega_{\min}$ as given by (27) for $\beta \approx 0.03$–0.04, one would expect $\omega_{\min}$ to go over into $\omega^{(1)}_{\min}$ as β increases. Only for values of Γ/ω_0 smaller by many times than those in the experimental device will $\omega_{\min}$ go over into $\omega^{(2)}_{\min}$.

It is evident from Fig. 3 that the theory well describes not only the experimental data for $\omega^{(1)}_{\min}$, $\omega_{\min}$, and $\omega^{(1)}_{\max}$ (especially if one allows for a systematic shift of the latter) at comparitively small β, but also the data for $\omega^{(2)}_{\max}$ at relatively large β, when dissipation-induced corrections to $Q(\omega)$ are small near the maximum of peak [see also Fig. 2(f)].

VI. CONCLUSION

The analogue experiments have fully vindicated the prediction, based (a) on an analytic theory[26] and (b) on matrix continued fraction expansion,[27(a)] that under certain conditions $Q(\omega)$ for the double-well Duffing oscillator will possess three separate maxima. The analytic theory[17,26] has been extended to provide a description of

the shape of the peak due to intrawell vibrations valid for arbitrary noise intensities. The wing of the zero-frequency peak at weak noise is described as well. The theoretical results concerning not only the general structure of the spectrum, but also the shapes, intensities, and positions of the peaks at finite frequencies are in good agreement with analogue experiment [and especially so if due account is taken of a small systematic shift in frequency in the latter caused by imperfect modeling of (1) by the circuit]. We note in particular the quantitative confirmation of the predicted[17] rather complicated shape of the peak due to underdamped intrawell vibrations even for small noise intensities. The fact that the predicted[17,16,26,27] evolution of $Q(\omega)$ with β has now been confirmed in a well-characterized "real physical system," albeit one that is somewhat contrived in character, can be taken to imply that very similar phenomena may confidently be anticipated in a very wide range of underdamped bistable systems occurring in nature.

We conclude by commenting on how remarkable it is that the peak at zero frequency has so consistently been ignored in previous theories, and especially so given that low-frequency hopping between potential wells in an underdamped bistable system was exactly the problem first considered by Kramers[30] in 1940. Certainly, noise or thermally induced hopping among bound or localized states is necessarily a beginning point for modeling the wide variety of noise encountered in nature.[36] It is our hope that the experimental results and theoretical discussion given above will have served to clarify a murky point that is nonetheless of wide ranging relevance and importance.

ACKNOWLEDGMENTS

We are grateful to G. Nicolis for a stimulating discussion and subsequent correspondence on bistable oscillator dynamics. This research was supported by the United Kingdom Science and Engineering Research Council.

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