Spectral Efficiency of CDMA with Random Spreading

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***Abstract*—The CDMA channel with randomly and independently chosen spreading sequences accurately models the situation where pseudonoise sequences span many symbol periods. Furthermore, its analysis provides a comparison baseline for CDMA channels with deterministic signature waveforms spanning one symbol period. We analyze the spectral efficiency (total capacity per chip) as a function of the number of users, spreading gain, and signal-to-noise ratio, and we quantify the loss in efficiency relative to an optimally chosen set of signature sequences and relative to multiaccess with no spreading. White Gaussian background noise and equal-power synchronous users are assumed. The following receivers are analyzed: a) optimal joint processing, b) single-user matched filtering, c) decorrelation, and d) MMSE linear processing.**

***Index Terms*—Channel capacity, code-division multiple access, Gaussian channels, multiuser detection, multiuser information theory, spread spectrum.**

I. Introduction and Summary of Results

A. Spectral Efficiency

DIRECT-Sequence Spread-Spectrum code-division multiple access (CDMA) has well-known desirable features: dynamic channel sharing, robustness to channel impairments, graceful degradation, ease of cellular planning, etc. These advantages result from the assignment of "signature waveforms" with large time–bandwidth products to every potential user of the system. Each signature can be viewed as a unit-norm vector in an N-dimensional signal space, where N is the spreading gain or number of chips per symbol. In the model considered in this paper, K users linearly modulate their signatures with the outputs of respective autonomous encoders. The central question we address is the capacity loss incurred by the imposition of such a structure on the transmitted signals, and by the imposition of several suboptimal, but practically appealing, receiver structures based on single-user decoding. Our analysis considers a white Gaussian channel with users constrained to have identical average received powers.

The fundamental figure of merit is the *spectral efficiency* C, defined as the total number of bits per chip that can be transmitted arbitrarily reliably. Since the bandwidth of the CDMA system is (roughly) equal to the reciprocal of the chip duration, the spectral efficiency can be viewed as the bits per second per hertz (bits/s/Hz) supported by the system.[1] Note that if the code rates (bits per symbol) employed by each individual user are identical and denoted by R, then the spectral efficiency is equal to the product

$$\mathsf{C} = \frac{K}{N} R. \tag{1}$$

In a system where no spreading is imposed, the encoders are able to control the symbols modulating each chip independently. Therefore, assuming chip-synchronism, the Cover-Wyner capacity region of the conventional Gaussian multiaccess channel [1] applies to this case and the spectral efficiency in the absence of spreading is given by

$$\mathsf{C}^* = \frac{1}{2}\log\left(1 + \frac{K}{N}\mathsf{SNR}\right). \tag{2}$$

where, for consistency with the results below, SNR denotes the energy per transmitted N chips divided by the Gaussian noise spectral level σ^2. This means that the energy per bit divided by $N_0 = 2\sigma^2$ is

$$\frac{E_b}{N_0} = \frac{\mathsf{SNR}}{2R}. \tag{3}$$

Once the spectral efficiency is determined, it is possible to obtain the minimum bandwidth necessary to transmit a predetermined information rate or the maximum information rate that can be supported by a given bandwidth. In order to compare different systems (with possibly different spreading gains and data rates), the spectral efficiency must be given as a function of $\frac{E_b}{N_0}$. According to (1) and (3), if the spectral efficiency of the system reaches the optimum level C^* in (2), then SNR can be substituted by

$$\mathsf{SNR} = \frac{2N}{K}\frac{E_b}{N_0}\mathsf{C}^*$$

so the maximum spectral efficiency $\mathsf{C}^*(\frac{E_b}{N_0})$ in the absence of spreading is the solution to

$$\mathsf{C}^*\left(\frac{E_b}{N_0}\right) = \frac{1}{2}\log\left(1 + 2\mathsf{C}^*\left(\frac{E_b}{N_0}\right)\frac{E_b}{N_0}\right) \tag{4}$$

[1] With quadrature orthogonal modulation, one bit per chip corresponds to essentially 2 bits/s/Hz. Without loss of conceptual scope, we focus on real-valued channels in this paper.

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or equivalently

$$\frac{4^{\mathsf{C}^*\left(\frac{E_b}{N_0}\right)}-1}{2\mathsf{C}^*\left(\frac{E_b}{N_0}\right)}=\frac{E_b}{N_0}.$$

Since (4) does not depend on K, when the transmitted signals are not constrained to the spread-spectrum format, the spectral efficiency is the same as in a single-user system with power equal to the sum of the powers.

The solution to (4) is well known [2] to be positive if and only if

$$\frac{E_b}{N_0}>\log_e 2=-1.6 \text{ dB}.$$

Furthermore, the asymptotic growth satisfies

$$\lim_{\frac{E_b}{N_0}\to\infty}\frac{\mathsf{C}^*\left(\frac{E_b}{N_0}\right)}{10\log_{10}\frac{E_b}{N_0}}=\frac{\log 10}{20}=0.166 \text{ bits/dB}. \tag{5}$$

Assuming maximum-likelihood decoding, the capacity of synchronous and asynchronous CDMA white Gaussian multiple-access channels was found in [3], [4] as a function of the assigned signature waveforms and signal-to-noise ratios. CDMA channel capacity depends on the signature waveforms through their crosscorrelations. For example, the spectral efficiency of a synchronous CDMA system where identical signature waveforms are assigned to all users is given by

$$\mathsf{C}^{\text{sgle}}=\frac{1}{2N}\log(1+K\,\mathsf{SNR}) \tag{6}$$

whereas in the case of orthogonal sequences the spectral efficiency is equal to

$$\mathsf{C}^{\text{orth}}=\frac{K}{2N}\log(1+\mathsf{SNR}), \qquad \text{if } K\le N. \tag{7}$$

Substituting

$$\mathsf{SNR}=\mathsf{C}^{\text{orth}}\frac{2N}{K}\frac{E_b}{N_0}$$

we obtain that if $K\le N$, then

$$\mathsf{C}^{\text{orth}}\left(\frac{K}{N},\frac{E_b}{N_0}\right)=\frac{K}{N}\mathsf{C}^*\left(\frac{E_b}{N_0}\right). \tag{8}$$

The equality of C^{orth} and C^* for $K=N$ is a consequence of the well-known fact [1] that orthogonal multiple access incurs no loss in capacity relative to unconstrained multiple access for equal-rate equal-power users in an additive Gaussian noise channel. It is also known [5] that even if $K>N$, there exist spreading codes that incur no loss in capacity relative to multiaccess with no spreading.

Despite their overlap in time and frequency, the K users can be completely separated at the receiver by means of a matched-filter front-end provided the signature waveforms are mutually orthogonal. In that case, single-user error-control coding and decoding is sufficient. Nonorthogonal CDMA arises whenever $K>N$ or the users are asynchronous. Moreover, channel distortion (such as multipath) and out-of-cell interference are common impairments that destroy the orthogonality of signature waveforms. Optimal spectral efficiency in nonorthogonal CDMA requires joint processing and decoding of users. As advocated in a number of recent works [6]–[24], it is sensible in terms of complexity–performance tradeoff to adopt as a front-end a (soft-output) *multiuser detector* [25] followed by autonomous single-user error-control decoders. In our analysis of spectral efficiency we consider, in addition to optimal decoding, some popular linear multiuser detector front-ends

- single-user matched filter,
- decorrelator,
- Linear Minimum Mean-Square-Error (MMSE).

In those three cases we study suboptimal single-user decoding of individual linear transformation outputs. Suboptimality results from two different simplifications: a) the output of only one linear transformation is used, and b) no attempt to exploit knowledge of the codebooks of interferers is made at each individual single-user decoder.

Unlike the aforementioned references [3], [4], our purpose is to evaluate the spectral efficiency of CDMA systems where signature waveforms are assigned at random. Denote the unit-norm signature of the kth user by

$$[c_{k1},\cdots,c_{kN}]$$

and assume that $c_{kj}\in\{-1/\sqrt{N},+1/\sqrt{N}\}$ are chosen equally likely and independent for all (k,j). (Nonbinary random signature sequence models are also analyzed in the paper.) The rationale for averaging capacity with respect to random signature waveforms is twofold.

- It accurately models CDMA systems (such as IS-95, [26], [27]) where pseudonoise sequences span many symbol periods.
- The spectral efficiency averaged with respect to the choice of signatures provides a lower bound to the optimum spectral efficiency achievable with a deterministic choice of signature waveforms.

Most analyses of multiuser detectors have focused on the bit-error rate of uncoded communication [25]. The results found in this paper for the decorrelator and MMSE receivers give the best achievable performance with error-control coding assuming random signature waveforms. As we mentioned, this serves as a lower bound to the performance achievable through design of signatures with favorable crosscorrelation properties. Furthermore, this analysis is directly applicable to multiuser detectors operating with spreading codes whose periodicity is much larger than the spreading gain (e.g., [28]–[30]).

B. Previous Results

We now summarize the main results available in the literature relevant to the problem considered here. Other than [3], [31], most existing capacity results pertain to the symbol-synchronous case.

1) Optimal Decoding: Optimal decoding can be performed by a bank of matched filters (which converts the received process to a discrete-time vector process) followed by joint maximum-likelihood decoding of the error control code (e.g., [32], [33]). The formula in [4] for capacity as a function of the signature waveforms was used in [5] to show that with *Welch-bound-equality* (WBE) signature waveforms the spectral efficiency of the CDMA system is equal to the case of no-spreading (2). A necessary condition for the existence of WBE signature waveforms is $K \geq N$. When the number of users is an integer multiple of the spreading gain $K = mN$, then an optimum signature sequence assignment is obtained by selecting a set of N orthogonal sequences and assigning each of them to m users. It is straightforward to check that the spectral efficiency of such a CDMA system is given by (2) if optimal decoding is used.

More generally, WBE signature waveforms with binary antipodal spreading are known to exist for many other choices of (K, N). For example, given a Hadamard matrix of size K one can take any $N \leq K$ rows of the matrix to form an $N \times K$ matrix of WBE signature vectors with binary antipodal spreading. Hadamard matrices of size $m2^s$, $m = 1, 2, 3 \cdots$, $s = k(m), k(m)+1, \cdots$, with $k(m) = [2\log_2(m-3)]$ are known to exist [34]. Thus for any $\beta \geq 1$, sequences of WBE signature waveforms whose K/N ratio converges to β are guaranteed to exist.[2]

It had been conjectured in [38] that as $K/N \to \infty$ and $\mathsf{SNR} \to \infty$ the loss incurred by a random choice of signatures vanishes. This was verified independently by Monte Carlo simulation in [21] and with an asymptotic $K \geq N \to \infty$ lower bound on the average capacity for random signature waveforms in [39], [40].

2) Single-User Matched Filter: The capacity of the single-user matched filter followed by single-user decoding has been previously analyzed approximating the multiaccess interference at the output of the matched filter by Gaussian noise. When the signatures are random and are antipodally modulated, then [41] (see also [42], [43]) found that the spectral efficiency as $K/N \to \infty$ goes to 0.5 nat/chip $= 0.72$ bit/chip.

3) Decorrelator: If the signature waveforms are linearly independent, a front-end consisting of a bank of decorrelators [44] incurs no loss of information since it is a one-to-one transformation of the sufficient statistics and eliminates multiaccess interference from each of its outputs. Optimal decoding still requires joint processing of all K outputs due to the correlation among the noise components. The point of studying capacity with a decorrelating front-end is that it lends itself naturally to a suboptimal approach in which single-user decoding is based on each individual decorrelator (unquantized) output. Since the output of each single-user decorrelator is uncontaminated by multiaccess interference, the analysis of the single-user decorrelator capacity requires the single-user capacity formula evaluated at the decorrelator output signal-to-noise ratio, which is equal to the maximum near–far resistance $\bar{\eta}$ [25]. The expected maximum near–far resistance with random binary sequences is shown in [45] to be lower-bounded by

$$E[\bar{\eta}] \geq 1 - \frac{K-1}{N}$$

for $K \leq N$. This bound is shown to be tight as $K \to \infty$ in [25] (see also [17]). An analogous problem in the asynchronous setting is considered in [46] (see also [12]). Bounds and Monte Carlo simulation of capacity using the decorrelator were given in [21] and [22]. Random sequences with complex-valued chips where sequences are uniformly distributed on the surface of the unit-radius N-dimensional sphere are considered in [20] and [23]. Those references find an expression for the density function of the maximum near–far resistance as a function of K and N, and an asymptotic $(K \to \infty)$ expression for E_b/N_0 as a function of the desired spectral efficiency.

4) MMSE: The linear MMSE receiver [47], [48] offers a compromise between the multiaccess interference suppression capabilities of the decorrelator and the optimal background-noise-combating capabilities of the single-user matched filter. Unlike the decorrelator, the MMSE filter is well-defined regardless of whether K is smaller or larger than N. As in the case of the decorrelator, we are interested in the spectral efficiency of the bank of MMSE linear transformations followed by single-user decoders. When the channel symbols are binary and binary decisions are made at the output of the linear transformation, [49] shows that the spectral efficiency (of both the matched filter and of the MMSE transformation) tends to 0.46 bits/chip as $K/N \to \infty$ in the synchronous case and to 0.69 bits/chip in the asynchronous case. Monte Carlo simulation of the expected MMSE capacity with binary sequences and (nonbinary) power-constrained codewords was given in [21]. Monte Carlo simulations with spherical random codes are also undertaken in [20]. Up to now, no analytical results existed on the asymptotic spectral efficiency of MMSE processing or on the optimal spreading gain as a function of the number users. Simultaneously to a conference version [50] of the present paper, [51] gives an equation satisfied by the large-K output signal-to-noise ratio of the MMSE receiver without assuming equal received powers.

C. Summary of Results

Next we summarize the main conclusions found in this paper on the capacity of spread-spectrum systems with random spreading. Since the spectral efficiency depends on the spreading sequences, it is a random variable itself. In our asymptotic (in K) analysis we do not just average spectral efficiency with respect to the spreading sequences, but we show convergence of the (random) spectral efficiencies to deterministic quantities. Such asymptotic determinism holds regardless of whether the period of the spreading sequence is equal or longer than the symbol interval. Fig. 1 shows the spectral efficiencies of the optimal receiver, the MMSE receiver, the decorrelator, and the single-user matched filter with random spreading and a fixed $\frac{E_b}{N_0}$. For comparison purposes, we show the spectral efficiencies achievable by an optimum joint decoder with no spreading and an orthogonal CDMA system for $K \leq N$.

Throughout the paper, the key ratio of number of users to number of dimensions is denoted by

$$\beta = \frac{K}{N}.$$

[2] Relaxing the condition that the signature waveforms are binary-valued, the construction of WBE signatures has been studied in [35] with equal powers and [36], [37] with arbitrary powers.

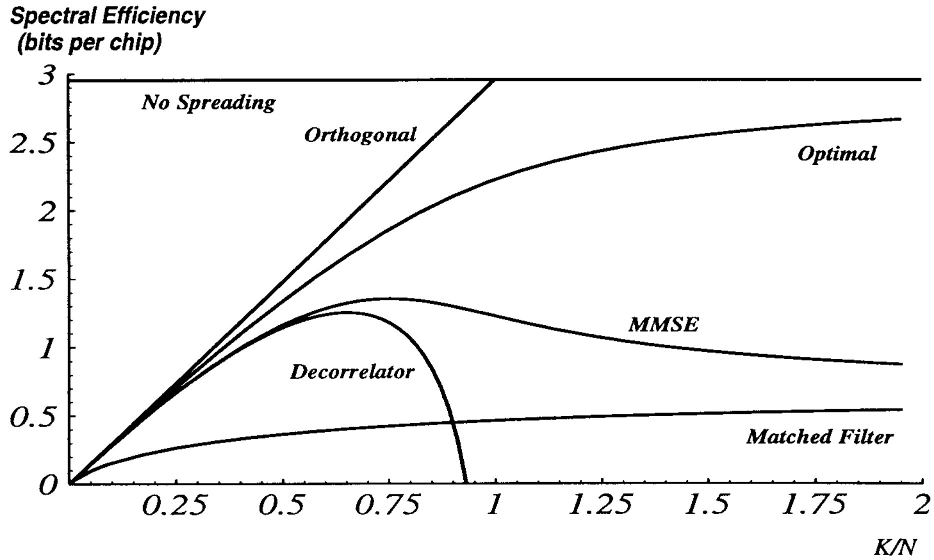

Fig. 1. Large-K spectral efficiencies for $E_b/N_0 = 10$ dB. No spreading (4); orthogonal (8). Random signatures: optimal (9), matched filter (10), decorrelator (11), MMSE (12).

1) Asymptotic Optimum Spectral Efficiency: The optimum spectral efficiency for $0 < \beta$ converges almost surely as $K \to \infty$ to

$$\lim_{K\to\infty} \mathsf{C}^{\mathrm{opt}} = \frac{\beta}{2}\log\left(1+\mathsf{SNR}-\frac{1}{4}\mathcal{F}(\mathsf{SNR},\beta)\right) + \frac{1}{2}\log\left(1+\mathsf{SNR}\beta-\frac{1}{4}\mathcal{F}(\mathsf{SNR},\beta)\right) - \frac{\log e}{8\mathsf{SNR}}\mathcal{F}(\mathsf{SNR},\beta) \tag{9}$$

where

$$\mathcal{F}(x,z) \stackrel{\mathrm{def}}{=} \left(\sqrt{x(1+\sqrt{z})^2+1}-\sqrt{x(1-\sqrt{z})^2+1}\right)^2.$$

2) Loss in Spectral Efficiency: When $K = N = 2$, binary random sequences achieve 75% of the spectral efficiency of orthogonal sequences. When K is large, the loss in spectral efficiency as a function of E_b/N_0 due to a random choice of sequences (as opposed to optimal) vanishes as $E_b/N_0 \to \infty$ or as $\beta \to \infty$. The maximum loss is 50% and occurs at $K = N$, $E_b/N_0 \downarrow \log_e 2$.

3) Matched-Filter Spectral Efficiency: The spectral efficiency of the single-user matched filter converges almost surely as $K \to \infty$ to

$$\lim_{K\to\infty} \mathsf{C}^{\mathrm{sumf}} = \frac{\beta}{2}\log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{SNR}\beta}\right). \tag{10}$$

The maximum (over K/N) spectral efficiency of the single-user matched filter receiver is

$$\lim_{\beta\to\infty} \mathsf{C}^{\mathrm{sumf}}\left(\beta,\frac{E_b}{N_0}\right) = \frac{\log_2 e}{2} - \frac{1}{2}\frac{N_0}{E_b}$$

for $\frac{E_b}{N_0} > \log_e 2$. Unless $\frac{E_b}{N_0}$ is relatively low and β is high, the use of random signatures as opposed to optimally chosen sequences brings about substantial losses in spectral efficiency for the single-user matched filter. For example, if $K = N$ random signatures achieve at most $1/3$ of the capacity of orthogonal signatures.

4) Decorrelator Spectral Efficiency: If $\beta \le 1$, the spectral efficiency of the decorrelator converges in mean-square sense as $K \to \infty$ to

$$\lim_{K\to\infty} \mathsf{C}^{\mathrm{deco}} = \frac{\beta}{2}\log(1+\mathsf{SNR}(1-\beta))$$

which yields

$$\mathsf{C}^{\mathrm{deco}}\left(\beta,\frac{E_b}{N_0}\right) = \beta\mathsf{C}^*\left((1-\beta)\frac{E_b}{N_0}\right). \tag{11}$$

5) MMSE Spectral Efficiency: If $\beta > 0$, the spectral efficiency of the linear MMSE transformation converges in mean-square sense as $K \to \infty$ to

$$\lim_{K\to\infty} \mathsf{C}^{\mathrm{mmse}} = \frac{\beta}{2}\log\left(1+\mathsf{SNR}-\frac{1}{4}\mathcal{F}(\mathsf{SNR},\beta)\right). \tag{12}$$

The difference between the optimum spectral efficiency and the MMSE spectral efficiency is equal to

$$\lim_{K\to\infty} \mathsf{C}^{\mathrm{opt}} - \lim_{K\to\infty} \mathsf{C}^{\mathrm{mmse}} = \Delta\left(\frac{\mathcal{F}(\mathsf{SNR},\beta)}{4\mathsf{SNR}}\right) \tag{13}$$

where

$$\Delta(x) \stackrel{\mathrm{def}}{=} \frac{1}{2}\log\left(\frac{1}{1-x}\right) - \frac{x}{2}\log e, \qquad 0 \le x < 1$$

and

$$\frac{\mathcal{F}(\mathsf{SNR},\beta)}{4\mathsf{SNR}} \le \min\{\beta, 1\}.$$

Since $\Delta(x) \to \infty$ as $x \uparrow 1$, the loss of spectral efficiency due to linear processing (followed by single-user decoding) grows without bound with SNR when $\beta \ge 1$.

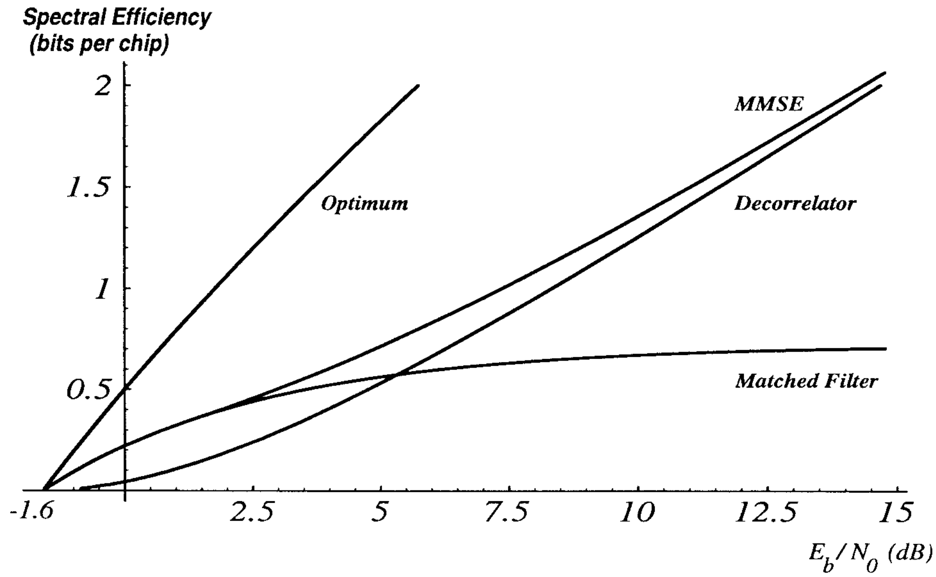

Fig. 2. Large-K spectral efficiencies with optimum K/N.

6) Optimum Coding–Spreading Tradeoff: When the spreading gain N is a free design parameter, it is of course interesting to solve for the value that optimizes the spectral efficiency with random spreading. The answer, as we can see in Fig. 1, depends heavily on the type of receiver. For either optimum processing or matched filtering followed by single-user decoding, spectral efficiency is maximized by letting $K/N \to \infty$. Thus for those receivers, the coding–spreading tradeoff favors coding: it is best to use error-correcting codes with very low rates (cf. (1)) and a negligible spreading gain with respect to the number of users. This conclusion was known to hold for the single-user matched filter [41] (although it may not extend to noncoherent demodulation models [52]). Note, however, that the behavior of optimum processing and the conventional single-user matched filter at $K/N \to \infty$ are quite different: the optimal spectral efficiency grows without bound with $\frac{E_b}{N_0}$, whereas the matched-filter efficiency approaches 0.72 bit/chip monotonically as $\frac{E_b}{N_0} \to \infty$.

For large K, the optimum choice of K/N for the decorrelator ranges from 0 for $\frac{E_b}{N_0} \downarrow -1.6$ dB to 1 for $\frac{E_b}{N_0} \to \infty$ (cf. Fig. 3). The optimum coding–spreading tradeoff of the decorrelator dictates using codes whose rates (bits/symbol) lie between 0 ($\frac{E_b}{N_0} \downarrow -1.6$ dB) and C^{deco} ($\frac{E_b}{N_0} \to \infty$). With an optimum choice of spreading gain, the decorrelator spectral efficiency with random signature waveforms is better than that of the single-user matched filter for $\frac{E_b}{N_0} > 5.2$ dB (Fig. 2). Unlike the single-user matched filter, the spectral efficiency of the decorrelator grows without bound as $\frac{E_b}{N_0} \to \infty$.

As far as the optimum coding–spreading tradeoff for the MMSE receiver, for low $\frac{E_b}{N_0}$ it favors making K/N very large in which case the MMSE receiver achieves essentially the same spectral efficiency as the single-user matched filter (Fig. 2). The optimum K/N reaches 1 at $\frac{E_b}{N_0}$ =4 dB, and reaches a minimum of 0.75 at $\frac{E_b}{N_0}$ =10 dB (cf. Fig. 3).

7) Dynamic Power Allocation: Assuming maximum-likelihood decoding and long spreading sequences, the gain in spectral efficiency achievable by allocating instantaneous power as a function of the instantaneous crosscorrelations is small enough not to warrant the required increase in complexity.

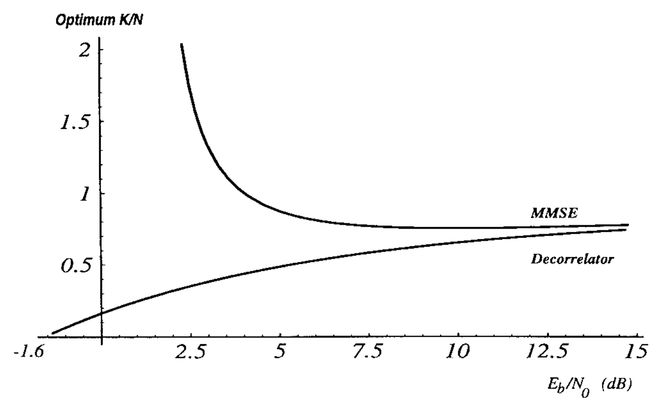

Fig. 3. Optimum K/N for large K.

II. Crosscorrelations of Random Sequences

The kth user sends the codeword

$$[b_k[1], \cdots, b_k[n]]$$

by transmitting

$$A_k \sum_{i=1}^{n} b_k[i] s_k(t - iT).$$

The signature waveform s_k has duration-T, unit energy, and lives in an N-dimensional space.

$$s_k(t) = \sum_{j=1}^{N} c_{kj} \psi_j(t)$$

where

$$\boldsymbol{c}_k = [c_{k1}, \cdots, c_{kN}]$$

is the *spreading code* assigned to the kth user. The chip waveforms are orthonormal[3]

$$\langle \psi_l, \psi_j \rangle = \delta_{lj}.$$

The crosscorrelations between the signature waveforms are denoted by

$$\begin{aligned}\rho_{kl} &= \langle s_k, s_l \rangle \\ &= \sum_{n=1}^{N} c_{kn} c_{ln}\end{aligned}$$

A. Binary Sequences

In the binary sequence model, the N-chip signatures assigned to the K users are independently equiprobably chosen from the vertices of an N-dimensional hypercube, i.e., for all $k = 1, \cdots, K$ and $n = 1, \cdots, N$, $\{c_{kn}\}$ are independent equally likely to be $-1/\sqrt{N}$ or $1/\sqrt{N}$. The $K \times K$ matrix $\boldsymbol{R}$ of crosscorrelations has unit diagonal elements and off-diagonal elements equal to

$$\rho_{kl} = \sum_{n=1}^{N} c_{kn} c_{ln} \tag{14}$$

$$= \frac{1}{N} \sum_{n=1}^{N} d_n \tag{15}$$

where $\{d_1, \cdots, d_N\}$ are independent equally likely to be $+1$ or -1. Thus $-1 \leq \rho_{kl} \leq 1$ is binomially distributed

$$P\left(\rho_{kl} = 1 - \frac{2i}{N}\right) = \binom{N}{i} 2^{-N}, \qquad i = 0, 1, \cdots, N \tag{16}$$

with the following moments:

$$E[\rho_{kl}] = 0 \tag{17}$$

$$E\left[\rho_{kl}^2\right] = \frac{1}{N} \tag{18}$$

$$E\left[\rho_{kl}^4\right] = \frac{3}{N^2} - \frac{2}{N^3}. \tag{19}$$

By the DeMoivre–Laplace central limit theorem, we have[4]

$$\sqrt{N} \rho_{kl} \stackrel{\text{dist}}{\rightarrow} \mathcal{N}(0, 1). \tag{20}$$

The crosscorrelations ρ_{kl} are pairwise independent but not jointly independent [25, p. 70].

When the ratio $K/N = \beta$ is kept constant (or converges to a constant), then the distribution of the eigenvalues of $\boldsymbol{R}$ converges according to the following result.

Proposition II.1 [53]: For random binary sequences the proportion of the K eigenvalues of $\boldsymbol{R}$ that lie below x converges (as $K \to \infty$) to the cumulative distribution function of the probability density function

$$f_\beta(x) = [1 - \beta^{-1}]^+ \delta(x) + \frac{\sqrt{[x - a(\beta)]^+ [b(\beta) - x]^+}}{2\pi\beta x} \tag{21}$$

[3] $\delta_{lj} = 1$ if $l = j$; $\delta_{lj} = 0$ if $l \neq j$.

[4] Convergence in distribution is denoted by $\stackrel{\text{dist}}{\rightarrow}$.

where $\delta(x)$ is a unit point mass at 0,

$$[z]^+ = \max\{0, z\},$$

and

$$\begin{aligned}a(\beta) &= (1 - \sqrt{\beta})^2 \\ b(\beta) &= (1 + \sqrt{\beta})^2.\end{aligned}$$

Furthermore, the distribution of the eigenvalues of

$$\frac{N}{K} \sum_{k=1}^{K} \boldsymbol{c}_k \boldsymbol{c}_k^T$$

converges to the cumulative distribution function of $f_{1/\beta}$.

It follows from either Proposition II.1 or [54] that if $\beta \leq 1$, then the probability that $\boldsymbol{R}$ is nonsingular goes to 1 as $K \to \infty$. Obviously, if $\beta > 1$, then $\boldsymbol{R}$ is singular.

B. Spherical Sequences

In the spherical random sequence model, the N-chip signatures are drawn uniformly from the surface of the unit N-sphere. Accordingly, the sequence assigned to the kth user admits the representation

$$[c_{k1}, \cdots, c_{kN}] = \frac{1}{\sqrt{\sum_{i=1}^{N} g_{ki}^2}} [g_{k1}, \cdots, g_{kN}]$$

where $[g_{k1}, \cdots, g_{kN}]$ are independent zero-mean Gaussian random variables with identical variance. By symmetry, the distribution of the crosscorrelation

$$\rho_{kl} = \sum_{n=1}^{N} c_{kn} c_{ln}$$

does not depend on $[c_{k1}, \cdots, c_{kN}]$. Thus $f_{\rho_{kn}}$, the density of ρ_{kn}, is the same as the density of

$$g_{l1} \Bigg/ \sqrt{\sum_{i=1}^{N} g_{li}^2}$$

which is [25, p. 72]

$$f_{\rho_{kn}}(x) = \frac{1}{C_N} (1 - x^2)^{(N-3)/2}, \qquad |x| \leq 1 \tag{22}$$

with

$$C_N = \begin{cases} \frac{\pi (N-3)!!}{(N-2)!!}, & N \text{ even} \\ \frac{2 (N-3)!!}{(N-2)!!}, & N \text{ odd.} \end{cases}$$

and

$$N!! = \begin{cases} 1, & N = -1, 0, 1 \\ N(N-2) \cdots 1, & N = 2k+1 \\ N(N-2) \cdots 2, & N = 2k. \end{cases}$$

The second moment of the crosscorrelation in the spherical model satisfies

$$\lim_{N \to \infty} N E\left[\rho_{kl}^2\right] = 1 \tag{23}$$

and using the weak law of large numbers [55, p. 285] it can be concluded that (20) also holds in the spherical model.

Regarding the behavior of the eigenvalues of the crosscorrelation matrix in the spherical sequence model, we note that Proposition II.1 remains true for any matrix whose coefficients are given by (14) where $\{c_{kl}\}$ are independent and identically distributed (i.i.d.) with finite variance [53]. This means that Proposition II.1 holds for a matrix $\tilde{\boldsymbol{R}}$ defined as

$$\tilde{\rho}_{kl} = \frac{1}{N}\sum_{n=1}^{N} g_{kn}g_{ln} \tag{24}$$

$$= \frac{||\boldsymbol{g}_k||\,||\boldsymbol{g}_l||}{N}\rho_{kl} \tag{25}$$

where $\boldsymbol{g}_k = [g_{k1}, \cdots, g_{kN}]$, and $||\boldsymbol{g}_k||$ is its Euclidean norm. In order to show that the asymptotic eigenvalue distributions of $\tilde{\boldsymbol{R}}$ and $\boldsymbol{R}$ coincide, it is enough to show [56] that the weak norm of their difference

$$|\tilde{\boldsymbol{R}} - \boldsymbol{R}| = \frac{\beta}{K^2}\sum_{k=1}^{K}\sum_{l=1}^{K}\frac{N\langle \boldsymbol{g}_k, \boldsymbol{g}_l\rangle^2}{||\boldsymbol{g}_k||^2||\boldsymbol{g}_l||^2}\left(1 - \frac{||\boldsymbol{g}_k||\,||\boldsymbol{g}_l||}{N}\right)^2$$

vanishes almost surely.

In general, analytical results are easier if unnormalized Gaussian sequences are considered. For example, the nonasymptotic eigenvalue distribution is known. If a model of long spreading sequences is considered, then the ergodicity of the Gaussian sequence implies that the average transmitted power is asymptotically deterministic even if the encoder does not take into account power fluctuations in the signature waveforms. As $N \to \infty$, the same spectral efficiency obtains as in the normalized spherical sequence model.

III. Optimum Decoding

A. Preliminaries

Throughout this paper we assume that codewords are power-constrained

$$\frac{1}{n}\sum_{i=1}^{n} b_k^2[i] \le 1.$$

Then, the total capacity (sum-rate) of the synchronous CDMA channel

$$y(t) = \sum_{k=1}^{K}\sum_{i=1}^{n} A_k b_k[i] s_k(t - iT) + n(t)$$

was found in [4] to be equal to

$$\frac{1}{2}\log\left(\det\left[\boldsymbol{I} + \sigma^{-2}\boldsymbol{ARA}\right]\right)$$

where $\boldsymbol{A} = \mathrm{diag}\,\{A_1, \cdots, A_K\}$, and σ^2 is the spectral level of the white Gaussian noise $n(t)$.

If the users have equal power, then $A_k = A$

$$\mathsf{SNR} = \frac{A^2}{\sigma^2}$$

and the optimum spectral efficiency is equal to

$$\mathsf{C}^{\mathrm{opt}}(\mathsf{SNR}, \boldsymbol{R}, K, N) = \frac{1}{2N}\log\left(\det\left[\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R}\right]\right). \tag{26}$$

The average with respect to $\boldsymbol{R}$ is denoted by

$$\bar{\mathsf{C}}_s^{\mathrm{opt}}(\mathsf{SNR}, K, N) = E\left[\frac{1}{2N}\log\left(\det\left[\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R}\right]\right)\right]. \tag{27}$$

Averaging with respect to $\boldsymbol{R}$ yields the capacity of long-code CDMA systems where the periodicity of the spreading code is much larger than the symbol duration. The reason is that as the blocklength goes to infinity, all realizations of $\boldsymbol{R}$ occur (many times) and the encoding/decoding system can treat the symbols corresponding to each realization as an independent subcodeword. Although this reasoning strictly applies to the case where $\boldsymbol{R}$ takes a finite number of values (e.g., binary sequence model), it can also be extended to encompass the spherical random sequence model.

Interestingly, in the setting of long spreading codes, we can do better than (27) by means of dynamic power allocation, namely, subcodewords corresponding to different realizations of $\boldsymbol{R}$ can be allocated different powers, as long as their average power remains intact. For example, we would expect that the optimal strategy will assign more power to the propitious times at which the signatures are orthogonal, than to the times at which all users are assigned the same signature. With dynamic power allocation, the average capacity becomes

$$\bar{\mathsf{C}}_d^{\mathrm{opt}}(\mathsf{SNR}, K, N) = \max E\left[\frac{1}{2N}\log\left(\det\left[\boldsymbol{I} + \sigma^{-2}\boldsymbol{A}(\boldsymbol{R})\boldsymbol{R}\boldsymbol{A}(\boldsymbol{R})\right]\right)\right] \tag{28}$$

where the maximum is over all mappings from $K \times K$ crosscorrelation matrices to positive diagonal matrices, such that

$$E[\boldsymbol{A}(\boldsymbol{R})] = A^2\boldsymbol{I}.$$

B. Two-User Channel

As usual, it is illustrative to consider the two-user case first. In this case

$$\boldsymbol{R} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

and the average static (27) and dynamic (28) spectral efficiencies particularize to

$$\bar{\mathsf{C}}_s^{\mathrm{opt}}(\mathsf{SNR}, 2, N) = E\left[\frac{1}{2N}\log(1 + 2\mathsf{SNR} + (1-\rho^2)\mathsf{SNR}^2)\right] \tag{29}$$

and

$$\bar{\mathsf{C}}_d^{\mathrm{opt}}(\mathsf{SNR}, 2, N) = \max E\left[\frac{1}{2N}\log\left(1 + 2\mathsf{SNR}(\rho) + (1-\rho^2)\mathsf{SNR}^2(\rho)\right)\right] \tag{30}$$

respectively, where the expectations are with respect to ρ and the maximum in (30) is with respect to the function $\mathsf{SNR}(\cdot)$ that satisfies

$$E[\mathsf{SNR}(\rho)] = \mathsf{SNR}. \tag{31}$$

The solution to this optimization problem is

$$\mathsf{SNR}(\rho) = (1-\rho^2)^{-1}\left(\frac{1}{\lambda}(1-\rho^2) - 1 + \sqrt{\frac{1}{\lambda^2}(1-\rho^2)^2+\rho^2}\right) \tag{32}$$

where $0 \leq \lambda \leq 2$ is the associated Lagrange multiplier chosen so that (31) is satisfied. It can be verified from (32) that

$$\mathsf{SNR}(0) = 2\mathsf{SNR}(1).$$

Let us now evaluate (29) and (30) in the case of binary sequences in which ρ is binomially distributed (16). If $N = 2$, then

$$P[\rho^2 = 1] = P[\rho = 0] = \frac{1}{2}$$

and (29), (30) become

$$\bar{\mathsf{C}}_s^{\mathsf{opt}}(\mathsf{SNR}, 2, 2) = \frac{1}{8}\log(1+2\mathsf{SNR}) + \frac{1}{4}\log(1+\mathsf{SNR}) \tag{33}$$

and

$$\bar{\mathsf{C}}_d^{\mathsf{opt}}(\mathsf{SNR}, 2, 2) = \frac{3}{8}\log\left(1+\frac{4}{3}\mathsf{SNR}\right). \tag{34}$$

Comparing (34) and (2) at $K = N = 2$ we see that

$$\bar{\mathsf{C}}_d^{\mathsf{opt}}\left(\frac{E_b}{N_0}, 2, 2\right) = \frac{3}{4}\mathsf{C}^*\left(\frac{E_b}{N_0}\right)$$

whereas orthogonal sequences achieve

$$\bar{\mathsf{C}}_d^{\mathsf{orth}}\left(\frac{E_b}{N_0}, 2, 2\right) = \mathsf{C}^*\left(\frac{E_b}{N_0}\right).$$

Interestingly, we can check that the gain due to dynamic power assignment is minute in this case. The maximum difference occurs for asymptotically high $\frac{E_b}{N_0}$ and is equal to

$$\lim_{\mathsf{SNR}\to\infty}\left\{\bar{\mathsf{C}}_d^{\mathsf{opt}}(\mathsf{SNR}, 2, 2) - \bar{\mathsf{C}}_s^{\mathsf{opt}}(\mathsf{SNR}, 2, 2)\right\} = \text{ 0.03 bit/chip.}$$

For all but very low $\frac{E_b}{N_0}$, the maximum relative gain is also very small. Its maximum value is attained for $\frac{E_b}{N_0} \downarrow \log_e 2$

$$\max_{\frac{E_b}{N_0}}\left\{\bar{\mathsf{C}}_d^{\mathsf{opt}}\left(\frac{E_b}{N_0}, 2, 2\right) \Big/ \bar{\mathsf{C}}_s^{\mathsf{opt}}\left(\frac{E_b}{N_0}, 2, 2\right)\right\} = \frac{9}{8}.$$

For a counterpart of the results in this subsection with the spherical random sequence model see Appendix I.

C. K-User Channel

Returning now to the general K-user channel, we will see that the very small gain realized by dynamic power assignment in the two-user case is even smaller for larger number of users. The reason is that the likelihood of atypically bad/good crosscorrelation matrices decreases with K (and also with N). Moreover, the maximum difference between dynamic-power and static-power capacity occurs at $\mathsf{SNR} \to \infty$, the reason being that constant power allocation is best to combat the background noise. Accordingly, it makes sense to focus in the asymptotic regime $\mathsf{SNR} \to \infty$ in our analysis of the difference between dynamic-power and static-power spectral efficiency. The following result (proved in Appendix II) lets the power-allocation strategy depend on the instantaneous crosscorrelation matrix but not on the user index. In the more general case, we conjecture that the asymptotically optimal strategy is to let

$$\mathsf{SNR}_k(\boldsymbol{R}) = \frac{C}{\sum_j 1\{|\rho_{jk}| = 1\}}$$

where the indicator function is denoted

$$1\{A\} = \begin{cases} 1, & \text{if } A \text{ is true} \\ 0, & \text{if } A \text{ is false.} \end{cases}$$

Proposition III.1: Consider the class of dynamic power allocation strategies where all users are constrained to use the same power $\mathsf{SNR}_k(\boldsymbol{R}) = \mathsf{SNR}(\boldsymbol{R})$. Then

$$\lim_{\mathsf{SNR}\to\infty}\bar{\mathsf{C}}_d^{\mathsf{opt}}(\mathsf{SNR}, K, N) - \bar{\mathsf{C}}_s^{\mathsf{opt}}(\mathsf{SNR}, K, N) = \frac{1}{2N}E\left[r(\boldsymbol{R})\log\frac{r(\boldsymbol{R})}{E[r(\boldsymbol{R})]}\right] \tag{35}$$

where $r(\boldsymbol{R})$ denotes the rank of $\boldsymbol{R}$.

The quantity in the right-hand side of (34) is very small. As we saw, it is equal to 0.03 bit/chip if $(K, N) = (2, 2)$, it equals 0.02 bit/chip if $(K, N) = (3, 3)$. As $K \to \infty$ the gain vanishes[5] because the probability that $\boldsymbol{R}$ is not full-rank goes to zero [54]. In view of these results we conclude that the very small gain in optimal spectral efficiency brought about by dynamic power allocation does not warrant the increase in complexity in encoding/decoding. Henceforth, we restrict attention to encoding with power allocation that does not depend on the instantaneous signature waveforms.

D. $K \to \infty$

The complexity of analytical results on spectral efficiency quickly grows with the number of users. Fortunately, as $K \to \infty$, not only do analytical results become feasible but, as the following result demonstrates, the randomness of spectral efficiency due to the random choice of signatures vanishes.

Proposition III.2: Suppose that the eigenvalue distribution of $\boldsymbol{R}$ converges to $F(x)$ almost surely for all $x > 0$. Then, the optimum spectral efficiency converges almost surely to

$$\lim_{K\to\infty}\mathsf{C}^{\mathsf{opt}}(\mathsf{SNR}, \boldsymbol{R}, K, K/\beta) = \frac{\beta}{2}E[\log(1+\mathsf{SNR}X)] \tag{36}$$

$$= \frac{\beta}{2}\int_0^\infty \frac{1-F(z)}{\frac{1}{\mathsf{SNR}}+z}\,dz \tag{37}$$

where the expectation of X is with respect to the distribution F. Thus for binary random sequences

$$\lim_{K\to\infty}\mathsf{C}^{\mathsf{opt}}(\mathsf{SNR}, \boldsymbol{R}, K, K/\beta) = \frac{1}{4\pi}\int_{a(\beta)}^{b(\beta)}\log(1+\mathsf{SNR}x)\sqrt{(b(\beta)-x)(x-a(\beta))}\,\frac{dx}{x} \tag{38}$$

where $a(\beta) = (\sqrt{\beta}-1)^2$, and $b(\beta) = (\sqrt{\beta}+1)^2$.

[5] This conclusion does not require asymptotically large SNR as we indicate in the next subsection.

Proof: Fix a $K \times K$ crosscorrelation matrix $\boldsymbol{R}$. The eigenvalues of the $K \times K$ matrix $\boldsymbol{R}$ will be denoted by $\lambda_1^{(K)}(\boldsymbol{R}) \le \cdots \le \lambda_K^{(K)}(\boldsymbol{R})$, and

$$Y_k^{(K)} \stackrel{\text{def}}{=} h\big(\lambda_k^{(K)}(\boldsymbol{R})\big)$$

where

$$h(x) \stackrel{\text{def}}{=} \frac{1}{2}\log(1+\mathsf{SNR}x).$$

Let us also define the cumulative distribution functions

$$F^{(K)}(x) \stackrel{\text{def}}{=} \frac{1}{K}\sum_{i=1}^{K} 1\{\lambda_i^{(K)}(\boldsymbol{R}) \le x\} \tag{39}$$

$$F_Y^{(K)}(x) \stackrel{\text{def}}{=} \frac{1}{K}\sum_{i=1}^{K} 1\{Y_i^{(K)} \le x\}. \tag{40}$$

Note that by monotonicity of the function h

$$F_Y^{(K)}(x) = F^{(K)}(h^{-1}(x)). \tag{41}$$

According to (26),

$$\frac{1}{\beta}\mathsf{C}^{\text{opt}}(\mathsf{SNR}, \boldsymbol{R}, K, N) = \frac{1}{K}\sum_{k=1}^{K} Y_k^{(K)} \tag{42}$$

$$= \int_0^\infty \big(1 - F_Y^{(K)}(x)\big)\,dx \tag{43}$$

$$= \int_0^\infty \big(1 - F^{(K)}(h^{-1}(x))\big)\,dx \tag{44}$$

$$= \frac{1}{2}\int_0^\infty \frac{1 - F^{(K)}(z)}{\frac{1}{\mathsf{SNR}} + z}\,dz \tag{45}$$

where (43) follows from (40), and (45) is a result of a simple change of integration variable.

Upon taking limits in (45) and using the bounded convergence theorem (e.g. [57]) to interchange limit and integration we obtain

$$\lim_{K\to\infty} \mathsf{C}^{\text{opt}}(\mathsf{SNR}, \boldsymbol{R}, K, N) = \frac{\beta}{2}\int_0^\infty \frac{1 - \lim_{K\to\infty} F^{(K)}(z)}{\frac{1}{\mathsf{SNR}} + z}\,dz.$$

Finally, Proposition II.1 and integration per parts can be used to verify (38). □

The closed-form expression for the optimal spectral efficiency as a function of β and SNR given in (9) is obtained by means of the identity (100) found in Section VI. This circumvents having to deal with the cumbersome definite integral in (38). Rapajic [58] solves the definite integral in (38) dealing with the cases $\beta < 1$ and $\beta > 1$ separately. Unlike (9), the expression found in [58] is not directly related to the MMSE spectral efficiency.

From (9) it is straightforward to show that

$$C^{\text{opt}}\left[\frac{1}{\beta}, \mathsf{SNR}\beta\right] = \frac{1}{\beta}C^{\text{opt}}[\beta, \mathsf{SNR}] \tag{46}$$

where $C^{\text{opt}}[\beta, \mathsf{SNR}]$ denotes the right-hand side of (9). The optimal spectral efficiency in terms of $\frac{E_b}{N_0}$ is the solution to

$$\mathsf{C}^{\text{opt}}\left(\beta, \frac{E_b}{N_0}\right) = C^{\text{opt}}\left[\beta, \frac{2}{\beta}\frac{E_b}{N_0}\mathsf{C}^{\text{opt}}\left(\beta, \frac{E_b}{N_0}\right)\right]. \tag{47}$$

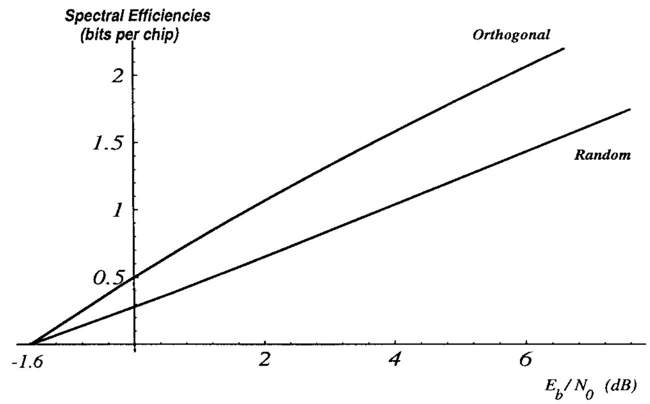

Fig. 4. Optimum spectral efficiencies with orthogonal and random sequences $K = N \to \infty$.

Using (46) several interesting analytical properties can be shown for the solution to (47).

$$\mathsf{C}^{\text{opt}}\left(\beta, \frac{E_b}{N_0}\right) = \beta\mathsf{C}^{\text{opt}}\left(\frac{1}{\beta}, \frac{E_b}{N_0}\right) \tag{48}$$

$$\lim_{\beta\to 0}\frac{1}{\beta}\mathsf{C}^{\text{opt}}\left(\beta, \frac{E_b}{N_0}\right) = \mathsf{C}^*\left(\frac{E_b}{N_0}\right) \tag{49}$$

$$\lim_{\beta\to\infty}\mathsf{C}^{\text{opt}}\left(\beta, \frac{E_b}{N_0}\right) = \mathsf{C}^*\left(\frac{E_b}{N_0}\right) \tag{50}$$

where (50) follows from (48) and (49).

It is straightforward to show that as $\frac{E_b}{N_0} \to \infty$ the slope of the spectral efficiency achievable with random sequences as a function of $\frac{E_b}{N_0}$ (dB) goes to

$$\lim_{\frac{E_b}{N_0}\to\infty} \frac{\mathsf{C}^{\text{opt}}\big(\beta, \frac{E_b}{N_0}\big)}{10\log_{10}\frac{E_b}{N_0}} = 0.166\ \min\{1, \beta\}\ \text{bits/dB} \tag{51}$$

which coincides with the optimum behavior (5) for $\beta \ge 1$.

In Fig. 4 we have shown $\mathsf{C}^{\text{opt}}(1, \frac{E_b}{N_0})$and $\mathsf{C}^{\text{orth}}(1, \frac{E_b}{N_0}) = \mathsf{C}^*(\frac{E_b}{N_0})$. The slopes of both curves with the logarithm of $\frac{E_b}{N_0}$ are asymptotically equal. However, there is a nonnegligible gap between both curves:

$$\mathsf{C}^{\text{opt}}\left(1, \frac{E_b}{N_0}\right) - \mathsf{C}^*\left(\frac{E_b}{N_0}\right) = \frac{1}{4\pi}\int_0^4 \log\left(\frac{1 + 2\frac{E_b}{N_0}\mathsf{C}^{\text{opt}}\big(1, \frac{E_b}{N_0}\big)x}{1 + 2\frac{E_b}{N_0}\mathsf{C}^*\big(\frac{E_b}{N_0}\big)}\right)\sqrt{\frac{4-x}{x}}\,dx \tag{52}$$

which can be as large as

$$\lim_{\frac{E_b}{N_0}\to\infty}\mathsf{C}^{\text{opt}}\left(1, \frac{E_b}{N_0}\right) - \mathsf{C}^*\left(\frac{E_b}{N_0}\right) = \frac{1}{4\pi}\int_0^4 (\log x)\sqrt{\frac{4-x}{x}}\,dx \tag{53}$$

$$= -\frac{1}{2}\log e = -\ 0.72\ \text{bit/chip} \tag{54}$$

where (53) follows from the fact that C^{opt} and C^* have the same slope with large $\frac{E_b}{N_0}$ (cf. (5) and (51)). A limiting result similar to (54) can be found in [40].

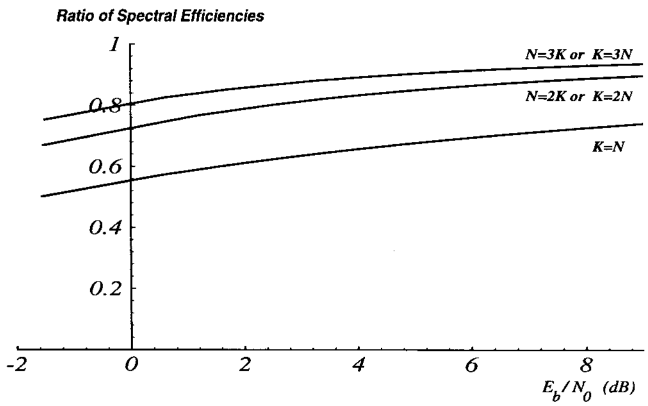

Fig. 5. Optimal processing. Spectral efficiency with random signatures divided by spectral efficiency with optimally chosen signatures.

If $\frac{E_b}{N_0}$ is close to its lower limit of $\log_e 2 = -1.6$ dB, then random sequences achieve only 50% of the spectral efficiency of orthogonal sequences. Fig. 5 displays the proportion of the spectral efficiency of optimum sequences which is achieved by random sequences, i.e., if $\beta \leq 1$, then

$$\gamma^{\mathrm{opt}}\left(\beta, \frac{E_b}{N_0}\right) \stackrel{\text{def}}{=} \frac{\mathsf{C}^{\mathrm{opt}}\left(\beta, \frac{E_b}{N_0}\right)}{C^{\mathrm{orth}}\left(\beta, \frac{E_b}{N_0}\right)} = \frac{\mathsf{C}^{\mathrm{opt}}\left(\beta, \frac{E_b}{N_0}\right)}{\beta C^*\left(\frac{E_b}{N_0}\right)}$$

and if $\beta > 1$, then

$$\gamma^{\mathrm{opt}}\left(\beta, \frac{E_b}{N_0}\right) \stackrel{\text{def}}{=} \frac{\mathsf{C}^{\mathrm{opt}}\left(\beta, \frac{E_b}{N_0}\right)}{C^*\left(\frac{E_b}{N_0}\right)} = \frac{\beta \mathsf{C}^{\mathrm{opt}}\left(\frac{1}{\beta}, \frac{E_b}{N_0}\right)}{C^*\left(\frac{E_b}{N_0}\right)}.$$

Therefore, we get the following identity:

$$\gamma^{\mathrm{opt}}\left(\beta, \frac{E_b}{N_0}\right) = \gamma^{\mathrm{opt}}\left(\frac{1}{\beta}, \frac{E_b}{N_0}\right).$$

Note that random sequences are asymptotically optimal (i.e., as good as orthogonal for $\beta \leq 1$ and as good as WBE for $\beta > 1$) under the following conditions (see Fig. 5).

- Fixed β and $\frac{E_b}{N_0} \to \infty$.
- Fixed $\frac{E_b}{N_0}$ and $\beta \to \infty$.
- Fixed $\frac{E_b}{N_0}$ and $\beta \to 0$.

Adhering to the suboptimal approach where all the users are constrained to have the same power as in Proposition III.1, we notice that (38) still holds with $\mathsf{SNR}(\boldsymbol{R})$ replacing SNR. Then, maximizing with respect to $\mathsf{SNR}(\boldsymbol{R})$, satisfying $E[\mathsf{SNR}(\boldsymbol{R})] = \mathsf{SNR}$ yields by the concavity of the logarithm that the optimum choice is $\mathsf{SNR}(\boldsymbol{R}) = \mathsf{SNR}$. For asymptotically large SNR, the result conforms with (35).

Another byproduct of the proof of Proposition III.2 is the practically relevant fact that without loss of optimality users can choose codebooks without regard to the assigned signature waveforms or to their evolution in a CDMA system with long codes.

IV. Single-User Matched Filter

The output of the matched filter of user 1 is the following discrete-time process:

$$y_1[i] = Ab_1[i] + A\sum_{j=2}^{K} \rho_{1j} b_j[i] + \sigma n_1[i] \tag{55}$$

where $\{n_1[i]\}$ is an independent Gaussian sequence with unit variance and $\{b_k[1], \cdots, b_k[n]\}$ is the input codeword of user k. The receiver under consideration in this section is suboptimal because its scalar observations are not sufficient statistics and because it treats the multiuser interference as noise without attempting to exploit possible knowledge of the codebooks of the interfering users. A rigorous analysis of the capacity of this important channel has not been undertaken previously. It is customary (e.g. [59]) to simply approximate the interference

$$A\sum_{j=2}^{K} \rho_{1j} b_j[i] + \sigma n_1[i]$$

as an independent Gaussian sequence. However, the problem is more subtle than may appear at first glance. The crosscorrelations are known at the receiver and the input distribution of $b_j[i]$ need not be Gaussian. Thus the single-user channel (55) is, in general, non-Gaussian, and its capacity depends on the crosscorrelations. Achieving the capacity of (55) requires that the receiver of user 1 knows the crosscorrelations and input distributions of all the interferers. However, the following result shows that that information becomes useless as the number of users grows without bound. Furthermore, the dependence of the capacity on the actual realization of signature waveforms vanishes asymptotically.

Proposition IV.1: Let $\mathsf{C}(\rho_{12}, \cdots, \rho_{1K}, p_{b_2}, \cdots, p_{b_K})$ denote the capacity of the single-user channel (55) subject to the following constraints:

- $\frac{1}{n}\sum_{i=1}^{n} b_1^2[i] \leq 1$.
- For $j = 2, \cdots K$, the random variables $\{b_j[i]\}$ are independent with distribution p_{b_j}.
- $E[b_j[i]] = 0$.
- $E[b_j^2[i]] = 1$.
- $\{n_1[i]\}$ is a memoryless Gaussian process with unit variance.

If the sequences are drawn according to either the binary or the spherical random models, then as $\beta N = K \to \infty$

$$\mathsf{C}(\rho_{12}, \cdots, \rho_{1K}, p_{b_2}, \cdots, p_{b_K}) \stackrel{\text{a.s.}}{\to} \frac{1}{2}\log\left(1 + \frac{A^2}{\sigma^2 + \beta A^2}\right). \tag{56}$$

Proof: It follows from well-known results on the capacity of non-Gaussian channels [60], [61] that

$$\begin{aligned} &\frac{1}{2}\log\left(1 + \frac{A^2}{\sigma^2 + A^2\sum_{j=2}^{K}\rho_{1j}^2}\right) \\ &\leq \mathsf{C}(\rho_{12}, \cdots, \rho_{1K}, p_{b_2}, \cdots, p_{b_K}) &(57)\\ &\leq \frac{1}{2}\log\left(1 + \frac{A^2}{\sigma^2 + A^2\sum_{j=2}^{K}\rho_{1j}^2}\right) \\ &\quad + D\left(\sigma n_1 + A\sum_{j=2}^{K}\rho_{1j}b_j \,\middle\|\, \mathcal{N}\left(0, \sigma^2 + A^2\sum_{j=2}^{K}\rho_{1j}^2\right)\right) &(58)\end{aligned}$$

where

$$D\left(\sigma n_1 + A\sum_{j=2}^{K}\rho_{1j}b_j \,\middle\|\, \mathcal{N}\left(0, \sigma^2 + A^2\sum_{j=2}^{K}\rho_{1j}^2\right)\right)$$

denotes the non-Gaussianness of the Gaussian plus interference noise quantified by its divergence from the Gaussian distribution with identical variance.

According to (57) and (58), it is enough to show that

$$\log\left(1 + \frac{1}{\gamma + \sum_{j=2}^{K}\rho_{1j}^2}\right) \overset{\text{a.s.}}{\to} \log\left(1 + \frac{1}{\gamma + \beta}\right) \tag{59}$$

for all $\gamma > 0$, and

$$D\left(\sigma n_1 + A\sum_{j=2}^{K}\rho_{1j}b_j \,\middle\|\, \mathcal{N}\left(0, \sigma^2 + A^2\sum_{j=2}^{K}\rho_{1j}^2\right)\right) \overset{\text{a.s.}}{\to} 0. \tag{60}$$

To show (59), we recall the behavior of the second moment of the crosscorrelations in either the binary model (18) or the spherical model (23) and we use a strong law of large numbers for independent and identically distributed random variables whose distribution may depend on the number of terms in the sum [57, Theorem 5.4.1] to show

$$\sum_{j=2}^{K}\rho_{1j}^2 = \beta\frac{1}{K}\sum_{j=2}^{K}N\rho_{1j}^2 \overset{\text{a.s.}}{\to} \beta$$

and (59) follows.

To show (60), we invoke the recent version of the central limit theorem with convergence in the sense of divergence under the Lindeberg–Feller condition[6] [62] (see also [63, p. 601]), which in our setting becomes

$$\lim_{K\to\infty}\sum_{j=2}^{K}E_b\left[\rho_{1j}^2 b_j^2 1\{\rho_{1j}^2 b_j^2 > \xi\} \mid \rho_{1j}\right] = 0 \tag{61}$$

in addition to

$$E_b\left[b_j\rho_{1j} \mid \rho_{1j}\right] = 0$$

which holds because p_{b_j} has zero mean. Thus we need to show that the set of sequences $\{\rho_{12}, \cdots, \rho_{1K}\}$ for which (61) holds has asymptotic unit probability for all $\xi > 0$.

Let us choose an arbitrary scalar $h > 0$, and let us bound each random variable in (61) by

$$\begin{aligned}\rho_{1j}^2 b_j^2 1\{\rho_{1j}^2 b_j^2 > \xi\}&\left[1\left\{\rho_{1j}^2 \le \frac{h}{N}\right\} + 1\left\{\rho_{1j}^2 > \frac{h}{N}\right\}\right] \\ &\le \frac{h}{N}b_j^2 1\left\{\frac{h}{N}b_j^2 > \xi\right\} + b_j^2\rho_{1j}^2 1\left\{\rho_{1j}^2 > \frac{h}{N}\right\}.\end{aligned} \tag{62}$$

Upon taking expectations with respect to b_j and summing over j we get that the left-hand side of (61) is upper-bounded by

$$\frac{(K-1)h}{N}E\left[b_2^2 1\{hb_2^2 > \xi N\}\right] + \sum_{j=2}^{K}\rho_{1j}^2 1\left\{\rho_{1j}^2 > \frac{h}{N}\right\}. \tag{63}$$

[6] Due to the convolution with a Gaussian random variable in the first distribution of the divergence in (60), the convergence in (60) can also be proven directly without invoking the general result in [62].

Since N grows without bound and the second moment of b_2 exists, the first term in (63) vanishes for all (h, ξ). Regarding the second term in (63) we multiply it by the constant factor N/K and note that by the independence of the crosscorrelations ρ_{1j}, we can apply the law of large numbers again to obtain

$$\frac{1}{K}\sum_{j=2}^{K}N\rho_{1j}^2 1\left\{\rho_{1j}^2 > \frac{h}{N}\right\} \overset{\text{a.s.}}{\to} \lim_{N\to\infty}E\left[N\rho_{1j}^2 1\{N\rho_{1j}^2 > h\}\right] \tag{64}$$

$$= E[Y^2 1\{Y^2 > h\}] \tag{65}$$

where Y is a standard Gaussian random variable; (65) follows from (20) and (18) (binary model) and (23) (spherical model). Since the choice of h is arbitrary we can make the right-hand side of (65) as small as desired, thereby concluding the proof. □

By focusing on asymptotics in K we have been able to circumvent the open problem of finding the capacity-achieving distribution when the input distributions of all K users are constrained to be identical. Furthermore, the result of Proposition IV.1 suggests that unless K is small the solution to that open problem cannot be very far from Gaussian.

We see from (56) that a CDMA system with random sequences, N chips per symbol, a single-user matched-filter front-end whose output signal-to-noise ratio in the absence of interfering users is SNR, and a target output signal-to-noise ratio of ρ^{sumf} can accommodate up to

$$K = N\left(\frac{1}{\rho^{\mathsf{sumf}}} - \frac{1}{\mathsf{SNR}}\right) \tag{66}$$

users—a result obtained independently in [51] without analyzing capacity.

Equation (56) gives the capacity per user and per symbol (N chips). To obtain the spectral efficiency, all we need to do is multiply by K and divide by N. Recalling that the energy per symbol divided by the noise spectral level is $\mathsf{SNR} = A^2/\sigma^2$, we obtain that the asymptotic spectral efficiency for the single-user matched filter as a function of $\beta = K/N$ and SNR is given by

$$\mathsf{C}^{\mathsf{sumf}} = \frac{\beta}{2}\log\left(1 + \frac{\mathsf{SNR}}{1 + \mathsf{SNR}\beta}\right). \tag{67}$$

Upon substitution of

$$\mathsf{SNR} = \mathsf{C}^{\mathsf{sumf}}\frac{2}{\beta}\frac{E_b}{N_0}$$

we obtain that the asymptotic spectral efficiency of the single-user matched filter is equal to

$$\mathsf{C}^{\mathsf{sumf}}\left(\beta, \frac{E_b}{N_0}\right) = \frac{\beta}{2}\log_2(1 + \nu) \tag{68}$$

where ν is the solution to

$$\left(\frac{1}{\nu} - \beta\right)\log_2(1 + \nu) = \frac{N_0}{E_b}.$$

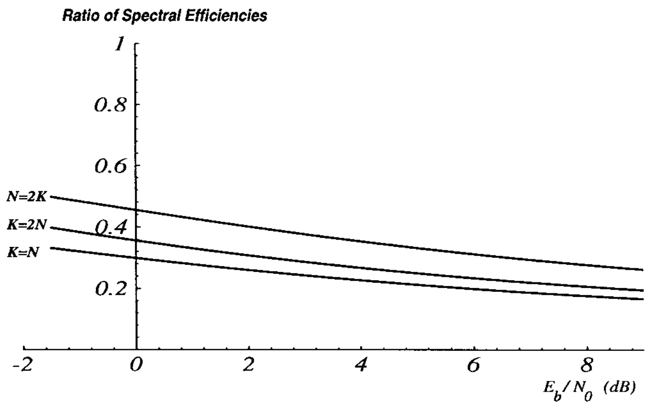

Fig. 6. Single-user matched filtering. Spectral efficiency with random signatures divided by spectral efficiency with optimally chosen signatures.

It can be shown from (68) that $\mathsf{C}^{\text{sumf}}(\beta, \frac{E_b}{N_0})$ is monotonically increasing with β, and

$$\lim_{\beta\to\infty} \mathsf{C}^{\text{sumf}}\left(\beta, \frac{E_b}{N_0}\right) = \frac{\log_2 e}{2} - \frac{1}{2}\frac{N_0}{E_b} \tag{69}$$

if $\frac{E_b}{N_0} > \log_e 2$. The asymptotic spectral efficiency for large $\frac{E_b}{N_0}$ can be seen (from either (67) or (68)) to be

$$\lim_{\frac{E_b}{N_0}\to\infty} \mathsf{C}^{\text{sumf}}\left(\beta, \frac{E_b}{N_0}\right) = \frac{\beta}{2}\log\left(1+\frac{1}{\beta}\right). \tag{70}$$

Thus (cf. Subsection IV-B)

$$\mathsf{C}^{\text{sumf}}\left(\beta, \frac{E_b}{N_0}\right) \le \frac{\log e}{2}$$

with asymptotic equality when both $\frac{E_b}{N_0} \to \infty$ and $\beta \to \infty$.

Fig. 6 shows the ratio of spectral efficiency of random spreading and single-user matched filtering to the spectral efficiency of optimally designed sequences and maximum-likelihood decoding. Recall that if $K \le N$, orthogonal sequences are optimal, and if $K = mN$, they remain optimal provided each sequence is assigned to m users. In either case, the single-user matched filter is an optimal front-end. However, in the latter case, the maximum-likelihood receiver is different from the receiver considered in this section which deals with the interferers as noise. The ratio of spectral efficiencies is monotonically decreasing with $\frac{E_b}{N_0}$. At $\frac{E_b}{N_0} \downarrow -1.6$ dB, the ratio is $1/3$ at $K = N$, and higher for any other K/N.

V. DECORRELATOR

In contrast to the single-user matched filter, the decorrelator for user k correlates the received signal with respect to the projection of the signature waveform s_k on the subspace orthogonal to the space spanned by the interfering waveforms [25]. When the signature waveforms are linearly independent ($\boldsymbol{R}$ is invertible) then such a projection can be expressed as

$$\tilde{s}_k(t) = \sum_{j=1}^{K} \boldsymbol{R}_{kj}^{-1} s_j(t) \tag{71}$$

where $\boldsymbol{R}_{kj}^{-1}$ denotes the (k,j) element of the inverse of the crosscorrelation matrix. Since

$$\langle s_j, \tilde{s}_k\rangle = \delta_{jk}$$

such a transformation succeeds in completely eliminating any interference from other users, and the decoder sees a single-user memoryless channel. As we mentioned in Section I, even if an optimum single-user encoder/decoder system is used, this receiver is not optimal because the output stream of the single-user decorrelator is not a sufficient statistic. The spectral efficiency is obtained by summing the individual capacities and dividing by N

$$\mathsf{C}^{\text{deco}} = \frac{1}{2N}\sum_{k=1}^{K} \log(1+\mathsf{SNR}\bar{\eta}_k) \tag{72}$$

where $\bar{\eta}_k$ is the optimum near–far resistance of the kth user [25]

$$\bar{\eta}_k = \frac{1}{\boldsymbol{R}_{kk}^{-1}}. \tag{73}$$

What if $\boldsymbol{R}$ is not invertible? Then, the decorrelator can still be defined through the Moore–Penrose generalized inverse of $\boldsymbol{R}$ [44], [25]. If s_k is not spanned by the interfering signature waveforms, then (72) and (73) still hold provided the inverse in (73) is replaced by the Moore–Penrose inverse. If s_k is spanned by the interfering signature waveforms, then $\bar{\eta}_k = 0$. In that case, the decorrelator for user k cannot tune out the interferers, but the capacity achievable by a single-user decoder is nonzero, as in the case of the single-user matched filter.

Proposition V.1: For $\beta \le 1$ and binary random spreading, the spectral efficiency of the decorrelator converges in mean square as $K \to \infty$ to

$$\lim_{K\to\infty} \mathsf{C}^{\text{deco}} = \frac{\beta}{2}\log(1+\mathsf{SNR}(1-\beta)). \tag{74}$$

Proof: The proof entails showing

$$\lim_{K\to\infty} E[\mathsf{C}^{\text{deco}}] = \frac{\beta}{2}\log(1+\mathsf{SNR}(1-\beta)) \tag{75}$$

and

$$\lim_{K\to\infty} \operatorname{var}(\mathsf{C}^{\text{deco}}) = 0. \tag{76}$$

The capacity achievable in the event that the crosscorrelation matrix is singular is bounded between 0 and $\frac{1}{2}\log(1+\mathsf{SNR})$. Since that event has vanishing probability (as $N \to \infty$), neither the mean nor the variance of C^{deco} will be affected asymptotically if we can change the distribution under which $\boldsymbol{R}$ is chosen by conditioning on it being nonsingular. In that case, the spectral efficiency is given by (72). By symmetry, the distribution of $\bar{\eta}_k$ is independent of k. Thus using (72) we obtain

$$E[\mathsf{C}^{\text{deco}}] = \frac{\beta}{2}E[\log(1+\mathsf{SNR}\bar{\eta}_K)] \tag{77}$$

and using the Cauchy–Schwarz inequality

$$\operatorname{var}(\mathsf{C}^{\text{deco}}) \le \frac{\beta^2}{4}\operatorname{var}(\log(1+\mathsf{SNR}\bar{\eta}_K)). \tag{78}$$

Note that had the random variables $\bar{\eta}_k$ been independent, then we could have claimed equality in (78) after dividing the right-hand side by K. Fortunately, the bound in (78) is good enough for our purposes. Consider the following result.

Proposition V.2 [25]: If $\beta \le 1$, then for all $k = 1, \cdots, K$ the maximum near–far resistance satisfies

$$\lim_{K\to\infty} \bar{\eta}_k = 1 - \beta$$

where the limit is in mean-square sense.

It is not true in general that $\mathrm{var}\,(X_K) \to 0$ implies $\mathrm{var}\,(\log(1 + X_K)) \to 0$. However, this implication can be shown to hold in our case because $X_K = \mathsf{SNR}\bar{\eta}_K$ is always nonnegative.

Now, we can evaluate (74) at the signal-to-noise ratio

$$\mathsf{SNR} = \mathsf{C}^{\text{deco}} \frac{2N}{K} \frac{E_b}{N_0}$$

to get the equation satisfied by the asymptotic spectral efficiency of the decorrelator

$$\mathsf{C}^{\text{deco}}\left(\beta, \frac{E_b}{N_0}\right) = \frac{\beta}{2} \log\left(1 + \mathsf{C}^{\text{deco}}\left(\beta, \frac{E_b}{N_0}\right) \frac{2}{\beta} \frac{E_b}{N_0}(1-\beta)\right)$$

which upon comparison to (4), yields

$$\mathsf{C}^{\text{deco}}\left(\beta, \frac{E_b}{N_0}\right) = \beta \mathsf{C}^*\left((1-\beta)\frac{E_b}{N_0}\right). \tag{79}$$

The result in (79) can be interpreted as the decorrelator achieving the same efficiency of orthogonal spreading (cf. (8)) except for a penalty in signal-to-noise ratio of $10\log_{10}(1-\beta)$ decibels. The system load $\beta^*(\frac{E_b}{N_0})$, that achieves the maximum of (79)

$$\mathsf{C}^{\text{deco}}\left(\beta^*\left(\frac{E_b}{N_0}\right), \frac{E_b}{N_0}\right) = \max_{\beta} \mathsf{C}^{\text{deco}}\left(\beta, \frac{E_b}{N_0}\right)$$

can be obtained as the solution to

$$4^{\mathsf{C}^*((1-\beta^*)\frac{E_b}{N_0})} \log_e 2 = \frac{E_b}{N_0}.$$

Notice that $\mathsf{C}^{\text{deco}} = 0$ if

$$\frac{E_b}{N_0} \le \frac{\log_e 2}{1-\beta}.$$

This means that the minimum $\frac{E_b}{N_0}$ necessary for reliable communication with the decorrelator is equal to -1.6 dB plus the noise enhancement factor in decibels. Therefore, for any given $\frac{E_b}{N_0}$, the spectral efficiency of the decorrelator becomes zero for a value of β that is strictly smaller than 1 (cf. Fig. 1).

When the system load β is large enough, the spectral efficiency of the decorrelator with random spreading degrades to the point that it is even lower than that of the single-user matched filter. In such a case, performance can be improved by neglecting the presence of a subset of users or, preferably, by using the linear transformation discussed in Section VI.

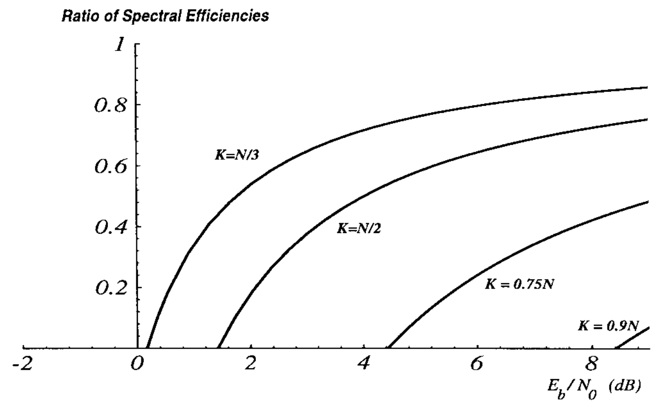

Fig. 7. Decorrelator. Spectral efficiency with random signatures divided by spectral efficiency with optimally chosen signatures.

Since the decorrelator is an optimum front-end in the case of orthogonal sequences, the loss due to the use of random sequences is given by (via (8) and (79))

$$\gamma^{\text{deco}}\left(\beta, \frac{E_b}{N_0}\right) \stackrel{\text{def}}{=} \frac{\mathsf{C}^{\text{deco}}\left(\beta, \frac{E_b}{N_0}\right)}{\mathsf{C}^{\text{orth}}\left(\beta, \frac{E_b}{N_0}\right)} = \frac{\mathsf{C}^*\left((1-\beta)\frac{E_b}{N_0}\right)}{\mathsf{C}^*\left(\frac{E_b}{N_0}\right)} \tag{80}$$

if $\beta \le 1$. Fig. 7 shows (80). Comparing this figure to Fig. 5 we can see that for high $\frac{E_b}{N_0}$ and low β the decorrelator almost achieves optimal spectral efficiency (see also Fig. 1). As should be expected and in contrast to the single-user matched filter, the suboptimality of the random choice decreases with $\frac{E_b}{N_0}$.

The results in this section hold verbatim if the received powers are different (but nonzero) since neither the decorrelator nor its output depend on the power of the interferers. We call attention to the fact that for the decorrelator (and MMSE receiver in the next section) dynamic power allocation is useful when β exceeds the optimum load β^* for a given $\frac{E_b}{N_0}$ (Fig. 1). To see this, let $\alpha = K^*/K$ be the largest multiple of $1/K$ not exceeding β^*/β. Every encoder can be forced to transmit energy only in a fraction α of symbols in a way that K^* users are simultaneously active at every symbol. This allows each user to boost its power by a factor $1/\alpha$ at the times at which it is active. If the decorrelator is changed from symbol to symbol so as to take into account only those users that have nonzero power, then the resulting spectral efficiency is equal to $\mathsf{C}^{\text{deco}}(\alpha\beta, \frac{E_b}{N_0})$, which can be as close as desired to the optimum $\mathsf{C}^{\text{deco}}(\beta^*, \frac{E_b}{N_0})$ for sufficiently large K.

VI. Linear MMSE Receiver

We start by recalling from [25] several elementary properties of the linear MMSE multiuser receiver. If all the received signal-to-noise ratios are identical, the MMSE receiver for the kth user correlates the incoming signal with

$$\tilde{s}_k(t) = \sum_{j=1}^{K} [\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R}]^{-1}_{kj} s_j(t). \tag{81}$$

This linear transformation does not eliminate multiaccess interference from its output but it achieves the maximum output signal-to-interference ratio given by

$$\frac{1}{\mathsf{M}_k(\mathsf{SNR})} - 1 = \frac{1}{[\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R}]^{-1}_{kk}} - 1 \tag{82}$$

where $\mathsf{M}_k(\mathsf{SNR})$ is the minimum mean-square error for the kth user when all users have signal-to-noise ratio equal to SNR. The maximum rate achievable by a single-user decoder depends on the distribution of the symbols transmitted by the interferers. Since the inputs are power-constrained, the minimax distribution for the interferers is Gaussian. Although, as noted before, this does not imply that Gaussian inputs are optimum if all input distributions are constrained to be equal, the MMSE spectral efficiency is lower-bounded by the spectral efficiency of a single-user channel with signal-to-noise ratio given by (82)

$$\mathsf{C}^{\mathsf{mmse}} \geq \frac{\beta}{2}\frac{1}{K}\sum_{k=1}^{K}\log\left(1/[\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R}]^{-1}_{kk}\right). \tag{83}$$

It has been observed in [64] that the Gaussian approximation for the output of the MMSE transformation is excellent even if there are very few binary-valued interferers. Moreover, as $K \to \infty$, the central-limit theorem proof in Proposition IV.1 can be extended to the current case to show that the spectral efficiency is not affected by the distribution of the symbols transmitted by the interferers. As $K \to \infty$, not only does the bound in (83) become tight but it admits a particularly interesting closed-form expression.

Proposition VI.1: For $\beta > 0$ and binary random spreading, the spectral efficiency of the MMSE receiver converges in mean square as $K \to \infty$ to

$$\lim_{K\to\infty}\mathsf{C}^{\mathsf{mmse}} = \frac{\beta}{2}\log\left(1 + \mathsf{SNR} - \frac{1}{4}\mathcal{F}(\mathsf{SNR},\beta)\right) \tag{84}$$

where

$$\mathcal{F}(x,z) \stackrel{\text{def}}{=} \left(\sqrt{x(1+\sqrt{z})^2+1} - \sqrt{x(1-\sqrt{z})^2+1}\right)^2. \tag{85}$$

Proof: Analogously to Proposition V.1, the result follows if we show the following convergence result for the minimum mean-square error

$$\begin{aligned}\mathsf{M}_k(\boldsymbol{R}) &= [\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R}]^{-1}_{kk} &(86)\\ &\stackrel{\text{m.s.}}{\to} 1 - \frac{1}{4\beta\mathsf{SNR}}\mathcal{F}(\mathsf{SNR},\beta) &(87)\\ &= \left[1 + \mathsf{SNR} - \frac{1}{4}\mathcal{F}(\mathsf{SNR},\beta)\right]^{-1}. &(88)\end{aligned}$$

Equation (88) follows from (85) after tedious algebra. To show the convergence result in (87) let us show first that it holds for the respective expectations. Consider the following chain:

$$\begin{aligned}E[\mathsf{M}_k(\boldsymbol{R})] &= E\left[[\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R}]^{-1}_{kk}\right]\\ &= \frac{1}{K}E[\mathrm{tr}\,[\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R}]^{-1}]\\ &= \frac{1}{K}E\left[\sum_{j=1}^{K}\lambda_j([\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R}]^{-1})\right]\\ &= E\left[\frac{1}{1+\mathsf{SNR}\lambda_k(\boldsymbol{R})}\right]\\ &\to \int_0^\infty \frac{1}{1+\mathsf{SNR}x}f_\beta(x)\,dx &(89)\\ &= 1 - \frac{1}{4\beta\mathsf{SNR}}\mathcal{F}(\mathsf{SNR},\beta) &(90)\end{aligned}$$

where the density in (89) was defined in (21).

To show mean-square convergence (86), we note that $0 < \mathsf{M}_k(\boldsymbol{R}) \leq 1$, and we follow [25] to express the normalized variance of $\mathsf{M}_K(\boldsymbol{R})$ as

$$\begin{aligned}&\frac{\mathrm{var}\,(\mathsf{M}_K(\boldsymbol{R}))}{E^2[\mathsf{M}_K(\boldsymbol{R})]}\\ &\quad\leq E\left[\frac{\mathsf{M}_K^2(\boldsymbol{R}) + E^2[\mathsf{M}_K(\boldsymbol{R})] - 2\mathsf{M}_K(\boldsymbol{R})E[\mathsf{M}_K(\boldsymbol{R})]}{\mathsf{M}_K(\boldsymbol{R})E^2[\mathsf{M}_K(\boldsymbol{R})]}\right]\\ &\quad= E\left[\frac{1}{\mathsf{M}_K(\boldsymbol{R})}\right] - \frac{1}{E[\mathsf{M}_K(\boldsymbol{R})]}. &(91)\end{aligned}$$

The proposition will follow upon showing that the right-hand side of (91) vanishes asymptotically. To that end, recall that $\boldsymbol{c}_k$ denotes the spreading code of the kth user and define the $N \times N$ matrix

$$\boldsymbol{\Xi} = \boldsymbol{I} + \mathsf{SNR}\sum_{k=1}^{K-1}\boldsymbol{c}_k\boldsymbol{c}_k^T.$$

It can be shown that [47], [25]

$$\frac{1}{\mathsf{M}_K(\boldsymbol{R})} - 1 = \mathsf{SNR}\,\boldsymbol{c}_K^T\boldsymbol{\Xi}^{-1}\boldsymbol{c}_K.$$

Taking expectations with respect to binary spreading codes we get

$$\begin{aligned}E\left[\frac{1}{\mathsf{M}_K(\boldsymbol{R})}\right] - 1 &= \mathsf{SNR}\,E\left[\boldsymbol{c}_K^T\boldsymbol{\Xi}^{-1}\boldsymbol{c}_K\right]\\ &= \frac{\mathsf{SNR}}{N}\sum_{n=1}^{N}E[(\boldsymbol{\Xi}^{-1})_{nn}] &(92)\\ &= \frac{\mathsf{SNR}}{N}E[\mathrm{tr}(\boldsymbol{\Xi}^{-1})]\\ &= \frac{\mathsf{SNR}}{N}E\left[\sum_{n=1}^{N}\lambda_n(\boldsymbol{\Xi}^{-1})\right]\\ &= \frac{\mathsf{SNR}}{N}E\left[\sum_{n=1}^{N}\frac{1}{\lambda_n(\boldsymbol{\Xi})}\right]\\ &= E\left[\frac{\mathsf{SNR}}{\lambda_1(\boldsymbol{\Xi})}\right]\\ &= E\left[\frac{\mathsf{SNR}}{1+\mathsf{SNR}\,\beta\lambda_1\left(\frac{1}{\beta}\sum_{k=1}^{K-1}\boldsymbol{c}_k\boldsymbol{c}_k^T\right)}\right]\\ &\to \int_0^{+\infty}\frac{\mathsf{SNR}}{1+\mathsf{SNR}\,\beta x}f_{1/\beta}(x)\,dx &(93)\\ &= \mathsf{SNR} - \frac{1}{4}\mathcal{F}(\mathsf{SNR},\beta) &(94)\end{aligned}$$

where (92) follows by averaging with respect to $\boldsymbol{c}_K$, whose components are independent and zero-mean; the limit (93) follows from Proposition II.1. □

It is well known that the maximum sum-rate of the Cover–Wyner capacity region of the one-dimensional additive white Gaussian noise multiple-access channel can be achieved by the technique of successive cancellation [1]. Although successive cancellation does not result in maximum-likelihood decisions (regardless of whether data are encoded), it becomes asymptotically optimum as the error probability of intermediate decisions vanishes with code blocklength. This implies that in a synchronous system where each user were assigned the same signature waveform, a successive canceler (each of whose stages consists of a matched filter followed by a single-user decoder which ignores previously decoded users and treats yet undecoded users as noise) would be also asymptotically optimum. Several recent references ([65]–[67]) have generalized this result to the K-user synchronous CDMA channel by noticing that under the assumption of perfect cancellation, the succesive canceler which uses an MMSE filter that ignores previously decoded users achieves the same capacity as the maximum-likelihood decoder. This is a direct consequence of the following identity (cf. (26) and (83)):

$$\log\left(\det\left[\boldsymbol{I}+\mathsf{SNR}\boldsymbol{R}\right]\right)=\sum_{k=1}^{K}\log\left(1/[\boldsymbol{I}+\mathsf{SNR}\boldsymbol{R}^{(k)}]_{kk}^{-1}\right) \quad (95)$$

where $\boldsymbol{R}^{(k)}$ denotes the kth principal minor (crosscorrelation matrix of users $1,\cdots,k$). Equation (95) is a special case of the elementary matrix identity

$$\det\left[\boldsymbol{A}^{(K)}\right]\prod_{k=1}^{K}\left[\boldsymbol{A}^{(k)}\right]_{kk}^{-1}=1. \quad (96)$$

As pointed out in [67], (95) can be used to express the optimum sum-capacity as an integral of the MMSE capacity we found in Proposition VI.1. Substituting (95) in (26) we obtain

$$\begin{aligned}
&\lim_{K\to\infty}\mathsf{C}^{\mathsf{opt}}(\mathsf{SNR},\boldsymbol{R},K,K/\beta)\\
&=\lim_{K\to\infty}\frac{\beta}{2K}\sum_{k=1}^{K}\log\left(1/\left[\boldsymbol{I}+\mathsf{SNR}\boldsymbol{R}^{(k)}\right]_{kk}^{-1}\right) &(97)\\
&=\lim_{K\to\infty}\frac{\beta}{2K}\sum_{k=1}^{K}\log\left(1+\mathsf{SNR}-\frac{1}{4}\mathcal{F}\left(\mathsf{SNR},\frac{k}{K}\beta\right)\right) &(98)\\
&=\frac{\beta}{2}\int_0^1\log\left(1+\mathsf{SNR}-\frac{1}{4}\mathcal{F}(\mathsf{SNR},x\beta)\right)dx &(99)\\
&=\frac{1}{2}\int_0^{\beta}\log\left(1+\mathsf{SNR}-\frac{1}{4}\mathcal{F}(\mathsf{SNR},z)\right)dz &(100)
\end{aligned}$$

where the limits in (97) are understood in probability, (98) follows from (88), and (99) follows from the definition of Riemann integral. The solution of the definite integral in (100) is given in (9). The expression for the difference between the optimum and MMSE spectral efficiency given in (13) can be checked from (9) and (88).

We emphasize that the capacity found in Proposition VI.1 holds regardless of whether the signature waveform changes from symbol to symbol (long pseudorandom codes) or stays constant. Even in the latter case, the randomness due to the choice of signatures vanishes as the number of users grows. Moreover, since the analysis shows convergence of the output signal-to-noise ratio of the MMSE receiver, the asymptotic determinism (and equivalence of long and short random spreading codes) applies to uncoded systems [25] and to the performance of suboptimal error-control codes. However, for symbol-synchronous systems with small number of users using certain simple error-correcting codes, the short-term averaging effect of long codes may be beneficial [15].

In parallel to (66), if a target output signal-to-noise ratio of ρ^{mmse} is desired for the MMSE receiver, then it can be verified from the expression for the output signal-to-noise ratio in (94) that the number of users that can be accommodated is (cf. [51])

$$K=N\left(\frac{1}{\rho^{\mathsf{mmse}}}+1\right)\left(1-\frac{\rho^{\mathsf{mmse}}}{\mathsf{SNR}}\right). \quad (101)$$

As usual, $\mathsf{C}^{\mathsf{mmse}}(\beta,\frac{E_b}{N_0})$ is obtained by substituting

$$\mathsf{SNR}=\frac{2}{\beta}\frac{E_b}{N_0}\mathsf{C}^{\mathsf{mmse}}\left(\beta,\frac{E_b}{N_0}\right)$$

into

$$\mathsf{C}^{\mathsf{mmse}}\left(\beta,\frac{E_b}{N_0}\right)=\frac{\beta}{2}\log\left(1+\mathsf{SNR}-\frac{1}{4}\mathcal{F}(\mathsf{SNR},\beta)\right). \quad (102)$$

Let us study the behavior of the spectral efficiency of the MMSE receiver for asymptotically large $\frac{E_b}{N_0}$. For $0<\beta<1$, the MMSE and decorrelator spectral efficiencies coincide (cf. Fig. 2)

$$\lim_{\frac{E_b}{N_0}\to\infty}\mathsf{C}^{\mathsf{mmse}}\left(\beta,\frac{E_b}{N_0}\right)-\mathsf{C}^{\mathsf{deco}}\left(\beta,\frac{E_b}{N_0}\right)=0. \quad (103)$$

Thus according to (5) and (103), if $\beta<1$, then

$$\lim_{\frac{E_b}{N_0}\to\infty}\frac{\mathsf{C}^{\mathsf{mmse}}\left(\beta,\frac{E_b}{N_0}\right)}{\mathsf{C}^{*}\left(\frac{E_b}{N_0}\right)}=\beta.$$

If $\beta=1$, then it can be shown that

$$\lim_{\frac{E_b}{N_0}\to\infty}\frac{\mathsf{C}^{\mathsf{mmse}}\left(1,\frac{E_b}{N_0}\right)}{\mathsf{C}^{*}\left(\frac{1}{2}\frac{E_b}{N_0}\right)}=\frac{1}{2}.$$

If $\beta>1$, then it can be shown that

$$\lim_{\frac{E_b}{N_0}\to\infty}\mathsf{C}^{\mathsf{mmse}}\left(\beta,\frac{E_b}{N_0}\right)=\frac{\beta}{2}\log\frac{\beta}{\beta-1}.$$

The asymptotic behavior of the spectral efficiency of the MMSE receiver with $\beta\to\infty$ is identical to that of the single-user matched filter

$$\lim_{\beta\to\infty}\mathsf{C}^{\mathsf{mmse}}\left(\beta,\frac{E_b}{N_0}\right)=\frac{\log_2 e}{2}-\frac{1}{2}\frac{N_0}{E_b} \quad (104)$$

for $\frac{E_b}{N_0}>\log_e 2$.

Fig. 8 depicts the function

$$\gamma^{\mathsf{mmse}}\left(\beta,\frac{E_b}{N_0}\right)\stackrel{\text{def}}{=}\frac{\mathsf{C}^{\mathsf{mmse}}\left(\beta,\frac{E_b}{N_0}\right)}{\mathsf{C}^{\mathsf{orth}}\left(\beta,\frac{E_b}{N_0}\right)}=\frac{\mathsf{C}^{\mathsf{mmse}}\left(\beta,\frac{E_b}{N_0}\right)}{\beta\mathsf{C}^{*}\left(\frac{E_b}{N_0}\right)}$$

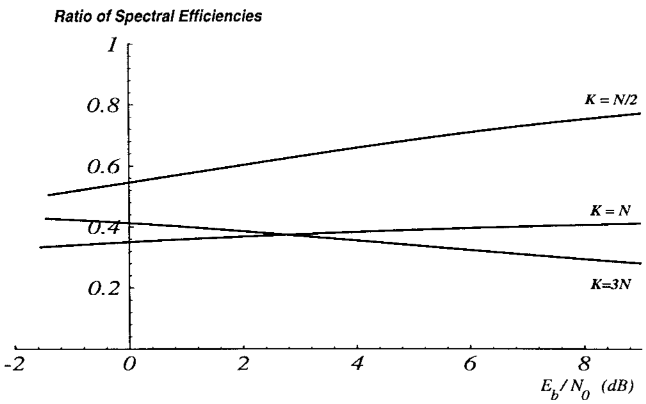

Fig. 8. MMSE receiver: Spectral efficiency with random signatures divided by spectral efficiency with optimally chosen signatures.

if $\beta \leq 1$, and

$$\gamma^{\mathrm{mmse}}\left(\beta, \frac{E_b}{N_0}\right) \stackrel{\text{def}}{=} \frac{\mathsf{C}^{\mathrm{mmse}}\left(\beta, \frac{E_b}{N_0}\right)}{\mathsf{C}^*\left(\frac{E_b}{N_0}\right)} \tag{105}$$

if $\beta > 1$. Comparing Fig. 8 to Fig. 5 we see that, unlike the optimal receiver, the MMSE receiver with random spreading suffers substantial losses for $K = 3N$. For $K = N$, and in the range of $\frac{E_b}{N_0}$ considered in Fig. 8, a random choice achieves around 40% of the spectral efficiency achieved by orthogonal sequences. As the spreading gain N increases, the MMSE detector loss is more important at low $\frac{E_b}{N_0}$ and approaches that in Figs. 5 and 7 for large $\frac{E_b}{N_0}$. The deleterious effect of low $\frac{E_b}{N_0}$ on the decorrelator (Fig. 7) is not suffered by the MMSE receiver. Relative to Fig. 6, we see that at low $\frac{E_b}{N_0}$ the MMSE and single-user matched filter behave similarly; at high $\frac{E_b}{N_0}$ the comparison depends heavily on K/N.

VII. Conclusion

A misconception that has arisen in the last few years claims that in CDMA systems with a large number of users, error-control coding, perfect power control, and long codes, little can be gained by exploiting the structure of the multiaccess interference at the receiver (cf. [18]). Our results have shown that exactly the opposite conclusion is true. Because of the deleterious effects of imperfect power control on the single-user matched filter, we would expect that the spectral inefficiency of that receiver to be even greater in that situation.

Another misconception predicts that multiuser detectors suffer from high sensitivity to the actual signature waveforms (cf. [18]). On the contrary, our convergence results have shown that, as the number of users grows, the variability in achievable signal-to-noise ratio and spectral efficiency due to the choice of signature waveforms vanishes.

With large K/N, random CDMA incurs negligible spectral efficiency loss relative to no-spreading if an optimum receiver is used. However, we have shown that linear multiuser detection is distinctly suboptimal for large K/N. This warrants the study of nonlinear suboptimal multiuser detection, such as decision-feedback schemes [66], [65] and iterative decoding procedures [68]–[70], which have already demonstrated very competitive performance with limited complexity. The capacity-achieving nature of error-free successive cancellation using single-user decoders with MMSE linear front ends (cf. Section VI) lends further motivation for analyzing practical approximations to that ideal scheme.

The optimum coding–spreading tradeoff favors negligible spreading (with respect to the number of users) for either optimum or single-user matched-filter processing. In contrast, nonnegligible spreading is optimum for linear multiuser detectors such as the decorrelator and the MMSE receiver. With an optimal choice of spreading factor, the spectral efficiencies of the decorrelator and MMSE receivers grow without bound as $\frac{E_b}{N_0}$ increases, in contrast to the single-user matched filter for which large signal-to-noise ratios offer little incentive (Fig. 2). For large K/N, even if the signal-to-noise ratio is very low, the spectral efficiency of the single-user matched filter is a fraction of the optimum one. So even though the background noise is dominant it pays to exploit the structure of the multiaccess interference because there are several users per degree of freedom. The loss in spectral efficiency due to a random choice of spreading sequences depends on $\frac{E_b}{N_0}$, K, N, and the type of receiver used. Interestingly, we have found that for the optimal receiver, the single-user matched filter, and the decorrelator, the maximal loss occurs at $K = N$.

We have focused exclusively on power-constrained inputs. If the channel symbols modulating the signature waveforms are restricted to be binary, then existing results on the capacity of single-user binary input Gaussian channels can be used to deal with the decorrelator, MMSE, and single-user matched filter. However, optimal spectral efficiency under such constraint is unknown, except when K/N is large in which case the symbol SNR is low and binary inputs are almost as good as Gaussian [4], [71].

For low K/N systems (such as state-of-the-art CDMA), either the decorrelator or the MMSE are excellent choices and little inefficiency results from random rather than orthogonal signatures.

Fading can be incorporated in the analysis, replacing $b_k(i)$ by $\tilde{b}_k(i) = \alpha_{ki}\, b_k(i)$, where α_{ki} are i.i.d. random variables known to the receiver but not to the transmitter. This framework can be used to model either a classical fading effect (independent from symbol to symbol because of interleaving) or to account for nonideal power control fluctuations. Our asymptotic-in-K results can be generalized to this setting and to nonequal deterministic received powers, using recent results on the spectral distribution of random matrices [72], [73][7].

The coding–spreading tradeoff considered in this paper is not limited to direct-sequence spread-spectrum systems; it can be interpreted in a general way, where degrees of freedom in time/frequency/space are used for coding and spreading purposes. For example, multicarrier CDMA [75] can be considered a dual (in frequency) to the direct-sequence format (in time) [25].

For illustration purposes, let us consider the homogeneous fading model [25] where chips are affected by identically distributed fading coefficients: $\tilde{c}_{kl} = \alpha_{kl} c_{kl}$. Let us also

[7] See [74] for further discussion.

assume that the fading coefficients have unit power and are known to the receiver. Under mild conditions on the distributions of $\{\tilde{c}_{kl}\}$ [72], the optimal spectral efficiency found in Proposition III.2 can be shown to extend to this case.

In the case of $K = N \to \infty$, it is interesting to compare the performance of such a frequency-division spreading scheme, to that where classical frequency division is used as an orthogonal channel accessing technique. The latter gives rise to the spectral efficiency (in bits per frequency slot)

$$C_{\mathsf{fdm}}^{\mathsf{orth}} = \frac{1}{2} E \log\left(1 + \alpha^2 \,\mathsf{SNR}\right) \tag{106}$$

where the expectation is taken with respect to the fading power random variable α^2 and where SNR is the individual signal-to-noise ratio (over all frequency slots). This is to be compared to the results of Proposition III.2. For low SNR both schemes yield the same behavior of $\frac{1}{2}\mathsf{SNR}\log e$, and for large SNR

$$C^{\mathsf{opt}} = \frac{1}{2}\log(\mathsf{SNR}) - \frac{1}{2}\log e + o(1) \tag{107}$$

while the result in (106) depends on the distribution of α^2. Certainly for no fading $C_{\mathsf{fdm}}^{\mathsf{opt}}$ is advantageous, being equivalent to the optimal accessing technique [8]. This advantage is also maintained for Rayleigh fading where α^2 is exponentially distributed [76] and

$$C_{\mathsf{fdm}}^{\mathsf{opt}} = \frac{1}{2}\log(\mathsf{SNR}) - \frac{\mathcal{C}}{2}\log e + o(1) \tag{108}$$

where $\mathcal{C} = 0.57721$ is the Euler constant. The comparison of (107) to (108) answers in part an open problem posed in [76] on the relative advantage of CDMA versus TDMA in a single-cell fading channel.

Our analysis has focused on symbol-synchronous CDMA channels. The generalization to symbol-asynchronous CDMA is nontrivial (cf. [3]), but highly interesting for many applications[8].

Appendix I

In this appendix we consider the two-user case with the spherical random sequence model under which the density of ρ is given by (22). Then, the spectral efficiency with static power allocation is

$$\begin{aligned}
&\hat{\mathsf{C}}_s^{\mathsf{opt}}(\mathsf{SNR}, 2, N)\\
&\quad= E\left[\frac{1}{2N}\log\left(1 + 2\mathsf{SNR} + (1-\rho^2)\mathsf{SNR}^2\right)\right] &(109)\\
&\quad= \frac{1}{N}\log(1+\mathsf{SNR}) + \frac{1}{NC_N}\int_0^{\pi/2}\cos^{N-2}(\theta)\\
&\qquad\times\log\left(1 - \left(\frac{\mathsf{SNR}}{1+\mathsf{SNR}}\right)^2 \sin^2\theta\, d\theta\right) &(110)
\end{aligned}$$

and the dynamic-power spectral efficiency is

$$\begin{aligned}
&\hat{\mathsf{C}}_d^{\mathsf{opt}}(\mathsf{SNR}, 2, N)\\
&\quad= \max E\left[\frac{1}{2N}\log\left(1 + 2\mathsf{SNR}(\rho) + (1-\rho^2)\mathsf{SNR}^2(\rho)\right)\right] &(111)\\
&\quad= \frac{1}{2N}\log\frac{2}{\lambda^2} + \frac{1}{NC_N}\int_0^{\pi/2}\cos^{N-2}(\theta)\\
&\qquad\times\log(\cos^2\theta + \sqrt{\cos^4\theta + \lambda^2\sin^2\theta})\, d\theta &(112)
\end{aligned}$$

where the Lagrange coefficient λ is specified by

$$\frac{1}{\lambda} - \frac{C_{N-2}}{C_N} + \frac{2}{\lambda C_N}\int_0^{\pi/2}\cos^{N-4}\theta\sqrt{\cos^4\theta + \lambda^2\sin^2\theta}\, d\theta = \mathsf{SNR}.$$

For low SNR, the following limiting behaviors can be verified from (110) and (112):

$$\hat{\mathsf{C}}_s^{\mathsf{opt}}(\mathsf{SNR}, 2, N) = \frac{\mathsf{SNR}}{N}\log e + o(\mathsf{SNR}) \tag{113}$$

$$\hat{\mathsf{C}}_d^{\mathsf{opt}}(\mathsf{SNR}, 2, N) = \frac{\mathsf{SNR}}{N}\log e + o(\mathsf{SNR}) \tag{114}$$

whereas at high SNR

$$\begin{aligned}
\hat{\mathsf{C}}_s^{\mathsf{opt}}(\mathsf{SNR}, 2, N) &= \frac{\log\mathsf{SNR}}{N} + (-1)^N\left[\sum_{k=1}^{N-2}\frac{(-1)^{k+1}}{k} - 1\right]\\
&\quad\times\log 2 + o(1) &(115)\\
\hat{\mathsf{C}}_d^{\mathsf{opt}}(\mathsf{SNR}, 2, N) &= \frac{\log\mathsf{SNR}}{N} + (-1)^N\left[\sum_{k=1}^{N-2}\frac{(-1)^{k+1}}{k} - 1\right]\\
&\quad\times\log 2 + o(1). &(116)
\end{aligned}$$

We see that at either extreme of SNR, the gains of dynamic power allocation with spherical sequences vanish.

In the case of $N = 2$, (109) becomes

$$\begin{aligned}
\hat{\mathsf{C}}_s^{\mathsf{opt}}(\mathsf{SNR}, 2, 2) &= \frac{1}{2}\log(1+\mathsf{SNR})\\
&\quad+\frac{1}{2}\log\left(1 + \sqrt{1 - \left(\frac{\mathsf{SNR}}{1+\mathsf{SNR}}\right)^2}\right)\\
&\quad-\frac{1}{2}\log 2.
\end{aligned}$$

Appendix II
Proof of Proposition III.1

For every realization of $\boldsymbol{R}$ we can write

$$\begin{aligned}
&\log\det[\boldsymbol{I} + \mathsf{SNR}(\boldsymbol{R})\boldsymbol{R}] - \log\det[\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R}]\\
&\quad= \log\det[(\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R})^{-1}(\boldsymbol{I} + \mathsf{SNR}(\boldsymbol{R})\boldsymbol{R})]\\
&\quad= \log\det[\boldsymbol{I} + (\mathsf{SNR}(\boldsymbol{R}) - \mathsf{SNR})(\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R})^{-1}\boldsymbol{R}]. &(117)
\end{aligned}$$

Let us consider the matrix that appears in (117)

$$\boldsymbol{M}(\mathsf{SNR}) = (\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R})^{-1}\boldsymbol{R}.$$

Note that if v is an eigenvector of $\boldsymbol{R}$ with eigenvalue λ, then v is an eigenvector of $\boldsymbol{M}(\mathsf{SNR})$ with eigenvalue equal to

$$\frac{\lambda}{1 + \mathsf{SNR}\lambda}.$$

[8] See [77] for a signal-to-noise ratio analysis of the MMSE receiver in the chip-synchronous case.

Since the determinant of a nonnegative-definite matrix is the product of its eigenvalues we have

$$\lim_{\sigma \to 0} \log \det [\boldsymbol{I} + (\mathsf{SNR}(\boldsymbol{R}) - \mathsf{SNR})(\boldsymbol{I} + \mathsf{SNR}\boldsymbol{R})^{-1}\boldsymbol{R}] = r(\boldsymbol{R}) \log \frac{\mathsf{SNR}(\boldsymbol{R})}{\mathsf{SNR}}. \quad (118)$$

The maximization of the expected value of (118) with respect to the power allocation is equivalent to

$$\max \sum_{j=1}^{K} P[r(\boldsymbol{R}) = j]\, j \log \alpha(j)$$

where the maximization is with respect to $\alpha(k) = \mathsf{SNR}(\boldsymbol{R})/\mathsf{SNR}$ if $r(\boldsymbol{R}) = k$ such that

$$\sum_{j=1}^{K} P[r(\boldsymbol{R}) = j]\alpha(j) = 1.$$

The method of Lagrange multipliers readily yields

$$\alpha(j) = \frac{j}{E[r(\boldsymbol{R})]}$$

or, equivalently,

$$\mathsf{SNR}(\boldsymbol{R}) = \mathsf{SNR}\frac{r(\boldsymbol{R})}{E[r(\boldsymbol{R})]}. \qquad \square$$

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