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SPECTRAL ESTIMATES AND ASYMPTOTICS FOR INTEGRAL OPERATORS ON SINGULAR SETS

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For singular numbers of integral operators of the form

$$u(x) \mapsto \int F_1(X)K(X, Y, X - Y)F_2(Y)u(Y)\mu(dY)$$

with a measure μ singular with respect to the Lebesgue measure in $\mathbb{R}^{\mathbf{N}}$ we obtain order-sharp estimates for the counting function. The kernel $K(X, Y, Z)$ is assumed to be smooth in X, Y, $Z \neq 0$ and to admit an asymptotic expansion in homogeneous functions in the Z variable as $Z \to 0$. The order in the estimates is determined by the leading homogeneity order in the kernel and geometric properties of the measure μ and involves integral norms of the weight functions F_1 and F_2. In the selfadjoint case, we obtain asymptotics of the eigenvalues of this integral operator provided that μ is the surface measure on a Lipschitz surface of some positive codimension $\mathfrak{d}$. Bibliography: 16 *titles.*

To Volodya Maz'ya with admiration

1 Introduction

Since the pathbreaking papers [1]–[3] by Birman and Solomyak published in the 1960s and 1970s it became a general wisdom that order-sharp eigenvalue and singular number estimates for operators of various kinds under weakest regularity conditions in terms of integral norms of coefficients open up the possibility of studying fine characteristics of the spectrum, in particular,

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the spectral asymptotics. This approach has been demonstrated many times, including the most recent developments [4, 5]. However, there are certain types of problems where such estimates are not yet known. The present paper is devoted to one of such missing cases. We consider integral operators with weak diagonal polarity of the kernel, acting in the L_2 space with a measure μ singular with respect to the Lebesgue measure in $\mathbb{R}^{\mathbf{N}}$ and containing integrable weight functions. Our main results are new spectral estimates for such operators. They are closely related to recent developments in the topic of spectral estimates for pseudodifferential operators with singular measures (see [4]–[7]), but the regularity conditions imposed in the cited papers can be reduced owing to specifics of integral operators. In the selfadjoint case, when μ is the surface measure on a compact Lipschitz surface of a certain codimension $\mathfrak{d} \geqslant 1$ in $\mathbb{R}^{\mathbf{N}}$, we find the eigenvalue asymptotics.

Our approach to operators involving singular measures follows [7] and [5], is based upon the fundamental trace and embedding theorems due to Maz'ya [8].

1.1. Setting. Let $K(X,Y,Z)$, $X,Y \in \mathbb{R}^{\mathbf{N}}$, $Z \in \mathbb{R}^{\mathbf{N}} \setminus 0$, be an integral kernel of potential type, smooth in X, Y, and in Z with $Z \neq 0$. The kernel is supposed to be polyhomogeneous, which means that $K(X,Y,Z)$ can be expanded in the asymptotic series

$$K(X,Y,Z) \sim \sum K_j(X,Y,Z), \quad j = 0, \dots, \quad Z \to 0. \tag{1.1}$$

The function $K_j(X,Y,Z)$ is positively homogeneous in Z of degree $\theta + j$, and the leading order is $\theta > -\mathbf{N}$, so the kernel possesses a weak singularity. If $\theta + j$ is an even nonnegative number, the symbol K_j can contain, in addition to the above homogeneous function denoted by $K_j^{(\mathrm{hom})}$, a term $K_j^{(\log)}(X,Y,Z)$ of the form $Q_j(X,Y,Z)\log|Z|$, where $Q_j(X,Y,Z)$ is a homogeneous polynomial of degree $\theta + j$ in Z and smooth in other variables. We note that the representation (1.1) for a given kernel is not unique since it can be changed by making the Taylor expansion in the $X - Y$ variable at the point X or Y and regrouping the resulting terms. We denote by $\mathbf{K}$ the integral operator with kernel K.

Let μ be a compactly supported Borel measure on $\mathbb{R}^{\mathbf{N}}$, without point masses, with support (the smallest closed set of full μ-measure) $\mathscr{M}$, and let $F_1(X)$ and $F_2(X)$ be μ-measurable functions on $\mathscr{M}$. (It is sometimes convenient to assume that F_ι vanish almost everywhere with respect to the measure μ.)

Under these conditions, we consider the integral operator $\mathbf{T} = \mathbf{T}[\mu, \mathbf{K}, F_1, F_2]$ in $L_{2,\mu}$,

$$(\mathbf{T}u)(X) = \int\limits_{\mathscr{M}} F_1(X)K(X,Y,X-Y)F_2(Y)u(Y)\mu(dY), \tag{1.2}$$

or, formally,

$$(\mathbf{T}u) = (F_1\mu)\mathbf{K}((F_2\mu)u).$$

The more general matrix case can be considered, where $K(X,Y,Z)$ is a $\mathbf{k} \times \mathbf{k}$ matrix-valued function subject to the above conditions and $F_1(X)$, $F_2(X)$ are μ-measurable $\mathbf{k}\times\mathbf{k}$ matrix-valued functions. Here, the operator $\mathbf{T}[\mu, K, F_1, F_2]$ formally acts as (1.2), but now u is a measurable $\mathbf{k}$-component vector-valued function on $\mathscr{M}$. Below, we specify the conditions on μ, K, F_1, and F_2 granting the boundedness of $\mathbf{T}$ in $L_{2,\mu}$. It is also possible to consider the even more general setting where K, F_1, F_2 are rectangular matrix-valued functions of proper size, so that the product in (1.2) makes sense, but this case does not involve any new ideas and we leave it to the interested reader)

Of special interest is the formally Hermitian case, where the kernel $K(X,Y,Z)$ is symmetric in the sense

$$K(X,Y,Z) = K(Y,X,-Z)^*,$$

the symbol * denotes the (complex or matrix) conjugation operation and $F_1(X) = F_2(X)^*$. Under these conditions, the operator $\mathbf{T}$ is considered in the space $L_{2,\mu}$. This operator is formally selfadjoint, and if it happens to be bounded, it is selfadjoint in $L_{2,\mu}$.

There are two alternative ways to present results on properties of integral operators of potential type. On one hand, we can fix a measure μ and describe how nice the kernel and weight functions should be. We take a somewhat different course of action. We fix a kernel and determine properties of the measure and the weight functions guaranteeing the boundedness of the integral operator and the required spectral estimates.

Let the operator (1.2) be selfadjoint and bounded. In this case, for $\lambda > 0$ we denote by $n_\pm(\lambda, \mathbf{T})$ the total multiplicity of the spectrum of $\pm\mathbf{T}$ in (λ, ∞). If there are infinitely many such eigenvalues or there are some points of the essential spectrum in this interval, then we set $n_\pm(\lambda, \mathbf{T}) = \infty$. In the general, not necessarily selfadjoint, case we study estimates for the singular numbers of the operator $\mathbf{T}$, i.e., the counting function

$$n(\lambda, \mathbf{T}) := n_+(\lambda^2, \mathbf{T}^*\mathbf{T}).$$

We are interested in finding estimates and, if possible, asymptotics of these counting functions as $\lambda \to 0$ in terms of integral characteristics of the weights $F_1(X)$ and $F_2(X)$, geometric characteristics of the singular measure μ, and properties of the kernel K.

We restrict ourselves to the case of a compactly supported measure μ. By the standard localization procedure, this problem reduces to a problem for operators acting on a set in a smooth compact Riemannian manifold, for example, on the torus $\mathbb{T}^\mathbf{N}$ with standard metric. In this case, it is assumed that the homogeneity condition on the kernel holds for $Z = X - Y$ close to zero, i.e., near the diagonal $X = Y$, while the kernel is smooth away from the diagonal. The case of a noncompactly supported measure μ presents certain complications, which we skip in this paper in order to avoid excessive technicalities.

1.2. Relation to earlier results. Spectral problems for weighted weakly polar integral operators were considered in [1, 3, 9, 10], where the main interest was in finding the weakest possible conditions on the kernel and weights such that the singular numbers admit estimates of the same order as for the nonweighted operator. In [1, 3, 9, 10], μ is the Lebesgue measure on $\mathbb{R}^\mathbf{N}$ restricted to the set $\mathscr{M}$. Moreover, an additional factor $\Phi(X,Y)$, a multiplier, in the kernel of the operator $\mathbf{T}$ was subject to certain milder regularity restrictions. In this setting, spectral estimates and asymptotics for $\mathbf{T}$ were established with an order depending on the dimension $\mathbf{N}$ and the homogeneity order θ. For some values of parameters one weight or both weights could be incorporated in the measure, so μ could be any finite Borel measure, not necessarily absolutely continuous with respect to the Lebesgue measure. However, the above-mentioned spectral estimates have the same order for all admissible weights and measures, and this order depends only on the homogeneity order of the kernel and the dimension $\mathbf{N}$ of the space.

However, certain applications require sharp eigenvalue estimates for a more general singular measure μ. For example, μ can be the Hausdorff measure on a fractal set or the surface measure on a nonsmooth surface. In these cases, the classical results in the above-mentioned papers give, if applicable, only o-small and, consequently, not sharp estimates compared with the absolutely continuous case, so they are not applicable, at least directly.

1.3. Specifics of the new approach and the main results. In the recent papers [4]–[7], an approach was developed for obtaining spectral estimates for a class of weighted pseudodifferential operators involving singular measures, the so-called Birman–Schwinger type operators. In some cases, in particular, for μ being the surface measure on a Lipschitz surface of some positive codimension in $\mathbb{R}^{\mathbf{N}}$, as well as for the Hausdorff measure on a uniformly rectifiable set, these estimates are order-sharp, which is confirmed by the eigenvalue asymptotic formulas obtained there. To prove these spectral estimates, we use the classical variational approach based upon piecewise polynomial approximations, initiated by Birman and Solomyak more than 50 years ago and adapted to our singular measure setting. Asymptotic formulas were derived by using perturbation ideas coming back to Birman and Solomyak as well and, for Lipschitz surfaces, arguments based on the results of the authors [11, 12]. One of crucial steps is a reduction of the spectral problem for a pseudodifferential operator with singular weight to an integral operator with singular measure.

In the present paper, we apply this relation to studying the spectrum of integral operators with (poly)homogeneous kernel, thus using the above reduction in the backwards direction. The conditions imposed on the measure μ similarly to [5], are formulated in terms of inequalities for the μ-measure of balls in $\mathbb{R}^{\mathbf{N}}$. For $0 < \alpha < \mathbf{N}$ we consider three classes of measures without point masses

$$\mu \in \mathscr{P}^{\alpha}_{+}, \quad \mu(B(X,r)) \leqslant \mathscr{A}(\mu) r^{\alpha}, \quad r > 0, \tag{1.3}$$

$$\mu \in \mathscr{P}^{\alpha}_{-}, \quad \mu(B(X,r)) \geqslant \mathscr{B}(\mu) r^{\alpha}, \quad 0 < r < \operatorname{diam} \mathscr{M}, \tag{1.4}$$

$$\mu \in \mathscr{P}^{\alpha}, \quad \mathscr{B}(\mu) r^{\alpha} \leqslant \mu(B(X,r)) \leqslant \mathscr{A}(\mu) r^{\alpha}, \quad 0 < r < \operatorname{diam} \mathscr{M}; \tag{1.5}$$

here, $B(X,r)$ is the (open) ball with radius r and center X.

Depending on the homogeneity order of the kernel and properties of the singular measure μ, we obtain estimates for the singular numbers involving certain integral norms of the weight functions. Unlike the case of absolutely continuous measures, considered in [9, 10, 1], where the order of estimates is determined only by the dimension $\mathbf{N}$ and the leading homogeneity order θ of the kernel, in our case, the order (the power of λ) of the eigenvalue estimate also depends on the exponent α in the characteristic (1.3), (1.4), or (1.5). The choice of which of these conditions is present in the corresponding formulations is determined by the relation between the dimension $\mathbf{N}$ and the order of singularity θ. We note that, when the two-sided condition (1.5) is imposed, a rather special case of the homogeneity degree $\theta = 0$ is covered. Here, the leading term in the kernel $K(X,Y,Z)$ is the sum of two terms, one of which, K_0^{hom}, is zero order positively homogeneous in Z for small Z and the other, $K_0^{\log}$, has the form $Q_0(X,Y)\log|Z|$ with smooth function Q_0. In this case, the order of eigenvalue estimates (and asymptotics, when it is proved) equals -1 and does not depend on α in (1.5), the effect noticed earlier in [7] for the corresponding spectral problem for pseudodifferential operators (see also the discussion in [4] concerning the noncommutative integration with respect to singular measures).

A special feature of the reduction to a pseudodifferential problem is the need for smoothness requirements, inherent to the pseudodifferential analysis. For integral operators considered in the paper, such smoothness requirements turn out to be excessive and they are considerably relaxed by means of introduction of Schur multipliers, similar to how this was done in the classical papers by Birman and Solomyak for absolutely continuous measures. In proving spectral estimates and asymptotics, we use the additional flexibility provided by the presence of weight functions.

When smooth, they can be incorporated in the kernel $K(X,Y,X-Y)$, while, in the opposite, a nonsmooth factor in the kernel can be incorporated in the weight functions F_1 and F_2. All this enables us to establish the eigenvalue estimates and asymptotics under somewhat milder regularity conditions than in [5] in the pseudodifferential setting.

2 Pseudodifferential and Integral Operators: Reduction

2.1. Initial definitions. It is known that an integral operator with homogeneous kernel which is smooth away from the diagonal can be understood as a pseudodifferential operator, and vice versa (see, for example, [3, 11, 13] and the references therein).

For a polyhomogeneous classical symbol $\mathfrak{k}(X,\Xi)$ of order $-l<0$ admitting the expansion in homogeneous functions

$$\mathfrak{k}(X,\Xi) \sim \sum_{j=0}^{\infty} \mathfrak{k}_j(X,\Xi), \quad \Xi \to \infty, \quad X \in \mathbb{R}^{\mathbf{N}}, \quad \Xi \in \mathbb{R}^{\mathbf{N}} \setminus \{0\},$$

with positively homogeneous terms $\mathfrak{k}_j(X,t\Xi) = t^{-l-j}\mathfrak{k}_j(X,\Xi)$, $t>0$, the pseudodifferential operator $\mathfrak{K}(X,D_X)$ with this symbol is defined by the usual local formula

$$(\mathfrak{K}(X,D_X)u)(X) = \mathscr{F}^{-1}_{\Xi\to X}\mathfrak{k}(X,\Xi)\mathscr{F}_{Y\to\Xi}u(Y),$$

where $\mathscr{F}$ is the Fourier transform. This operator can be equivalently represented as an integral operator with kernel $K(X,Y,X-Y)$ admitting the asymptotic expansion

$$K(X,Y,X-Y) \sim \sum_{j=0}^{\infty} K_j(X,X-Y), \quad X-Y \to 0, \tag{2.1}$$

in homogeneous or log-homogeneous functions. The kernel $K_j(X,(X-Y))$ is the properly regularized Fourier transform of the symbol $\mathfrak{k}_j(X,\Xi)$ in Ξ (see [11, 13] for details and further references). We again note that the representation (2.1) is not uniquely defined: expanding the kernel $K(X,Y,X-Y)$ at the point $Y=X$ in the $X-Y$ variable and further regrouping the resulting terms, we arrive at a different composition of the kernel. This relation goes through without complications as long as the symbol and kernel are infinitely smooth. There is an extensive, rather technically advanced literature devoted to the treatment of the case of symbols with finite smoothness (see [14, 15] and the references therein). Our approach based upon the study of finitely smooth integral kernels is more elementary.

The point of our interest is operators of the form $\mathbf{T} = \mathbf{T}[\mu,\mathfrak{K},F_1,F_2]$ formally described as $\mathbf{T} = (F_1\mu)\mathfrak{K}(F_2\mu)$, where μ is a singular measure and F_1, F_2 are μ-measurable functions. We need to explain how such operators are rigorously defined. The definition will be based upon a fixed factorization of $\mathfrak{K}$ as $\mathfrak{K} = \mathfrak{K}_1\mathfrak{K}_2$, where $\mathfrak{K}_\iota$ are pseudodifferential operators of order $-\gamma_\iota < 0$, $\gamma_1+\gamma_2 = l$. The case of our special interest is the case where the pseudodifferential operator $\mathfrak{K}$ is selfadjoint nonnegative and has the form $\mathfrak{K} = \mathfrak{L}\mathfrak{L}^*$, where $\mathfrak{L}$ is a pseudodifferential operator of order $-\gamma = -l/2 < 0$.

2.2. Definition of operators. Following [5, 7], we distinguish three cases, where the conditions on the measure μ and weights F_ι are formulated in different ways. These cases are determined by the relation between the order γ and dimension $\mathbf{N}$.

- *Subcritical*: $2\gamma_\iota < \mathbf{N}$. The measure μ is supposed to belong to $\mathscr{P}^\alpha_+$ with $\alpha > \mathbf{N} - 2\gamma_\iota$.

- *Critical*: $2\gamma_\iota = \mathbf{N}$. The measure μ is supposed to belong to $\mathscr{P}^\alpha$ with some α, $0 < \alpha < \mathbf{N}$.
- *Supercritical*: $2\gamma_\iota > \mathbf{N}$. The measure μ is supposed to belong to $\mathscr{P}_-^\alpha$ with some $\alpha > 0$.

Since $\mathfrak{K}$ is, in fact, an integral operator with polyhomogeneous kernel $K(X, Y, X - Y)$, $\mathbf{T}$ can be formally described as the integral operator

$$(\mathbf{T}u)(X) = (\mathbf{T}[\mu, \mathfrak{K}, F_1, F_2]u)(X) = F_1(X) \int K(X, Y, X - Y)u(Y)F_2(Y)\mu(dY). \tag{2.2}$$

Now, we explain how this operator is rigorously defined. In the next section, we find conditions for the boundedness of this operator. Our reasoning follows the natural rule: if all objects are rigorously defined, the resulting construction coincides with the one obtained by formal manipulations.

First, let the pseudodifferential operator $\mathfrak{K}$ be factorized as $\mathfrak{K} = \mathfrak{K}_1\mathfrak{K}_2$, where $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are pseudodifferential operators of negative orders $-\gamma_\iota < 0$ and $\gamma_1 + \gamma_2 = l$ respectively.

Suppose that we have defined *bounded* operators $\mathbf{T}_1 = F_1\mathbf{t}_{\mathscr{M}}\mathfrak{K}_1 : L_2(\mathbb{T}^{\mathbf{N}}) \to L_{2,\mu}$ and $\mathbf{T}_2 = F_2{}^*\mathbf{t}_{\mathscr{M}}\mathfrak{K}_2^* : L_2(\mathbb{T}^{\mathbf{N}}) \to L_{2,\mu}$, where $\mathbf{t}_{\mathscr{M}}$ is the restriction operator of functions in $H^{\gamma_\iota}(\mathbb{T}^{\mathbf{N}})$ to $\mathscr{M}$. Then the operator $\mathbf{T}$ in $L_{2,\mu}$ is defined by

$$\mathbf{T}[\mu, \mathfrak{K}, F_1, F_2] = \mathbf{T}_1\mathbf{T}_2^*.$$

Once we have found some estimates for the singular numbers of the operator $\mathbf{T}_\iota$, we can apply the Ky Fan inequality

$$n(\lambda_1\lambda_2, \mathbf{T}_1\mathbf{T}_2^*) \leqslant n(\lambda_1, \mathbf{T}_1) + n(\lambda_2, \mathbf{T}_2) \tag{2.3}$$

with conveniently chosen λ_1, λ_2, $\lambda_1\lambda_2 = \lambda$, to find estimates for the singular numbers of $\mathbf{T}$.

Thus, we are reduced to the task of defining bounded operators $\mathbf{T}_\iota$, $\iota = 1, 2$, and proving estimates for their singular numbers. This topic was already covered by the considerations in [8] and further in [7] and [5]. In fact, the boundedness of $\mathbf{T}_\iota$ is equivalent to the boundedness of the operator $\mathbf{S}_\iota = \mathbf{T}_\iota^*\mathbf{T}_\iota$ in $L_2(\mathbb{T}^{\mathbf{N}})$. The latter operator has the quadratic form

$$(\mathbf{S}_\iota f, f)_{L_2(\mathbb{T}^{\mathbf{N}})} = (\mathbf{T}_\iota^*\mathbf{T}_\iota f, f)_{L_2(\mathbb{T}^{\mathbf{N}})} = (\mathbf{T}_\iota f, \mathbf{T}_\iota f)_{L_{2,\mu}} = \int |F_\iota(X)|^2 |(\mathfrak{K}_\iota f)(X)|^2 \mu(dX). \tag{2.4}$$

Operators of this type were considered in [7] and [5], where conditions were established for the boundedness of the quadratic form (2.4), thus justifying the reasoning above and the spectral estimates for the corresponding operators. To understand the action of the operator $\mathbf{T}_\iota$, we consider its sesquilinear form for $u \in L_2(\mathbb{T}^{\mathbf{N}})$, $v \in L_{2,\mu}$,

$$(\mathbf{T}_\iota u, v)_{L_{2,\mu}} = (F_\iota \mathbf{t}_{\mathscr{M}}\mathfrak{K}_\iota u, v)_{L_{2,\mu}} = (\mathbf{t}_{\mathscr{M}}\mathfrak{K}_\iota u, F_\iota^* v)_{L_{2,\mu}}. \tag{2.5}$$

For $u \in L_2(\mathbb{T}^{\mathbf{N}})$ the function $\mathfrak{K}_\iota u$ belongs to the Sobolev space $H^{\gamma_\iota}(\mathbb{T}^{\mathbf{N}})$. So, to assign sense to the expression in (2.5), we need to understand $F_\iota^* v$ as an element in the adjoint space $H^{-\gamma_\iota}(\mathbb{T}^{\mathbf{N}})$ of distributions. In the supercritical case $2\gamma_\iota > \mathbf{N}$, the space $H^{\gamma_\iota}(\mathbb{T}^{\mathbf{N}})$ is embedded in $C(\mathbb{T}^{\mathbf{N}})$. Therefore, for $F_\gamma^* \in L_{2,\mu}$ the product $F_\iota^* v$ belongs to $L_{1,\mu}$ and such a function defines a continuous functional on $C(\mathbb{T}^{\mathbf{N}})$, which means that it is an element of $C(\mathbb{T}^{\mathbf{N}})' \subset H^{-\gamma_\iota}(\mathbb{T}^{\mathbf{N}})$. In the subcritical case $2\gamma_\iota < \mathbf{N}$, there is no embedding of the Sobolev space $H^{\gamma_\iota}(\mathbb{T}^{\mathbf{N}})$ into $C(\mathbb{T}^{\mathbf{N}})$, so the restriction of $f = \mathfrak{K}_\iota u$ to the support of μ does not make immediate sense. However, for the measure μ in

$\mathscr{P}_+^\alpha$ with some $\alpha > \mathbf{N} - 2\gamma_\iota$ and for $F_\iota^* \in L_{2\sigma_\iota}$ with $2\sigma_\iota = \frac{\alpha}{2\gamma_\iota - \mathbf{N} + \alpha}$, for any fixed $v \in L_{2,\mu}$ the functional

$$\psi_{F_\iota^* \overline{v(X)}\mu}(f) = \int f(X)(F_\iota^*(X)\overline{v(X)})\mu(dX) \tag{2.6}$$

defined initially on $H^{\gamma_\iota}(\mathbb{T}^\mathbf{N}) \cap C(\mathbb{T}^\mathbf{N})$ is continuous in the $H^{\gamma_\iota}(\mathbb{T}^\mathbf{N})$-norm by Theorem 11.8 in [8] and thus can be extended by continuity to the whole space $H^{\gamma_\iota}(\mathbb{T}^\mathbf{N})$.

A similar reasoning goes through in the critical case $2\gamma_\iota = \mathbf{N}$. If μ belongs to $\mathscr{P}_+^\alpha$ with some $\alpha > 0$, then the functional (2.6) defined first on continuous functions in $H^{\gamma_\iota}(\mathbb{T}^\mathbf{N})$ extends to a continuous functional on $H^{\gamma_\iota}(\mathbb{T}^\mathbf{N})$ as soon as $|F_\iota^*|^2$ belongs to the Orlicz space $L \log L(\mu)$, again by Theorem 11.8 and Corollary 11.8 (2) in [8]. In both latter cases, by duality, for a fixed $f \in H^{\gamma_\iota}(\mathbb{T}^\mathbf{N})$ the expression (2.6) defines a continuous antilinear functional of $v \in L_{2,\mu}$. Therefore, $F_\iota \mathbf{t}_\mathscr{M} f$ is an element in $L_{2,\mu}$.

In all three cases, the operator $\mathbf{T}_\iota$ can be represented by the diagram

$$\mathbf{T}_\iota : L_2(\mathbb{T}^\mathbf{N}) \xrightarrow{\mathfrak{K}_\iota} H^{\gamma_\iota}(\mathbb{T}^\mathbf{N}) \xrightarrow{F_\iota \mathbf{t}_\mathscr{M}} L_{2,\mu}$$

with all arrows determining continuous operators. Therefore, $\mathbf{T} = \mathbf{T}_1 \mathbf{T}_2^*$ is a bounded operator in $L_{2,\mu}$.

3 Estimates for Singular Numbers

3.1. Estimates for singular numbers of the operator $\mathbf{T}_\iota$. We use the fact that the singular numbers of the operator $\mathbf{T}_\iota$, i.e., the eigenvalues of the operator $(\mathbf{T}_\iota \mathbf{T}_\iota^*)^{1/2}$ in $L_{2,\mu}$ coincide with the eigenvalues of the operator $(\mathbf{T}_\iota^* \mathbf{T}_\iota)^{1/2}$ in $L_2(\mathbb{T}^\mathbf{N})$. The latter operators were considered in [7, 5]. According to the results of [7], the order $\lambda^{-2\sigma_\iota}$ of the spectral estimates for the operator $\mathbf{T}_\iota = \mathbf{T}[\mu, \mathfrak{K}_\iota, F_\iota]$ is determined by the parameters γ_i, $\mathbf{N}$, α as follows:

$$\sigma_\iota = \frac{\alpha}{2\gamma_\iota - \mathbf{N} + \alpha}.$$

Hence $\sigma_\iota > 1$ in the subcritical case, $\sigma_\iota = 1$ in the critical case, and $\sigma_\iota < 1$ in the supercritical case.

The assertion to follow is the combination of Theorem 2.3 in [7] (in the critical case) and Theorems 3.3 and 3.8 in [5] (in the noncritical cases.) Namely, the operator $\mathbf{T}_\iota^* \mathbf{T}_\iota$, of the form

$$((F_\iota \mu)\mathbf{t}_\mathscr{M} \mathfrak{K}_\iota)^*((F_\iota \mu)\mathbf{t}_\mathscr{M} \mathfrak{K}_\iota),$$

thus defined in $L_2(\mathbb{T}^\mathbf{N})$ by the quadratic form (2.4), exactly fits in the setting of these theorems, with $V = |F_\iota|^2$. We collect the corresponding results.

Theorem 3.1. *Let the measure μ satisfy the conditions* (1.3), (1.4), *or* (1.5). *Suppose that for the weight function F_ι the function $|F_\iota|^2$ belongs to the space $L_{\sigma_\iota,\mu}$ in the subcritical case, the space $L_{1,\mu}$ in the supercritical case, and the Orlicz class $L^{\Psi,\mu}$, $\Psi(t) = (1+t)\log(1+t) - t$, in the critical case. Then the operator $\mathbf{T}_\iota = \mathbf{T}[\mu, \mathfrak{K}_\iota, F_\iota]$ is bounded as acting from $L_2(\mathbb{T}^\mathbf{N})$ to $L_{2,\mu}$ and the following estimates hold for the singular numbers of $\mathbf{T}_\iota$:*

$$\begin{aligned} &n(\lambda, \mathbf{T}_\iota) \equiv n(\lambda^2, \mathbf{T}_\iota^* \mathbf{T}_\iota) \lesssim \mathbf{C}_{\mathrm{sub}}(\mu, \mathfrak{K}_\iota, F_\iota)\lambda^{-2\sigma_\iota}, \\ &\mathbf{C}_{\mathrm{sub}}(\mu, \mathfrak{K}_\iota, F_\iota) = C_\iota \| |F_\iota|^2 \|_{L_{\sigma_\iota,\mu}}^{\sigma_\iota}, \quad \sigma_\iota > 1, \end{aligned} \tag{3.1}$$

$$n(\lambda, \mathbf{T}_\iota) \lesssim \mathbf{C}_{\text{crit}}(\mu, \mathfrak{K}_\iota, F_\iota)\lambda^{-2},$$
$$\mathbf{C}_{\text{crit}}(\mu, \mathfrak{K}_\iota, F_\iota) = C_\iota \||F_\iota|^2\|^{(\Psi,\mu,av)}, \quad \sigma_\iota = 1, \tag{3.2}$$

$$n(\lambda, \mathbf{T}_\iota) \lesssim \mathbf{C}_{\sup}(\mu, \mathfrak{K}_\iota, F_\iota)\lambda^{-2\sigma_\iota},$$
$$\mathbf{C}_{\sup}(\mu, \mathfrak{K}_\iota, F_\iota) = C_\iota \||F_\iota|^2\|_{L_{1,\mu}}^{\sigma_\iota} \mu(\mathscr{M})^{2-2\sigma_\iota}\lambda^{-2\sigma_\iota}, \quad \sigma_\iota < 1. \tag{3.3}$$

Here, $a(\lambda) \lesssim b(\lambda)$ means $\limsup_{\lambda\to 0} a(\lambda)b(\lambda)^{-1} \leqslant 1$ and $\| \cdots \|^{(\Psi,\mu,av)}$ in (3.2) *is the averaged Ψ-Orlicz norm with respect to the measure μ (see the definition in* [7, formula (2.1)]). *The constant C_ι in* (3.1)–(3.3) *depends on the dimension $\mathbf{N}$, the orders $-\gamma_\iota$ of operators $\mathfrak{K}_\iota$, the characteristic α of the measure μ, and the constants $\mathscr{A}(\mu)$, $\mathscr{B}(\mu)$ in* (1.3), (1.4), (1.5) *as well as on the operator $\mathfrak{K}_\iota$, but not on the weight function F_ι.*

3.2. Spectral estimates for the operator T. Having the weighted integral operator $\mathbf{T} = \mathbf{T}[\mu, \mathfrak{K}, F_1, F_2]$ with polyhomogeneous kernel of order $-\theta > -N$ or, which is equivalent, with the pseudodifferential operator $\mathfrak{K}$ of order $-l = -N + \theta < 0$, and given weight functions F_1, F_2 and measure μ, we are free to choose a factorization of the pseudodifferential operator $\mathfrak{K} = \mathfrak{K}_1\mathfrak{K}_2$. For $\mathfrak{K}_1$ we can take $\mathfrak{K}_1 = (1-\Delta)^{-\gamma_1/2}$. Therefore, $\mathfrak{K}_2 = (1-\Delta)^{\gamma_1/2}\mathfrak{K}$, where Δ is the Laplacian on the torus $\mathbb{T}^N$. This is equivalent to factorizing the integral operator $\mathbf{K}$ in the form $\mathbf{K} = \mathbf{K}_1\mathbf{K}_2$ with orders $\theta_1 = -\mathbf{N} + \gamma_1$ and $\theta_2 = -\mathbf{N} + \gamma_2$. If the factorization, namely, the choice of the orders γ_1 and γ_2, $\gamma_1 + \gamma_2 = \gamma$, is made, this determines the required properties of μ, F_1, F_2, to be described later on. We arrive at the factorization of the operator $\mathbf{T} = \mathbf{T}[\mu, \mathfrak{K}, F_1, F_1]$

$$\mathbf{T} = \mathbf{T}_1\mathbf{T}_2^*, \quad \mathbf{T}_\iota = \mathbf{T}[\mu, \mathfrak{K}_\iota, F_\iota], \quad \iota = 1, 2.$$

By the inequality (2.3) for the singular numbers of the product of operators, the estimates (3.1)–(3.3) imply the estimate for the singular numbers of the operator $\mathbf{T}$. To obtain this estimate, we set in (2.3)

$$\lambda_1 = \mathbf{a}\lambda^{\frac{\sigma_2}{\sigma_1+\sigma_2}}, \quad \lambda_2 = \mathbf{a}^{-1}\lambda^{\frac{\sigma_1}{\sigma_1+\sigma_2}}, \quad \lambda_1\lambda_2 = \lambda,$$

with $\mathbf{a} = (\mathbf{C}_*(\mu, \mathfrak{K}_1, F_1)\mathbf{C}_*(\mu, \mathfrak{K}_2, F_2)^{-1})^{1/(\sigma_1+\sigma_2)}$. Applying the Ky Fan inequality, we find

$$n(\lambda, \mathbf{T}) \lesssim \mathbf{C}(\mu, \mathfrak{K}_1, \mathfrak{K}_2, F_1, F_2)\lambda^{-2\sigma}, \tag{3.4}$$

$$\mathbf{C}(\mu, \mathfrak{K}_1, \mathfrak{K}_2, F_1, F_2) = \mathbf{C}_{\divideontimes}(\mu, \mathfrak{K}_1, F_1)^{\frac{\sigma_2}{\sigma_1+\sigma_2}}\mathbf{C}_{\divideontimes}(\mu, \mathfrak{K}_2, F_2)^{\frac{\sigma_1}{\sigma_1+\sigma_2}}, \tag{3.5}$$

where $\divideontimes$ stands for the proper subscript in (3.1)–(3.3).

From the expression for σ_1, σ_2 we see that the exponent $-2\sigma = -2\frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}$ in (3.4) equals, in fact, $-\frac{2\alpha}{l+2\alpha}$, which means that the order in the estimates for the singular numbers does not depend on the way how the factorization of $\mathfrak{K}$ is chosen. This choice, however, determines the conditions imposed on the measure μ and weight functions F_1 and F_2.

In the following assertion, we collect possible combinations of orders in factorization and describe the corresponding conditions on μ and F_1, F_2.

Theorem 3.2. *Let $\mathfrak{K}$ be a pseudodifferential operator of order $-l < 0$ factorized as $\mathfrak{K} = \mathfrak{K}_1\mathfrak{K}_2$, where $\mathfrak{K}_\iota$, $\iota = 1, 2$ are pseudodifferential operators of order $-\gamma_\iota < 0$, $\iota = 1, 2$, $\gamma_1 + \gamma_2 = l$. Then for the operator $\mathbf{T} = \mathbf{T}[\mu, \mathfrak{K}, F_1, F_1]$ the estimate* (3.4) *with* (3.5) *holds provided that μ and F_ι satisfy the following conditions*:

(1) *if* $2\gamma_1, 2\gamma_2 < \mathbf{N}$, *then* $\mu \in \mathscr{P}^\alpha_+$, $2\alpha > \mathbf{N} - 2\gamma_\iota$, $F_\iota \in L_{2\sigma_i,\mu}$, $\sigma_\iota = \frac{\alpha}{2\gamma_i - \mathbf{N} + 2\alpha}$,

(2) *if* $2\gamma_1 < \mathbf{N}$, $2\gamma_2 = \mathbf{N}$, *then* $\mu \in \mathscr{P}^\alpha$, $2\alpha > \mathbf{N} - 2\gamma_1$, $F_1 \in L_{2\sigma_1,\mu}$, $\sigma_1 = \frac{\alpha}{2\gamma_1 - \mathbf{N} + 2\alpha}$, $|F_2|^2 \in L^{\Psi,\mu}$,

(3) *if* $2\gamma_1 < \mathbf{N}$, $2\gamma_2 > \mathbf{N}$, *then* $\mu \in \mathscr{P}^\alpha$, $2\alpha > \mathbf{N} - 2\gamma_1$ $F_1 \in L_{2\sigma_1,\mu}$, $F_2 \in L_{2,\mu}$,

(4) *if* $2\gamma_1 = \mathbf{N}$, $2\gamma_2 = \mathbf{N}$, *then* $\mu \in \mathscr{P}^\alpha$, $\alpha > 0$, $|F_\iota|^2 \in L^{\Psi,\mu}$, $\iota = 1, 2$,

(5) *if* $2\gamma_1 = \mathbf{N}$, $2\gamma_2 > \mathbf{N}$, *then* $\mu \in \mathscr{P}^\alpha$, $\alpha > 0$ $|F_1|^2 \in L^{\Psi,\mu}$, $F_2 \in L_{2,\mu}$,

(6) *if* $2\gamma_1, 2\gamma_2 > \mathbf{N}$, *then* $\mu \in \mathscr{P}^\alpha_-$, $\alpha > 0$ $F_\iota \in L_{2,\mu}$, $\iota = 1, 2$.

3.3. Estimates for lower order operators. In our analysis, we need estimates for singular numbers in the case where the pseudodifferential operator $\mathfrak{K}$ is replaced by an operator of a lower order.

For weights F_1 and F_2 satisfying the assumptions of Theorem 3.1 with a factorization as in Theorem 3.2 for certain fixed measure μ and order $-l < 0$, we replace the pseudodifferential operator $\mathfrak{K}$ of order $-l$ by another operator $\mathfrak{K}'$ of order $-l' < -l$. It is natural to expect that the singular numbers of the operator $\mathbf{T}' = \mathbf{T}[\mu, \mathfrak{K}', F_1, F_2]$ decay faster than the singular numbers of the operator $\mathbf{T} = \mathbf{T}[\mu, \mathfrak{K}, F_1, F_2]$. This property is not quite trivial since it can happen that while $\mathbf{T}$ (or some of its factors) belongs to the subcritical case, the operator $\mathbf{T}'$ can get into the critical or supercritical case, so the conditions imposed on the measure μ and weights F_1 and F_2 might change. The following assertion states that, probably, not sharp, spectral estimates hold without changing the conditions on the measure μ.

Corollary 3.1. *Let the assumptions of Theorems* 3.1 *and* 3.2 *be satisfied for some factorization of* $\mathfrak{K}$ *and* μ, F_1, F_2, $\mathfrak{K}$. *Let* $\mathfrak{K}'$ *be a pseudodifferential operator of order* $-l' < -l$. *Then*

$$n(\lambda, \mathbf{T}') = o(\lambda^{-2\sigma}), \quad \lambda \to 0, \quad \mathbf{T}' = \mathbf{T}[\mu, \mathfrak{K}', F_1, F_2]. \tag{3.6}$$

Proof. Let $\mathfrak{K} = \mathfrak{K}_1\mathfrak{K}_2$ be the factorization presented in Theorem 3.1 with operators $\mathfrak{K}_1$ and $\mathfrak{K}_2$ of order $-\gamma_1, -\gamma_2$ respectively and $\gamma_1 + \gamma_2 = l$ so that μ, F_1, F_2 satisfy the assumptions of Theorem 3.1. We factorize $\mathfrak{K}'$ in the following way:

$$\mathfrak{K}' = \mathfrak{K}_1'\mathfrak{K}_3'\mathfrak{K}_2', \tag{3.7}$$

where

$$\mathfrak{K}_1' = (1 - \Delta)^{-\gamma_1/2}, \quad \mathfrak{K}_2' = (1 - \Delta)^{-\gamma_2/2}, \quad \mathfrak{K}_3' = (1 - \Delta)^{\gamma_1/2}\mathfrak{K}'(1 - \Delta)^{\gamma_2/2}.$$

The factorization (3.7) leads to the factorization of the operator $\mathbf{T}'$:

$$\mathbf{T}' = \mathbf{T}_1\mathfrak{K}_3'\mathbf{T}_2^*, \quad \mathbf{T}_1 = F_1\mathfrak{K}_1', \quad \mathbf{T}_2 = F_2^*\mathfrak{K}_2'.$$

The estimates for the singular numbers of the operators $\mathbf{T}_1$ and $\mathbf{T}_2$ are the same as in Theorem 3.2, while $\mathfrak{K}_3'$ is a pseudodifferential operator on $\mathbb{T}^{\mathbf{N}}$ of negative order $(l - l') < 0$ having decaying singular numbers, $n(\lambda, \mathfrak{K}_3') = O(\lambda^{(l-l')/\mathbf{N}})$. By the Ky Fan inequality, (3.6) holds. □

Remark 3.1. One can consider a more general setting, namely, an operator of the form (2.2), but acting from $L_{2,\mu}$ to $L_{2,\mu'}$, where μ' is a different (possibly, singular) measure. Here, extensive complications arise since the spectral estimate depends not only on properties of μ and μ' taken separately, but also on their relative position in $\mathbb{R}^{\mathbf{N}}$.

4 Schur Multipliers and Spectral Estimates

In their works, Birman and Solomyak have studied properties of transformations in Schatten classes of integral operators generated by Schur multipliers. We recall that for a function $\Phi(X,Y)$ the Schur multiplier transformation $\mathbf{M}[\Phi]$ associates with an integral operator with kernel $K(X,Y)$ the integral operator with kernel $\Phi(X,Y)K(X,Y)$. Explicit analytic conditions on Φ granting that $\mathbf{M}[\Phi]$ transforms any integral operator in a certain Schatten class to an operator in the same class (or in some other prescribed Schatten class) have been found. In particular, the results of [9] on multipliers in weak Schatten classes were used to obtain estimates for eigenvalues of operators with kernel of the form $\Phi(X,Y)K(X,Y,X-Y)$, where K is an integral kernel with weak singularity as $X-Y\to 0$. More general results were obtained in [10, Sections 8 and 10] and [16] for operators in L_2 spaces with measure. For kernels with a relatively strong singularity (of order $-\theta\leqslant -\mathbf{N}/2$) on the diagonal, these measures were supposed to be absolutely continuous with respect to the Lebesgue measure, with prescribed properties of densities, while for a weaker singularity the results on singular number estimates hold for any finite measures, thus admitting singular measures. Simultaneously, the weaker is the polarity of kernels of integral operators on the diagonal, the more smoothness is required for the multiplier $\Phi(X,Y)$ to generate a transformation in the proper Schatten class. The conditions imposed on $\Phi(X,Y)$ in [10, 16] are expressed in rather complicated terms. We give a simplified version sufficient for our needs.

Proposition 4.1. *Let $\Phi(X,Y)$ be a function on $\mathbf{Q}\times\mathbf{Q}$, where $\mathbf{Q}$ is a cube in $\mathbb{R}^{\mathbf{N}}$. Then there exists $\mathbf{m}(\mathbf{N},q)$ such that for $m>\mathbf{m}(\mathbf{N},q)$ any function $\Phi(X,Y)$ possessing continuous partial derivatives of order up to m in all the variables is a Schur multiplier in the space of integral operators $\mathbf{T}: L_2(\mathbf{Q},\mu_2)\to L_2(\mathbf{Q},\mu_1)$ in the weak Schatten class Σ_q (of operators with singular numbers satisfying $n(\lambda,T)=O(\lambda^{-\frac{1}{q}})$) for any finite measures μ_1 and μ_2.*

We also use the fact that for any finite measures μ_1 and μ_2 operators with smooth kernel have arbitrarily rapid eigenvalue decay rate as soon as the smoothness is sufficiently high. This statement is a particular case of Proposition 2.1 in [10], where more complicated, but less restrictive conditions are imposed.

Proposition 4.2. *Let μ_1 and μ_2 be finite Borel measures on a cube $\mathbf{Q}\subset\mathbb{R}^{\mathbf{N}}$. Suppose that $U(X,Y)\in C^m(\mathbf{Q}\times\mathbf{Q})$ and $m>2\mathbf{N}$. Then the operator $\mathbf{U}: L_{2,\mu_2}\to L_{2,\mu_1}$*

$$(\mathbf{U}u)(X)=\int U(X,Y)u(Y)\mu_2(dY)$$

has singular numbers satisfying $n(\lambda,\mathbf{U})=O(\lambda^{-1/q})$, where $q=\dfrac{\mathbf{N}+2m}{2\mathbf{N}}$.

It is clear that these results automatically hold for operators defined on the torus $\mathbb{T}^{\mathbf{N}}$ instead of a cube. In our applications, the role of μ_1 and μ_2 are played by $F_1\mu$ and $F_2\mu$, where μ is a singular measure satisfying one of the conditions in (3.1)–(3.3) and F_1 and F_2 are weight functions. Thus, using Proposition 4.2, we arrive at our result on multipliers in the set of operators with singular measures.

We denote by $\mathscr{K}[\mathbf{N},\theta,\alpha]$ the space of integral operators satisfying one of the assumptions of Theorem 3.1.

Theorem 4.1. *Let* $\mathbf{K}$ *be an integral operator in* $\mathscr{K}[\mathbf{N},\theta,\alpha]$ *with some weakly polar kernel* $K(X,Y,X-Y)$ *with the singularity order* $-\theta$ *at the diagonal and the measure* μ *and weight functions* F_1 *and* F_2 *as above. Suppose that* Φ *is a function on* $\mathbb{T}^{\mathbf{N}}\times\mathbb{T}^{\mathbf{N}}$ *belonging to* $C^{2m}(\mathbb{T}^{\mathbf{N}}\times\mathbb{T}^{\mathbf{N}})$, $m>\mathbf{m}(\mathbf{N},2\sigma)$. *Then for the integral operator* $\mathbf{H}$ *with the same measure* μ *and weight functions and the kernel* $H(X,Y)=\Phi(X,Y)K(X,Y,X-Y)$ *the following estimate for the singular numbers holds*:

$$n(\lambda,\mathbf{H})\lesssim C(\Phi)\mathbf{C}(\mu,\mathfrak{K}_1,\mathfrak{K}_2,F_1,F_2)\lambda^{-2\sigma},$$

where the constant $\mathbf{C}(\mu,\mathfrak{K}_1,\mathfrak{K}_2,F_1,F_2)$ *is determined by the parameters in Theorem* 3.2, *thus depending on the factorization of the operator* $\mathfrak{K}$, *the parameter of the measure* μ, *and the proper integral norms of the weight functions* F_1 *and* F_2, *whereas* $C(F)$ *depends on the bounds of the derivatives of* Φ.

Proof. For every fixed $X\in\mathscr{M}$ we consider the starting fragment of the Taylor expansion of the function $\Phi(X,Y)$ at the point (X,X) in powers of $Y-X$:

$$\begin{aligned}\Phi(X,Y)&=\sum_{|\mathbf{n}|<m}(\mathbf{n}!)^{-1}\partial_Y^{\mathbf{n}}\Phi(X,X)(Y-X)^{\mathbf{n}}+\Phi_{(m)}(X,Y)\\&\equiv\sum_{|\mathbf{n}|<m}\Phi_{\mathbf{n}}(X)(Y-X)^{\mathbf{n}}+\Phi_{(m)}(X,Y).\end{aligned}\tag{4.1}$$

We first consider the leading term in $\mathbf{H}$ corresponding to the first term in (4.1), $\mathbf{n}=0$,

$$H_0(X,Y,X-Y)=\Phi(X,X)K(X,Y,X-Y)\equiv\Phi_0(X)K(X,Y,X-Y).$$

For given F_1 and F_2 the integral operator $\mathbf{H}_0$ with kernel $H_0(X,Y,X-Y)$ acts as

$$(\mathbf{H}_0u)(X)=\int F_1(X)\Phi(X,X)K(X,Y,X-Y)u(Y)F_2(Y)\mu(dY).$$

This operator is of the same kind as the initial one, but with the weight function F_1 replaced by the weight function $F_1(X)\Phi(X,X)$. We also note that $\Phi(X,X)$ is a bounded function and, consequently, $F_1(X)\Phi(X,X)$ belongs to the same space of weights as F_1, the space required by Theorems 3.1 and 3.2. Therefore, incorporating $\Phi(X,X)$ into the weight function, we obtain the same estimate for the singular numbers of the operator $\mathbf{H}_0$.

Next, we consider the operator $\mathbf{H}_{\mathbf{n}}$ corresponding to some term in (4.1) with $|\mathbf{n}|>0$. The kernel $H_{\mathbf{n}}(X,Y,X-Y)=K(X,Y,X-Y)(X-Y)^{\mathbf{n}}$ is smooth for $X\neq Y$ and has homogeneity of order $\theta+|\mathbf{n}|$ in $X-Y$, so it is larger than the homogeneity order which K has. Since the function $\Phi_{\mathbf{n}}(X)$ is bounded, the weight function $F_1(X)\Phi_{\mathbf{n}}(X)$ belongs to the same space of weights as F_1. By Corollary 3.1, this gives us the spectral estimate

$$n(\lambda,\mathbf{H}_{\mathbf{n}})=o(\lambda^{-\frac{2\alpha}{2l-N+2\alpha}}).$$

Finally, we consider the remainder term for $\mathbf{H}$; namely,

$$H_{(m)}(X,Y,X-Y)=\Phi_{(m)}(X,Y)K(X,Y,X-Y)$$

in (4.1). By our assumptions, the function $\Phi_{(m)}$ is $C^{\mathbf{m}}$ smooth outside the diagonal $X=Y$. Therefore, the same is valid for the product $\Phi_{(m)}K$. As for the diagonal $X=Y$, the function

$\Phi_{(m)}(X,Y)$ has zero of order m as $X \to Y$. Therefore, the product $H_{(m)} = \Phi_{(m)}K$ has zero of order not less than $\mathbf{N} + |\theta|$, and Proposition 4.2 gives the required singular number estimate as soon as m is taken large enough. $\square$

In the matrix case, we can consider both left and right multipliers:

$$\mathbf{M}[\Phi_\ell, \Phi_r]K = \Phi_\ell(X,Y)K(X,Y,X-Y)\Phi_r(X,Y),$$

with matrix functions $\Phi_\ell(X,Y)$ and $\Phi_r(X,Y)$ of proper size. The reasoning above carries over to this case automatically.

Here, we used the arbitrariness in the choice of weights in the proof above: we could incorporate a not sufficiently smooth factor in the kernel into the weight function. We return to this pattern below, when we consider multipliers in asymptotic formulas.

5 Eigenvalue Asymptotics

Following the basic strategy of using perturbation approach, we can establish asymptotic formulas. We consider the selfadjoint case, so the entries in the operator $\mathbf{K}$ in (1.2) satisfy the following conditions:

$$K(X,Y,X-Y) = K(Y,X,Y-X)^*, \quad F_1(X) = F_2(X)^*,$$

where the symbol * denotes the complex conjugation in the scalar case and the matrix conjugation in the vector case.

We suppose that the Hermitian kernel $K(X,Y,X-Y)$ is smooth for $Y \neq X$ and admits the asymptotic expansion in homogeneous functions as in Section 2 with the leading homogeneity order $-\theta > -\mathbf{N}$. Let $\Gamma \subset \mathbb{T}^{\mathbf{N}}$ be a Lipschitz surface in $\mathbb{T}^{\mathbf{N}}$ of dimension $d < \mathbf{N}$ and codimension $\mathfrak{d} = \mathbf{N} - d$. For a measure μ we take the Hausdorff measure $\mathscr{H}^d$ on the surface Γ, so $\mathscr{M} = \Gamma$. We assume that the surface is locally described by the equation $y = \varphi(x)$ in the coordinates $X = (x,y)$, where $x \in \mathscr{U} \subset \mathbb{R}^d$, $y \in \mathbb{R}^{\mathfrak{d}}$, and φ is a vector-valued function with $\mathfrak{d}$ components. In the coordinates $X = (x,y)$, the measure μ is described by

$$\mu(dx) = \det(\mathbf{1} + (\nabla\varphi)^*(\nabla\varphi))^{1/2}dx.$$

The measure μ belongs to $\mathscr{P}^\alpha$, $\alpha = d$ with constants $\mathscr{A}(\mu)$ and $\mathscr{B}(\mu)$ determined by the surface Γ globally. We suppose that the kernel $K(X,Y,X-Y)$ with leading homogeneity order $-\theta$ satisfies the assumptions of Theorems 3.1. We also suppose that the weight function $F = F_1 = F_2^*$ satisfies the assumptions of Theorem 3.2 under some factorization as well.

To write asymptotic formulas for eigenvalues, it is convenient to use the pseudodifferential representation: the integral operator with kernel $K(X,Y,X-Y)$ in $\mathbb{R}^{\mathbf{N}}$ is a pseudodifferential operator $\mathfrak{K}$ of order $-l = -\mathbf{N} + \theta$. Following [5], we suppose that $l > \mathfrak{d}$. The principal symbol of $\mathfrak{K}$, $\mathfrak{k}_0(X,\Xi)$, is the regularized Fourier transform of the leading term $K_0(X,X,X-Y)$ of the kernel in the last variable (see details in [3, 10, 11] and Section 2).

We recall the expression for the coefficient in the asymptotic formula (see [12, 5]). By the Rademacher theorem, at $\mathscr{H}^d$-almost all points X on Γ, there exists the tangent d-dimensional space $\mathrm{T}_X\Gamma$. We identify $\mathrm{T}_X\Gamma$ with the cotangent space $\mathrm{T}^*_X\Gamma$. Similarly, $\mathrm{N}_X\Gamma$ denotes the normal

$\mathfrak{d}$-dimensional space to Γ (identified with the conormal one). For such points X we define the symbol of order $-l+\mathfrak{d}<0$ on Γ

$$\mathbf{r}_0(X,\xi) = (2\pi)^{-\mathfrak{d}} \int\limits_{\mathrm{N}_X\Gamma} \mathfrak{k}_0(X;\xi,\eta)d\eta, \quad (X,\xi) \in \mathrm{T}^*_X\Gamma,$$

and the density

$$\rho^{\pm}_{\mathfrak{K}}(X) = \int\limits_{\mathrm{S}_X\Gamma} |F(X)|^{2\sigma}\mathbf{r}_0(X,\xi)^{\sigma}_{\pm}d\xi, \quad \sigma = \frac{d}{l-\mathfrak{d}}, \tag{5.1}$$

where $\mathbf{r}_0(X,\xi)_\pm$ are the positive and, respectively, negative parts of the symbol $\mathbf{r}_0(X,\xi)$,

$$\mathbf{r}_0(X,\xi)_\pm = \frac{1}{2}(|\mathbf{r}_0(X,\xi)| \pm \mathbf{r}_0(X,\xi)).$$

The eigenvalue asymptotics result for a smooth kernel is the following.

Theorem 5.1. *Under the above conditions, for the eigenvalues of* $\mathbf{T} = \mathbf{T}[\mu,\mathfrak{K},F,F^*]$ *the following asymptotic formulas hold*:

$$n_\pm(\lambda,\mathbf{T}) \sim \lambda^{-2\sigma}\mathbf{A}_\pm, \quad \lambda \to 0, \tag{5.2}$$

with coefficient

$$\mathbf{A}_\pm = \frac{1}{d(2\pi)^{d-1}} \int\limits_{\Gamma} \rho^{\pm}_{\mathfrak{K}}(X)\mu(dX). \tag{5.3}$$

Proof. The proof is similar to that of Theorem 6.2 in [5] or Theorem 6.4 in [12]. We explain the main steps, not going into technical details.

We first suppose that the weight function F is the restriction to Γ of a function $\widetilde{F}$ defined and smooth in $\mathbb{T}^{\mathbf{N}}$. Then we can incorporate F and F^* into the kernel K, keeping it smooth. Then the result is contained in [12, Theorem 6.4]. Further, for a general weight F satisfying the assumptions of the theorem, we can approximate it in the proper integral norm on Γ (namely, $L_{2\sigma,\mu}, L^{2\Psi,\mu}, L_{2,\mu}$ depending on the case in Theorem 3.2) by a function $F_{(\varepsilon)}$ admitting an extension as a smooth function in $\mathbb{T}^{\mathbf{N}}$. The construction of such an approximation is described in [7, Lemma 6.1]. By Theorem 3.1, this approximation leads to the smallness of the coefficient in the asymptotic eigenvalue estimates for the difference $\mathbf{T}[\mu,\mathfrak{K},F,F^*] - \mathbf{T}[\mu,\mathfrak{K},F_{(\varepsilon)},F^*_{(\varepsilon)}]$. By the standard application of the asymptotic perturbation lemma (see, for example, Theorem 4.1 in [2] or Lemma 6.1 in [12]), we pass to the limit in the eigenvalue asymptotic formula for $F_{(\varepsilon)}$ as $\varepsilon \to 0$, which gives (5.1). $\square$

Remark 5.1. One might be tempted to refer directly to the eigenvalue asymptotics results in [5]. However, the situation is not that simple. The eigenvalue asymptotics in [5] was proved for operators of the form $\mathfrak{A}^*(V\mu)\mathfrak{A}$, where $\mathfrak{A}$ is a negative order pseudodifferential operator and μ is a singular measure. Passing to integral operators, as in the present paper, we arrive at the study of eigenvalues for $(V^{1/2}\mu)\mathfrak{A}\mathfrak{A}^*(V^{1/2}\mu)$. Here, $\mathfrak{K} = \mathfrak{A}\mathfrak{A}^*$ is a nonnegative pseudodifferential operator. Therefore, the analysis of the operator $(V^{1/2}\mu)\mathfrak{K}(V^{1/2}\mu)$ when $\mathfrak{K}$ is not sign-definite does not pass through. Therefore, we need to refer to the results in [12] and perform an additional approximation.

Remark 5.2. The approach we use in Theorem 5.1 can be applied to operators acting in the spaces of vector-valued functions. Formally, the required results in [11, 12, 5] are not presented in the vector case, but they automatically follow without additional consideration. What is important here is that the basic results on the eigenvalue asymptotics for negative order pseudodifferential operators are stated and proved in [10] in the vector case. The asymptotic formulas in the vector case coincide with (5.2), but the expressions (5.3) for the asymptotic coefficients $\mathbf{A}_{\pm}$ should be replaced with their matrix versions. Namely, in the matrix case, the expression for the density in (5.1) should be replaced with

$$\rho^{\pm}_{\mathfrak{K}}(X) = \int\limits_{\mathrm{S}_X\Gamma} \mathrm{Tr}\{[F(X)\mathbf{r}_0(X,\xi)F^*(X)]^{\sigma}_{\pm}\}d\xi, \tag{5.4}$$

with further calculation of the coefficients $\mathbf{A}_{\pm}$ by means of (5.3). In (5.4), the subscript $\pm$ means the positive (respectively, negative) part of the corresponding matrix, so the expression on the right-hand side means that the proper (positive or negative) part of the Hermitian matrix $F(X)\mathbf{r}_0(X,\xi)F^*(X)$ is first found, then it is raised to the power σ, and then the trace of the resulting matrix is calculated to be further integrated over the cotangent sphere $\mathrm{S}_X\Gamma$.

6 Multipliers in Eigenvalue Asymptotics

In Section 5, the conditions imposed on the kernel $K(X,Y,X-Y)$ require it to be infinitely smooth for $X \neq Y$. The following reasoning involving multipliers allows us to reduce these smoothness conditions. Our results are not optimal since the regularity conditions imposed on the multiplier Φ may be weakened by using the full strength of multiplier theory due to Birman and Solomyak. We restrict ourselves to a rather technically simple setting to keep the paper more elementary.

Let $K(X,Y,X-Y)$ be a polyhomogeneous kernel of order $-\theta > -\mathbf{N}$, smooth for $X \neq Y$. Suppose that Γ is a Lipschitz surface in $\mathbb{T}^{\mathbf{N}}$ of dimension $d \geqslant 1$ and codimension $\mathfrak{d} \geqslant 1$ with surface measure μ. Let $F(X)$, $X \in \Gamma$ be a weight function satisfying the assumptions of Theorem 5.1. We assume that the multiplier $\Phi(X,Y)$ is a function of class C^{2m} in a neighborhood of $\Gamma \times \Gamma$, $2m > 2\mathbf{m} + |\theta|$. We consider the transformed Hermitian kernel

$$K_{\Phi}(X,Y,X-Y) = \Phi(X,Y)K(X,Y,X-Y)\Phi(Y,X)^*.$$

Theorem 6.1. *Under the above conditions, for the integral operator*

$$\mathbf{K}_{\Phi} : u(X) \mapsto \int\limits_{\Gamma} F(X)K_{\Phi}(X,Y,X-Y)F(Y)^*u(Y)\mu(dY)$$

the eigenvalue asymptotics (5.2) *holds with the coefficient* $\mathbf{A}_{\pm}$ *calculated by means of* (5.3) *with* $F(X)$ *replaced by* $F(X)\Phi(X,X)$.

Proof. The reasoning follows the pattern used in Theorem 4.1. We consider the starting fragment of the Taylor expansion of the multiplier $\Phi(X,Y)$ in $X-Y$ at the point $Y = X$

$$\Phi(X,Y) = \Phi(X,X) + \sum_{1 \leqslant |\mathbf{n}_Y| \leqslant m} (\mathbf{n}_X!)^{-1}\Phi^{(\mathbf{n}_Y)}_Y(X,X)(Y-X)^{\mathbf{n}_Y} + \Phi_{(m)}(X,Y), \tag{6.1}$$

where the remainder $\Phi_{(m)}(X,Y)$ is a function in C^m satisfying $\Phi_{(N)}(X,Y) = o(|X-Y|^{m+|\theta|})$, as $X-Y \to 0$. Similarly, $\Phi(Y,X)^*$ is expanded as

$$\Phi(Y,X)^* = \Phi(Y,Y)^* + \sum_{1 \leqslant |\mathbf{n}_X| \leqslant m} (\mathbf{n}_X!)^{-1} \Phi_X^{(\mathbf{n}_X)}(Y,Y)^*(X-Y)^{\mathbf{n}_X} + \Phi_{(m)}(Y,X)^*. \tag{6.2}$$

Using (6.1) and (6.2), we represent $K_\Phi(X,Y,X-Y)$ as the sum of three terms

$$K_\Phi(X,Y,X-Y) = \Phi(X,X)K(X,Y,X-Y)\Phi(Y,Y)^* + K_\Phi^{(m)} + \widetilde{K_\Phi^{(m)}}, \tag{6.3}$$

where

$$\begin{aligned} &K_\Phi^{(m)}(X,Y,X-Y) \\ &= \sum_{\substack{|\mathbf{n}_X|,|\mathbf{n}_Y| \leqslant m, \\ |\mathbf{n}_X|+|\mathbf{n}_Y|>0}} \frac{1}{\mathbf{n}_X!} \frac{1}{\mathbf{n}_Y!} \Phi_Y^{(\mathbf{n}_Y)}(X,X)(Y-X)^{\mathbf{n}_Y} K(X,Y,X-Y)(\Phi^*)_X^{(\mathbf{n}_X)}(Y,Y)(Y-X)^{\mathbf{n}_X} \end{aligned} \tag{6.4}$$

and $\widetilde{K_\Phi^{(m)}}$ contains the remainder terms in the expansions (6.1) and (6.2). The first term in (6.4), the product $F(X)\Phi(X,X)K(X,Y,X-Y)\Phi(Y,Y)^*F^*(Y)$ can be regrouped as

$$(F(X)\Phi(X,X))K(X,Y,X-Y)(\Phi(Y,Y)^*F(Y)^*).$$

This means that the corresponding integral operator can be considered as the one with the same smooth kernel $K(X,Y,X-Y)$, but with a different weight function $F(X)\Phi(X,X)$ instead of $F(X)$. Since the function $\Phi(X,X)$ is bounded, the new weight $F(X)\Phi(X,X)$ belongs to the same space of integrable functions as $F(X)$. Therefore, the eigenvalue asymptotics theorem can be applied to the integral operator under consideration, which gives the declared expression for the asymptotic coefficients.

In the second term in (6.3), a single summand has up to a constant the form

$$\Phi_Y^{(\mathbf{n}_Y)}(X,X)(Y-X)^{\mathbf{n}_Y}K(X,Y,X-Y)(\Phi^*)_X^{(\mathbf{n}_X)}(Y,Y)(X-Y)^{\mathbf{n}_X}. \tag{6.5}$$

Here, the product $K(X,Y,X-Y)(Y-X)^{\mathbf{n}_X+\mathbf{n}_Y}$ is smooth for $X \neq Y$ and has a weaker singularity than K as $X-Y \to 0$. The derivatives $\Phi_Y^{(\mathbf{n}_Y)}(X,X)$ and $\Phi_X^{(\mathbf{n}_X)}(Y,Y)$ are bounded and can be incorporated in the weight functions $F(X)$ and $F^*(Y)$. Therefore, for the corresponding integral operator, by Corollary 3.1, the singular number estimate holds with a faster eigenvalue decay rate.

Finally, in the third term in (6.3), in each summand, the remainder in the Taylor expansion of the multiplier Φ is present. Such a remainder has zero of high order at $X = Y$, together with derivatives. Therefore, the product of $K(X,Y,X-Y)$ with such a remainder has bounded derivatives of sufficiently high order and, by Proposition 4.2, the corresponding operator has arbitrarily fast decaying singular numbers as soon as the order of the derivatives is sufficiently high. $\square$

Remark 6.1. Similar to Remark 5.2, the above result on the multipliers in eigenvalue asymptotics can be automatically extended to the vector case. Here, the multiplier $\Phi(X,Y)$ is a $\mathbf{k} \times \mathbf{k}$ matrix and the transformed kernel has the form

$$K_\Phi(X,Y,X-Y) = \Phi(X,Y)K(X,Y,X-Y)\Phi(Y,X)^*. \tag{6.6}$$

The reasoning in proving the asymptotic formula for eigenvalues is the same as in Theorem 6.1.

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