Spectral Estimates for High-Frequency Sampled CARMA Processes

Vicky Fasen *‡ Florian Fuchs †‡

In this paper we consider a continuous-time autoregressive moving average (CARMA) process driven by either a symmetric α-stable Lévy process with $\alpha \in (0,2)$ or a symmetric Lévy process with finite second moments. In the asymptotic framework of high-frequency data within a long time interval, we establish a consistent estimate for the normalized power transfer function by applying a smoothing filter to the periodogram of the CARMA process. We use this result to propose an estimator for the parameters of the CARMA process and exemplify the estimation procedure by a simulation study.

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1 Introduction

In this paper we investigate continuous-time ARMA (CARMA) processes $Y = (Y_t)_{t\in\mathbb{R}}$ in the spectral domain and propose an estimator for the model parameters. For an overview and a comprehensive list of references on CARMA processes and their applications in several fields such as signal processing and control, econometrics and financial mathematics, we refer to [2, 9, 13]. The driving force of a CARMA process is a Lévy process $(L_t)_{t\in\mathbb{R}}$. A Lévy process $(L_t)_{t\geq 0}$ is defined (cf. [28]) to satisfy $L_0 = 0$ a.s., $(L_t)_{t\geq 0}$ has independent and stationary increments and the paths of $(L_t)_{t\geq 0}$ are stochastically continuous. An extension of a Lévy process $(L_t)_{t\geq 0}$ from the positive to the hole real line is given by $L_t := L_t \mathbb{1}_{\{t\geq 0\}} - \widetilde{L}_{-t-}\mathbb{1}_{\{t<0\}}$ for $t \in \mathbb{R}$, where $(\widetilde{L}_t)_{t\geq 0}$ is an independent copy of $(L_t)_{t\geq 0}$. Prominent examples are Brownian motions and stable Lévy processes. In this paper we restrict our attention to *symmetric* stable Lévy processes and *symmetric* Lévy processes with finite second moments. Then a CARMA process can be interpreted (its formal definition is given in Section 2) as a solution to the p-th order stochastic differential equation

$$a(\mathrm{D})\,Y_t = c(\mathrm{D})\,\mathrm{D}L_t, \quad t \in \mathbb{R}, \tag{1.1}$$

where D denotes the differential operator with respect to t and

$$a(z) := z^p + a_1 z^{p-1} + \ldots + a_p \qquad \text{and} \qquad c(z) := c_0 z^q + c_1 z^{q-1} + \ldots + c_q$$

*Institute of Stochastics, Karlsruhe Institute of Technology (KIT), Kaiserstraße 89, 76133 Karlsruhe, Germany. *Email:* vicky.fasen@kit.edu

†Institute of Mathematical Finance, Ulm University, Helmholtzstraße 18, 89081 Ulm and International School of Applied Mathematics, Technische Universität München, Boltzmannstraße 3, 85748 Garching, Germany. *Email:* ffuchs@ma.tum.de

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are the autoregressive and the moving average polynomial, respectively. Hence, CARMA processes can be seen as the continuous-time analog of (discrete-time) ARMA processes. From a statistical point of view, the so-called *power transfer function*

$$\Psi(\omega) := \frac{|c(i\omega)|^2}{|a(i\omega)|^2}, \quad \omega \in \mathbb{R}, \tag{1.2}$$

which corresponds (up to a constant) to the classical spectral density in the finite-variance case, is of central interest since it determines the model completely. The zeros of Ψ contain the zeros of $c(i\cdot)$, and hence, provided that the sign of the real part of any zero of $c(\cdot)$ is supposed to be known, one can identify uniquely the coefficients of the moving average polynomial from the power transfer function Ψ. Likewise the zeros of Ψ^{-1} characterize completely the coefficients of the autoregressive polynomial if one assumes to know the sign of the real parts of the zeros of $a(\cdot)$. From this it is obvious that, under *causality* and *invertibility* assumptions on the CARMA process, estimators for the power transfer function can be used to construct estimators for the coefficients of a and c.

The empirical version of the power transfer function (spectral density) is in the finite second moment case the *periodogram*. In [15] we have investigated the limit behavior of normalized and self-normalized versions of the periodogram of high-frequency sampled symmetric α-stable CARMA processes. In this paper we assume again that we observe the CARMA process Y only at equidistant time points $\{0, \Delta_n, 2\Delta_n, \ldots, n\Delta_n\}$ where $\Delta_n > 0$ is small, as used for modelling high-frequency data appearing in turbulence and finance (cf. [7, 13]), and $n \in \mathbb{N}$ is the total number of observations. More precisely, our asymptotic results hold under

Assumption 1. *We suppose that simultaneously $\Delta_n \to 0$ and $n\Delta_n \to \infty$ as $n \to \infty$.*

The normalized periodogram of the sampled sequence $Y^{\Delta_n} := (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ at frequency $\omega \in [-\pi, \pi]$ is given by

$$I_{n,Y^{\Delta_n}}(\omega) = \left| n^{-1/\alpha} \sum_{k=1}^{n} Y_{k\Delta_n} \mathrm{e}^{-i\omega k} \right|^2, \tag{1.3}$$

where for finite-variance CARMA processes we have $\alpha = 2$ and for α-stable CARMA processes α is the index of stability. A self-normalized alternative, no longer depending on α, is given for $\omega \in [-\pi, \pi]$ by

$$\widehat{I}_{n,Y^{\Delta_n}}(\omega) = \frac{I_{n,Y^{\Delta_n}}(\omega)}{n^{-2/\alpha} \sum_{k=1}^{n} Y_{k\Delta_n}^2} = \frac{\left| \sum_{k=1}^{n} Y_{k\Delta_n} \mathrm{e}^{-i\omega k} \right|^2}{\sum_{k=1}^{n} Y_{k\Delta_n}^2}. \tag{1.4}$$

As stated in [15, Theorems 3.5 and 3.10], both the normalized as well as the self-normalized periodogram are not consistent estimators for the power transfer function if the Lévy process is α-stable, $\alpha \in (0, 2]$. The limit distribution is a function of an α-stable random vector which reduces in the finite-variance case to an exponential distribution. We will generalize these results to finite-variance CARMA processes and to a very general high-frequency grid distance Δ_n. The limit results for high-frequency sampled finite-variance CARMA processes are analog to the results for finite-variance CARMA processes sampled at an equidistant time grid as derived in [14].

However, by applying linear smoothing filters to the periodogram consistent estimators for the (normalized) power transfer function can be constructed which is the main topic of this paper. We will consider the class of estimators of the form

$$T_{n,Y^{\Delta_n}}(\omega) = \sum_{|k| \le m_n} W_n(k) I_{n,Y^{\Delta_n}}(\omega_k) \tag{1.5}$$

and

$$\widehat{T}_{n,Y^{\Delta_n}}(\omega) = \sum_{|k| \le m_n} W_n(k) \widehat{I}_{n,Y^{\Delta_n}}(\omega_k) \tag{1.6}$$

where

$$\omega_k = \omega + \frac{k}{n}, \qquad |k| \le m_n, \tag{1.7}$$

and $(m_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ satisfying

Assumption 2. *We suppose that simultaneously $m_n \to \infty$ and $\frac{m_n}{n\Delta_n} \to 0$ as $n \to \infty$.*

The sequence of weight functions $W_n : \mathbb{Z} \to \mathbb{R}$ is specified by

$$W_n(k) = W_n(-k), \qquad W_n(k) \geq 0, \qquad \forall k \in \mathbb{N}, \tag{1.8a}$$

$$\sum_{|k|\leq m_n} W_n(k) = 1, \qquad \forall n \in \mathbb{N}, \tag{1.8b}$$

$$\max_{|k|\leq m_n} W_n^2(k) = o\left(\frac{1}{m_n}\right) \quad \text{as } n \to \infty. \tag{1.8c}$$

On the one hand, we will show that the sequence of smoothed self-normalized periodograms $\Delta_n \widehat{T}_{n,Y^{\Delta_n}}(\omega\Delta_n)$ is a consistent estimator for the normalized power transfer function. This result is in analogy to the one for ARMA models in discrete time obtained by Klüppelberg and Mikosch in [22]. On the other hand, for finite-variance CARMA processes the smoothed normalized periodograms $\Delta_n T_{n,Y^{\Delta_n}}(\omega\Delta_n)$ provide consistent estimators for the 2π-multiple of the spectral density, as well.

Thereafter, these results are used to develop an estimator for the parameters of the CARMA process. Our heuristic will basically consist of a constrained nonlinear least squares problem where the constraints come from the (necessary) additional assumption of causality and invertibility of the CARMA process. The estimator is then given as the best, in terms of least squares, (normalized) rational approximation for the smoothed periodogram values. It is an alternative to the ones presented in [4, 5, 18] working for both finite-variance and stable CARMA processes with infinite second moments. The Gaussian quasi-maximum-likelihood estimation has been derived in [5, 29] for Lévy-driven (multivariate) CARMA processes with finite second moments. In [18] a heuristic study of the estimation of stable CARMA$(2,1)$ processes is presented. A nonparametric estimator for the kernel function of a CARMA process is proposed in [6], and for Ornstein-Uhlenbeck processes, which are CARMA$(1,0)$ processes, an efficient estimator for the mean reversion parameter of the Ornstein-Uhlenbeck model has been obtained in [4, 19] by using methods of [11]. Compared to the other estimators the new contribution of this paper is that the estimator performs well for both finite-variance and infinite-variance models and we are able to estimate both the autoregressive and the moving average polynomial.

The paper is structured in the following way. In Section 2 we recall the formal definition of a Lévy-driven CARMA process and present some assumptions and notations of the paper. The main results are stated in Section 3. These include the asymptotic behavior of the different smoothed periodogram versions and of the periodogram itself. The topic of Section 4 is then the statistical inference for the model parameters of a CARMA process, illustrated by a simulation study for a CARMA$(2,1)$ process. Finally, Section 5 contains the proofs.

Notation

We use $\mathbb{N}^*$ and $\mathbb{R}^*$ for the natural and real numbers, respectively, excluding zero and $\mathbb{Z}$ for the integers. For the minimum of two real numbers $a, b \in \mathbb{R}$ we write shortly $a \wedge b$ and for the maximum $a \vee b$. The real and imaginary part of a complex number $z \in \mathbb{C}$ is written as $\Re(z)$ and $\Im(z)$, respectively, and its complex conjugate as $\overline{z}$. For two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we say $a_n \sim b_n$ as $n \to \infty$ if $\lim_{n\to\infty} a_n/b_n = 1$. The transpose of a matrix M is written as M^T and the m-dimensional identity matrix shall be denoted by I_m. On $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ the Euclidean norm is denoted by $|\cdot|$ whereas on $\mathbb{K}^m$ it will be usually written as $\|\cdot\|$. For two random variables X and Y the notation $X \stackrel{\mathscr{D}}{=} Y$ means equality in distribution. If we consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$, we denote convergence in probability to some random variable X by $X_n \stackrel{\mathbb{P}}{\to} X$ as $n \to \infty$ and convergence in distribution by $X_n \stackrel{\mathscr{D}}{\to} X$ as $n \to \infty$.

2 Preliminaries

2.1 Lévy-driven CARMA Processes

We recall the definition of an α-stable random variable and then present the notation which we use throughout the paper for the underlying driving Lévy process.

Definition 2.1. *A real-valued random variable Z is called* symmetric α-stable *($S\alpha S$) with index of stability $\alpha \in (0,2]$, if its characteristic function is of the form*

$$\Phi_Z(u) = \mathbb{E}\left[\exp\{iuZ\}\right] = \exp\left\{-\sigma^\alpha |u|^\alpha\right\}, \quad u \in \mathbb{R},$$

for some $\sigma \geq 0$. The parameter σ is called scale parameter. *A* symmetric α-stable Lévy process $(L_t)_{t\in\mathbb{R}}$ *with scale parameter σ_L is a Lévy process where L_1 is $S\alpha S$ with scale parameter σ_L.*

In particular a $S2S$ random variable is normally distributed and a 2-stable Lévy process is a Brownian motion. For the driving Lévy process we use the following notation.

Definition 2.2. *Let $\alpha \in (0,2]$ and $\sigma_L \geq 0$. By $L(\alpha, \sigma_L)$ we denote a* symmetric *Lévy process that is either*

(i) α-stable with scale parameter σ_L if $\alpha \in (0,2)$, or

(ii) satisfies $\mathbb{E}\left[L_1^2\right] = \sigma_L^2$ if $\alpha = 2$.

A CARMA process driven by $L(\alpha, \sigma_L)$ is then defined as follows. Assume that we have given $p, q \in \mathbb{N}$, $p > q$, and $a_1, \ldots, a_p, c_0, \ldots, c_q \in \mathbb{R}$, $a_p, c_0 \neq 0$, let

$$A := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ -a_p & -a_{p-1} & \ldots & \ldots & -a_1 \end{pmatrix} \in \mathbb{R}^{p\times p}$$

and let $(X_t)_{t\in\mathbb{R}}$ be a strictly stationary solution to the stochastic differential equation

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + e_p\,\mathrm{d}L_t, \quad t \in \mathbb{R}, \tag{2.1a}$$

where e_p denotes the p-th unit vector in $\mathbb{R}^p$. Then the process

$$Y_t := c^T X_t, \quad t \in \mathbb{R}, \tag{2.1b}$$

with $c = (c_q, c_{q-1}, \ldots, c_{q-p+1})^T$ (where we use the convention $c_j = 0$ for $j < 0$) is said to be a *CARMA process* of order (p, q). Necessary and sufficient conditions for the existence of a strictly stationary CARMA process are given in [8]. In this paper we will suppose

Assumption 3. *The eigenvalues $\lambda_1, \ldots, \lambda_p$ of A are distinct and possess strictly negative real parts.*

Under this assumption, the solution for the state process in (2.1a) is unique, strictly stationary, causal and can be written as

$$X_t = \int_{-\infty}^{t} \mathrm{e}^{(t-s)A} e_p\,\mathrm{d}L_s, \quad t \in \mathbb{R}. \tag{2.2a}$$

Hence, the CARMA process Y can also be expressed as a Lévy-driven *moving average process* $Y_t = \int_{-\infty}^{\infty} g(t-s)\,\mathrm{d}L_s$, $t \in \mathbb{R}$, with kernel function

$$g(t) = c^T \mathrm{e}^{tA} e_p\,\mathbb{1}_{[0,\infty)}(t). \tag{2.2b}$$

Notably the CARMA process can be interpreted as solution of the stochastic differential equation given in (1.1).

2.2 Decomposition of the Smoothed (Self-Normalized) Periodogram

Before stating the main results, we derive a series representation of the sampled sequence Y^{Δ_n} driven by a Lévy process $L(\alpha,\sigma_L)$ as in Definition 2.2. We use this representation for a suitable decomposition of the Fourier transform of Y^{Δ_n} and its associated smoothed (self-normalized) periodogram. Recall that the discrete Fourier transform is given by $F_{n,Y^{\Delta_n}}(\omega) := n^{-1/\alpha}\sum_{k=1}^{n} Y_{k\Delta_n}\,\mathrm{e}^{-i\omega k}$ for any $\omega\in[-\pi,\pi]$.

It is well known that every solution to (2.1a) satisfies

$$X_t = \mathrm{e}^{(t-s)A}X_s + \int_s^t \mathrm{e}^{(t-u)A}e_p\,\mathrm{d}L_u, \qquad \forall s,t\in\mathbb{R},\ s<t.$$

Then, under Assumption 3, we have by iteration that the state process X at time point $k\Delta_n$ can be written in the *series representation*

$$X_{k\Delta_n} = \sum_{j=0}^{\infty} \mathrm{e}^{j\Delta_n A}\,\xi^*_{n,k-j}\,e_p, \qquad k\in\mathbb{Z}, \tag{2.3}$$

with the $\mathbb{R}^{p\times p}$-valued noise sequence

$$\xi^*_{n,k} := \int_{(k-1)\Delta_n}^{k\Delta_n} \mathrm{e}^{(k\Delta_n - s)A}\mathrm{d}L_s, \qquad n\in\mathbb{N},\, k\in\mathbb{Z}. \tag{2.4}$$

We define, for any $\omega\in[-\pi,\pi]$,

$$\begin{aligned}
U_{n,j}(\omega) &:= \sum_{k=1-j}^{n-j} \xi^*_{n,k}\,\mathrm{e}^{-i\omega k} - \sum_{k=1}^{n}\xi^*_{n,k}\,\mathrm{e}^{-i\omega k},\\
K_{n,\Delta_n}(\omega) &:= \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n A - i\omega \mathrm{I}_p)}U_{n,j}(\omega),\\
M_{n,\Delta_n}(\omega) &:= \left(\mathrm{I}_p - \mathrm{e}^{-i\omega}\cdot \mathrm{e}^{\Delta_n A}\right)^{-1}\sum_{k=1}^{n}\xi^*_{n,k}\,\mathrm{e}^{-i\omega k},\\
J_{n,\Delta_n}(\omega) &:= c^T M_{n,\Delta_n}(\omega)\,e_p \qquad \text{and}\\
R_{n,\Delta_n}(\omega) &:= J_{n,\Delta_n}(\omega)\cdot\overline{c^T K_{n,\Delta_n}(\omega)\,e_p} + c^T K_{n,\Delta_n}(\omega)\,e_p\cdot\overline{J_{n,\Delta_n}(\omega)} + \left|c^T K_{n,\Delta_n}(\omega)\,e_p\right|^2.
\end{aligned} \tag{2.5}$$

The series representation of the state process X in Eq. (2.3) then immediately yields the following decomposition for the Fourier transform of the sampled sequence Y^{Δ_n}

$$\begin{aligned}
n^{\frac{1}{\alpha}}F_{n,Y^{\Delta_n}}(\omega) &= \sum_{k=1}^{n} Y_{k\Delta_n}\,\mathrm{e}^{-i\omega k} = c^T\left(\sum_{k=1}^{n}\sum_{j=0}^{\infty}\mathrm{e}^{j\Delta_n A}\,\xi^*_{n,k-j}\,\mathrm{e}^{-i\omega k}\right)e_p\\
&= c^T\left(\sum_{j=0}^{\infty}\mathrm{e}^{j(\Delta_n A - i\omega\mathrm{I}_p)}\sum_{k=1}^{n}\xi^*_{n,k}\,\mathrm{e}^{-i\omega k}\right)e_p + c^T\left(\sum_{j=0}^{\infty}\mathrm{e}^{j(\Delta_n A - i\omega\mathrm{I}_p)}U_{n,j}(\omega)\right)e_p\\
&= c^T M_{n,\Delta_n}(\omega)\,e_p + c^T K_{n,\Delta_n}(\omega)\,e_p = J_{n,\Delta_n}(\omega) + c^T K_{n,\Delta_n}(\omega)\,e_p
\end{aligned} \tag{2.6}$$

and hence, we may split the smoothed (self-normalized) periodogram in a main, limit-determining, part and a vanishing rest term (cf. upcoming Propositions 3.3 and 3.4):

$$\begin{aligned}
\widehat{T}_{n,Y^{\Delta_n}}(\omega) &= \sum_{|k|\le m_n} W_n(k)\,\frac{\left|\sum_{u=1}^{n} Y_{u\Delta_n}\,\mathrm{e}^{-i\omega_k u}\right|^2}{\sum_{u=1}^{n} Y^2_{u\Delta_n}} \overset{(2.6)}{=} \sum_{|k|\le m_n} W_n(k)\,\frac{\left|J_{n,\Delta_n}(\omega_k) + c^T K_{n,\Delta_n}(\omega_k)\,e_p\right|^2}{\sum_{u=1}^{n} Y^2_{u\Delta_n}}\\
&= \sum_{|k|\le m_n} W_n(k)\,\frac{\left|J_{n,\Delta_n}(\omega_k)\right|^2}{\sum_{u=1}^{n} Y^2_{u\Delta_n}} + \sum_{|k|\le m_n} W_n(k)\,\frac{R_{n,\Delta_n}(\omega_k)}{\sum_{u=1}^{n} Y^2_{u\Delta_n}}.
\end{aligned} \tag{2.7}$$

3 Limit Behavior of the Smoothed Periodogram

Our main limit theorem is the following:

Theorem 3.1. *Suppose $\alpha \in (0,2]$, $\sigma_L > 0$ and let $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ denote the sampled sequence of a non-trivial CARMA(p,q) process driven by the Lévy process $L(\alpha, \sigma_L)$ as in Definition 2.2. Moreover, Assumptions 1 to 3 may hold, and assume that the weight functions W_n satisfy* (1.8). *Then the smoothed self-normalized periodogram as in Eq.* (1.6) *satisfies for any $\omega \in \mathbb{R}^*$,*

$$\Delta_n \widehat{T}_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathbb{P}} \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} \qquad \text{as } n \to \infty,$$

where g is the kernel function of the CARMA process (see (2.2b)*), i.e., the smoothed self-normalized periodogram is a weakly consistent estimator for the normalized power transfer function.*

For $\alpha = 2$ the normalization $n^{-1}\sum_{k=1}^n Y_{k\Delta_n}^2$ converges in probability, as $n \to \infty$, to $\int_0^\infty g^2(s)\,\mathrm{d}s \cdot \sigma_L^2$ (cf. [13, Theorem 5.5(a)]) such that a direct conclusion is

Corollary 3.2. *Under the same assumptions as in Theorem 3.1, suppose in addition that $\alpha = 2$. Then the smoothed normalized periodogram as in Eq.* (1.5) *satisfies for any $\omega \in \mathbb{R}^*$,*

$$\Delta_n T_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathbb{P}} \sigma_L^2 \cdot \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \qquad \text{as } n \to \infty,$$

i.e., the smoothed normalized periodogram is a weakly consistent estimator for the 2π-multiple of the spectral density.

The proof of Theorem 3.1 will be divided into two parts. The first one shows that the main part in (2.7) converges to the normalized power transfer function as $n \to \infty$.

Proposition 3.3. *Under the same assumptions as in Theorem 3.1 we have for any $\omega \in \mathbb{R}^*$,*

$$\Delta_n \sum_{|k|\le m_n} W_n(k) \frac{\left|J_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2}{\sum_{u=1}^n Y_{u\Delta_n}^2} \xrightarrow{\mathbb{P}} \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} \qquad \text{as } n \to \infty,$$

where $J_{n,\Delta_n}(\,\cdot\,)$ has been defined in (2.5).

The second part shows that the rest term in (2.7) vanishes as $n \to \infty$.

Proposition 3.4. *Suppose $\alpha \in (0,2]$, $\sigma_L > 0$ and let $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ denote the sampled sequence of a non-trivial CARMA(p,q) process driven by the Lévy process $L(\alpha, \sigma_L)$ as in Definition 2.2. Moreover, Assumptions 1 to 3 may hold, and assume that the weight functions W_n satisfy* (1.8a) *and* (1.8b). *Then we have for any $\omega \in \mathbb{R}^*$,*

$$\Delta_n \sum_{|k|\le m_n} W_n(k) \frac{R_{n,\Delta_n}((\omega\Delta_n)_k)}{\sum_{u=1}^n Y_{u\Delta_n}^2} \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty,$$

where $R_{n,\Delta_n}(\,\cdot\,)$ has been defined in (2.5).

In Theorem 3.1 we have shown that the smoothed self-normalized periodogram provides consistent estimates for the (normalized) power transfer function of symmetric α-stable as well as finite-variance CARMA processes. Recall that normalized and self-normalized periodogram versions have been investigated in [15] under more restrictive assumptions on Δ_n than here. Moreover, in that paper only the Gaussian case has been studied but not the general finite-variance setting. Therefore, the following theorem should be seen as an extension of the results in [15]. It concerns the limit behavior of the normalized periodogram including finite-variance CARMA processes.

Theorem 3.5. *Suppose* $\alpha \in (0,2]$, $\sigma_L > 0$ *and let* $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ *denote the sampled sequence of a non-trivial CARMA(p,q) process driven by the Lévy process* $L(\alpha, \sigma_L)$ *as in Definition 2.2. Moreover, Assumptions 1 and 3 may hold. Then the periodogram as in* (1.3) *satisfies for any* $\omega \in \mathbb{R}^*$,

$$\Delta_n^{2-\frac{2}{\alpha}} I_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathscr{D}} \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \cdot \left| \int_{[0,1)} \mathrm{e}^{2\pi i s}\,\mathrm{d}L_s^* \right|^2 \qquad \text{as } n \to \infty,$$

where $(L_t^*)_{t\geq 0}$ *is a symmetric* α*-stable Lévy process with scale parameter* σ_L *if* $\alpha \in (0,2)$ *and for* $\alpha = 2$ *it is a symmetric Brownian motion with* $\mathrm{Var}(L_1^*) = \sigma_L^2$.

A direct conclusion is the asymptotic behavior of the periodogram for finite-variance CARMA processes.

Corollary 3.6. *Let the assumptions of Theorem 3.5 hold and suppose* $\alpha = 2$. *Then the normalized periodogram as in* (1.3) *satisfies for any* $\omega \in \mathbb{R}^*$,

$$\Delta_n I_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathscr{D}} \sigma_L^2 \cdot \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \cdot \left(\frac{N_1^2}{2} + \frac{N_2^2}{2} \right) \qquad \text{as } n \to \infty,$$

where N_1 *and* N_2 *are i.i.d. standard normal random variables, and the self-normalized periodgram as in* (1.4) *satisfies for any* $\omega \in \mathbb{R}^*$,

$$\Delta_n \widehat{I}_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathscr{D}} \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} \cdot \left(\frac{N_1^2}{2} + \frac{N_2^2}{2} \right) \qquad \text{as } n \to \infty,$$

where g is the kernel function of the CARMA process (see (2.2b)*).*

From Proposition 3.4 we know already that the rest term in (2.7) with $W_n(0) = 1$ and $W_n(k) = W_n(-k) = 0$ for $k \in \mathbb{N}$ vanishes. These weights do not satisfy (1.8c), but obviously (1.8a) and (1.8b). The next proposition investigates the main part.

Proposition 3.7. *Under the same assumptions as in Theorem 3.5 we have for any* $\omega \in \mathbb{R}^*$,

$$\Delta_n^{2-\frac{2}{\alpha}} n^{-\frac{2}{\alpha}} \left| J_{n,\Delta_n}(\omega\Delta_n) \right|^2 \xrightarrow{\mathbb{P}} \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \cdot \left| \int_{[0,1)} \mathrm{e}^{2\pi i s}\,\mathrm{d}L_s^* \right|^2 \qquad \text{as } n \to \infty,$$

where $J_{n,\Delta_n}(\,\cdot\,)$ *has been defined in* (2.5) *and* $(L_t^*)_{t\geq 0}$ *is as in Theorem 3.5.*

Remark 3.8.

(i) As already mentioned above, in [15] the general finite-variance case has not been considered. In this spirit, Theorem 3.5 and Corollary 3.6 should be seen as an extension of [15, Theorems 3.5 and 3.10], although we have stated only the univariate limit distributions for the normalized and self-normalized periodogram here. However, it seems to be possible to derive also the limit behavior for different frequencies. In this case, the limit depends again on the dependence structure of those frequencies if $\alpha < 2$, cf. [15, Section 2.2]. As in the Gaussian case (cf. [15, Remark 3.6(ii)]) different periodogram ordinates of finite-variance CARMA models are asymptotically independent.

(ii) Theorem 3.5 (and its proof) confirms our conjecture in [15, Remark 3.7], namely that the assumption $n\Delta_n^\beta \to \infty$ with $\beta = \max\{1+\delta, \alpha(p-1)+\max\{0, 1-\alpha\}\}$ for some $\delta > 0$ is not necessary for the limit results of normalized and self-normalized periodogram versions of symmetric α-stable CARMA processes. Instead, supposing $n\Delta_n \to \infty$ as $n \to \infty$ is already sufficient. Note that, anyway, the partition of the periodogram used in [15] provides deeper insight into structural properties of CARMA processes in the frequency domain and therein lay the necessity for the stronger condition on the observation grid (cf. also [15, Proof of Proposition 3.2 and Remark 3.7]).

(iii) We want to compare the limit results for the high-frequency sampled finite-variance CARMA process Y^{Δ_n} with the results for a finite-variance CARMA process sampled at an equidistant time grid $Y^\Delta =$

$(Y_{k\Delta})_{k\in\mathbb{Z}}$ for some $\Delta > 0$ fixed. For that, let $f_{Y^{\Delta_n}}$ denote the spectral density of Y^{Δ_n}, f_{Y^Δ} the spectral density of Y^Δ and finally

$$f_Y(\omega) = \frac{\sigma_L^2}{2\pi} \frac{|c(i\omega)|^2}{|a(i\omega)|^2}, \quad \omega \in \mathbb{R},$$

the spectral density of the continuous-time process Y. Moreover, the periodogram of the sampled sequence Y^Δ is denoted by $I_{n,Y^\Delta}(\omega) = \left|n^{-1/2}\sum_{k=1}^n Y_{k\Delta}\,\mathrm{e}^{-i\omega k}\right|^2$ for $\omega \in [-\pi,\pi]$. A conclusion of [14, Theorem 3.1] for the equidistant sampling is that for any $\omega \in (-\pi/\Delta, 0) \cup (0, \pi/\Delta)$,

$$\frac{I_{n,Y^\Delta}(\omega\Delta)}{f_{Y^\Delta}(\omega\Delta)} \xrightarrow{\mathscr{D}} 2\pi\left(\frac{N_1^2}{2} + \frac{N_2^2}{2}\right) \qquad \text{as } n\to\infty,$$

and of Corollary 3.6 and [15, Eq. (1.5)] for the high-frequency time sampling that for any $\omega \in \mathbb{R}^*$,

$$\frac{I_{n,Y^{\Delta_n}}(\omega\Delta_n)}{f_{Y^{\Delta_n}}(\omega\Delta_n)} \xrightarrow{\mathscr{D}} 2\pi\left(\frac{N_1^2}{2} + \frac{N_2^2}{2}\right) \qquad \text{as } n\to\infty.$$

Surprisingly the structure of the limit results is the same and will be of advantage for statistical inference. The similarities suggest that the rate of convergence of Δ_n has no influence on the asymptotic behavior. □

4 Estimation of the CARMA Parameters

In this section we propose a (spectral) estimation procedure for the autoregressive (AR) and moving average (MA) parameters of a CARMA process, based on Theorem 3.1 and Corollary 3.2. We will exemplify our method by a simulation study for the CARMA$(2,1)$ case.

Let $\alpha \in (0,2]$, $\sigma_L > 0$ and $Y^{\Delta_n} = (Y_{k\Delta_n})_{k\in\mathbb{Z}}$ be the sampled sequence of a non-trivial CARMA(p,q) process driven by the Lévy process $L(\alpha,\sigma_L)$ as in Definition 2.2. W.l.o.g. we assume in the following that $c_0 = 1$ (note that multiplying the MA polynomial by constants is equivalent to multiplying the scale parameter σ_L of the underlying Lévy process by the same factor). Thus its MA and AR polynomials are given by

$$c(z) := z^q + c_1 z^{q-1} + \ldots + c_q = \prod_{k=1}^q (z-\mu_k) \quad \text{and} \quad a(z) := z^p + a_1 z^{p-1} + \ldots + a_p = \prod_{j=1}^p (z-\lambda_j),$$

where $\mu_1,\ldots,\mu_q$ denote the zeros of c and $\lambda_1,\ldots,\lambda_p$, as usual, the zeros of a. The corresponding normalized power transfer function (cf. (1.2)) can be written as

$$C\cdot\Psi(\omega) = C\cdot\frac{\prod_{k=1}^q\left(\omega^2 - 2\,\Im(\mu_k)\,\omega + |\mu_k|^2\right)}{\prod_{j=1}^p\left(\omega^2 - 2\,\Im(\lambda_j)\,\omega + |\lambda_j|^2\right)} = C\cdot\frac{\prod_{k=1}^q(\omega + i\,\mu_k)(\omega - i\,\overline{\mu_k})}{\prod_{j=1}^p(\omega + i\,\lambda_j)(\omega - i\,\overline{\lambda_j})}, \qquad \omega\in\mathbb{R},$$

with $C^{-1} = \int_0^\infty g^2(s)\,\mathrm{d}s$ (where g is as in (2.2b)).

The following example illustrates this relationship in the case of a CARMA$(2,1)$ process.

Example 4.1 (CARMA$(2,1)$ process)**.** Consider a CARMA$(2,1)$ process which is the strictly stationary solution to

$$\left(\mathrm{D}^2 + a_1\mathrm{D} + a_2\right)Y_t = (\mathrm{D}+\mu)\,\mathrm{D}L_t, \quad t\in\mathbb{R},$$

i.e. $c(z) = z+\mu$ and $a(z) = z^2 + a_1 z + a_2 = (z-\lambda_1)(z-\lambda_2)$. In this case the kernel g in (2.2b) is given by

$$g(t) = \frac{\lambda_1+\mu}{\lambda_1-\lambda_2}\,\mathrm{e}^{t\lambda_1} + \frac{\lambda_2+\mu}{\lambda_2-\lambda_1}\,\mathrm{e}^{t\lambda_2}, \quad t\ge 0,$$

and the normalized power transfer function can be written as

$$\frac{\Psi(\omega)}{\int_0^\infty g^2(s)\,\mathrm{d}s} = \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} = C(a_1,a_2,\mu)\cdot\frac{\omega^2+\mu^2}{\omega^4+(a_1^2-2a_2)\omega^2+a_2^2}, \quad \omega\in\mathbb{R},$$

with $C(a_1,a_2,\mu) = \left(\int_0^\infty g^2(s)\,\mathrm{d}s\right)^{-1} = -2\lambda_1\lambda_2\frac{\lambda_1+\lambda_2}{\mu^2+\lambda_1\lambda_2} = 2\frac{a_1a_2}{\mu^2+a_2}$. □

Hence, we observe that the zeros of

$$\widetilde{\Psi}(\omega) := C\cdot\Psi(-i\omega) = C\cdot(-1)^{p-q}\frac{\prod_{k=1}^q(\omega-\mu_k)(\omega+\overline{\mu_k})}{\prod_{j=1}^p(\omega-\lambda_j)(\omega+\overline{\lambda_j})}$$

are given by μ_k and $-\overline{\mu_k}$, $k\in\{1,\ldots,q\}$, and the poles of $\widetilde{\Psi}$ (i.e. the zeros of $\widetilde{\Psi}^{-1}$) are λ_j and $-\overline{\lambda_j}$, $j\in\{1,\ldots,p\}$. Consequently, we will have to suppose

Assumption 4. *The zeros $\mu_1,\ldots,\mu_q$ of the moving average and the zeros $\lambda_1,\ldots,\lambda_p$ of the autoregressive polynomial are all distinct and possess strictly negative real parts*

in order to be able to identify the parameters of the CARMA process from its normalized power transfer function.

Remark 4.2. Note that Assumption 3 is included in Assumption 4. Moreover, requiring that the AR zeros $\lambda_1,\ldots,\lambda_p$ possess strictly negative real parts is a standard assumption that ensures causality of the CARMA process. The analog condition on the MA zeros guarantees invertibility. □

It is clear that Assumption 4 will lead to a constraint for the parameter vector $\theta := (a_1,\ldots,a_p,c_1,\ldots,c_q)^T$, i.e. θ has to be an element of some subset $\Theta\subseteq\mathbb{R}^{p+q}$. The power transfer function is henceforth denoted by Ψ_θ and its normalization by C_θ.

Our estimation heuristic is the following. Suppose we have observed the CARMA(p,q) process on the time grid $\{\Delta_n,\ldots,n\Delta_n\}$ and let $m\in\mathbb{N}^*$. Then we choose m different frequencies $\omega_j\in(0,\pi/\Delta_n)$, $j=1,\ldots,m$, and solve the *constrained nonlinear least squares problem*

$$\widehat{\theta} := \operatorname*{argmin}_{\theta\in\Theta}\sum_{j=1}^m\left|\log\left(C_\theta\cdot\Psi_\theta(\omega_j)\right)-\log\left(\Delta_n\widehat{T}_{n,Y^{\Delta_n}}(\omega_j\Delta_n)\right)\right|^2. \tag{4.1}$$

Remark 4.3. Under the additional assumption of a finite fourth moment of the driving Lévy process, the asymptotic behavior of the variance of the smoothed periodogram for ARMA models in discrete time [3, Theorem 10.4.1] and the proof of Theorem 3.5 suggest that for $\omega\in\mathbb{R}^*$,

$$\lim_{n\to\infty}\mathbb{E}\left[\Delta_n I_{n,Y^{\Delta_n}}(\omega\Delta_n)\right] = \sigma_L^2\Psi(\omega) \qquad \text{and} \tag{4.2}$$

$$\lim_{n\to\infty}\left(\sum_{|k|\le m_n}W_n(k)^2\right)^{-1}\Delta_n^2\operatorname{Var}(T_{n,Y^{\Delta_n}}(\omega\Delta_n)) = \sigma_L^4\Psi^2(\omega). \tag{4.3}$$

Eq. (4.3) implies that the variance of the smoothed periodogram is higher for frequencies with a high and lower for frequencies with a low power transfer function, respectively. Together with (4.2) this suggests to use the logarithmic transformation as a *variance stabilizing technique* (see also [26, Sections 2.9.1 and 6.2.4]). We have observed in our simulation study that also in the α-stable case, this transformation made the results more reliable. □

Methods for constrained optimization and (non)linear least squares problems are discussed, for instance, in the monographs [1, 16, 25]. We have decided to use the solver MINOS and as interface the modeling language AMPL (see [17, 24] for the MINOS user's guide and a general introduction to AMPL, respectively). In the presence of linear constraints (which will be the case in our setting) MINOS solves (4.1) using a *reduced-gradient* algorithm combined with a *quasi-Newton* algorithm that is described in [23].

In our example of a CARMA(2,1) process, the optimization problem (4.1) becomes the following.

Example 4.4 (CARMA$(2,1)$ process)**.** We consider again the CARMA$(2,1)$ process as in Example 4.1. Assumption 4 yields immediately that $a_1, a_2, \mu > 0$ must hold. Hence, the (unknown) parameter vector $\theta = (a_1, a_2, \mu)^T$ is an element of $\Theta := (0,\infty)^3$. The optimization problem in (4.1) then becomes

$$\begin{pmatrix}\widehat{a_1}\\ \widehat{a_2}\\ \widehat{\mu}\end{pmatrix} = \underset{(a_1,a_2,\mu)^T\in(0,\infty)^3}{\operatorname{argmin}} \sum_{j=1}^{m} \left| \log\left(\frac{2a_1a_2}{\mu^2+a_2}\right) + \log\left(\frac{\omega_j^2+\mu^2}{\omega_j^4+(a_1^2-2a_2)\omega_j^2+a_2^2}\right) - \log\left(\Delta_n \widehat{T}_{n,Y^{\Delta_n}}(\omega_j\Delta_n)\right)\right|^2. \quad (4.4)$$

□

Simulation Study

As announced at the beginning of this section, we will carry out a simulation study for a CARMA$(2,1)$ process in order to show how the estimation heuristic (4.4) performs in the finite-variance as well as in the stable case. Our simulation study should be compared to the one in [18, Chapter 4]. Therefore, we have chosen not only similar values of α but also comparable CARMA parameters.

For each α taking on the values 2, 1.8, 1.6, 1.4 and 1.25, we have simulated 250 different sample paths of an α-stable CARMA$(2,1)$ process with parameters $a_1 = 2$, $a_2 = 0.1$ and $\mu = 0.2$. In the Gaussian case (i.e. $\alpha = 2$) we have chosen the standard deviation of the underlying Lévy process to be $\sigma_L = 1.5$ and in the other scenarios we have fixed the same value as the scale parameter for the driving process. Every CARMA sample path is simulated by means of an Euler approximation of the corresponding SDE in its state space representation (cf. (2.1)). The mesh of the simulation time grid has been set to 0.01 and the number of total time steps is equal to 150000. The observed CARMA sample, however, is chosen to be only every 10th simulated value, i.e. the CARMA process has been observed at time points $\{\Delta_n, 2\Delta_n, \ldots, n\Delta_n\}$ with $\Delta_n = 0.1$ and $n = 15000$.

Note that in the Gaussian case, we can easily reformulate (4.4) as

$$\begin{pmatrix}\widehat{\sigma_L}\\ \widehat{a_1}\\ \widehat{a_2}\\ \widehat{\mu}\end{pmatrix} = \underset{(\sigma_L,a_1,a_2,\mu)^T\in(0,\infty)^4}{\operatorname{argmin}} \sum_{j=1}^{m} \left| 2\log(\sigma_L) + \log\left(\frac{\omega_j^2+\mu^2}{\omega_j^4+(a_1^2-2a_2)\omega_j^2+a_2^2}\right) - \log\left(\Delta_n T_{n,Y^{\Delta_n}}(\omega_j\Delta_n)\right)\right|^2$$

by virtue of Corollary 3.2. Thus, by using the normalized smoothed periodogram in the Gaussian case, we shall get an estimate for the standard deviation σ_L of the underlying Lévy process on top.

For each realized time series, we computed then smoothed periodogram values for 300 equidistant frequencies ω_j in the interval $[0.005, 2\pi]$, i.e. $\omega_j = 0.005 + (j-1)/299 \cdot (2\pi - 0.005)$, $j = 1, \ldots, 300$. Our smoothing filter has $m_n = \lfloor\sqrt{n\Delta_n}\rfloor = 38$ nodes with equal weights $W_n(k) = 1/(2m_n+1) = 1/77$ for any $|k| \le m_n = 38$. Concerning several aspects of these (necessary) specifications in practice, we refer the reader, for instance, to [26, Chapter 7].

Our results are reported in Table 1. As in [18], we observe that the estimates of the CARMA parameters become better in terms of the standard deviation when α decreases. However, in terms of the bias no evident relationship is visible. In Figure 1, we plotted smoothed periodogram values for some selected time series in order to show the effect of the logarithmic transformation we have used (cf. also Remark 4.3).

5 Proofs

We start with three lemmata that we will need for the proofs of our main results. The third one is the "Ornstein-Uhlenbeck version" of Proposition 3.4.

Lemma 5.1. *Under the same assumptions as in Theorem 3.1 we have for any $\omega \in \mathbb{R}^*$,*

$$\frac{1}{\Delta_n}\sum_{|k|\le m_n} W_n(k) \frac{\left|J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}} \xrightarrow{\mathbb{P}} \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} \quad \text{as } n\to\infty,$$

		σ_L	a_1	a_2	μ
	True	1.5	2.0	0.1	0.2
	Mean	1.5127	2.0859	0.1182	0.2159
$\alpha = 2$	Bias	0.0127	0.0859	0.0182	0.0159
	Std. dev.	0.0392	0.1204	0.0358	0.0366
	Mean	-	2.0580	0.1108	0.2185
$\alpha = 1.8$	Bias	-	0.0580	0.0108	0.0185
	Std. dev.	-	0.1240	0.0372	0.0378
	Mean	-	2.0626	0.1079	0.2127
$\alpha = 1.6$	Bias	-	0.0626	0.0079	0.0127
	Std. dev.	-	0.1130	0.0315	0.0361
	Mean	-	2.0659	0.1101	0.2129
$\alpha = 1.4$	Bias	-	0.0659	0.0101	0.0129
	Std. dev.	-	0.1151	0.0311	0.0329
	Mean	-	2.0776	0.1140	0.2149
$\alpha = 1.25$	Bias	-	0.0776	0.0140	0.0149
	Std. dev.	-	0.0928	0.0286	0.0307

Table 1: Simulation study for different values of α, based on 250 sample paths each: mean, bias and standard deviation of the estimates for the CARMA parameters.

where $J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k) := c^T(i\omega\mathrm{I}_p - A)^{-1}e_p\left(\sum_{u=1}^n \Delta L(u\Delta_n)\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right)$ *with* $\Delta L(u\Delta_n) := L_{u\Delta_n} - L_{(u-1)\Delta_n}$ *for any* $u \in \{1,\ldots,n\}$ *and* $n \in \mathbb{N}$.

Proof. First, we note from [10, Lemma 3.1] that $c^T(i\omega\mathrm{I}_p - A)^{-1}e_p = c(i\omega)a(i\omega)^{-1}$ for any $\omega \in \mathbb{R}$ and from [15, Proposition 3.8(ii)] we obtain

$$\sum_{u=1}^n Y^2_{u\Delta_n} = \sum_{j=0}^\infty g^2(j\Delta_n)\cdot\sum_{u=1}^n \Delta L(u\Delta_n)^2 + o_P\left(\Delta_n^{-1}(n\Delta_n)^{\frac{2}{\alpha}}\right) \quad \text{as } n\to\infty.$$

Thus, we deduce

$$\frac{1}{\Delta_n}\sum_{|k|\le m_n} W_n(k)\frac{\left|J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}} = \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s\cdot|a(i\omega)|^2}\sum_{|k|\le m_n} W_n(k)\frac{\left|\sum_{u=1}^n \Delta L(u\Delta_n)\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2}{\sum_{u=1}^n \Delta L(u\Delta_n)^2}\cdot(1+o_P(1))$$

and it is sufficient to show that

$$\sum_{|k|\le m_n} W_n(k)\frac{\left|\sum_{u=1}^n \Delta L(u\Delta_n)\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2}{\sum_{u=1}^n \Delta L(u\Delta_n)^2} \xrightarrow{\mathbb{P}} 1 \quad \text{as } n\to\infty. \tag{5.1}$$

Define $Z_{n,u} := \Delta_n^{-1/\alpha}\Delta L(u\Delta_n)$ for any $n\in\mathbb{N}$, $u\in\mathbb{Z}$. If $\alpha\in(0,2)$, $(Z_{n,u})_{u\in\mathbb{Z}}$ is a sequence of i.i.d. symmetric α-stable random variables with scale parameter σ_L, and in the case $\alpha = 2$ they are symmetric satisfying $\mathbb{E}[Z^2_{n,u}] = \sigma_L^2$ for any $n\in\mathbb{N}$, $u\in\mathbb{Z}$. Then we write as in the proof of [21, Lemma 6.1]

$$\begin{aligned}\sum_{|k|\le m_n} W_n(k)\frac{\left|\sum_{u=1}^n \Delta L(u\Delta_n)\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2}{\sum_{u=1}^n \Delta L(u\Delta_n)^2} &= \sum_{|k|\le m_n} W_n(k)\frac{\left|\sum_{u=1}^n Z_{n,u}\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2}{\sum_{u=1}^n Z^2_{n,u}}\\ &\overset{(1.8b)}{=} 1 + \sum_{1\le u\ne s\le n}\frac{Z_{n,u}Z_{n,s}}{\sum_{r=1}^n Z^2_{n,r}}\sum_{|k|\le m_n} W_n(k)\cos((\omega\Delta_n)_k(u-s))\end{aligned}$$

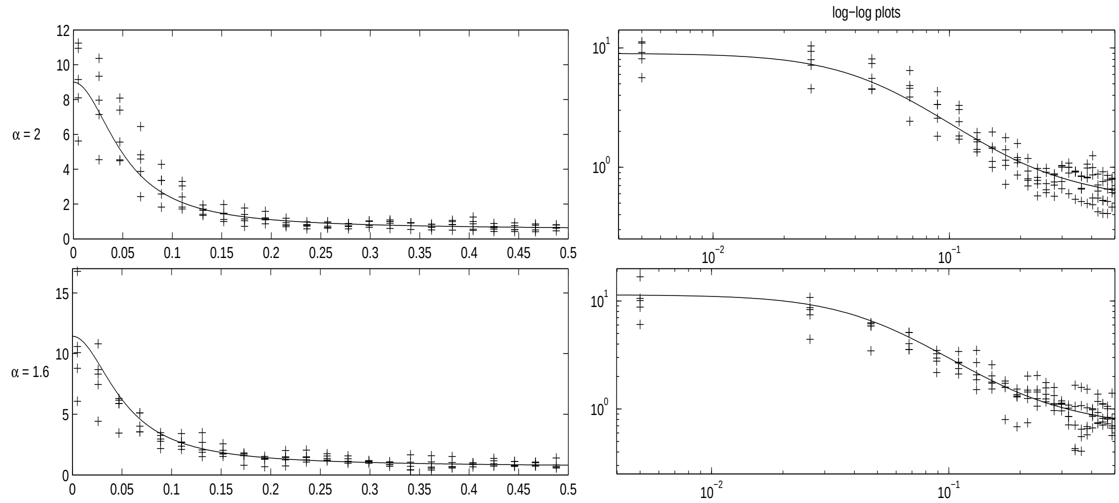

Figure 1: Smoothed periodogram values plotted against frequencies for five selected time series (pluses) in the Gaussian case (on top) and the 1.6-stable case (below). The true spectral density and normalized power transfer function is plotted as a solid line, respectively. The two graphs on the RHS are the left ones on a log-log scale.

$$=: 1 + \sum_{1 \le u \ne s \le n} a_{us}(\omega\Delta_n) \cdot \frac{Z_{n,u} Z_{n,s}}{\sum_{r=1}^{n} Z_{n,r}^2}.$$

Now,

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{1 \le u \ne s \le n} a_{us}(\omega\Delta_n) \cdot \frac{Z_{n,u} Z_{n,s}}{\sum_{r=1}^{n} Z_{n,r}^2}\right)^2\right] &= 2\,\mathbb{E}\left[\frac{Z_{n,1}^2 Z_{n,2}^2}{\left(\sum_{u=1}^{n} Z_{n,u}^2\right)^2}\right] \sum_{1 \le u \ne s \le n} a_{us}^2(\omega\Delta_n) \\ &= O\left(n^{-2} \sum_{1 \le u \ne s \le n} a_{us}^2(\omega\Delta_n)\right) \end{aligned}$$

as $n \to \infty$, where for the first inequality we used that $(Z_{n,u})_{u \in \mathbb{Z}}$ is a sequence of i.i.d. symmetric random variables. The second equality follows from [21, Lemma 5.8] if $\alpha \in (0,2)$ and from the SLLN together with the Dominated Convergence Theorem if $\alpha = 2$, respectively. Hence, in order to show (5.1), it remains to prove that for any $\omega \in \mathbb{R}^*$,

$$\sum_{1 \le u \ne s \le n} a_{us}^2(\omega\Delta_n) = o(n^2) \quad \text{as } n \to \infty. \tag{5.2}$$

By virtue of [21, Lemma 5.9(iv)] and Eq. (1.7) we obtain for some $C > 0$

$$\begin{aligned} \sum_{1 \le u \ne s \le n} a_{us}^2(\omega\Delta_n) &= \sum_{|k_1|,|k_2| \le m_n} W_n(k_1) W_n(k_2) \sum_{1 \le u \ne s \le n} \cos\left((\omega\Delta_n)_{k_1}(u-s)\right) \cdot \cos\left((\omega\Delta_n)_{k_2}(u-s)\right) \\ &= \frac{1}{2} \sum_{-m_n \le k_1 \ne k_2 \le m_n} W_n(k_1) W_n(k_2) \left\{ \frac{\sin^2\left(\frac{k_1-k_2}{2}\right)}{\sin^2\left(\frac{k_1-k_2}{2n}\right)} + \frac{\sin^2\left(\frac{k_1+k_2}{2} + \omega n \Delta_n\right)}{\sin^2\left(\frac{k_1+k_2}{2n} + \omega\Delta_n\right)} - 2n \right\} \\ &\quad + \frac{1}{2} \sum_{k=-m_n}^{m_n} W_n^2(k) \left\{ n^2 + \frac{\sin^2(\omega n \Delta_n + k)}{\sin^2(\omega\Delta_n + \frac{k}{n})} - 2n \right\} \end{aligned}$$

$$\leq C\cdot n^2\Bigg\{\sum_{-m_n\leq k_1\neq k_2\leq m_n} W_n(k_1)\,W_n(k_2)\left[(k_1-k_2)^{-2}+(2\omega n\Delta_n+k_1+k_2)^{-2}\right] + \sum_{k=-m_n}^{m_n} W_n^2(k)\left[1+(\omega n\Delta_n+k)^{-2}\right]\Bigg\},$$

if n is only sufficiently large. Since $m_n\to\infty$ and $n\Delta_n m_n^{-1}\to\infty$ as $n\to\infty$ (see Assumption 2), we deduce that, for any $k_1\neq k_2\in\{-m_n,\dots,m_n\}$ and $\omega\in\mathbb{R}^*$, the term $(2\omega n\Delta_n+k_1+k_2)^{-2}$ can be bounded by $(k_1-k_2)^{-2}$ and $(\omega n\Delta_n+k)^{-2}$, $|k|\leq m_n$, can be bounded by 1, respectively, for all sufficiently large n. Hence, we have

$$n^{-2}\sum_{1\leq u\neq s\leq n} a_{us}^2(\omega\Delta_n)\leq 2C\cdot\max_{|k|\leq m_n} W_n^2(k)\cdot\left[2\sum_{j=1}^{2m_n}(2m_n-j+1)\frac{1}{j^2}+O(m_n)\right]=2C\cdot\max_{|k|\leq m_n} W_n^2(k)\cdot O(m_n) \stackrel{(1.8c)}{=} o(1)\quad\text{as } n\to\infty,$$

which completes the proof of the lemma. □

Remark 5.2. If we assume only (1.8a) and (1.8b) on the weight functions W_n, Eq. (5.2) is no longer valid. However, a slight modification of the proof above shows that $\sum_{1\leq u\neq s\leq n} a_{us}^2(\omega\Delta_n)=O(n^2)$ as $n\to\infty$ in this case. Hence, we still have, as $n\to\infty$,

$$\sum_{|k|\leq m_n} W_n(k)\frac{\left|\sum_{u=1}^n \Delta L(u\Delta_n)\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2}{\sum_{u=1}^n \Delta L(u\Delta_n)^2}=1+O_P(1) \tag{5.3}$$

and consequently also

$$\frac{1}{\Delta_n}\sum_{|k|\leq m_n} W_n(k)\frac{\left|J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2}{\sum_{u=1}^n Y_{u\Delta_n}^2}=\frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s\cdot|a(i\omega)|^2}\cdot(1+O_P(1)) \tag{5.4}$$

as $n\to\infty$, if we drop assumption (1.8c) on the weight functions. We will use these facts in the upcoming proofs of Lemmata 5.3 and 5.4 and Proposition 3.4. □

Lemma 5.3. *Under the same assumptions as in Proposition 3.4 we have for any $\omega\in\mathbb{R}^*$,*

$$\Delta_n^{-\frac{2}{\alpha}}\sum_{|k|\leq m_n} W_n(k)\left|\frac{\Delta_n}{n^{1/\alpha}}J_{n,\Delta_n}((\omega\Delta_n)_k)-\frac{1}{n^{1/\alpha}}J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2\stackrel{\mathbb{P}}{\to}0\quad\text{as } n\to\infty,$$

where $J^{(1)}_{n,\Delta_n}(\,\cdot\,)$ is as in Lemma 5.1.

Proof. We split the proof in two parts. First, we will establish

$$(n\Delta_n)^{-\frac{2}{\alpha}}\sum_{|k|\leq m_n} W_n(k)\left|J^{(2)}_{n,\Delta_n}((\omega\Delta_n)_k)-J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2\stackrel{\mathbb{P}}{\to}0\quad\text{as } n\to\infty, \tag{5.5}$$

where $J^{(2)}_{n,\Delta_n}((\omega\Delta_n)_k):=c^T(i\omega\mathrm{I}_p-A)^{-1}\left(\sum_{u=1}^n\xi^*_{n,u}\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right)e_p$ and $\xi^*_{n,u}$ is as in Eq. (2.4). Thereafter, we will show that also

$$\Delta_n^{-\frac{2}{\alpha}}\sum_{|k|\leq m_n} W_n(k)\left|\frac{\Delta_n}{n^{1/\alpha}}J_{n,\Delta_n}((\omega\Delta_n)_k)-\frac{1}{n^{1/\alpha}}J^{(2)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2\stackrel{\mathbb{P}}{\to}0\quad\text{as } n\to\infty. \tag{5.6}$$

Note that Equations (5.5) and (5.6) together imply the claim of the lemma.

As to (5.5), we observe first that, due to Assumption 3, the eigenvalues of A are supposed to be distinct. Hence, there exists an invertible matrix $D\in\mathbb{C}^{p\times p}$ such that $A=D\,\mathrm{diag}(\lambda_1,\ldots,\lambda_p)\,D^{-1}$ and thus,

$$\mathrm{e}^{A}=D\,\mathrm{diag}\left(\mathrm{e}^{\lambda_1},\ldots,\mathrm{e}^{\lambda_p}\right)D^{-1}. \tag{5.7}$$

Setting

$$\widehat{\xi}_{n,u}:=D^{-1}\,\xi^{*}_{n,u}\,D\overset{(5.7)}{=}\mathrm{diag}\left(\int_{(u-1)\Delta_n}^{u\Delta_n}\mathrm{e}^{(u\Delta_n-s)\lambda_1}\mathrm{d}L_s,\ldots,\int_{(u-1)\Delta_n}^{u\Delta_n}\mathrm{e}^{(u\Delta_n-s)\lambda_p}\mathrm{d}L_s\right), \tag{5.8}$$

we obtain

$$\begin{aligned}J^{(2)}_{n,\Delta_n}((\omega\Delta_n)_k)-J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)&=c^T(i\omega\mathrm{I}_p-A)^{-1}\sum_{u=1}^{n}\left(\xi^{*}_{n,u}-\Delta L(u\Delta_n)\,\mathrm{I}_p\right)\mathrm{e}^{-i(\omega\Delta_n)_k u}\,e_p\\&\overset{(5.8)}{=}c^T(i\omega\mathrm{I}_p-A)^{-1}D\left[\sum_{u=1}^{n}\left(\widehat{\xi}_{n,u}-\Delta L(u\Delta_n)\,\mathrm{I}_p\right)\mathrm{e}^{-i(\omega\Delta_n)_k u}\right]D^{-1}\,e_p\end{aligned}$$

and hence, for some $C>0$,

$$\begin{aligned}&(n\Delta_n)^{-\frac{2}{\alpha}}\sum_{|k|\le m_n}W_n(k)\left|J^{(2)}_{n,\Delta_n}((\omega\Delta_n)_k)-J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2\\&\le C\,(n\Delta_n)^{-\frac{2}{\alpha}}\sum_{j=1}^{p}\sum_{|k|\le m_n}W_n(k)\left|\sum_{u=1}^{n}\mathrm{e}^{-i(\omega\Delta_n)_k u}\int_{(u-1)\Delta_n}^{u\Delta_n}\left(\mathrm{e}^{(u\Delta_n-s)\lambda_j}-1\right)\mathrm{d}L_s\right|^2\\&\le 2C\,(n\Delta_n)^{-\frac{2}{\alpha}}\sum_{j=1}^{p}\sum_{|k|\le m_n}W_n(k)\left[\left|\sum_{u=1}^{n}\mathrm{e}^{-i(\omega\Delta_n)_k u}\int_{(u-1)\Delta_n}^{u\Delta_n}\left(\mathrm{e}^{(u\Delta_n-s)\Re(\lambda_j)}\cos((u\Delta_n-s)\Im(\lambda_j))-1\right)\mathrm{d}L_s\right|^2\right.\\&\qquad\left.+\left|\sum_{u=1}^{n}\mathrm{e}^{-i(\omega\Delta_n)_k u}\int_{(u-1)\Delta_n}^{u\Delta_n}\mathrm{e}^{(u\Delta_n-s)\Re(\lambda_j)}\sin((u\Delta_n-s)\Im(\lambda_j))\,\mathrm{d}L_s\right|^2\right]\\&=:2C\,(n\Delta_n)^{-\frac{2}{\alpha}}\sum_{j=1}^{p}I_1^{(j)}+I_2^{(j)}.\end{aligned} \tag{5.9}$$

Again we define $Z_{n,u}:=\Delta_n^{-1/\alpha}\Delta L(u\Delta_n)$ for any $n\in\mathbb{N}$, $u\in\mathbb{Z}$, and for any $j\in\{1,\ldots,p\}$, we set

$$d^{(j)}_{\Delta_n}:=\left(\int_0^{\Delta_n}\left|\mathrm{e}^{s\Re(\lambda_j)}\cos(s\Im(\lambda_j))-1\right|^{\alpha}\mathrm{d}s\right)^{1/\alpha}\quad\text{and}\quad f^{(j)}_{\Delta_n}:=\left(\int_0^{\Delta_n}\left|\mathrm{e}^{s\Re(\lambda_j)}\sin(s\Im(\lambda_j))\right|^{\alpha}\mathrm{d}s\right)^{1/\alpha}.$$

We will use that $\lim_{n\to\infty}\Delta_n^{-1/\alpha}d^{(j)}_{\Delta_n}=\lim_{n\to\infty}\Delta_n^{-1/\alpha}f^{(j)}_{\Delta_n}=0$ for any $j\in\{1,\ldots,p\}$ (cf. the proof of [15, Lemma 2.1(ii)]). Now, for any $j\in\{1,\ldots,p\}$,

$$(n\Delta_n)^{-\frac{2}{\alpha}}I_1^{(j)}\overset{\mathscr{D}}{=}\Delta_n^{-\frac{2}{\alpha}}\left(d^{(j)}_{\Delta_n}\right)^2\cdot\sum_{|k|\le m_n}W_n(k)\left|n^{-\frac{1}{\alpha}}\sum_{u=1}^{n}Z_{n,u}\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2\quad\text{and}$$

$$(n\Delta_n)^{-\frac{2}{\alpha}}I_2^{(j)}\overset{\mathscr{D}}{=}\Delta_n^{-\frac{2}{\alpha}}\left(f^{(j)}_{\Delta_n}\right)^2\cdot\sum_{|k|\le m_n}W_n(k)\left|n^{-\frac{1}{\alpha}}\sum_{u=1}^{n}Z_{n,u}\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2.$$

Since $n^{-2/\alpha}\sum_{u=1}^{n}Z^2_{n,u}$ converges weakly as $n\to\infty$, respectively, to a (positive) $\alpha/2$-stable random variable if $\alpha\in(0,2)$ and to σ_L^2 if $\alpha=2$, we deduce from (5.3) that both $(n\Delta_n)^{-2/\alpha}I_1^{(j)}$ and $(n\Delta_n)^{-2/\alpha}I_2^{(j)}$ converge to 0 in probability as $n\to\infty$ for any $j\in\{1,\ldots,p\}$. This implies that the right-hand side of (5.9) converges to 0 in probability and completes the proof of Eq. (5.5).

As to (5.6), for any $k \in \{-m_n, \dots, m_n\}$ and n sufficiently large, the inequality

$$
\begin{aligned}
&\left\| \Delta_n \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n \left(A - i\left(\omega + \frac{k}{n\Delta_n}\right)\mathrm{I}_p \right)} \right)^{-1} - (i\omega \mathrm{I}_p - A)^{-1} \right\| \\
&\quad \leq \left\| \Delta_n \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n \left(A - i\left(\omega + \frac{k}{n\Delta_n}\right)\mathrm{I}_p \right)} \right)^{-1} \right\| \cdot \left\| (i\omega \mathrm{I}_p - A)^{-1} \right\| \cdot \left\| i\omega \mathrm{I}_p - A - \Delta_n^{-1} \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n \left(A - i\left(\omega + \frac{k}{n\Delta_n}\right)\mathrm{I}_p \right)} \right) \right\| \\
&\quad \overset{(5.7)}{\leq} \text{const.} \cdot \sum_{j=1}^{p} \Delta_n \left| 1 - \mathrm{e}^{\Delta_n \left(\lambda_j - i\left(\omega + \frac{k}{n\Delta_n}\right) \right)} \right|^{-1} \cdot \Bigg[\left\| i\omega \mathrm{I}_p - A - \Delta_n^{-1} \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n (A - i\omega \mathrm{I}_p)} \right) \right\| \\
&\qquad\qquad + \left\| \Delta_n^{-1} \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n (A - i\omega \mathrm{I}_p)} \right) \right\| \cdot \left| 1 - \mathrm{e}^{-i\frac{k}{n}} \right| \Bigg] \\
&\quad \leq \text{const.} \cdot \sum_{j=1}^{p} 2 \left| \lambda_j - i \left(\omega + \frac{k}{n\Delta_n} \right) \right|^{-1} \cdot \mathrm{e}^{\Delta_n \| A - i\omega \mathrm{I}_p \|} \cdot \left[\frac{\Delta_n}{2} \left\| A - i\omega \mathrm{I}_p \right\|^2 + \left\| A - i\omega \mathrm{I}_p \right\| \cdot \left| 1 - \mathrm{e}^{-i\frac{k}{n}} \right| \right] \\
&\quad \leq \text{const.} \cdot \sum_{j=1}^{p} \left(\left| \lambda_j - i\omega \right| - \frac{m_n}{n\Delta_n} \right)^{-1} \cdot \mathrm{e}^{\Delta_n \| A - i\omega \mathrm{I}_p \|} \cdot \left[\frac{\Delta_n}{2} \left\| A - i\omega \mathrm{I}_p \right\|^2 + \left\| A - i\omega \mathrm{I}_p \right\| \cdot \frac{m_n}{n} \right] \overset{n\to\infty}{\to} 0
\end{aligned}
$$

holds, where the last convergence result follows from Assumptions 1 to 3. Thus, define

$$
\varepsilon_n := \max_{|k| \leq m_n} \left\| \Delta_n \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n \left(A - i\left(\omega + \frac{k}{n\Delta_n}\right)\mathrm{I}_p \right)} \right)^{-1} - (i\omega \mathrm{I}_p - A)^{-1} \right\|,
$$

where, for any $\omega \in \mathbb{R}^*$, we have $\varepsilon_n \searrow 0$ as $n \to \infty$. Then, for some $C > 0$,

$$
\begin{aligned}
&\Delta_n^{-\frac{2}{\alpha}} \sum_{|k| \leq m_n} W_n(k) \left| \frac{\Delta_n}{n^{1/\alpha}} J_{n,\Delta_n}((\omega\Delta_n)_k) - \frac{1}{n^{1/\alpha}} J^{(2)}_{n,\Delta_n}((\omega\Delta_n)_k) \right|^2 \\
&\quad = \Delta_n^{-\frac{2}{\alpha}} \sum_{|k| \leq m_n} W_n(k) \left| c^T \left[\Delta_n \left(\mathrm{I}_p - \mathrm{e}^{\Delta_n \left(A - i\left(\omega + \frac{k}{n\Delta_n}\right)\mathrm{I}_p \right)} \right)^{-1} - (i\omega \mathrm{I}_p - A)^{-1} \right] n^{-\frac{1}{\alpha}} \left(\sum_{u=1}^{n} \xi^*_{n,u} \mathrm{e}^{-i(\omega\Delta_n)_k u} \right) e_p \right|^2 \\
&\quad \overset{(5.8)}{\leq} C \varepsilon_n \sum_{j=1}^{p} \Delta_n^{-\frac{2}{\alpha}} \sum_{|k| \leq m_n} W_n(k) \left| n^{-\frac{1}{\alpha}} \sum_{u=1}^{n} \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \mathrm{e}^{(u\Delta_n - s)\lambda_j} \, \mathrm{d}L_s \right|^2 \\
&\quad \leq 2C \varepsilon_n \sum_{j=1}^{p} \Delta_n^{-\frac{2}{\alpha}} \sum_{|k| \leq m_n} W_n(k) \Bigg[\left| n^{-\frac{1}{\alpha}} \sum_{u=1}^{n} \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \left(\mathrm{e}^{(u\Delta_n - s)\lambda_j} - 1 \right) \mathrm{d}L_s \right|^2 \\
&\qquad\qquad + \left| n^{-\frac{1}{\alpha}} \sum_{u=1}^{n} \Delta L(u\Delta_n) \, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \Bigg]. \qquad (5.10)
\end{aligned}
$$

Now, having in mind that $\varepsilon_n \searrow 0$ as $n \to \infty$, the same arguments as used above give that the right-hand side of (5.10) converges to 0 in probability as $n \to \infty$. This completes the proof of Eq. (5.6) and hence, Lemma 5.3 is shown. □

Lemma 5.4. *Suppose $\alpha \in (0,2]$, $\sigma_L > 0$ and define a family of sequences of i.i.d. random variables $(Z_{n,u})_{u\in\mathbb{Z}}$ such that, if $\alpha \in (0,2)$, $(Z_{n,u})_{u\in\mathbb{Z}} = (S_u)_{u\in\mathbb{Z}}$ for all $n \in \mathbb{N}$ with independent symmetric α-stable random variables S_u each with scale parameter σ_L and in the case $\alpha = 2$ the random variables $Z_{n,u}$ are symmetric and satisfy $\mathbb{E}\left[Z^2_{n,u}\right] = \sigma_L^2$ for any $n \in \mathbb{N}$ and $u \in \mathbb{Z}$. Moreover, Assumptions 1 to 3 may hold, and assume that the weight functions W_n satisfy* (1.8a) *and* (1.8b). *Then we have for any $\omega \in \mathbb{R}^*$ and $r \in \{1, \dots, p\}$,*

$$
\frac{\Delta_n^2}{n^{2/\alpha}} \sum_{|k| \leq m_n} W_n(k) \left| \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n \lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n} \right] Z_{n,u} \, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \overset{\mathbb{P}}{\to} 0 \qquad \textit{as } n \to \infty.
$$

Proof. We follow along the lines of [21, Lemmata 6.2 and 6.3]. Setting

$$U^Z_{n,j}(\omega) := \sum_{u=1-j}^{n-j} Z_{n,u}\,\mathrm{e}^{-i\omega u} - \sum_{u=1}^{n} Z_{n,u}\,\mathrm{e}^{-i\omega u}$$

we first observe that

$$\begin{aligned}
& n^{-\frac{2}{\alpha}} \left| \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n}\right] Z_{n,u}\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2 \\
&\leq 2\left[\left|n^{-\frac{1}{\alpha}} \sum_{j=n+1}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)}\, U^Z_{n,j}((\omega\Delta_n)_k)\right|^2 + \left|n^{-\frac{1}{\alpha}} \sum_{j=0}^{n} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)}\, U^Z_{n,j}((\omega\Delta_n)_k)\right|^2\right] \\
&=: 2\left(A^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k) + A^{(2)}_{n,\Delta_n}((\omega\Delta_n)_k)\right).
\end{aligned}$$

We start with the proof of

$$\Delta_n^2 \sum_{|k|\leq m_n} W_n(k) A^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k) \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n\to\infty. \tag{5.11}$$

We have

$$\begin{aligned}
\sum_{|k|\leq m_n} W_n(k) A^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k) &\leq 2n^{-\frac{2}{\alpha}} \Bigg\{ \sum_{|k|\leq m_n} W_n(k) \left|\sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k} \sum_{u=1-j}^{n-j} Z_{n,u}\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2 \\
&\quad + \sum_{|k|\leq m_n} W_n(k) \left|\sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k}\right|^2 \cdot \left|\sum_{u=1}^{n} Z_{n,u}\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2 \Bigg\} \\
&=: 2n^{-\frac{2}{\alpha}}(V_1 + V_2)
\end{aligned}$$

and

$$\Delta_n^2\, n^{-\frac{2}{\alpha}} V_2 \leq \Delta_n^2 \left(\sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^2 \cdot \sum_{|k|\leq m_n} W_n(k) \frac{\left|\sum_{u=1}^{n} Z_{n,u}\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right|^2}{\sum_{u=1}^{n} Z^2_{n,u}} \cdot n^{-\frac{2}{\alpha}} \sum_{u=1}^{n} Z^2_{n,u}$$

where the second term is equal to $1 + O_P(1)$ as $n\to\infty$ (this is a simple consequence of Eq. (5.3)). The third term converges, if $\alpha\in(0,2)$, weakly to a positive $\alpha/2$-stable random variable (see, for instance, [27, Theorem 7.1]) and for $\alpha = 2$ we know due to the WLLN that $n^{-1}\sum_{u=1}^{n} Z^2_{n,u} \xrightarrow{\mathbb{P}} \sigma_L^2$ as $n\to\infty$. The first term satisfies

$$\Delta_n \sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n\Re(\lambda_r)} = \Delta_n \frac{\mathrm{e}^{(n+1)\Delta_n\Re(\lambda_r)}}{1-\mathrm{e}^{\Delta_n\Re(\lambda_r)}} \overset{n\to\infty}{\sim} -\frac{1}{\Re(\lambda_r)} \mathrm{e}^{n\Delta_n\Re(\lambda_r)} \to 0 \qquad \text{as } n\to\infty \tag{5.12}$$

by virtue of Assumptions 1 and 3 and hence, $\Delta_n^2 n^{-2/\alpha} V_2 \xrightarrow{\mathbb{P}} 0$.

As to V_1, we get

$$\begin{aligned}
V_1 \leq 2\Bigg\{ &\sum_{|k|\leq m_n} W_n(k) \left|\sum_{u=-n}^{-1} Z_{n,u}\,\mathrm{e}^{-i(\omega\Delta_n)_k u} \sum_{j=n+1}^{n-u} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k}\right|^2 \\
&+ \sum_{|k|\leq m_n} W_n(k) \left|\sum_{u=-\infty}^{-n-1} Z_{n,u}\,\mathrm{e}^{-i(\omega\Delta_n)_k u} \sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k}\right|^2 \Bigg\} =: 2(V_{11} + V_{12})
\end{aligned}$$

and

$$V_{11} = \sum_{u=-n}^{-1} Z_{n,u}^2 \sum_{|k|\le m_n} W_n(k) \left| \sum_{j=n+1}^{n-u} \mathrm{e}^{j\Delta_n \lambda_r - ij(\omega\Delta_n)_k} \right|^2$$

$$+ \sum_{-n\le u_1 \ne u_2 \le -1} Z_{n,u_1} Z_{n,u_2} \sum_{|k|\le m_n} W_n(k) \sum_{j_1=n+1}^{n-u_1} \sum_{j_2=n+1}^{n-u_2} \mathrm{e}^{j_1\Delta_n\lambda_r + j_2\Delta_n\overline{\lambda_r} - i(\omega\Delta_n)_k\cdot(u_1-u_2+j_1-j_2)}$$

$$=: V_{111} + V_{112}.$$

As above, we know that $n^{-2/\alpha}\sum_{u=-n}^{-1} Z_{n,u}^2$ converges in distribution as $n\to\infty$. Together with (5.12) this yields

$$\Delta_n^2\, n^{-\frac{2}{\alpha}}\, V_{111} \le n^{-\frac{2}{\alpha}} \sum_{u=-n}^{-1} Z_{n,u}^2 \cdot \underbrace{\sum_{|k|\le m_n} W_n(k)}_{\overset{(1.8b)}{=}1} \cdot \left(\Delta_n \sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n \Re(\lambda_r)} \right)^2 \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n\to\infty. \tag{5.13}$$

For any $\varepsilon > 0$ a conditional application of Bonami's inequality (cf. [21, Section 5.2]) yields a $C(\varepsilon) > 0$ such that

$$\begin{aligned}
&\mathbb{P}\left(\Delta_n^2\, n^{-\frac{2}{\alpha}} |V_{112}| > \varepsilon \right) \\
&\le \mathbb{E}\Bigg[\exp\Bigg\{ -C(\varepsilon)\,\varepsilon\,\Delta_n^{-2} n^{\frac{2}{\alpha}} \Bigg[\sum_{-n\le u_1\ne u_2\le -1} Z_{n,u_1}^2 Z_{n,u_2}^2 \Bigg(\sum_{|k|\le m_n} W_n(k) \\
&\qquad \sum_{j_1=n+1}^{n-u_1} \sum_{j_2=n+1}^{n-u_2} \Re\left(\mathrm{e}^{j_1\Delta_n\lambda_r + j_2\Delta_n\overline{\lambda_r} - i(\omega\Delta_n)_k\cdot(u_1-u_2+j_1-j_2)} \right) \Bigg)^2 \Bigg]^{-1/2} \Bigg\} \Bigg] \\
&\overset{(1.8b)}{\le} \mathbb{E}\Bigg[\exp\Bigg\{ -C(\varepsilon)\,\varepsilon\,\Delta_n^{-2} n^{\frac{2}{\alpha}} \Bigg[\sum_{-n\le u_1\ne u_2\le -1} Z_{n,u_1}^2 Z_{n,u_2}^2 \Bigg(\sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n\Re(\lambda_r)} \Bigg)^4 \Bigg]^{-1/2} \Bigg\} \Bigg] \\
&\le \mathbb{E}\Bigg[\exp\Bigg\{ -C(\varepsilon)\,\varepsilon \left(n^{-\frac{2}{\alpha}} \sum_{u=-n}^{-1} Z_{n,u}^2 \right)^{-1} \left(\Delta_n \sum_{j=n+1}^{\infty} \mathrm{e}^{j\Delta_n\Re(\lambda_r)} \right)^{-2} \Bigg\} \Bigg]
\end{aligned}$$

and the right-hand side converges to 0 as $n\to\infty$ by virtue of Eq. (5.13) and Lebesgue dominated convergence.

Hence, $\Delta_n^2 n^{-2/\alpha} V_{11} \xrightarrow{\mathbb{P}} 0$ as $n\to\infty$ is shown. Concerning V_{12} we proceed similarly. We write

$$V_{12} = \sum_{u=-\infty}^{-n-1} Z_{n,u}^2 \sum_{|k|\le m_n} W_n(k) \left| \sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k} \right|^2$$

$$+ \sum_{-\infty\le u_1\ne u_2\le -n-1} Z_{n,u_1} Z_{n,u_2} \sum_{|k|\le m_n} W_n(k) \sum_{j_1=1-u_1}^{n-u_1} \sum_{j_2=1-u_2}^{n-u_2} \mathrm{e}^{j_1\Delta_n\lambda_r + j_2\Delta_n\overline{\lambda_r} - i(\omega\Delta_n)_k\cdot(u_1-u_2+j_1-j_2)}$$

$$=: V_{121} + V_{122}$$

and observe that $V_{121} \le \sum_{u=-\infty}^{-n-1} Z_{n,u}^2 \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\Re(\lambda_r)} \right)^2$. We prove that, for any $\delta \ge 0$,

$$f_n(\delta) := \mathbb{E}\left[\exp\left\{ -\frac{\delta^2}{2}\Delta_n^2\, n^{-\frac{2}{\alpha}} \sum_{u=-\infty}^{-n-1} Z_{n,u}^2 \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\Re(\lambda_r)} \right)^2 \right\} \right] \to 1 \qquad \text{as } n\to\infty. \tag{5.14}$$

Note that this implies $\Delta_n^2 n^{-2/\alpha} V_{121} \stackrel{\mathbb{P}}{\to} 0$ as $n \to \infty$. Let $(N_u)_{u\in\mathbb{Z}}$ be i.i.d. $N(0,1)$-random variables with characteristic function $\mathbb{E}[\exp(i\theta N_u)] = \exp(-\theta^2/2)$ independent of $(Z_{n,u})_{u\in\mathbb{Z}}$ for any $n \in \mathbb{N}$. Then, we have for $\alpha \in (0,2)$

$$\begin{aligned} f_n(\delta) &= \mathbb{E}\left[\exp\left\{-\frac{\delta^2}{2}\Delta_n^2\, n^{-\frac{2}{\alpha}} \sum_{u=-\infty}^{-n-1} S_u^2 \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^2\right\}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\exp\left\{i\delta\Delta_n\, n^{-\frac{1}{\alpha}} \sum_{u=-\infty}^{-n-1} S_u N_u \sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right\}\middle|\, (S_u)_{u\in\mathbb{Z}}\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\exp\left\{i\delta\Delta_n\, n^{-\frac{1}{\alpha}} \sum_{u=-\infty}^{-n-1} S_u N_u \sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right\}\middle|\, (N_u)_{u\in\mathbb{Z}}\right]\right] \\ &= \mathbb{E}\left[\exp\left\{-\sigma_L^\alpha \delta^\alpha \frac{\Delta_n^\alpha}{n} \sum_{u=-\infty}^{-n-1} |N_u|^\alpha \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^\alpha\right\}\right] \end{aligned}$$

and $\mathbb{E}\big[\Delta_n^\alpha n^{-1} \sum_{u=-\infty}^{-n-1} |N_u|^\alpha \big(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\Re(\lambda_r)}\big)^\alpha\big] = \mathbb{E}[|N_1|^\alpha] \cdot \Delta_n^\alpha n^{-1} \sum_{u=-\infty}^{-n-1} \big(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\Re(\lambda_r)}\big)^\alpha \to 0$ as $n \to \infty$, since $\mathbb{E}[|N_1|^\alpha] < \infty$ and

$$\begin{aligned} \frac{\Delta_n^\alpha}{n} \sum_{u=-\infty}^{-n-1} \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n \Re(\lambda_r)}\right)^\alpha &= \frac{\Delta_n^\alpha}{n} \cdot \left(\frac{\mathrm{e}^{\Delta_n \Re(\lambda_r)}\left(1-\mathrm{e}^{n\Delta_n\Re(\lambda_r)}\right)}{1-\mathrm{e}^{\Delta_n\Re(\lambda_r)}}\right)^\alpha \cdot \frac{\mathrm{e}^{(n+1)\Delta_n\Re(\lambda_r)\alpha}}{1-\mathrm{e}^{\Delta_n\Re(\lambda_r)\alpha}} \\ &\stackrel{n\to\infty}{\sim} \left(-\frac{1}{\Re(\lambda_r)}\right)^\alpha \cdot \frac{\mathrm{e}^{n\Delta_n\Re(\lambda_r)\alpha}}{-n\Delta_n\Re(\lambda_r)\alpha} \stackrel{n\to\infty}{\to} 0 \end{aligned} \tag{5.15}$$

due to Assumptions 1 and 3. Lebesgue dominated convergence then obviously gives $f_n(\delta) \to 1$ for any $\delta \geq 0$, i.e. Eq. (5.14) is shown for $\alpha \in (0,2)$. If $\alpha = 2$, we first write as above

$$f_n(\delta) = \mathbb{E}\left[\exp\left\{i\delta\Delta_n\, n^{-\frac{1}{2}} \sum_{u=-\infty}^{-n-1} Z_{n,u} N_u \sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\Re(\lambda_r)}\right\}\right].$$

Then, using the independence of $(N_u)_{u\in\mathbb{Z}}$ and $(Z_{n,u})_{u\in\mathbb{Z}}$, we obtain

$$\mathbb{E}\left[\left(\frac{\Delta_n}{n^{1/2}} \sum_{u=-\infty}^{-n-1} Z_{n,u} N_u \sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\Re(\lambda_r)}\right)^2\right] = \frac{\Delta_n^2}{n} \sum_{u=-\infty}^{-n-1} \underbrace{\mathbb{E}\left[Z_{n,u}^2 N_u^2\right]}_{=\sigma_L^2} \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\Re(\lambda_r)}\right)^2 \stackrel{n\to\infty}{\to} 0,$$

where the latter can be shown as in the case $\alpha \in (0,2)$ above (cf. (5.15)). We can apply again the Dominated Convergence Theorem and deduce that $f_n(\delta) \to 1$ for any $\delta \geq 0$ also in the case $\alpha = 2$. Hence, (5.14) and $\Delta_n^2 n^{-2/\alpha} V_{121} \stackrel{\mathbb{P}}{\to} 0$ as $n \to \infty$ is shown.

Analogously to V_{112} above, we obtain for V_{122} with $\varepsilon > 0$

$$\begin{aligned} &\mathbb{P}\left(\Delta_n^2\, n^{-\frac{2}{\alpha}} |V_{122}| > \varepsilon\right) \\ &\leq \mathbb{E}\left[\exp\left\{-C(\varepsilon)\,\varepsilon\,\Delta_n^{-2}\, n^{\frac{2}{\alpha}} \left[\sum_{-\infty\leq u_1\neq u_2\leq -n-1} Z_{n,u_1}^2 Z_{n,u_2}^2 \left(\sum_{j_1=1-u_1}^{n-u_1} \mathrm{e}^{j\Delta_n\Re(\lambda_r)}\right)^2 \left(\sum_{j_2=1-u_2}^{n-u_2} \mathrm{e}^{j\Delta_n\Re(\lambda_r)}\right)^2\right]^{-1/2}\right\}\right] \\ &\leq \mathbb{E}\left[\exp\left\{-C(\varepsilon)\,\varepsilon\left(\Delta_n^2\, n^{-\frac{2}{\alpha}} \sum_{u=-\infty}^{-n-1} Z_{n,u}^2 \left(\sum_{j=1-u}^{n-u} \mathrm{e}^{j\Delta_n\Re(\lambda_r)}\right)^2\right)^{-1}\right\}\right] \to 0 \quad \text{as } n\to\infty \end{aligned}$$

due to (5.14) and, once more, Lebesgue dominated convergence.

Hence, also $\Delta_n^2 n^{-2/\alpha} V_{122} \xrightarrow{\mathbb{P}} 0$ and $\Delta_n^2 n^{-2/\alpha} V_{12} \xrightarrow{\mathbb{P}} 0$ holds as $n \to \infty$. Note at this point that Equation (5.11) has been shown.

Thus, it remains to prove that also

$$\Delta_n^2 \sum_{|k| \le m_n} W_n(k) A^{(2)}_{n,\Delta_n}((\omega\Delta_n)_k) \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty. \tag{5.16}$$

First,

$$\begin{aligned}
\sum_{|k| \le m_n} W_n(k) A^{(2)}_{n,\Delta_n}((\omega\Delta_n)_k) \le 2n^{-\frac{2}{\alpha}} \Bigg\{ & \sum_{|k| \le m_n} W_n(k) \left| \sum_{j=1}^{n} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k} \sum_{u=1-j}^{0} Z_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \\
& + \sum_{|k| \le m_n} W_n(k) \left| \sum_{j=1}^{n} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k} \sum_{u=n-j+1}^{n} Z_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \Bigg\} \\
=: 2n^{-\frac{2}{\alpha}} (\widetilde{V}_1 + \widetilde{V}_2) &
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{V}_1 &= \sum_{|k| \le m_n} W_n(k) \left| \sum_{u=1-n}^{0} Z_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \sum_{j=1-u}^{n} \mathrm{e}^{j\Delta_n\lambda_r - ij(\omega\Delta_n)_k} \right|^2 \\
&\overset{(1.8b)}{\le} \sum_{u=1-n}^{0} Z^2_{n,u} \left(\sum_{j=1-u}^{n} \mathrm{e}^{j\Delta_n \Re(\lambda_r)} \right)^2 \\
&\qquad + \sum_{1-n \le u_1 \ne u_2 \le 0} Z_{n,u_1} Z_{n,u_2} \sum_{|k| \le m_n} W_n(k) \sum_{j_1=1-u_1}^{n} \sum_{j_2=1-u_2}^{n} \mathrm{e}^{j_1\Delta_n\lambda_r + j_2\Delta_n\overline{\lambda_r} - i(\omega\Delta_n)_k(u_1-u_2+j_1-j_2)} \\
&=: \widetilde{V}_{11} + \widetilde{V}_{12}.
\end{aligned}$$

Now, $\widetilde{V}_{11}$ can be dealt with like V_{121} above and one observes that in order to show $\Delta_n^2 n^{-2/\alpha} \widetilde{V}_{11} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, it is sufficient to prove

$$\lim_{n\to\infty} \frac{\Delta_n^\alpha}{n} \sum_{u=1-n}^{0} \left(\sum_{j=1-u}^{n} \mathrm{e}^{j\Delta_n \Re(\lambda_r)} \right)^\alpha = 0. \tag{5.17}$$

This follows from [15, Lemma 2.2(iii)] by setting $p = 1$ (note that in this case $\Psi_j^{\Delta_n} = \mathrm{e}^{j\Delta_n\lambda_r}$). The proof of $\Delta_n^2 n^{-2/\alpha} \widetilde{V}_{12} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$ is then completely analog to the one of $\Delta_n^2 n^{-2/\alpha} V_{122} \xrightarrow{\mathbb{P}} 0$ above.

Finally,

$$\begin{aligned}
\widetilde{V}_2 &\le \sum_{u=1}^{n} Z^2_{n,u} \left(\sum_{j=n+1-u}^{n} \mathrm{e}^{j\Delta_n \Re(\lambda_r)} \right)^2 \\
&\qquad + \sum_{1 \le u_1 \ne u_2 \le n} Z_{n,u_1} Z_{n,u_2} \sum_{|k| \le m_n} W_n(k) \sum_{j_1=n+1-u_1}^{n} \sum_{j_2=n+1-u_2}^{n} \mathrm{e}^{j_1\Delta_n\lambda_r + j_2\Delta_n\overline{\lambda_r} - i(\omega\Delta_n)_k(u_1-u_2+j_1-j_2)} \\
&=: \widetilde{V}_{21} + \widetilde{V}_{22}
\end{aligned}$$

and in order to show $\Delta_n^2 n^{-2/\alpha} \widetilde{V}_{21} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, it is, as for $\widetilde{V}_{11}$, sufficient to prove

$$\lim_{n\to\infty} \frac{\Delta_n^\alpha}{n} \sum_{u=1}^{n} \left(\sum_{j=n+1-u}^{n} \mathrm{e}^{j\Delta_n \Re(\lambda_r)} \right)^\alpha = 0.$$

However, this is exactly Eq. (5.17). As for V_{122} and $\widetilde{V}_{12}$, one obtains that $\Delta_n^2\, n^{-2/\alpha}\, \widetilde{V}_{22} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, as well. This completes the proof of (5.16).

Equations (5.11) and (5.16) together yield the statement of the lemma. □

Proof of Proposition 3.3. In Lemma 5.1 it has been shown that, for any $\omega \in \mathbb{R}^*$,

$$\frac{1}{\Delta_n} \sum_{|k|\leq m_n} W_n(k) \frac{\left|J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}} \xrightarrow{\mathbb{P}} \frac{|c(i\omega)|^2}{\int_0^\infty g^2(s)\,\mathrm{d}s \cdot |a(i\omega)|^2} \quad \text{as } n \to \infty,$$

with $J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k) = c^T(i\omega \mathrm{I}_p - A)^{-1} e_p \left(\sum_{u=1}^n \Delta L(u\Delta_n)\,\mathrm{e}^{-i(\omega\Delta_n)_k u}\right)$. Since $\Delta_n\,(n\Delta_n)^{-2/\alpha} \sum_{u=1}^n Y^2_{u\Delta_n}$ converges in distribution as $n \to \infty$, respectively, to $\int_0^\infty g^2(s)\,\mathrm{d}s \cdot [L,L]_1$ with $([L,L]_t)_{t\geq 0}$ being the quadratic variation process of $(L_t)_{t\geq 0}$ if $\alpha \in (0,2)$ and to $\int_0^\infty g^2(s)\,\mathrm{d}s \cdot \sigma_L^2$ if $\alpha = 2$ (cf. [13, Theorem 5.5(a)]), a straightforward application of the Cauchy-Schwarz inequality shows that it is sufficient to prove

$$\Delta_n^{-\frac{2}{\alpha}} \sum_{|k|\leq m_n} W_n(k) \left| \frac{\Delta_n}{n^{1/\alpha}} J_{n,\Delta_n}((\omega\Delta_n)_k) - \frac{1}{n^{1/\alpha}} J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k) \right|^2 \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty. \tag{5.18}$$

However, (5.18) is a consequence of Lemma 5.3. □

Proof of Proposition 3.4. First, by virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\Delta_n \sum_{|k|\leq m_n} W_n(k) \frac{|R_{n,\Delta_n}((\omega\Delta_n)_k)|}{\sum_{u=1}^n Y^2_{u\Delta_n}} \\ &\leq 2 \left(2\Delta_n \sum_{|k|\leq m_n} W_n(k) \frac{\left|J_{n,\Delta_n}((\omega\Delta_n)_k) - \frac{1}{\Delta_n} J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2 + \left|\frac{1}{\Delta_n} J^{(1)}_{n,\Delta_n}((\omega\Delta_n)_k)\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}} \right)^{1/2} \\ &\quad \times \left(\Delta_n \sum_{|k|\leq m_n} W_n(k) \frac{\left|c^T K_{n,\Delta_n}((\omega\Delta_n)_k)\, e_p\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}} \right)^{1/2} + \Delta_n \sum_{|k|\leq m_n} W_n(k) \frac{\left|c^T K_{n,\Delta_n}((\omega\Delta_n)_k)\, e_p\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}}, \end{aligned}$$

where $K_{n,\Delta_n}(\,\cdot\,)$ and $J_{n,\Delta_n}(\,\cdot\,)$ are as in Eq. (2.5) and $J^{(1)}_{n,\Delta_n}(\,\cdot\,)$ has been defined in Lemma 5.1.

Since $\Delta_n\,(n\Delta_n)^{-2/\alpha} \sum_{u=1}^n Y^2_{u\Delta_n}$ converges in distribution, respectively, to $\int_0^\infty g^2(s)\,\mathrm{d}s \cdot [L,L]_1$ as $n \to \infty$ if $\alpha \in (0,2)$ with $([L,L]_t)_{t\geq 0}$ being the quadratic variation process of $(L_t)_{t\geq 0}$ and g the kernel function in (2.2b) and to $\int_0^\infty g^2(s)\,\mathrm{d}s \cdot \sigma_L^2$ if $\alpha = 2$ (cf. [13, Theorem 5.5(a)]), we can combine Lemma 5.3 and (5.4) in order to deduce that

$$\begin{aligned} &\Delta_n \sum_{|k|\leq m_n} W_n(k) \frac{|R_{n,\Delta_n}((\omega\Delta_n)_k)|}{\sum_{u=1}^n Y^2_{u\Delta_n}} \\ &\leq O_P(1) \cdot \left(\Delta_n \sum_{|k|\leq m_n} W_n(k) \frac{\left|c^T K_{n,\Delta_n}((\omega\Delta_n)_k)\, e_p\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}} \right)^{1/2} + \Delta_n \sum_{|k|\leq m_n} W_n(k) \frac{\left|c^T K_{n,\Delta_n}((\omega\Delta_n)_k)\, e_p\right|^2}{\sum_{u=1}^n Y^2_{u\Delta_n}} \end{aligned}$$

as $n \to \infty$. Therefore, it is sufficient to prove the following:

$$\Delta_n^{2-\frac{2}{\alpha}}\, n^{-\frac{2}{\alpha}} \sum_{|k|\leq m_n} W_n(k) \left|c^T K_{n,\Delta_n}((\omega\Delta_n)_k)\, e_p\right|^2 \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty. \tag{5.19}$$

To this end, we define

$$\widehat{U}_{n,j}(\omega) := \sum_{u=1-j}^{n-j} \widehat{\xi}_{n,u}\, \mathrm{e}^{-i\omega u} - \sum_{u=1}^{n} \widehat{\xi}_{n,u}\, \mathrm{e}^{-i\omega u} \qquad \text{and}$$

$$\widehat{K}_{n,\Delta_n}(\omega) := \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n \mathrm{diag}(\lambda_1,\ldots,\lambda_p) - i\omega \mathrm{I}_p)}\, \widehat{U}_{n,j}(\omega), \quad -\pi \le \omega \le \pi,$$

where $\widehat{\xi}_{n,u}$ is given by (5.8). Then

$$K_{n,\Delta_n}(\omega) = D \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n \mathrm{diag}(\lambda_1,\ldots,\lambda_p) - i\omega \mathrm{I}_p)}\, \widehat{U}_{n,j}(\omega)\, D^{-1} = D\, \widehat{K}_{n,\Delta_n}(\omega)\, D^{-1},$$

which implies

$$\begin{aligned}
\Delta_n^{2-\frac{2}{\alpha}} n^{-\frac{2}{\alpha}} \sum_{|k|\le m_n} W_n(k) \left| c^T K_{n,\Delta_n}((\omega\Delta_n)_k)\, e_p \right|^2 &\le \text{const.} \cdot \Delta_n^{2-\frac{2}{\alpha}} n^{-\frac{2}{\alpha}} \sum_{|k|\le m_n} W_n(k) \left\| \mathrm{vec}\big(\widehat{K}_{n,\Delta_n}((\omega\Delta_n)_k)\big) \right\|^2 \\
&= \text{const.} \cdot \sum_{r,s=1}^{p} \Delta_n^{2-\frac{2}{\alpha}} n^{-\frac{2}{\alpha}} \sum_{|k|\le m_n} W_n(k) \left| \widehat{K}^{(r,s)}_{n,\Delta_n}((\omega\Delta_n)_k) \right|^2 \\
&= \text{const.} \cdot \sum_{r=1}^{p} \Delta_n^{2-\frac{2}{\alpha}} n^{-\frac{2}{\alpha}} \sum_{|k|\le m_n} W_n(k) \left| \widehat{K}^{(r,r)}_{n,\Delta_n}((\omega\Delta_n)_k) \right|^2, \qquad (5.20)
\end{aligned}$$

since $\widehat{K}_{n,\Delta_n}(\,\cdot\,) = \Big(\widehat{K}^{(r,s)}_{n,\Delta_n}(\,\cdot\,)\Big)_{r,s\in\{1,\ldots,p\}}$ is diagonal.

Now, for any $r \in \{1,\ldots,p\}$,

$$\begin{aligned}
\widehat{K}^{(r,r)}_{n,\Delta_n}((\omega\Delta_n)_k) &= e_r^T \widehat{K}_{n,\Delta_n}((\omega\Delta_n)_k)\, e_r \\
&= \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n} \right] \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \mathrm{e}^{(u\Delta_n - s)\lambda_r} \mathrm{d}L_s \\
&= \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n} \right] \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \Re\left(\mathrm{e}^{(u\Delta_n - s)\lambda_r} \right) \mathrm{d}L_s \\
&\quad + i \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n} \right] \mathrm{e}^{-i(\omega\Delta_n)_k u} \int_{(u-1)\Delta_n}^{u\Delta_n} \Im\left(\mathrm{e}^{(u\Delta_n - s)\lambda_r} \right) \mathrm{d}L_s.
\end{aligned}$$

We define $(Z_{n,u})_{u\in\mathbb{Z}} := \Delta_n^{-1/\alpha}(\Delta L(u\Delta_n))_{u\in\mathbb{Z}}$ such that, if $\alpha \in (0,2)$, $(Z_{n,u})_{u\in\mathbb{Z}}$ are i.i.d. symmetric α-stable random variables with scale parameter σ_L, and in the case $\alpha = 2$ it is an i.i.d. symmetric sequence satisfying $\mathbb{E}[Z_{n,u}^2] = \sigma_L^2$ for any $n \in \mathbb{N}$, $u \in \mathbb{Z}$.

Note that

$$\left(\int_{(u-1)\Delta_n}^{u\Delta_n} \Re\left(\mathrm{e}^{(u\Delta_n - s)\lambda_r} \right) \mathrm{d}L_s \right)_{u\in\mathbb{Z}} \overset{\mathscr{D}}{=} \left(\int_0^{\Delta_n} \left| \Re\big(\mathrm{e}^{s\lambda_r}\big) \right|^{\alpha} \mathrm{d}s \right)^{\frac{1}{\alpha}} \cdot (Z_{n,u})_{u\in\mathbb{Z}} =: C_n^{(r)} \cdot (Z_{n,u})_{u\in\mathbb{Z}} \qquad \text{and likewise}$$

$$\left(\int_{(u-1)\Delta_n}^{u\Delta_n} \Im\left(\mathrm{e}^{(u\Delta_n - s)\lambda_r} \right) \mathrm{d}L_s \right)_{u\in\mathbb{Z}} \overset{\mathscr{D}}{=} \left(\int_0^{\Delta_n} \left| \Im\big(\mathrm{e}^{s\lambda_r}\big) \right|^{\alpha} \mathrm{d}s \right)^{\frac{1}{\alpha}} \cdot (Z_{n,u})_{u\in\mathbb{Z}} =: \widetilde{C}_n^{(r)} \cdot (Z_{n,u})_{u\in\mathbb{Z}}.$$

Since $C_n^{(r)} \sim \Delta_n^{1/\alpha}$ and $\Delta_n^{-1/\alpha} \widetilde{C}_n^{(r)} \to 0$ as $n \to \infty$ for any $r \in \{1,\ldots,p\}$ (cf. [15, Lemma 2.1(ii) and its proof]) and since, for any $r \in \{1,\ldots,p\}$,

$$\Delta_n^2\, n^{-\frac{2}{\alpha}} \sum_{|k|\le m_n} W_n(k) \left| \sum_{j=0}^{\infty} \mathrm{e}^{j(\Delta_n\lambda_r - i(\omega\Delta_n)_k)} \left[\sum_{u=1-j}^{n-j} - \sum_{u=1}^{n} \right] Z_{n,u}\, \mathrm{e}^{-i(\omega\Delta_n)_k u} \right|^2 \overset{\mathbb{P}}{\to} 0$$

as $n \to \infty$ (see Lemma 5.4), we obtain that the right-hand side of Eq. (5.20) converges to 0 in probability as $n \to \infty$ which in turn yields (5.19) and hence, completes the proof of the proposition. □

Proof of Proposition 3.7. Note first that we can understand the self-normalized periodogram as a special case of the smoothed one by choosing the weight functions W_n as $W_n(0) = 1$ and $W_n(k) = 0$ for any $k \neq 0$. These weights do not satisfy (1.8c), but obviously (1.8a) and (1.8b). With that "degenerated" choice of weight functions and Lemma 5.3, we deduce immediately that it is sufficient to prove the following:

$$(n\Delta_n)^{-2/\alpha} \left| J^{(1)}_{n,\Delta_n}(\omega\Delta_n) \right|^2 \xrightarrow{\mathscr{D}} \frac{|c(i\omega)|^2}{|a(i\omega)|^2} \cdot \left| \int_{[0,1)} \mathrm{e}^{2\pi i x}\, \mathrm{d}L^*_x \right|^2 \quad \text{as } n \to \infty,$$

for any $\omega \in \mathbb{R}^*$, where $J^{(1)}_{n,\Delta_n}(\,\cdot\,)$ has been defined in Lemma 5.1. Now, it is clearly enough to show that

$$(n\Delta_n)^{-2/\alpha} \left| \sum_{u=1}^{n} \Delta L(u\Delta_n)\, \mathrm{e}^{-i\omega\Delta_n u} \right|^2 \xrightarrow{\mathscr{D}} \left| \int_{[0,1)} \mathrm{e}^{2\pi i x}\, \mathrm{d}L^*_x \right|^2 \quad \text{as } n \to \infty. \tag{5.21}$$

Let $(Z_{n,u})_{u\in\mathbb{Z}} := \Delta_n^{-1/\alpha}(\Delta L(u\Delta_n))_{u\in\mathbb{Z}}$ for $n \in \mathbb{N}$. Then (5.21) follows, by virtue of the Continuous Mapping Theorem (see, for instance, [20, Theorem 13.25]), from

$$n^{-1/\alpha} \left(\sum_{u=1}^{n} Z_{n,u} \cos(\omega\Delta_n u),\, \sum_{u=1}^{n} Z_{n,u} \sin(\omega\Delta_n u) \right) \xrightarrow{\mathscr{D}} \left(\int_0^1 \cos(2\pi x)\, \mathrm{d}L^*_x,\, \int_0^1 \sin(2\pi x)\, \mathrm{d}L^*_x \right) \quad \text{as } n \to \infty,$$

which, in turn, is equivalent to

$$n^{-1/\alpha} \sum_{u=1}^{n} \underbrace{Z_{n,u}\, (b_1 \cos(\omega\Delta_n u) + b_2 \sin(\omega\Delta_n u))}_{=:X_{n,u}} \xrightarrow{\mathscr{D}} \int_0^1 [b_1 \cos(2\pi x) + b_2 \sin(2\pi x)]\, \mathrm{d}L^*_x \quad \text{as } n \to \infty, \tag{5.22}$$

for any $(b_1, b_2)^T \in \mathbb{R}^2$.

First, we prove (5.22) for $\alpha \in (0,2)$. Since $(Z_{n,u})_{u\in\mathbb{Z}}$ are an i.i.d. sequence of symmetric α-stable random variables with scale paramater σ_L, the random variable $n^{-1/\alpha}\sum_{u=1}^{n} X_{n,u}$ is again symmetric α-stable with scale parameter $\sigma_{n,L}$ where

$$\sigma^{\alpha}_{n,L} = \frac{\sigma^{\alpha}_L}{n} \sum_{u=1}^{n} |b_1 \cos(\omega\Delta_n u) + b_2 \sin(\omega\Delta_n u)|^{\alpha}.$$

Moreover, writing $\omega = 2\pi\eta$, we have

$$\sigma^{\alpha}_{n,L} = \frac{\sigma^{\alpha}_L}{n} \sum_{u=1}^{n} \left| b_1 \cos(2\pi\{\eta\Delta_n u\}) + b_2 \sin(2\pi\{\eta\Delta_n u\}) \right|^{\alpha} \overset{n\to\infty}{\to} \sigma^{\alpha}_L \cdot \int_0^1 |b_1 \cos(2\pi x) + b_2 \sin(2\pi x)|^{\alpha}\, \mathrm{d}x$$

where the convergence can be shown as in the proof of [15, Proposition 2.6, (4.11)]. This results in (5.22) for $\alpha \in (0,2)$.

For $\alpha = 2$ we prove (5.22) with the Lindeberg-Feller Theorem (see, e.g., [12, p. 114]). Obviously, for each n, the random variables $X_{n,u}$, $1 \le u \le n$, are independent with $\mathbb{E}[X_{n,u}] = 0$ since $Z_{n,u}$ are supposed to be symmetric. Moreover, writing again $\omega = 2\pi\eta$, we have

$$\begin{aligned}\frac{1}{n}\sum_{u=1}^{n} \mathrm{Var}(X_{n,u}) &= \frac{\sigma^2_L}{n} \sum_{u=1}^{n} \big(b_1 \cos(2\pi\{\eta\Delta_n u\}) + b_2 \sin(2\pi\{\eta\Delta_n u\})\big)^2 \\ &\overset{n\to\infty}{\to} \sigma^2_L \cdot \int_0^1 (b_1 \cos(2\pi x) + b_2 \sin(2\pi x))^2\, \mathrm{d}x = \sigma^2_L \cdot \left(\frac{b_1^2}{2} + \frac{b_2^2}{2} \right),\end{aligned}$$

where the convergence can be shown again as in the proof of [15, Proposition 2.6, (4.11)]. Since, for any $\varepsilon > 0$,

$$\frac{1}{n}\sum_{u=1}^{n} \mathbb{E}\left[X^2_{n,u}\, \mathbb{1}_{\{|X_{n,u}|>\varepsilon\sqrt{n}\}} \right] \le (|b_1| + |b_2|)^2 \cdot \mathbb{E}\left[Z^2_{n,1}\, \mathbb{1}_{\left\{|Z_{n,1}|>\frac{\varepsilon\sqrt{n}}{|b_1|+|b_2|}\right\}} \right] \overset{n\to\infty}{\to} 0,$$

we can apply the Lindeberg-Feller Theorem and deduce

$$
\begin{aligned}
n^{-1/2} \sum_{u=1}^{n} Z_{n,u} \left(b_1 \cos(\omega\Delta_n u) + b_2 \sin(\omega\Delta_n u)\right) &\xrightarrow{\mathscr{D}} \sqrt{\sigma_L^2 \cdot \left(\frac{b_1^2}{2} + \frac{b_2^2}{2}\right)} \cdot N(0,1) \\
&\stackrel{\mathscr{D}}{=} \sigma_L \left(\frac{b_1}{\sqrt{2}} N_1 + \frac{b_2}{\sqrt{2}} N_2\right) \\
&\stackrel{\mathscr{D}}{=} \int_0^1 \left[b_1 \cos(2\pi x) + b_2 \sin(2\pi x)\right] \mathrm{d}L_x^*,
\end{aligned} \tag{5.23}
$$

where N_1, N_2 are i.i.d. $N(0,1)$ random variables. This shows (5.22) and completes the proof. □

Proof of Theorem 3.5. Since we can understand the (self-)normalized periodogram as a special case of the smoothed one by choosing the weight functions W_n as $W_n(0) = 1$ and $W_n(k) = 0$ for any $k \neq 0$, which do not satisfy (1.8c), but obviously (1.8a) and (1.8b), we can use the same partition as in Eq. (2.7) and apply Proposition 3.4 together with Proposition 3.7 to obtain the statement. □

Proof of Corollary 3.6. Follows from Theorem 3.5, (5.23) and $n^{-1}\sum_{u=1}^{n} Y_{u\Delta_n}^2 \xrightarrow{\mathbb{P}} \int_0^\infty g^2(s)\,\mathrm{d}s \cdot \sigma_L^2$ if $\alpha = 2$ (cf. [13, Theorem 5.5(a)]). □

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