

SPECTRAL MEASURE OF HEAVY TAILED BAND AND COVARIANCE RANDOM MATRICES

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ABSTRACT. We study the asymptotic behavior of the appropriately scaled and possibly perturbed spectral measure $\hat{\mu}$ of large random real symmetric matrices with heavy tailed entries. Specifically, consider the $N \times N$ symmetric matrix \mathbf{Y}_N^σ whose (i, j) entry is $\sigma(\frac{i}{N}, \frac{j}{N})x_{ij}$ where $(x_{ij}, 1 \leq i \leq j < \infty)$ is an infinite array of i.i.d real variables with common distribution in the domain of attraction of an α -stable law, $\alpha \in (0, 2)$, and σ is a deterministic function. For random diagonal \mathbf{D}_N independent of \mathbf{Y}_N^σ and with appropriate rescaling a_N , we prove that $\hat{\mu}_{a_N^{-1}\mathbf{Y}_N^\sigma + \mathbf{D}_N}$ converges in mean towards a limiting probability measure which we characterize. As a special case, we derive and analyze the almost sure limiting spectral density for empirical covariance matrices with heavy tailed entries.

1. INTRODUCTION

We study the asymptotic behavior of the spectral measure of large band random real symmetric matrices with independent (apart from symmetry) heavy tailed entries. Specifically, with $(x_{ij}, 1 \leq i \leq j < \infty)$ an infinite array of i.i.d real variables, let \mathbf{X}_N denote the $N \times N$ symmetric matrix given by

$$X_N(i, j) = x_{ij} \text{ if } i \leq j, x_{ji} \text{ otherwise.}$$

Fixing $\sigma : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, a (uniformly over $1/N$ -lattice grids) square integrable measurable function such that $\sigma(x, y) = \sigma(y, x)$, we denote by \mathbf{Y}_N^σ the $N \times N$ symmetric matrix with entries $Y_N^\sigma(i, j) = \sigma(\frac{i}{N}, \frac{j}{N})x_{ij}$. These matrices are sometime called “band matrices” after the choice of $\sigma(x, y) = \mathbf{1}_{|x-y| \leq b}$ for some $0 < b < 1$ (c.f. Remark 1.9). Another important special case, $\sigma(x, y) = \mathbf{1}_{(x-1/2)(1/2-y) > 0}$ yields the spectral measure of empirical covariance matrices $\mathbf{X}_N \mathbf{X}_N^t$ (as shown in Section 5.1).

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For i.i.d. entries $(x_{ij}, 1 \leq i \leq j \leq N)$ of finite second moment, it was proved by Berezin that the spectral measure of $\mathbf{A}_N^\sigma := N^{-1/2} \mathbf{Y}_N^\sigma$ converges almost surely weakly (see a rigorous proof in [7]). More precisely, for any $z \in \mathbb{C} \setminus \mathbb{R}$ the matrices $\mathbf{G}_N(z) := (z\mathbf{I}_N - \mathbf{A}_N^\sigma)^{-1}$ are such that for any bounded continuous function ϕ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \phi\left(\frac{i}{N}\right) G_N(z)_{ii} = \int_0^1 \phi(u) K_u^\sigma(z) du \quad \text{a.s.}$$

with $K_x^\sigma(z)$ the unique solution of $K_x^\sigma(z) = (z - \int_0^1 |\sigma(x, v)|^2 K_v^\sigma(z) dv)^{-1}$ such that $z \mapsto \int_0^1 \phi(u) K_u^\sigma(z) du$ is analytic in $\mathbb{C} \setminus \mathbb{R}$. In particular, taking constant $\phi(\cdot)$ we have the almost sure convergence of the spectral measure of \mathbf{A}_N^σ to the probability measure μ_2^σ whose Cauchy-Stieltjes transform is

$$(1.1) \quad G_2^\sigma(z) = \int \frac{1}{z - \lambda} d\mu_2^\sigma(\lambda) = \int_0^1 K_v^\sigma(z) dv.$$

We consider here the case of heavy tailed entries, where the common distribution of the absolute values of the x_{ij} 's is in the domain of attraction of an α -stable law, for $\alpha \in]0, 2[$. That is, there exists a slowly varying function $L(\cdot)$ such that for any $u > 0$,

$$(1.2) \quad \mathbb{P}(|x_{ij}| \geq u) = L(u)u^{-\alpha}.$$

The normalizing constants

$$(1.3) \quad a_N := \inf\{u : \mathbb{P}[|x_{ij}| \geq u] \leq \frac{1}{N}\},$$

are then such that $a_N = L_0(N)N^{1/\alpha}$ for some (other) slowly varying function $L_0(\cdot)$.

Hereafter, let \mathbf{A}_N^σ denote the normalized matrix $\mathbf{A}_N^\sigma := a_N^{-1} \mathbf{Y}_N^\sigma$ having eigenvalues $(\lambda_1, \dots, \lambda_N)$ and the corresponding spectral measure $\hat{\mu}_{\mathbf{A}_N^\sigma} := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$ (and when the choice of $\sigma(\cdot)$ is clear we also use the notations \mathbf{Y}_N and \mathbf{A}_N for \mathbf{Y}_N^σ and \mathbf{A}_N^σ , respectively). Predictions about the limiting spectral measure in case $\sigma(\cdot, \cdot) \equiv 1$ (the heavy tail analog of Wigner's theorem) have been made in [2] and rigorously verified in [1] (c.f. [1, Section 8]). We follow here the approach of [1], which consists of proving the convergence of the resolvent, i.e. of the mean of the Cauchy-Stieltjes transform of the spectral measure, outside of the real line, by proving tightness and characterizing uniquely the possible limit points. In the latter task, for each $\alpha \in (0, 2)$ the limiting spectral measure of \mathbf{A}_N^σ is characterized in terms of the entire functions

$$(1.4) \quad g_\alpha(y) := \int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \exp\{-t^{\frac{\alpha}{2}} y\} dt,$$

$$(1.5) \quad h_\alpha(y) := \int_0^\infty e^{-t} \exp\{-t^{\frac{\alpha}{2}} y\} dt = 1 - \frac{\alpha}{2} y g_\alpha(y).$$

We define for any $\alpha \in (0, 2)$ the usual branch of the power function $x \mapsto x^\alpha$, which is the analytic function on $\mathbb{C} \setminus \mathbb{R}^-$ such that $(i)^\alpha = e^{i\frac{\pi\alpha}{2}}$. This amounts to choosing $x^\alpha = r^\alpha e^{i\alpha\theta}$ when $x = re^{i\theta}$ with $\theta \in]-\pi, \pi[$. We also adopt throughout the notation $x^{-\alpha}$ for $(x^{-1})^\alpha$. With these notations in place, recall [1, Theorem 1.4] that in case $\sigma(\cdot, \cdot) \equiv 1$, the limiting spectral measure μ_α for Wigner matrices with entries in the domain of attraction of an α -stable law has for $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, the

Cauchy-Stieltjes transform

$$(1.6) \quad G_\alpha(z) := \int \frac{1}{z-x} d\mu_\alpha(x) = \frac{1}{z} h_\alpha(Y(z)),$$

where $Y(z)$ is the unique analytic on \mathbb{C}^+ solution of

$$(1.7) \quad z^\alpha Y(z) = C_\alpha g_\alpha(Y(z))$$

tending to zero at infinity, and $C_\alpha := i^\alpha \Gamma(1 - \frac{\alpha}{2}) / \Gamma(\frac{\alpha}{2})$. In [1, Theorem 1.6] it is further shown that μ_α has a smooth symmetric density ρ_α outside a compact set of capacity zero, and that $t^{\alpha+1} \rho_\alpha(t) \rightarrow \alpha/2$ as $t \rightarrow \infty$.

In addition to considering the more general case of band matrices, we devote some effort to the analysis of the limiting Cauchy-Stieltjes transform as $\Im(z) \rightarrow 0$ and its consequences on existence and regularity of the limiting density. For example, as a by product of our analysis we prove the following about μ_α of [1], showing in particular that it has a uniformly bounded density.

Proposition 1.1. *The unique analytic on \mathbb{C}^+ solution $Y(z)$ of (1.7) tending to zero at infinity takes values in the set $\mathcal{K}_\alpha := \{Re^{i\theta} : |\theta| \leq \frac{\alpha\pi}{2}, R \geq 0\}$ on which $g_\alpha(\cdot)$ is uniformly bounded. Its continuous extension to $\mathbb{R} \setminus \{0\}$ is analytic except possibly at the finite set $\mathcal{D}_\alpha = \{0, \pm t : t^\alpha = C_\alpha g'_\alpha(y) > 0, y \in \mathcal{K}_\alpha, g_\alpha(y) = yg'_\alpha(y)\}$. Further, the symmetric uniformly bounded density of μ_α is*

$$(1.8) \quad \rho_\alpha(t) = -\frac{1}{\pi t} \Im(h_\alpha(Y(t))) = \frac{\alpha|t|^{\alpha-1}}{2|C_\alpha|^\pi} \Im(i^{-\alpha} Y(|t|^2)),$$

continuous at $t \neq 0$, real-analytic outside \mathcal{D}_α and non-vanishing on any open interval.

Remark 1.2. *It is noted in [1, Remark 1.5] that $\alpha \mapsto \mu_\alpha$ is continuous on $(0, 2)$ with respect to weak convergence of probability measures. We further show in Lemma 5.2 that as $\alpha \rightarrow 2$ the measures μ_α converge to the semi-circle law μ_2 .*

Let \mathcal{C}_\star denote the set of piecewise constant functions $\sigma(x, y)$ such that for some finite q , some $0 = b_0 < b_1 < \dots < b_q = 1$ and a $q \times q$ symmetric matrix of entries $\{\sigma_{rs}, 1 \leq r, s \leq q\}$,

$$(1.9) \quad \sigma(x, y) = \sigma_{rs} \quad \text{for all } (x, y) \in (b_{r-1}, b_r] \times (b_{s-1}, b_s].$$

Our next result provides the weak convergence of the spectral measures for \mathbf{A}_N^σ and characterizes the Cauchy-Stieltjes transform of their limit, in case $\sigma \in \mathcal{C}_\star$. Even for $\sigma(\cdot, \cdot) \equiv 1$ it goes beyond the results of [1] by strengthening the weak convergence of the expected spectral measures $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N}]$ to the weak convergence of $\hat{\mu}_{\mathbf{A}_N}$ holding with probability one. A special interesting case of σ is when $q = 2$ and $\sigma_{rs} = \mathbf{1}_{|r-s|=1}$, out of which we get the spectral measure of the empirical covariance matrices $a_N^{-2} \mathbf{X}_N \mathbf{X}_N^t$ (c.f. Theorem 1.10 and its proof in Section 5).

Theorem 1.3. *Fixing $\sigma \in \mathcal{C}_\star$, let $\Delta_r = b_r - b_{r-1}$ for $r = 1, \dots, q$. With probability one, the sequence $\hat{\mu}_{\mathbf{A}_N^\sigma}$ converges weakly towards the non-random, symmetric probability measure μ^σ . The limiting measure has a continuous density ρ^σ on $\mathbb{R} \setminus \{0\}$ which is bounded off zero, and its Cauchy-Stieltjes transform is, for any $z \in \mathbb{C}^+$,*

$$(1.10) \quad G_{\alpha, \sigma}(z) := \int \frac{1}{z-x} d\mu^\sigma(x) = \frac{1}{z} \sum_{s=1}^q \Delta_s h_\alpha(Y_s(z)),$$

where $\underline{Y}(z) \equiv (Y_r(z), 1 \leq r \leq q)$ is the unique solution of

$$(1.11) \quad z^\alpha Y_r(z) = C_\alpha \sum_{s=1}^q |\sigma_{rs}|^\alpha \Delta_s g_\alpha(Y_s(z)),$$

composed of functions that are analytic on $z \in \mathbb{C}^+$ and tend to zero as $|z| \rightarrow \infty$. Moreover, $z^\alpha \underline{Y}(z)$ is uniformly bounded on \mathbb{C}^+ , both $G_{\alpha, \sigma}(z)$ and $\underline{Y}(z) \in (\mathcal{K}_\alpha)^q$ have continuous, algebraic extensions to $\mathbb{R} \setminus \{0\}$, and for some $R = R(\sigma)$ finite the mapping $\underline{Y}(z)$ extends analytically through the subset (R, ∞) where $\rho^\sigma(t) = -\frac{1}{\pi t} \sum_{s=1}^q \Delta_s \Im(h_\alpha(Y_s(t)))$ is real-analytic. Finally, the map $z \mapsto \underline{Y}(z)$ is injective whenever $\sigma \neq 0$.

Remark 1.4. The measure μ^σ may have an atom at zero when $q > 1$. Indeed, Theorem 1.10 provides one such example in case $q = 2$.

Remark 1.5. While we do not pursue it here, similarly to [1, Section 9], one can apply the moment method developed by Zakharevich [9], to characterize μ^σ as the weak limit $B \rightarrow \infty$ of the limiting spectral measures for appropriately truncated matrices $\mathbf{A}_N^{\sigma, B}$. As done in Lemma 5.2 for $\sigma \equiv 1$, we expect this to yield the continuity of μ^σ with respect to $\alpha \rightarrow 2$, for each fixed $\sigma \in \mathcal{C}_*$, i.e. to connect the limiting measures of Theorem 1.3 to μ_σ^2 of (1.1).

Let $L_\star^2([0, 1]^2)$ denote the space of equivalence classes with respect to the semi-norm

$$\|f\|_\star := \limsup_{n \rightarrow \infty} \|f(n^{-1}\lceil nx \rceil, n^{-1}\lceil ny \rceil)\|_2,$$

on the space of functions on $[0, 1]^2$ for which $\|\cdot\|_\star$ is finite. For each measurable $f : [0, 1]^2 \mapsto \mathbb{R}$ let $\|f\| := \|\int_0^1 |f(x, v)| dv\|_\infty$ denote the associated operator norm, where $\|\cdot\|_\infty$ denotes hereafter the usual (essential-sup) norm of $L^\infty((0, 1])$. We consider the subset \mathcal{F}_α of those symmetric measurable functions $\sigma \in L_\star^2([0, 1]^2)$ with $\|\sigma\|^\alpha$ finite which are each the L_\star^2 -limit of some sequence $\sigma_p \in \mathcal{C}_*$ such that

$$(1.12) \quad \lim_{p \rightarrow \infty} \|\sigma_p\|^\alpha - \|\sigma\|^\alpha = 0.$$

In fact, to verify that $\sigma \in \mathcal{F}_\alpha$ it suffices to check that $\|\sigma\|^\alpha$ is finite and find L_\star^2 -approximation of $\sigma(\cdot, \cdot)$ by bounded continuous symmetric functions $\sigma_p(\cdot, \cdot)$ for which (1.12) holds. Obviously \mathcal{F}_α contains all bounded continuous symmetric functions on $[0, 1]^2$ (but for example $\sigma(x, y) = 1/\sqrt{x+y} \in L_\star^2([0, 1]^2)$ is not in \mathcal{F}_α).

Remark 1.6. Things are a bit simpler if in the definition of the matrix \mathbf{Y}_N^σ one replaces the sample $\sigma(\frac{i}{N}, \frac{j}{N})$ by the average of $\sigma(\cdot, \cdot)$ with respect to Lebesgue measure on $(\frac{i-1}{N}, \frac{i}{N}] \times (\frac{j-1}{N}, \frac{j}{N}]$, for then we can replace throughout this paper the semi-norm $\|\cdot\|_\star$ and the space $L_\star^2([0, 1]^2)$ by the usual L^2 -norm and space.

We further say that $\sigma \in \mathcal{F}_\alpha$ is equivalent to $\tilde{\sigma} \in \mathcal{C}_*$ if for the relevant finite partition $0 = b_0 < b_1 < \dots < b_q = 1$ we have for any $1 \leq r, s \leq q$ that

$$\int_{b_{s-1}}^{b_s} |\sigma(x, v)|^\alpha dv = |\tilde{\sigma}_{rs}|^\alpha \quad \text{for all } x \in (b_{r-1}, b_r].$$

Extending Theorem 1.3 we next characterize the Cauchy-Stieltjes transform of μ^σ for any $\sigma \in \mathcal{F}_\alpha$.

Theorem 1.7. *Given $\sigma \in \mathcal{F}_\alpha$, the sequence $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\sigma}]$ converges weakly towards the symmetric probability measure μ^σ such that for some $R = R(\sigma)$ finite,*

$$(1.13) \quad \int \frac{1}{z-x} d\mu^\sigma(x) = \frac{1}{z} \int_0^1 h_\alpha(Y_v^\sigma(z)) dv$$

and Y^σ is the unique analytic mapping $Y^\sigma : \mathbb{C}^+ \mapsto L^\infty((0,1]; \mathcal{K}_\alpha)$ such that if $|z| \geq R$ then for almost every $x \in (0,1]$

$$(1.14) \quad z^\alpha Y_x^\sigma(z) = C_\alpha \int_0^1 |\sigma(x,v)|^\alpha g_\alpha(Y_v^\sigma(z)) dv.$$

The measure μ^σ has a density ρ^σ on $\mathbb{R} \setminus \{0\}$ which is bounded off zero and such that $t^{\alpha+1} \rho^\sigma(t) \rightarrow \frac{\alpha}{2} \int |\sigma(x,v)|^\alpha dx dv$ as $t \rightarrow \infty$.

Further, if $\sigma \in \mathcal{F}_\alpha$ is equivalent to $\tilde{\sigma} \in \mathcal{C}_*$ then $\mu^\sigma = \mu^{\tilde{\sigma}}$.

Remark 1.8. *A similar invariance applies in case of entries with bounded variance, where the kernel $K_x^\sigma(z)$ that characterizes the limit law in (1.1) is the same across each equivalence class of \mathcal{F}_2 . Also note that for $\alpha = 2$ we have $C_2 = -1$ and $g_2(y) = h_2(y) = 1/(y+1)$ is well defined when $\Re(y) > -1$. Plugging the latter expressions into (1.13) and (1.14) indeed coincide with (1.1) upon setting $zK_x^\sigma(z) = g_2(Y_x^\sigma(z)) = 1/(1+Y_x^\sigma(z))$, whereas (1.6) and (1.7) result for $\alpha = 2$ with $Y(z) = -\frac{1}{z}G_2(z)$ and the Cauchy-Stieltjes transform $G_2(z) = (z - \sqrt{z^2 - 4})/2$ of the semi-circle law μ_2 (upon properly choosing the branch of the square root).*

Remark 1.9. *The equivalence between $\sigma \in \mathcal{F}_\alpha$ and $\tilde{\sigma} \in \mathcal{C}_*$ is often quite useful. For example, if $\varphi : [-1,1] \rightarrow \mathbb{R}$ is any even, periodic function of period one and finitely many jump discontinuities then $\sigma(x,y) = \varphi(x-y) \in \mathcal{F}_\alpha$ and is equivalent to the constant $\tilde{\sigma} = [\int_0^1 |\varphi(v)|^\alpha dv]^{1/\alpha}$. Consequently, in this case μ^σ equals $\mu_\alpha(\tilde{\sigma} \cdot)$ of [1] and hence has the symmetric, uniformly bounded, continuous off zero, density $\tilde{\sigma}^{-1} \rho_\alpha(t/\tilde{\sigma})$ with respect to Lebesgue measure on \mathbb{R} .*

Consider next the empirical covariance matrices $\mathbf{W}_{N,M} = a_{N+M}^{-2} \mathbf{X}_{N,M} \mathbf{X}_{N,M}^t$ where $\mathbf{X}_{N,M}$ is an $N \times M$ matrix with heavy tailed entries x_{ij} , $1 \leq i \leq N$, $1 \leq j \leq M$, the law of which satisfies (1.2) (and \mathbf{B}^t denotes throughout the transpose of the matrix \mathbf{B}). Taking $N \rightarrow \infty$ and $M/N \rightarrow \gamma \in (0,1]$ the scaling constant a_N is chosen per (1.3) (so from (1.2) we have that $a_{N+M}^2 \sim N^{\frac{2}{\alpha}}(1+\gamma)^{2/\alpha} L_1(N)$ for some slowly varying function $L_1(\cdot)$). In this setting we show the following about the limiting spectral measure of $\mathbf{W}_{N,M}$.

Theorem 1.10. *If $N \rightarrow \infty$ and $\frac{M}{N} \rightarrow \gamma \in (0,1]$ then with probability one, the spectral measures $\hat{\mu}_{\mathbf{W}_{N,M}}$ converge to a non-random probability measure μ_α^γ . The probability measure μ_α^1 is absolutely continuous with the density*

$$\rho_\alpha^1(t) = 2^{1/\alpha} t^{-1/2} \rho_\alpha(2^{1/\alpha} \sqrt{t})$$

on $(0, \infty)$. Fixing $\gamma \in (0,1)$ let $(Y_1(z), Y_2(z))$ denote the unique analytic functions of $z \in \mathbb{C}^+$ tending to zero at infinity, such that

$$(1.15) \quad z^\alpha Y_1(z) = \frac{\gamma}{1+\gamma} C_\alpha g_\alpha(Y_2(z)), \quad z^\alpha Y_2(z) = \frac{1}{1+\gamma} C_\alpha g_\alpha(Y_1(z)).$$

The functions $Y_1(z)$ and $Y_2(z)$ extend continuously to functions on $(0, \infty)$ that are analytic through (R, ∞) for some finite $R = R_\alpha^\gamma$. The probability measure μ_α^γ then

has an atom at zero of mass $1 - \gamma$ and the continuous density

$$(1.16) \quad \rho_\alpha^\gamma(t) = -\frac{1}{\pi t} \Im(h_\alpha(Y_1(\sqrt{t}))),$$

on $(0, \infty)$ which is real-analytic on (R, ∞) , bounded off zero, does not vanish in any neighborhood of zero and such that $t^{1+\alpha/2} \rho_\alpha^\gamma(t) \rightarrow \frac{\alpha\gamma}{2(1+\gamma)}$ as $t \rightarrow \infty$.

Remark 1.11. Note the contrast between the non-vanishing near zero density ρ_α^γ and the Pastur-Marchenko law μ_2^γ which vanishes throughout $[0, 1 - \gamma]$ (c.f. [8]).

We also consider diagonal perturbations of heavy tailed matrices. Namely, the limit of the spectral measures $\hat{\mu}_{\mathbf{A}_N^\sigma + \mathbf{D}_N}$ where \mathbf{D}_N is a diagonal $N \times N$ matrix, whose entries $\{D_N(k, k), 1 \leq k \leq N\}$ are real valued, independent of the random variables $(x_{ij}, 1 \leq i \leq j < \infty)$ and identically distributed, of law $\mu^{\mathbf{D}}$ which has a finite second moment. In this setting we have the following extension of Theorem 1.3 and Theorem 1.7.

Theorem 1.12. Let $\hat{\mathcal{K}}_\alpha := \{R_0 e^{i\varphi} : -\frac{\alpha\pi}{2} \leq \varphi \leq 0, R_0 \geq 0\}$. Given $\sigma \in \mathcal{F}_\alpha$, the sequence $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\sigma + \mathbf{D}_N}]$ converges weakly towards the probability measure $\mu^{\sigma, \mathbf{D}}$ whose Cauchy-Stieltjes transform at $z \in \mathbb{C}^+$ is

$$(1.17) \quad G_{\alpha, \sigma}^{\mathbf{D}}(z) = \int \frac{1}{z - \lambda} d\mu^{\mathbf{D}}(\lambda) \int_0^1 h_\alpha((\lambda - z)^{-\frac{\alpha}{2}} \hat{X}_v^\sigma(z)) dv,$$

for some $R = R(\sigma)$ finite and the unique analytic mapping $\hat{X}^\sigma : \mathbb{C}^+ \mapsto L^\infty((0, 1]; \hat{\mathcal{K}}_\alpha)$ such that if $\Im(z) \geq R(\sigma)$ then for almost every $x \in (0, 1]$

$$(1.18) \quad \hat{X}_x^\sigma(z) = \bar{C}_\alpha \int_0^1 |\sigma(x, v)|^\alpha \int (\lambda - z)^{-\frac{\alpha}{2}} g_\alpha((\lambda - z)^{-\frac{\alpha}{2}} \hat{X}_v^\sigma(z)) d\mu^{\mathbf{D}}(\lambda) dv.$$

If $\sigma \in \mathcal{C}_*$ then $\hat{X}_x^\sigma(z)$ takes the same value $\hat{X}_r(z)$ for all $x \in (b_{r-1}, b_r]$, where $(\hat{X}_r(z), 1 \leq r \leq q)$ is the unique collection of analytic functions from \mathbb{C}^+ to $\hat{\mathcal{K}}_\alpha$ such that

$$(1.19) \quad \hat{X}_r(z) = \bar{C}_\alpha \sum_{s=1}^q |\sigma_{rs}|^\alpha \Delta_s \int (\lambda - z)^{-\frac{\alpha}{2}} g_\alpha((\lambda - z)^{-\frac{\alpha}{2}} \hat{X}_s(z)) d\mu^{\mathbf{D}}(\lambda)$$

and $|\hat{X}_r(z)| \leq c(\Im(z))^{-\frac{\alpha}{2}}$ for some finite c and all $r \in \{1, \dots, q\}$.

Remark 1.13. The substitution of $g_2(y) = h_2(y) = 1/(1+y)$ in (1.18) and (1.19) leads to the prediction $G_{2, \sigma}^{\mathbf{D}}(z) = \int (\lambda - z - \hat{X}_v^\sigma(z))^{-1} dv d\mu^{\mathbf{D}}(\lambda)$ with $\hat{X}_x^\sigma(z) = \int |\sigma(x, v)|^2 (\lambda - z - \hat{X}_v^\sigma(z))^{-1} dv d\mu^{\mathbf{D}}(\lambda)$ which in particular for $\sigma(\cdot, \cdot) \equiv 1$ results with $\hat{X}_x^\sigma(z) = G_2^{\mathbf{D}}(z)$ independent of x that corresponds to the celebrated free-convolution of $\mu^{\mathbf{D}}$ and μ_2 . Namely, $G_2^{\mathbf{D}}(z) = \int (\lambda - z - G_2^{\mathbf{D}}(z))^{-1} d\mu^{\mathbf{D}}(\lambda)$.

While beyond the scope of this paper, it is of interest to study the behavior of the eigenvectors of large random matrices of heavy tailed entries (such as \mathbf{A}_N^σ or $\mathbf{W}_{N, M}$), and in particular, to find out if they concentrate on indices associated with the entries of extreme values or are rather “spread-out”.

After devoting the next section to the truncation and approximation tools used in our work, we proceed to prove our main results, starting with the proof of Theorem 1.3 in Section 3. This is followed by the proof of Theorem 1.7 in Section 4, the specialization to covariance matrices (i.e. proof of Theorem 1.10) in Section 5 and the generalization to diagonal perturbations (i.e. proof of Theorem 1.12) in Section 6.

2. TRUNCATION, TIGHTNESS AND APPROXIMATIONS

As the second moment of entries of our random matrices is infinite, we start by providing appropriate truncated matrices, whose spectral measures approximate well (in the limit $N \rightarrow \infty$) the spectral measures $\hat{\mu}_{\mathbf{A}_N}$. Specifically, let \mathbf{Y}_N^B denote the $N \times N$ symmetric matrix with entries $\sigma(\frac{i}{N}, \frac{j}{N})x_{ij}\mathbf{1}_{|x_{ij}| < Ba_N}$ for $B > 0$. We further consider the $N \times N$ symmetric matrix \mathbf{Y}_N^κ with entries $\sigma(\frac{i}{N}, \frac{j}{N})x_{ij}\mathbf{1}_{|x_{ij}| < N^\kappa a_N}$ for $\kappa > 0$, and the corresponding normalized matrices,

$$\mathbf{A}_N^B := a_N^{-1}\mathbf{Y}_N^B, \quad \mathbf{A}_N^\kappa := a_N^{-1}\mathbf{Y}_N^\kappa.$$

It is easy to adapt the proof of [1, Lemma 2.4] to our setting and deduce that for every $\epsilon > 0$, there exists $B(\epsilon)$ finite and $\delta(\epsilon, B) > 0$ when $B > B(\epsilon)$, such that

$$\mathbb{P}(\text{rank}(\mathbf{Y}_N - \mathbf{Y}_N^B) \geq \epsilon N) \leq e^{-\delta(\epsilon, B)N}.$$

Likewise, for $\kappa > 0$, and $a \in]1 - \alpha\kappa, 1[$ there exists a finite constant $C = C(\alpha, \kappa, a)$ such that

$$\mathbb{P}(\text{rank}(\mathbf{Y}_N - \mathbf{Y}_N^\kappa) \geq N^a) \leq e^{-CN^a \log N}$$

(and both bounds are independent of $\sigma(\cdot, \cdot)$). By Lidskii's theorem it then readily follows that

$$(2.1) \quad \mathbb{P}\left(d_1(\hat{\mu}_{\mathbf{A}_N}, \hat{\mu}_{\mathbf{A}_N^B}) \geq 2\epsilon\right) \leq e^{-\delta(\epsilon, B)N},$$

$$(2.2) \quad \mathbb{P}\left(d_1(\hat{\mu}_{\mathbf{A}_N}, \hat{\mu}_{\mathbf{A}_N^\kappa}) \geq 2N^{a-1}\right) \leq e^{-CN^a \log N},$$

where the metric

$$d_1(\mu, \nu) := \sup_{\|f\|_{\mathbf{BL}} \leq 1, f \uparrow} \left| \int f d\nu - \int f d\mu \right|$$

on the set $\mathcal{P}(\mathbb{R})$ of Borel probability measures on \mathbb{R} is compatible with the topology of weak convergence (for example, see [1, Lemma 2.1]), and throughout $\|f\|_{\mathbf{BL}}$ denotes the standard Bounded Lipschitz norm on \mathbb{R} .

Just as in [1, Lemmas 3.1], we have the following tightness result.

Lemma 2.1. *The sequence $(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N}]; N \in \mathbb{N})$ is tight for the topology of weak convergence on $\mathcal{P}(\mathbb{R})$. Further, for every $B > 0$ and $\kappa > 0$, the sequences $(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^B}]; N \in \mathbb{N})$ and $(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\kappa}]; N \in \mathbb{N})$ are also tight in this topology.*

Proof. Recall that

$$(2.3) \quad \mathbb{E}\left[\frac{1}{N}\text{tr}((\mathbf{A}_N^B)^2)\right] = \frac{1}{Na_N^2} \sum_{i,j=1}^N \sigma\left(\frac{i}{N}, \frac{j}{N}\right)^2 \mathbb{E}[|x_{ij}|^2 \mathbf{1}_{|x_{ij}| < Ba_N}]$$

As the latter expectation does not depend on i, j and using the key estimate

$$(2.4) \quad \mathbb{E}[|x_{ij}|^\zeta \mathbf{1}_{|x_{ij}| < Ba_N}] \sim \frac{\alpha}{\zeta - \alpha} B^{\zeta - \alpha} a_N^\zeta N^{-1},$$

for any $\zeta > \alpha$, we deduce that since σ is in $L_*^2([0, 1]^2)$,

$$(2.5) \quad \lim_{N \rightarrow \infty} \mathbb{E}\left[\frac{1}{N}\text{tr}((\mathbf{A}_N^B)^2)\right] \leq \frac{\alpha}{2 - \alpha} B^{2 - \alpha} \|\sigma\|_*^2 < \infty.$$

This implies the tightness of $(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^B}], N \in \mathbb{N})$ which upon using (2.1) and (2.2) provides also the tightness of $(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N}], N \in \mathbb{N})$ and $(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\kappa}], N \in \mathbb{N})$, respectively (for more details, see the proof of [1, Lemma 3.1]). \square

We next show that it suffices to prove the convergence of the spectral measures $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\sigma}]$ for $\sigma(\cdot, \cdot)$ in any given dense subset of $L_\star^2([0, 1]^2)$.

Proposition 2.2. *Suppose that a sequence $(\sigma_p, p \in \mathbb{N})$ converges in $L_\star^2([0, 1]^2)$ towards σ and that for all $p \in \mathbb{N}$*

$$(2.6) \quad \lim_{N \rightarrow \infty} \mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{\sigma_p}}] = \mu^{\sigma_p}.$$

Then, μ^{σ_p} converges weakly as $p \rightarrow \infty$ towards some Borel probability measure μ^σ and $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\sigma}]$ converges weakly towards μ^σ as $N \rightarrow \infty$.

Proof. Note that for some finite constant $c = c(\alpha, B)$, independent of N and σ ,

$$(2.7) \quad \mathbb{E}[d_1(\hat{\mu}_{\mathbf{A}_N^{B, \sigma}}, \hat{\mu}_{\mathbf{A}_N^{B, \sigma_p}})]^2 \leq \mathbb{E}\left[\frac{1}{N} \text{tr}\left((\mathbf{A}_N^{B, \sigma} - \mathbf{A}_N^{B, \sigma_p})^2\right)\right] \leq c^2 \|\sigma - \sigma_p\|_\star^2.$$

Indeed, the leftmost inequality is based on Lidskii's theorem (see [3, (2.16)]), whereas the rightmost one is obtained by an application of (2.3)–(2.5) with σ replaced by $\sigma - \sigma_p$. Next, from the triangle inequality for the d_1 -metric, we have that

$$\begin{aligned} d_1(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\sigma}], \mu^{\sigma_p}) &\leq d_1(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\sigma}], \mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{B, \sigma}}]) + d_1(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{B, \sigma}}], \mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{B, \sigma_p}}]) \\ &\quad + d_1(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{B, \sigma_p}}], \mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{\sigma_p}}]) + d_1(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{\sigma_p}}], \mu^{\sigma_p}). \end{aligned}$$

By our hypothesis (2.6), the last term converges to zero as $N \rightarrow \infty$. Further, by (2.1) and the boundedness and convexity of d_1 , we find that for some $\epsilon(B) \rightarrow 0$ as $B \rightarrow \infty$, independently of σ and σ_p ,

$$\limsup_{N \rightarrow \infty} d_1(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\sigma}], \mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{B, \sigma}}]) + \limsup_{N \rightarrow \infty} d_1(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{B, \sigma_p}}], \mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{\sigma_p}}]) \leq 8\epsilon(B).$$

Moreover, by the convexity of d_1 and (2.7), we have that

$$\limsup_{N \rightarrow \infty} d_1(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{B, \sigma}}], \mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{B, \sigma_p}}]) \leq c(\alpha, B) \|\sigma - \sigma_p\|_\star.$$

Upon combining these estimates we deduce that for any $p \in \mathbb{N}$ and $B > 0$,

$$(2.8) \quad \limsup_{N \rightarrow \infty} d_1(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\sigma}], \mu^{\sigma_p}) \leq 8\epsilon(B) + c(\alpha, B) \|\sigma - \sigma_p\|_\star.$$

In particular, we get the bound

$$\sup_{p, q \geq r} d_1(\mu^{\sigma_p}, \mu^{\sigma_q}) \leq 16\epsilon(B) + 2c(\alpha, B) \delta(r)^2,$$

where by hypothesis $\delta(r)^2 := \sup_{p \geq r} \|\sigma - \sigma_p\|_\star$ converges to zero as $r \rightarrow \infty$. Taking r and B going to infinity such that $c(\alpha, B) \leq \delta(r)^{-1}$ we conclude that $(\mu^{\sigma_p}, p \in \mathbb{N})$ is d_1 -Cauchy and hence converges to some $\mu^\sigma \in \mathcal{P}(\mathbb{R})$ (recall that $\epsilon(B)$ and $c(\alpha, B)$ are independent of σ). By this convergence, combining (2.8) and the triangle inequality for the d_1 -metric, we deduce upon taking $p \rightarrow \infty$ and then $B \rightarrow \infty$, that $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\sigma}]$ also converges towards μ^σ as $N \rightarrow \infty$. \square

Remark 2.3. *By our assumptions, when dealing with $\sigma \in \mathcal{F}_\alpha$ we may and shall take in Proposition 2.2 some $\sigma_p \in \mathcal{C}_\star$. Since the rank of the matrix $\mathbb{E}[\mathbf{A}_N^{\kappa, \sigma_p}]$ is then uniformly bounded in N , as in [1, Remark 2.5] we may and shall recenter $\mathbf{A}_N^{\kappa, \sigma_p}$ without changing its limiting spectral distribution.*

We conclude by showing an interpolation property of $\hat{\mu}_{\mathbf{A}_N^\sigma}$ in case $\sigma \in \mathcal{C}_\star$. That is, the weak convergence of $\hat{\mu}_{\mathbf{A}_N^\sigma}$ follows once we have it along a suitable subsequence $\phi(n)$.

Lemma 2.4. *Suppose $\sigma \in \mathcal{C}_*$ and the increasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is such that $\phi(n-1)/\phi(n) \rightarrow 1$. If $\hat{\mu}_{\mathbf{A}_{\phi(n)}^\sigma}$ converges weakly to some probability measure μ^σ then so does $\hat{\mu}_{\mathbf{A}_N^\sigma}$.*

Proof. For any $N \in (\phi(n-1), \phi(n)]$ set $M = \phi(n)$ and let $\hat{\mathbf{A}}_N^\sigma$ denote the $M \times M$ dimensional matrix whose upper left $N \times N$ corner equals $(a_N/a_M)\mathbf{A}_N^\sigma$ and having zero entries everywhere else. Letting $0 = b_0 < b_1 < \dots < b_q = 1$ denote the partition that corresponds to $\sigma \in \mathcal{C}_*$, observe that $\hat{A}_N^\sigma(i, j) = A_M^\sigma(i, j)$ unless either $i \in (b_r N, b_r M]$ or $j \in (b_r N, b_r M]$ for some $r = 0, 1, \dots, q$. As the latter applies for at most $(q+1)(M-N+1)$ values of $1 \leq i \leq M$ and at most $(q+1)(M-N+1)$ values of $1 \leq j \leq M$, it follows that

$$\text{rank}(\hat{\mathbf{A}}_N^\sigma - \mathbf{A}_M^\sigma) \leq 2(q+1)(M-N+1),$$

so by Lidskii's theorem

$$d_1(\hat{\mu}_{\hat{\mathbf{A}}_N^\sigma}, \hat{\mu}_{\mathbf{A}_M^\sigma}) \leq 4(q+1)\left(1 - \frac{N-1}{M}\right) \leq 4(q+1)\left(1 - \frac{\phi(n-1)}{\phi(n)}\right),$$

which converges to zero as $N \rightarrow \infty$ (hence $n \rightarrow \infty$). Therefore, by the triangle inequality for the d_1 -metric, our assumption that $d_1(\hat{\mu}_{\mathbf{A}_{\phi(n)}^\sigma}, \mu^\sigma) \rightarrow 0$ implies that $d_1(\hat{\mu}_{\hat{\mathbf{A}}_N^\sigma}, \mu^\sigma) \rightarrow 0$ as $N \rightarrow \infty$. Next note that the eigenvalues of $\hat{\mathbf{A}}_N^\sigma$ are those of $(a_N/a_M)\mathbf{A}_N^\sigma$ augmented by $M-N$ zero eigenvalues. Fixing a monotone non-decreasing bounded Lipschitz function $f(\cdot)$, we have thus seen that

$$(2.9) \quad \int f d\hat{\mu}_{\hat{\mathbf{A}}_N^\sigma} = \left(1 - \frac{N}{M}\right)f(0) + \frac{N}{M} \int f(\beta_N x) d\hat{\mu}_{\mathbf{A}_N^\sigma}(x) \rightarrow \int f d\mu^\sigma,$$

when $N \rightarrow \infty$, where $1 \geq \beta_N := a_N/a_M \geq a_{\phi(n-1)}/a_{\phi(n)}$ (as both $\phi(\cdot)$ and a_k are non-decreasing, see (1.3)). Since $\phi(n-1)/\phi(n) \rightarrow 1$ the same applies for $N/M \in (\phi(n-1)/\phi(n), 1]$. Further, $a_k = L_0(k)k^{1/\alpha}$ with $L_0(\cdot)$ a slowly varying function, hence also $a_{\phi(n-1)}/a_{\phi(n)} \rightarrow 1$ when $n \rightarrow \infty$ and consequently $\beta_N \rightarrow 1$ as $N \rightarrow \infty$. Fixing $\epsilon > 0$, since $f(\cdot)$ is monotone and bounded, there exists $K = K(\epsilon)$ finite such that $|f(x) - f(y)| \leq \epsilon$ whenever $\min(x, y) \geq K$ or $\max(x, y) \leq -K$. Thus, for any $\beta \in (0, 1]$,

$$\sup_{\nu \in \mathcal{P}(\mathbb{R})} \int |f(x) - f(\beta x)| d\nu(x) \leq \epsilon + \frac{K}{\beta}(1-\beta)\|f\|_{\mathcal{L}}.$$

In particular, since $\beta_N \rightarrow 1$, for any $\epsilon > 0$,

$$\lim_{N \rightarrow \infty} \left| \int f d\hat{\mu}_{\mathbf{A}_N^\sigma} - \int f(\beta_N x) d\hat{\mu}_{\mathbf{A}_N^\sigma}(x) \right| \leq \epsilon,$$

which in view of (2.9) results with $\int f d\hat{\mu}_{\mathbf{A}_N^\sigma} \rightarrow \int f d\mu^\sigma$. This holds for each monotone non-decreasing bounded Lipschitz function $f(\cdot)$, which is equivalent to our thesis that $\hat{\mu}_{\mathbf{A}_N^\sigma}$ converges weakly to μ^σ . \square

3. INDUCTION AND THE LIMITING EQUATIONS

We consider throughout this section $\sigma \in \mathcal{C}_*$. That is, there exist $0 = b_0 < b_1 < \dots < b_q = 1$ and a $q \times q$ symmetric matrix of entries σ_{rs} for $1 \leq r, s \leq q$ such that

$$(3.1) \quad \sigma(x, y) = \sigma_{rs} \quad \text{for all } (x, y) \in (b_{r-1}, b_r] \times (b_{s-1}, b_s].$$

Associated with such σ are the random matrix \mathbf{A}_N^σ and the $N \times N$ piecewise constant matrix $\boldsymbol{\sigma}^N$ of entries $\sigma^N(i, j) = \sigma_{rs}$ for $[Nb_{r-1}] < i \leq [Nb_r]$ and $[Nb_{s-1}] < j \leq [Nb_s]$.

3.1. Characterization of limit points. For each $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ we define, as in [1, Section 4], the matrices $\mathbf{G}_N(z) := (z\mathbf{I}_N - \mathbf{A}_N)^{-1}$ and the probability measure L_N^z on \mathbb{C} such that for $f \in \mathcal{C}_b(\mathbb{C})$,

$$(3.2) \quad L_N^z(f) = \mathbb{E} \left[\frac{1}{N} \sum_{k=1}^N f(G_N(z)_{kk}) \right].$$

It is useful for our purpose to represent L_N^z as a weighted sum $L_N^z = \sum_{r=1}^q \Delta_{N,r} L_{N,r}^z$ where $L_{N,r}^z$ are the probability measures on \mathbb{C} given by

$$(3.3) \quad L_{N,r}^z(f) := \mathbb{E} \left[\frac{1}{[Nb_r] - [Nb_{r-1}]} \sum_{k=[Nb_{r-1}]+1}^{[Nb_r]} f(G_N(z)_{kk}) \right],$$

and $\Delta_{N,r} := N^{-1}([Nb_r] - [Nb_{r-1}]) \rightarrow \Delta_r$ as $N \rightarrow \infty$. Since each term $G_N(z)_{kk}$ belongs to the compact set $\mathbb{K}(z) := \{x \in \mathbb{C}^- : |x| \leq |\Im(z)|^{-1}\}$, the probability measures $L_{N,r}^z$ are supported on $\mathbb{K}(z)$ for all $N \in \mathbb{N}$ and $1 \leq r \leq q$.

We denote by $\mathbf{G}_N^\kappa(z)$ and $L_{N,r}^{z,\kappa}$ the corresponding objects when \mathbf{A}_N is replaced by the truncated matrix \mathbf{A}_N^κ . Similarly to [1, Lemma 4.4] we next show that

Lemma 3.1. *For $0 < \kappa < \frac{1}{2(2-\alpha)}$, any $1 \leq r \leq q$ and Lipschitz function f on $\mathbb{K}(z)$,*

$$\lim_{N \rightarrow \infty} \left| \mathbb{E}[L_{N,r}^{z,\kappa}(f)] - \mathbb{E} \left[f \left(\left(z - \sum_{k=1}^N \tilde{A}_N^\kappa([Nb_r], k)^2 G_N^\kappa(z)_{kk} \right)^{-1} \right) \right] \right| = 0,$$

where $\tilde{\mathbf{A}}_N^\kappa$ is an independent copy of \mathbf{A}_N^κ .

Proof. Without loss of generality, it suffices to prove the lemma for $r = 1$ (the general case follows by permuting indices). To this end, let $\bar{\mathbf{A}}_{N+1}^\kappa$ denote an $(N+1) \times (N+1)$ symmetric matrix obtained by adding to \mathbf{A}_N^κ a first row and column $\tilde{A}_N^\kappa(0, k) = \tilde{A}_N^\kappa(k, 0)$ such that $(\tilde{A}_N^\kappa(0, k), k \geq 1)$ is an independent copy of $(A_N^\kappa(1, k), k \geq 1)$ and $\tilde{A}_N^\kappa(0, 0) = \sigma^N(1, 1) a_N^{-1} x_{00} \mathbf{1}_{|x_{00}| < N^\kappa a_N}$. Next, consider the matrix $\bar{\mathbf{G}}_{N+1}^\kappa(z) = (z\mathbf{I}_{N+1} - \bar{\mathbf{A}}_{N+1}^\kappa)^{-1}$ and let $\bar{L}_{N+1,1}^{z,\kappa}$ denote the empirical measure of $\{\bar{G}_{N+1}^\kappa(z)_{kk}, 0 \leq k \leq [Nb_1]\}$. The invariance of the law of $\bar{\mathbf{A}}_{N+1}^\kappa$ with respect to symmetric permutations of its first $[Nb_1] + 1$ rows and columns implies that $\{\bar{G}_{N+1}^\kappa(z)_{kk}, 0 \leq k \leq [Nb_1]\}$ are identically distributed, hence for any $f \in \mathcal{C}_b(\mathbb{K}(z))$,

$$(3.4) \quad \mathbb{E}[\bar{L}_{N+1,1}^{z,\kappa}(f)] = \mathbb{E}[f(\bar{G}_{N+1}^\kappa(z)_{00})].$$

As in [1], the key to our proof is Schur's complement formula

$$\bar{G}_{N+1}^\kappa(z)_{00} = \left(z - \tilde{A}_N^\kappa(0, 0) - \sum_{k,l=1}^N \tilde{A}_N^\kappa(0, k) \tilde{A}_N^\kappa(l, 0) G_N^\kappa(z)_{kl} \right)^{-1},$$

from which we thus get that

$$(3.5) \quad \mathbb{E}[\bar{L}_{N+1,1}^{z,\kappa}(f)] = \mathbb{E} \left[f \left(\left(z - \tilde{A}_N^\kappa(0, 0) - \sum_{k,l=1}^N \tilde{A}_N^\kappa(0, k) \tilde{A}_N^\kappa(l, 0) G_N^\kappa(z)_{kl} \right)^{-1} \right) \right].$$

Recall that the entries of $\tilde{\mathbf{A}}_N^\kappa$ are centered (see Remark 2.3), and independent of the matrix $\mathbf{G}_N^\kappa(z)$. Further, as the entries of the matrix $\boldsymbol{\sigma}^N$ are uniformly bounded, the statement and proof of [1, Lemma 4.3] extends readily to our setting, showing that the off diagonal terms in the right hand side of (3.5) are small with overwhelming probability (this is simply based on a computation of the variance of this term, which is possible thanks to the cut-off κ). As shown in the proof of [1, Lemma 4.4], this allows us to neglect the terms $\tilde{A}_N^\kappa(0, 0)$ and $\sum_{k \neq l} \tilde{A}_N^\kappa(0, k) \tilde{A}_N^\kappa(l, 0) G_N^\kappa(z)_{kl}$ in (3.5), resulting with

$$(3.6) \quad \lim_{N \rightarrow \infty} \left| \mathbb{E}[\bar{L}_{N+1,1}^{z,\kappa}(f)] - \mathbb{E}\left[f\left(z - \sum_{k=1}^N \tilde{A}_N^\kappa(0, k)^2 G_N^\kappa(z)_{kk}^{-1}\right)\right] \right| = 0.$$

Further, with $\boldsymbol{\sigma}^N$ uniformly bounded, adapting the proof of [1, Lemma 4.1] to our setting, we deduce that

$$\lim_{N \rightarrow \infty} \mathbb{P}(d_1(L_{N,1}^{z,\kappa}, \bar{L}_{N+1,1}^{z,\kappa}) > N^{-\eta}) = 0,$$

for any $0 < \eta < \frac{1}{2}(1 - \kappa(2 - \alpha))$. Consequently, $|\mathbb{E}[\bar{L}_{N+1,1}^{z,\kappa}(f)] - \mathbb{E}[L_{N,1}^{z,\kappa}(f)]| \rightarrow 0$ as $N \rightarrow \infty$ and (3.6) finishes the proof of the lemma. \square

Identifying \mathbb{C} with \mathbb{R}^2 , recall [1, Definition 5.1]. Namely,

Definition 3.2. *Given $\alpha \in (0, 2)$ and a compactly supported probability measure μ on \mathbb{C} , let P^μ denote the probability measure on \mathbb{C} whose characteristic function at $\mathbf{t} \in \mathbb{R}^2$ is*

$$\int_{\mathbb{R}^2} e^{i\langle \mathbf{t}, \mathbf{x} \rangle} dP^\mu(\mathbf{x}) = \exp[-v_{\mu, \frac{\alpha}{2}}(\mathbf{t})^{\frac{\alpha}{2}} (1 - i\beta_{\mu, \frac{\alpha}{2}}(\mathbf{t}) \tan(\frac{\pi\alpha}{4}))],$$

where

$$\begin{aligned} v_{\mu, \alpha}(\mathbf{t}) &= [v_\alpha^{-1} \int |\langle \mathbf{t}, \mathbf{z} \rangle|^\alpha d\mu(\mathbf{z})]^{1/\alpha}, \\ v_\alpha^{-1} &= \int_0^\infty \frac{\sin x}{x^\alpha} dx = \frac{\Gamma(2 - \alpha) \cos(\frac{\pi\alpha}{2})}{1 - \alpha}, \\ \beta_{\mu, \alpha}(\mathbf{t}) &= \frac{\int |\langle \mathbf{t}, \mathbf{z} \rangle|^\alpha \text{sign}(\langle \mathbf{t}, \mathbf{z} \rangle) d\mu(\mathbf{z})}{\int |\langle \mathbf{t}, \mathbf{z} \rangle|^\alpha d\mu(\mathbf{z})}, \end{aligned}$$

and $\beta_{\mu, \alpha}(\mathbf{t}) = 0$ whenever $v_{\mu, \alpha}(\mathbf{t}) = 0$. In particular, if μ is supported in the closure of \mathbb{C}^- , then so does P^μ .

Equipped with this definition, our next proposition characterizes the set of possible limit points of $\{\mathbb{E}[L_{N,r}^{z,\kappa}], 1 \leq r \leq q\}$.

Proposition 3.3. *For $0 < \kappa < \frac{1}{2(2-\alpha)}$ and $z \in \mathbb{C}^+$, any limit point $(\mu_r^z, 1 \leq r \leq q)$ of the sequence $\{\mathbb{E}[L_{N,r}^{z,\kappa}], 1 \leq r \leq q\}$, $N \in \mathbb{N}$ consists of probability measures on $\mathbb{K}(z)$ that satisfy the system of equations*

$$(3.7) \quad \int f d\mu_r^z = \int f \left(z - \sum_{s=1}^q \sigma_{rs}^2 \Delta_s^{\frac{2}{\alpha}} x_s \right)^{-1} \prod_{s=1}^q dP^{\mu_s^z}(x_s)$$

for $r \in \{1, \dots, q\}$ and every bounded continuous function f on $\mathbb{K}(z)$.

The following concentration result is key to the proof of Proposition 3.3.

Lemma 3.4. For $\kappa \in (0, \frac{1}{2-\alpha})$ let $\epsilon = 1 - \kappa(2 - \alpha) > 0$. There exists $c < \infty$ so that for $z \in \mathbb{C}^+$, $s \in \{1, \dots, q\}$, $\delta > 0$, $N \in \mathbb{N}$ and any Lipschitz function f on $\mathbb{K}(z)$,

$$\mathbb{P}\left(\left|L_{N,s}^{z,\kappa}(f) - \mathbb{E}[L_{N,s}^{z,\kappa}(f)]\right| \geq \delta\right) \leq \frac{c\|f\|_{\mathbf{BL}}^2}{|\Im(z)|^4 \delta^2} N^{-\epsilon},$$

with $\|f\|_{\mathbf{BL}}$ denoting here the Bounded Lipschitz norm of f restricted to $\mathbb{K}(z)$.

Proof. Fixing $s \in \{1, \dots, q\}$ and $z \in \mathbb{C}^+$, note that the value of f outside the compact set $\mathbb{K}(z)$ on which all probability measures $L_{N,s}^{z,\kappa}$ are supported, is irrelevant. We thus assume without loss of generality that f is bounded and continuously differentiable and as in the proof of [1, Lemma 5.4], let

$$F_N(\mathbf{A}) := L_{N,s}^{z,\kappa}(f) = \frac{1}{N} \sum_{k=[b_{s-1}N]+1}^{[b_s N]} f(G_N^\kappa(z)_{kk}),$$

a smooth function of the $n = N(N-1)/2$ independent, centered, random variables $A_N^\kappa(k, l)$ for $1 \leq k \leq l \leq N$. By a classical martingale decomposition we see that

$$(3.8) \quad \mathbb{E}[(F_N - \mathbb{E}[F_N])^2] \leq \sum_{1 \leq i < j \leq N} \|\partial_{A(i,j)} F_N\|_\infty^2 \mathbb{E}[(A_N^\kappa(i, j) - \mathbb{E}[A_N^\kappa(i, j)])^2].$$

Moreover, similarly to the proof of [1, Lemma 5.4] we have here that

$$\begin{aligned} \partial_{A(m,l)} F_N(\mathbf{A}) &= \frac{1}{N} \sum_{k=[Nb_{s-1}]+1}^{[Nb_s]} f'(G_N^\kappa(z)_{kk})(G_N^\kappa(z)_{kl}G_N^\kappa(z)_{mk} + G_N^\kappa(z)_{km}G_N^\kappa(z)_{lk}) \\ &= \frac{1}{N} ([G_N^\kappa(z)D_s(f')G_N^\kappa(z)]_{ml} + [G_N^\kappa(z)D_s(f')G_N^\kappa(z)]_{lm}) \end{aligned}$$

with $D_s(f')$ the N -dimensional diagonal matrix of entries

$$D_s(f')_{kk} := f'(G_N^\kappa(z)_{kk})\mathbf{1}_{[Nb_{s-1}] < k \leq [Nb_s]}.$$

As the spectral radius of $G_N^\kappa(z)D_s(f')G_N^\kappa(z)$ is bounded by $\|f'\|_\infty/|\Im(z)|^2$, the same applies for each entry of this matrix. By the preceding, such bounds imply that

$$\sup_{i,j} \|\partial_{A(i,j)} F_N\|_\infty \leq 2\|f\|_{\mathbf{BL}}(N|\Im(z)|^2)^{-1}.$$

Further, with σ^N uniformly bounded, from (2.4) (for $\zeta = 2$), we get that for some c_0 finite and all N ,

$$\sup_{1 \leq i < j \leq N} \mathbb{E}[|A_N^\kappa(i, j)|^2] \leq c_0 N^{\kappa(2-\alpha)-1}.$$

As $\epsilon = 1 - \kappa(2 - \alpha) > 0$, substituting these bounds into (3.8) we find that

$$\mathbb{E}[(F_N - \mathbb{E}[F_N])^2] \leq 4c_0\|f\|_{\mathbf{BL}}^2|\Im(z)|^{-4}N^{-\epsilon},$$

and conclude the proof by Chebychev's inequality. \square

Proof of Proposition 3.3. The sequence of q -tuples of probability measures $(\mathbb{E}[L_{N,r}^{z,\kappa}], 1 \leq r \leq q)_{N \in \mathbb{N}}$, each supported in the compact set $\mathbb{K}(z)$, is clearly tight. Considering a subsequence $(\mathbb{E}[L_{\phi(N),r}^{z,\kappa}], 1 \leq r \leq q)_{N \in \mathbb{N}}$ that converges weakly to a limit point $(\mu_r^z, 1 \leq r \leq q)$, passing to a further subsequence still denoted $\phi(N)$ we have by Lemma 3.4 that $(L_{\phi(N),r}^{z,\kappa}, 1 \leq r \leq q)_{N \in \mathbb{N}}$ also converges almost surely to $(\mu_r^z, 1 \leq r \leq q)$, a q -tuple of probability measures on $\mathbb{K}(z)$.

By Lemma 3.1, fixing $r \in \{1, \dots, q\}$, it suffices to show that

$$U_N(z, r) := \sum_{k=1}^N \tilde{A}_N^\kappa([b_r N], k)^2 G_N^\kappa(z)_{kk}$$

is such that $U_{\phi(N)}(z, r)$ converges in law towards $\sum_{s=1}^q \sigma_{rs}^2 \Delta_s^{2/\alpha} x_s$ where $(x_s, 1 \leq s \leq q)$ are independent, with $x_s \in \mathbb{C}$ distributed according to $P^{\mu_s^z}$ for $s = 1, \dots, q$.

Note that $U_N(z, r) = \sum_{s=1}^q \sigma_{rs}^2 W_N(z, s)$, where

$$W_N(z, s) := \sum_{k=[Nb_{s-1}]+1}^{[Nb_s]} \hat{A}_N^\kappa([b_r N], k)^2 G_N^\kappa(z)_{kk},$$

and the i.i.d. random variables $\hat{A}_N^\kappa([b_r N], k) = \tilde{A}_N^\kappa([b_r N], k)/\sigma_{rs}$ are independent of $G_N^\kappa(z)$ and correspond to taking $\sigma \equiv 1$. Next let

$$a_N(s) = \inf\{u : \mathbb{P}(|x_{ij}| \geq u) \leq \frac{1}{N\Delta_{N,s}}\},$$

noting that by (1.2),

$$(3.9) \quad \lim_{N \rightarrow \infty} \frac{a_N(s)}{a_N} = \Delta_s^{1/\alpha}.$$

Further, applying [1, Theorem 10.4] for $X_k = \tilde{x}_{[Nb_r]k}^2$, $\tilde{a}_N = a_N(s)^2$ and $\ell(N) = (a_N/a_N(s))^2 N^{2\kappa} \rightarrow \infty$, on the subsequence $\phi(N)$ and subject to the event that $L_{\phi(N),s}^{z,\kappa}$ converges to μ_s^z , we deduce that $(a_N/a_N(s))^2 W_N(z, s)$ converges in law to $P^{\mu_s^z}$. By the conditional independence of $W_N(z, s)$ for $1 \leq s \leq q$ (per fixed $G_N^\kappa(z)$), and (3.9) we arrive at the stated convergence in law of $U_{\phi(N)}(z, r)$. \square

We next derive the analog of [1, Theorem 5.5].

Proposition 3.5. *For $0 < \kappa < \frac{1}{2(2-\alpha)}$ any subsequence of the functions $(X_{N,r}(z) := \mathbb{E}[L_{N,r}^{z,\kappa}(x^{\alpha/2})], 1 \leq r \leq q)$ from \mathbb{C}^+ to \mathbb{C}^q has at least one limit point $(X_r(z), 1 \leq r \leq q)$ such that $z \mapsto X_r(z)$ are analytic in \mathbb{C}^+ , $|X_r(z)| \leq (\Im(z))^{-\alpha/2}$ and for all $z \in \mathbb{C}^+$,*

$$(3.10) \quad X_r(z) = C(\alpha) \int_0^\infty t^{-1} (it)^{\frac{\alpha}{2}} e^{itz} \exp\{-(it)^{\frac{\alpha}{2}} \hat{X}_r(z)\} dt,$$

with $C(\alpha) = \frac{e^{-i\frac{\pi\alpha}{2}}}{\Gamma(\frac{\alpha}{2})}$ and

$$(3.11) \quad \hat{X}_r(z) := \Gamma(1 - \frac{\alpha}{2}) \sum_{s=1}^q |\sigma_{rs}|^\alpha \Delta_s X_s(z).$$

Proof. The proof is an easy adaptation of [1, Theorem 5.5]. In fact, for each $1 \leq r \leq q$, the analytic functions $X_{N,r}(z)$ on \mathbb{C}^+ are uniformly bounded by $(\Im(z))^{-\alpha/2}$ (hence uniformly bounded on compacts). Consequently, by Montel's theorem, any subsequence $(X_{\phi(N),r}(z), 1 \leq r \leq q)$ has a limit point $(X_r(z), 1 \leq r \leq q)$ (with respect to uniform convergence on compacts), consisting of analytic functions on \mathbb{C}^+ (c.f. [4, Theorem 17.21]), that obviously are also bounded by $(\Im(z))^{-\alpha/2}$. Fixing $z \in \mathbb{C}^+$ and passing to a further sub-subsequence along which the compactly supported probability measures $\mathbb{E}[L_{N,r}^{z,\kappa}]$ converge weakly to μ_r^z for all $1 \leq r \leq q$, it

follows by definition that $X_r(z) = \int x^{\frac{\alpha}{2}} d\mu_r^z(x)$ (as $x \mapsto x^{\alpha/2}$ is in $\mathcal{C}_b(\mathbb{K}(z))$). Next, we prove (3.10) by applying [1, Lemma 5.6] which states that for all $z \in \mathbb{C}^+$,

$$(3.12) \quad z^{-\frac{\alpha}{2}} = C(\alpha) \int_0^\infty t^{-1} (it)^{\frac{\alpha}{2}} e^{itz} dt.$$

Indeed, combining (3.7) and (3.12) we see that

$$\begin{aligned} X_r(z) &= \int (z - \sum_{s=1}^q \sigma_{rs}^2 \Delta_s^{\frac{2}{\alpha}} x_s)^{-\frac{\alpha}{2}} \prod_{s=1}^q dP^{\mu_s^z}(x_s) \\ &= C(\alpha) \int \int_0^\infty t^{-1} (it)^{\frac{\alpha}{2}} \exp\{it(z - \sum_{s=1}^q \sigma_{rs}^2 \Delta_s^{\frac{2}{\alpha}} x_s)\} dt \prod_{s=1}^q dP^{\mu_s^z}(x_s). \end{aligned}$$

Recall [1, Theorem 10.5] that for $\alpha \in (0, 2)$ and any probability measure ν compactly supported in the closure of \mathbb{C}^- ,

$$(3.13) \quad \int e^{-itx} dP^\nu(x) = \exp(-\Gamma(1 - \frac{\alpha}{2})(it)^{\frac{\alpha}{2}} \int x^{\frac{\alpha}{2}} d\nu(x)).$$

Since $z \in \mathbb{C}^+$ and $\Im(x_s) \leq 0$, by Fubini's theorem and (3.13) we deduce that

$$\begin{aligned} X_r(z) &= C(\alpha) \int_0^\infty t^{-1} (it)^{\frac{\alpha}{2}} e^{itz} \prod_{s=1}^q \left(\int \exp\{-it\sigma_{rs}^2 \Delta_s^{\frac{2}{\alpha}} x_s\} dP^{\mu_s^z}(x_s) \right) dt \\ &= C(\alpha) \int_0^\infty t^{-1} (it)^{\frac{\alpha}{2}} e^{itz} \prod_{s=1}^q \exp\{-\Gamma(1 - \frac{\alpha}{2})(it)^{\frac{\alpha}{2}} |\sigma_{rs}|^\alpha \Delta_s X_s(z)\} dt, \end{aligned}$$

as claimed. \square

3.2. Properties of the functions ($X_r, 1 \leq r \leq q$). We provide now key information about $X_r(z)$ of Proposition 3.5.

Lemma 3.6. *For $0 < \kappa < \frac{1}{2(2-\alpha)}$, $z \in \mathbb{C}^+$, if $X_s(z)$ is as in Proposition 3.5 and a_s are non-negative for $s \in \{1, \dots, q\}$, then $(-z)^{-\frac{\alpha}{2}} \sum_{s=1}^q a_s X_s(z)$ is in the set $\mathcal{K}_\alpha := \{Re^{i\theta} : |\theta| \leq \frac{\alpha\pi}{2}, R \geq 0\}$ on which for each $\beta > 0$, the entire function*

$$(3.14) \quad g_{\alpha,\beta}(y) := \int_0^\infty t^{\frac{\beta}{2}-1} e^{-t} \exp\{-t^{\frac{\alpha}{2}} y\} dt,$$

is uniformly bounded. In particular, this applies to $g_\alpha = g_{\alpha,\alpha}$, to $h_\alpha = g_{\alpha,2}$ and their derivatives of all order.

Proof. Recall that for $z \in \mathbb{C}^+$ the measures $L_{N,s}^{z,\kappa}$ are each supported on \mathbb{C}^- . Hence, by definition each of the functions $X_{N,s}(z)$ is in the closed cone

$$(3.15) \quad \widehat{\mathcal{K}}_\alpha := \{R_0 e^{i\hat{\theta}} : -\frac{\alpha\pi}{2} \leq \hat{\theta} \leq 0, R_0 \geq 0\},$$

and thus so is any limit point $X_s(z)$ of $X_{N,s}(z)$. Setting $w := (-z)^{-\frac{\alpha}{2}} \sum_{s=1}^q a_s X_s(z)$, it thus follows that for any $z \in \mathbb{C}^+$ and non-negative a_s ,

$$(3.16) \quad 0 \leq \arg(w) + \frac{\alpha}{2} \arg(z) \leq \frac{\alpha\pi}{2}.$$

In particular, $w \in \mathcal{K}_\alpha$, as claimed. Key to the boundedness of $g_{\alpha,\beta}(\cdot)$ on this set is the identity of [1, equation (40)], where it is shown that

$$(3.17) \quad (-z)^{-\beta/2} g_{\alpha,\beta}(y) = \int_0^\infty t^{-1} (it)^{\frac{\beta}{2}} e^{itz} \exp[-(-z)^{\frac{\alpha}{2}} (it)^{\frac{\alpha}{2}} y] dt,$$

for any $z \in \mathbb{C}^+$ and $y \in \mathbb{C}$. Indeed, for each $\alpha \in (0, 2)$ set $\eta = \eta(\alpha) \in (0, \pi/2]$ small enough so

$$\varphi := \frac{\pi\alpha}{4} + \frac{\alpha}{2}\eta < \frac{\pi}{2}$$

and let $z = e^{i\eta} \in \mathbb{C}^+$ when $\Im(y) \geq 0$ while $z = e^{i(\pi-\eta)} \in \mathbb{C}^+$ otherwise. Either way, $\Im(z) = \sin(\eta) > 0$ and if $y = Re^{i\theta} \in \mathcal{K}_\alpha$, that is $|\theta| \leq \alpha\pi/2$, then

$$\Re\left((-z)^{\frac{\alpha}{2}}(i)^{\frac{\alpha}{2}}y\right) = R \cos(|\theta| - \frac{\pi\alpha}{4} + \frac{\alpha}{2}\eta) \geq R \cos(\varphi) > 0.$$

Setting $\xi := \xi(\alpha) = \cos(\varphi)/(\sin(\eta))^{\alpha/2} > 0$ we thus deduce from (3.17) that for any $\beta > 0$,

$$\begin{aligned} |g_{\alpha,\beta}(y)| &\leq \int_0^\infty t^{\frac{\beta}{2}-1} e^{-t\sin(\eta)} \exp[-t^{\frac{\alpha}{2}}|y| \cos(\varphi)] dt \\ (3.18) \quad &= (\sin(\eta))^{-\beta/2} g_{\alpha,\beta}(\xi|y|) \leq (\sin(\eta))^{-\beta/2} g_{\alpha,\beta}(0), \end{aligned}$$

is uniformly bounded on \mathcal{K}_α . \square

Recall that a mapping $\underline{f} : \mathbb{U} \mapsto \mathbb{C}^q$ defined on some open $\mathbb{U} \subseteq \mathbb{C}^n$ is holomorphic on \mathbb{U} if each of its coordinates admits a convergent power series expansion around each point of \mathbb{U} . Proposition 3.5 suggests viewing $(X_r(z), 1 \leq r \leq q)$ as an implicit mapping from \mathbb{C}^+ into \mathbb{C}^q that is defined in terms of the zero set of the holomorphic $\underline{f} = (f_r(z, w_1, \dots, w_q), 1 \leq r \leq q)$, where

$$f_r(z, w_1, \dots, w_q) = w_r - C(\alpha) \int_0^\infty t^{-1} (it)^{\frac{\alpha}{2}} e^{itz} \exp\left\{- (it)^{\frac{\alpha}{2}} \sum_{s=1}^q c_{rs} w_s\right\} dt,$$

and $c_{rs} = \Gamma\left(1 - \frac{\alpha}{2}\right) |\sigma_{rs}|^\alpha \Delta_s$. Key properties of $(X_r(z), 1 \leq r \leq q)$ are then consequences of the rich theory of zero sets of holomorphic mappings. We shall employ this strategy, but for $\underline{Y}(z) \equiv (Y_1(z), \dots, Y_q(z))$ where $Y_r(z) := (-z)^{-\frac{\alpha}{2}} \widehat{X}_r(z)$ and $\widehat{X}_r(z)$ is given by (3.11). Indeed, our next result, extending [1, Theorem 6.1], characterizes $\underline{Y}(z)$ as implicitly defined for $u = z^{-\alpha}$ via $u \mapsto \underline{V}(u)$ such that

$$(3.19) \quad \underline{F}(u, \underline{V}(u)) = \underline{0}.$$

With $a_{rs} = C_\alpha |\sigma_{rs}|^\alpha \Delta_s$, the holomorphic mapping $\underline{F} : \mathbb{C} \times \mathbb{C}^q \mapsto \mathbb{C}^q$ is given for $u \in \mathbb{C}$ and $\underline{y} = (y_1, \dots, y_q) \in \mathbb{C}^q$ by

$$(3.20) \quad F_r(u, \underline{y}) = y_r - u \sum_{s=1}^q a_{rs} g_\alpha(y_s) \quad 1 \leq r \leq q$$

Proposition 3.7. *Setting $\mathcal{E}_\alpha := \{u \in \mathbb{C} : -\pi\alpha < \arg(u) < 0\}$, there exist $\varepsilon = \varepsilon(\sigma) > 0$ and a unique analytic solution $\underline{y} = \underline{V}(u)$ of $\underline{F}(u, \underline{y}) = \underline{0}$ on the open set $\mathcal{E}_{\alpha,\varepsilon} := \mathcal{E}_\alpha \cup \mathbb{B}(0, \varepsilon)$. Further, there exists a unique collection of analytic functions $(X_r(z), 1 \leq r \leq q)$ on \mathbb{C}^+ such that $|X_r(z)| \leq (\Im(z))^{-\frac{\alpha}{2}}$ and for which (3.10) holds. The functions $Y_r(z) = (-z)^{-\frac{\alpha}{2}} \widehat{X}_r(z)$ are then the unique solution of (1.11) analytic on $z \in \mathbb{C}^+$ and each tending to zero as $|z| \rightarrow \infty$. Moreover, $Y_r(z) = V_r(z^{-\alpha}) \in \mathcal{K}_\alpha$ are for $r = 1, \dots, q$ such that $Y_r(-\bar{z}) = \overline{Y_r(z)}$ and have an analytic continuation through (R, ∞) for some finite $R = R(\sigma)$, whereas $z^{\frac{\alpha}{2}} X_r(z)$ (hence $z^\alpha Y_r(z)$), are uniformly bounded on \mathbb{C}^+ .*

Proof. First, with $(-z)^{\frac{\alpha}{2}}(-z)^{-\frac{\alpha}{2}} = 1$, we deduce from (3.17) that (3.10) is equivalent to

$$(3.21) \quad X_r(z) = C(\alpha)(-z)^{-\frac{\alpha}{2}}g_{\alpha,\alpha}(Y_r(z)),$$

which in combination with (3.11) shows that $(Y_r(z), 1 \leq r \leq q)$ satisfies (1.11). The existence of analytic solutions $(X_r(z), 1 \leq r \leq q)$ and $(Y_r(z), 1 \leq r \leq q)$ such that $|X_r(z)| \leq (\Im(z))^{-\frac{\alpha}{2}}$ is thus obvious from Proposition 3.5. This solution of (1.11) consists by Lemma 3.6 of analytic functions from \mathbb{C}^+ to \mathcal{K}_α . Further, by the boundedness of $g_\alpha(\cdot)$ on \mathcal{K}_α we know that $|X_r(z)| \leq \kappa|z|^{-\alpha/2}$ and $|Y_r(z)| \leq \kappa|z|^{-\alpha}$ for some finite constant κ , all $z \in \mathbb{C}^+$ and $r \in \{1, \dots, q\}$.

We turn to prove the uniqueness of the analytic solution of (1.11) tending to zero as $\Im(z) \rightarrow \infty$ (hence the uniqueness of such solutions tending to zero as $|z| \rightarrow \infty$). To this end, considering \underline{F} of (3.20) note that $\underline{F}(0, \underline{0}) = \underline{0}$ and the complex Jacobian matrix of $\underline{y} \mapsto \underline{F}(0, \underline{y})$ at $\underline{y} = \underline{0}$ has a non-zero determinant (since $\partial_{y_s} F_r(0, \underline{0}) = \delta_{rs}$, with determinant one). Consequently, by the local implicit function theorem there are positive constants ε, δ and an analytic solution $\underline{y} = \underline{V}(u)$ of $\underline{F}(u, \underline{y}) = \underline{0}$ on $\mathbb{B}(0, \varepsilon)$ which for any $|u| < \varepsilon$ is also the unique solution with $\|\underline{y}\| < \delta$. Identifying \mathbb{C}^+ with \mathcal{E}_α via the analytic function $u = z^{-\alpha}$, note that $\underline{Y}(z)$ solves (1.11) for $z \in \mathbb{C}^+$ if and only if $\underline{V}(u) = \underline{Y}(z)$ satisfies (3.19) for $u \in \mathcal{E}_\alpha$. Consequently, setting $R = \varepsilon^{-1/\alpha}$ finite, any two solutions $\underline{Y}_i(z), i = 1, 2$ of (1.11) that tend to zero as $\Im(z) \rightarrow \infty$ coincide once $\Im(z) > R$ is large enough to assure that $\max_{i=1,2} \|\underline{Y}_i(z)\| < \delta$. The uniqueness of the analytic solution $z \mapsto \underline{Y}(z)$ of (1.11) on \mathbb{C}^+ tending to zero as $\Im(z) \rightarrow \infty$ then follows by the identity theorem. By (3.21) this implies also the uniqueness of the solution of (3.10) which is analytic and bounded by $(\Im(z))^{-\frac{\alpha}{2}}$ throughout \mathbb{C}^+ . Moreover, by the identity theorem, $u \mapsto \underline{V}(u)$ extends uniquely to an analytic solution of (3.19) on $\mathcal{E}_{\alpha,\varepsilon}$ and $\underline{Y}(z) = \underline{V}(z^{-\alpha})$ has an analytic extension through (R, ∞) .

Next, recall that $\mathbf{A}_N^{\kappa,-\sigma} = -\mathbf{A}_N^{\kappa,\sigma}$ are real-valued matrices, hence by definition $\mathbf{G}_N^{\kappa,-\sigma}(z) = -\overline{\mathbf{G}_N^{\kappa,\sigma}(-\bar{z})}$ for any $z \in \mathbb{C}^+$, implying by (3.3) that $L_{N,s}^{z,\kappa,-\sigma}(f(x)) = L_{N,s}^{-\bar{z},\kappa,\sigma}(f(-\bar{x}))$. If $x \in \mathbb{K}(z)$ then so is $-\bar{x}$ and $\overline{x^{\alpha/2}} = i^\alpha(-\bar{x})^{\alpha/2}$. It thus follows from Proposition 3.5 that $\overline{X_s^\sigma}(z) = i^\alpha X_s^\sigma(-\bar{z})$ for any $z \in \mathbb{C}^+$ and $1 \leq s \leq q$. Since $(X_r^\sigma(z), 1 \leq r \leq q)$ are uniquely determined by the equations (3.10) which are invariant under $\sigma \mapsto -\sigma$ and $(-z)^{\alpha/2} = i^\alpha(\bar{z})^{\alpha/2}$ for all $z \in \mathbb{C}^+$, we thus deduce from (3.11) that $\overline{Y_r}(z) = Y_r(-\bar{z})$ for all $1 \leq r \leq q$ and $z \in \mathbb{C}^+$. \square

To recap, for some $\varepsilon > 0$ we got the existence of a unique analytic solution $\underline{y} = \underline{V}(u)$ of $\underline{F}(u, \underline{y}) = \underline{0}$ on $\mathcal{E}_{\alpha,\varepsilon}$ for the holomorphic mapping $\underline{F} : \mathbb{C} \times \mathbb{C}^q \mapsto \mathbb{C}^q$ of (3.20). We proceed to show that $\underline{V}(u)$ has a continuous algebraic extension to $\overline{\mathcal{E}_{\alpha,\varepsilon}}$, and in particular to $(0, \infty)$ (by algebraic extension we mean that (3.19) holds throughout $\overline{\mathcal{E}_{\alpha,\varepsilon}}$). As we show in the sequel, this yields the claimed continuity of the density ρ^σ in Theorem 1.3.

To this end, recall that $\mathbb{M} \subseteq \mathbb{C}^n$ is an embedded complex manifold (in short, a manifold), of dimension p if for each $\underline{a} \in \mathbb{M}$ there exist a neighborhood \mathbb{U} of \underline{a} in \mathbb{C}^n and a holomorphic mapping $\underline{f} : \mathbb{U} \mapsto \mathbb{C}^{n-p}$ such that $\mathbb{M} \cap \mathbb{U} = \{\underline{z} \in \mathbb{U} : \underline{f}(\underline{z}) = \underline{0}\}$ and the complex Jacobian matrix of $\underline{f}(\cdot)$ is of rank $n-p$ at \underline{a} (in short, $\text{rank}_{\underline{a}}(\underline{f}) = n-p$, c.f [5, Definition 2, Section A.2.2]). Indeed, our claim is merely an application of the following general extension result for the mapping \underline{F} of (3.20), taking $u_0 = 0$ in the nonempty open simply connected set $\mathcal{O} = \mathcal{E}_{\alpha,\varepsilon}$ of piecewise smooth boundary.

Proposition 3.8. *Suppose $\underline{F} : \mathbb{C} \times \mathbb{C}^q \mapsto \mathbb{C}^q$ is a holomorphic mapping and $\underline{F}(u, \underline{V}(u)) = \underline{0}$ for analytic $\underline{V} : \mathcal{O} \mapsto \mathbb{C}^q$ and a nonempty open connected $\mathcal{O} \subseteq \mathbb{C}$. Suppose further that the graph*

$$(3.22) \quad \mathbb{V} := \{(u, \underline{V}(u)) : u \in \mathcal{O}\}$$

of \underline{V} is a one-dimensional complex manifold and the Jacobian determinant $\det[\partial_{\underline{y}}\underline{F}]$ is non-zero at some $\underline{v}_0 = (u_0, \underline{V}(u_0))$ with $u_0 \in \mathcal{O}$. Then, $\underline{V}(\cdot)$ has a continuous extension at boundary points $x \in \overline{\mathcal{O}}$ where \mathcal{O} is locally connected and \underline{V} is locally uniformly bounded (i.e. $\mathcal{O} \cup \{x\}$ admits a local basis of connected relative neighborhoods and \underline{V} is uniformly bounded on $U \cap \mathcal{O}$ for some neighborhood U of x in \mathbb{C}). Moreover, $\underline{F}(x, \underline{V}(x)) = \underline{0}$ at any such point.

Deferring the proof of Proposition 3.8 to the end of this section, we next collect all properties needed for applying it in our setting.

Lemma 3.9. *Assuming $\sigma \neq 0$, the mapping $u \mapsto \underline{V}(u)$ of Proposition 3.7 is injective on $\mathcal{E}_{\alpha, \varepsilon}$ (and consequently, so is the map $z \mapsto \underline{V}(z) = \underline{V}(z^{-\alpha})$). Further, in this case $\mathbb{V} := \{(u, \underline{V}(u)) : u \in \mathcal{E}_{\alpha, \varepsilon}\}$ is a one-dimensional complex manifold containing the point $(0, \underline{0})$ where $[\partial_{\underline{y}}\underline{F}]$ is the identity matrix, and $\|\underline{V}(u)\|_2 \leq K|u|$ for some finite constant $K = K(\sigma)$ and all $u \in \mathcal{E}_{\alpha, \varepsilon}$.*

Proof. First note that if $\underline{F}(u, \underline{y}) = \underline{F}(\tilde{u}, \underline{y}) = \underline{0}$ for some $\underline{y} \neq \underline{0}$ then by (3.20) necessarily $u = \tilde{u}$. Further, by excluding $\sigma \equiv 0$ we made sure that if $\underline{F}(u, \underline{0}) = \underline{0}$ then $u = 0$ (since $g_\alpha(0) > 0$ and $\sum_s a_{rs} \neq 0$ for some r). In particular, $u \mapsto \underline{V}(u)$ is injective. By the same reasoning, $\underline{V}'(u) \neq \underline{0}$. Indeed, (3.19) amounts to

$$(3.23) \quad V_r(u) - u \sum_{s=1}^q a_{rs} g_\alpha(V_s(u)) = 0 \quad 1 \leq r \leq q$$

and differentiating this identity in u , we see that if $\underline{V}'(u) = \underline{0}$ then necessarily

$$(3.24) \quad \sum_{s=1}^q a_{rs} g_\alpha(V_s(u)) = 0 \quad 1 \leq r \leq q.$$

Clearly, if (3.24) holds then it follows from (3.23) that $\underline{V}(u) = \underline{0}$ and as we have already seen, for $\sigma \neq 0$ it is then impossible for (3.24) to hold.

Next we show that $\mathbb{V} \subseteq \mathbb{C} \times \mathbb{C}^q$ is a complex one-dimensional manifold, by finding for any point $u \in \mathcal{E}_{\alpha, \varepsilon}$, a suitable holomorphic mapping from a neighborhood \mathbb{U} of $\underline{v} = (u, \underline{V}(u))$ in \mathbb{C}^{q+1} to \mathbb{C}^q having a Jacobian of rank q at \underline{v} . Indeed, as it is not possible to have $V'_1(u) = \dots = V'_q(u) = 0$, we may assume without loss of generality that, for a given u , $V'_q(u) \neq 0$. Then, by the inverse function theorem there exists a neighborhood $U \subseteq \mathcal{E}_{\alpha, \varepsilon}$ of u with $V_q(\cdot)$ having an analytic inverse on the neighborhood $V_q(U)$ of $V_q(u)$. Thus, on the neighborhood $\mathbb{U} = U \times \mathbb{C}^{q-1} \times V_q(U)$ of \underline{v} in \mathbb{C}^{q+1} we have the holomorphic mapping $\underline{f} : \mathbb{U} \mapsto \mathbb{C}^q$ where $f_r(w, \underline{y}) = y_r - V_r(V_q^{-1}(y_q))$ for $1 \leq r \leq q-1$ and $f_q(w, \underline{y}) = V_q(w) - y_q$. Clearly, $\underline{f}(w, \underline{y}) = \underline{0}$ for $(w, \underline{y}) \in \mathbb{U}$ if and only if $\underline{y} = \underline{V}(w)$ and $w \in U$, hence $\{(w, \underline{y}) \in \mathbb{U} : \underline{f}(w, \underline{y}) = \underline{0}\}$ is precisely $\mathbb{V} \cap \mathbb{U}$. Further, since $\partial_{y_r} f_s = \delta_{rs}$ for $1 \leq r \leq q-1$ and $\partial_w f_s = V'_q(w) \delta_{qs}$, the Jacobian determinant at \underline{v} of $\underline{f}(\cdot, y_q)$ with y_q fixed is $V'_q(u) \neq 0$. We conclude that $\text{rank}_{\underline{v}}(\underline{f}) = q$ and \mathbb{V} is a one dimensional complex manifold, as claimed.

Finally, while proving Proposition 3.7 we found that $\det[\partial_{\underline{y}}\underline{F}](0, \underline{0}) = 1$, that $\underline{V}(u) \in (\mathcal{K}_\alpha)^q$ for all $u \in \mathcal{E}_\alpha$ and that $\underline{V}(\cdot)$ is uniformly bounded on $\mathbb{B}(0, \varepsilon)$. With

$g_\alpha(\cdot)$ uniformly bounded on \mathcal{K}_α (and on compacts), it follows from (3.23) that $\|\underline{V}(u)\|_2 \leq K|u|$ for some finite constant $K = K(\sigma)$ and all $u \in \mathcal{E}_{\alpha,\varepsilon}$. \square

Remark 3.10. *The assumptions of Proposition 3.8 do not yield a unique extension of \underline{V} around boundary points of \mathcal{O} . That is, the extension provided there may well be non-analytic. For example, the Cauchy-Stieltjes transform $y = G_2(z)$ of the semi-circle law μ_2 at $z = u^{-1}$ is specified in terms of zeros of the holomorphic function $F(u, y) = y - u(y^2 + 1)$ on \mathbb{C}^2 . It is not hard to check that for any positive $\varepsilon < 1/2$ the unique analytic solution $y = V(u)$ of $F(u, y) = 0$ on $\mathcal{E}_{1,\varepsilon}$ is then $V(u) = (1 - \sqrt{1 - 4u^2})/(2u)$ for $u \neq 0$ and $V(0) = 0$. Following the arguments of Lemma 3.9, one finds that this injective function is uniformly bounded in the neighborhood of any boundary point of $\mathcal{E}_{1,\varepsilon}$ and its graph \mathbb{V} is a one-dimensional manifold containing the origin (where $\partial F/\partial y = 1$). However, $V(x)$ does not have an analytic extension at $x = 1/2$ as the corresponding density $\rho_2(t)$ is not real-analytic at $t = \pm 2$.*

For the convenience of the reader, we summarize, following the reference [5], the terminology and results about analytic functions of several complex variables which we use in proving Proposition 3.8.

A (local) analytic set is a subset \mathbb{A} of a complex manifold \mathbb{M} such that for any $\underline{a} \in \mathbb{A}$ there exists a neighborhood \mathbb{U} of \underline{a} in \mathbb{M} and a holomorphic mapping $\underline{f} : \mathbb{U} \mapsto \mathbb{C}^n$ such that $\mathbb{A} \cap \mathbb{U} = \{\underline{z} \in \mathbb{U} : \underline{f}(\underline{z}) = \underline{0}\}$ (in contrast with a manifold, there is no condition on the rank of the Jacobian of the mapping \underline{f}). We call $\mathbb{A} \subseteq \mathbb{M}$ an analytic subset of the complex manifold \mathbb{M} if this further applies at all $\underline{a} \in \mathbb{M}$ (and not only at the points \underline{a} in \mathbb{A}), and say that \mathbb{A} is a *proper* analytic subset of \mathbb{M} if $\mathbb{A} \neq \mathbb{M}$. In particular, any embedded complex manifold is an analytic set (of \mathbb{C}^q), but, unless it is closed in \mathbb{C}^q , it cannot be an analytic subset of \mathbb{C}^q . For example, $\mathbb{H} = \{\underline{z} \in \mathbb{C}^q : \|\underline{z}\|_2 < 1, z_1 = 0\}$ is a manifold (of dimension $q - 1$), a (local) analytic set in \mathbb{C}^q , but not an analytic subset of \mathbb{C}^q . However, as observed in [5, Section 1.2.1], every (local) analytic set on a complex manifold \mathbb{M} is an analytic subset of a certain neighborhood of \mathbb{M} (for example, \mathbb{H} is an analytic subset of the open unit ball in \mathbb{C}^q).

A point of an analytic set \mathbb{A} (on \mathbb{C}^q) is called *regular* if it has a neighborhood \mathbb{U} (in \mathbb{C}^q) so that $\mathbb{A} \cap \mathbb{U}$ is a manifold in \mathbb{C}^q . Clearly, the set $\text{reg}\mathbb{A}$ of regular points of an analytic set \mathbb{A} is a union of manifolds (alternatively, an analytic set is a manifold around each of its regular points). Topologically, most points of an analytic set are regular. That is, for an arbitrary analytic set \mathbb{A} the set $\text{reg}\mathbb{A}$ of regular points is everywhere dense in \mathbb{A} (c.f. [5, Section 1.2.3]). Thus, the dimension $\dim_{\underline{a}}\mathbb{A}$ of \mathbb{A} at a point $\underline{a} \in \mathbb{A}$ is defined as the dimension of the manifold around \underline{a} if $\underline{a} \in \text{reg}\mathbb{A}$ and in general by

$$\dim_{\underline{a}}\mathbb{A} = \limsup_{\underline{z} \rightarrow \underline{a}, \underline{z} \in \text{reg}\mathbb{A}} \dim_{\underline{z}}\mathbb{A}.$$

The dimension of the analytic set \mathbb{A} , denoted $\dim\mathbb{A}$ is then the largest such number when \underline{a} runs through \mathbb{A} and an analytic set \mathbb{A} is called p -dimensional if $\dim\mathbb{A} = p$ (see [5, Section 1.2.4]).

An essential ingredient of our proof is the notion of *irreducibility* and of irreducible components for analytic sets [5, Section 1.5.3]. An analytic subset \mathbb{A} of a complex manifold \mathbb{M} is reducible in \mathbb{M} if there exist two analytic subsets $\mathbb{A}_1, \mathbb{A}_2$

of \mathbb{M} so that $\mathbb{A} = \mathbb{A}_1 \cup \mathbb{A}_2$ and $\mathbb{A}_1 \neq \mathbb{A} \neq \mathbb{A}_2$. Otherwise \mathbb{A} is called irreducible. For example, $\mathbb{A} = \{\underline{z} \in \mathbb{C}^3: z_1 z_2 = z_1 z_3 = 0\}$ is a reducible set, being the union of $\mathbb{A}_1 = \{\underline{z} \in \mathbb{C}^3: z_2 = z_3 = 0\}$ (a one dimensional manifold), and $\mathbb{A}_2 = \{\underline{z} \in \mathbb{C}^3: z_1 = 0\}$ (a two dimensional manifold). An irreducible analytic subset \mathbb{A}' of an analytic set \mathbb{A} is called an *irreducible component* of \mathbb{A} if every analytic subset \mathbb{A}'' of \mathbb{A} such that $\mathbb{A}' \subsetneq \mathbb{A}''$ is reducible. It is known [5, Theorem, Section 1.5.4] that any analytic subset \mathbb{A} of a complex manifold \mathbb{M} has a unique decomposition into countably (or finitely) many irreducible components $\overline{\mathbb{S}}_j$, which are the closures (in \mathbb{M}) of the partition $\{\mathbb{S}_j\}$ of $\text{reg}\mathbb{A}$ into disjoint connected components. Further, $\dim\overline{\mathbb{S}}_j = \dim\mathbb{S}_j$ [5, Theorem, Section 1.5.1] and by definition of regular points each connected component \mathbb{S}_j is a manifold (in case $\mathbb{M} = \mathbb{C}^q$).

The importance of irreducibility for us resides in the following 'uniqueness' result: if \mathbb{A}, \mathbb{A}' are analytic subsets of a complex manifold, \mathbb{A} is irreducible and $\mathbb{A} \not\subseteq \mathbb{A}'$, then $\dim\mathbb{A} \cap \mathbb{A}' < \dim\mathbb{A}$ [5, Section 1.5.3, Corollary 1].

Topological properties simplify considerably when a set \mathbb{A} is contained in a proper analytic subset of a *connected* complex manifold \mathbb{M} . Indeed, in this context $\mathbb{M} \setminus \mathbb{A}$ is arc-wise connected and in case of a one dimensional manifold \mathbb{M} (that is, a *Riemann surface*), we further have that \mathbb{A} is locally finite i.e. $\mathbb{A} \cap \mathbb{K}$ is a finite set for any compact $\mathbb{K} \subseteq \mathbb{M}$ (see [5, Section 1.2.2]).

Proof of Proposition 3.8. Clearly, \mathbb{V} is a connected set (being the graph of a continuous function on the connected set \mathcal{O}). Further, by our assumptions, the connected one-dimensional complex manifold \mathbb{V} is contained in the analytic subset

$$\mathbb{A} = \{(u, \underline{y}) \in \mathbb{C} \times \mathbb{C}^q: \underline{F}(u, \underline{y}) = \underline{0}\},$$

of $\mathbb{C} \times \mathbb{C}^q$ given by the zeros of the holomorphic mapping \underline{F} .

We proceed to show the crux of our argument, that the closure $\overline{\mathbb{V}}$ of \mathbb{V} (in \mathbb{C}^{q+1}) is part of a one-dimensional irreducible component of \mathbb{A} . To this end, consider the analytic subset

$$\mathbb{D} = \{(u, \underline{y}) \in \mathbb{C} \times \mathbb{C}^q: \det[\partial_{\underline{y}}\underline{F}](u, \underline{y}) = 0\}$$

of $\mathbb{C} \times \mathbb{C}^q$. By definition, $\mathbb{V} \cap \mathbb{D}$ is an analytic subset of \mathbb{V} , and it is a proper subset, for we know that $\underline{v}_0 = (u_0, \underline{V}(u_0)) \in \mathbb{V} \setminus \mathbb{D}$. Further, by the implicit function theorem, the analytic subset \mathbb{A} is regular at any $\underline{a} \in \mathbb{A} \setminus \mathbb{D}$, so $\mathbb{V} \setminus \text{reg}\mathbb{A}$ is contained in the proper analytic subset $\mathbb{V} \cap \mathbb{D}$ of the Riemann surface \mathbb{V} . Consequently, by [5, Proposition 2, Section 1.2.2], we deduce that \mathbb{V} is an 'almost regular' part of \mathbb{A} . That is, $\mathbb{V} \setminus \text{reg}\mathbb{A}$ is locally finite and consequently the closure $\overline{\mathbb{V}}$ of \mathbb{V} in \mathbb{C}^{q+1} is the same as the closure of $\mathbb{V} \cap \text{reg}\mathbb{A}$. Further, by [5, Proposition 3, Section 1.2.2], $\mathbb{V} \cap \text{reg}\mathbb{A}$ is arc-wise connected, hence included in *one* connected component \mathbb{S} of $\text{reg}\mathbb{A}$, with $\overline{\mathbb{V}} = \overline{\mathbb{V} \cap \text{reg}\mathbb{A}}$ thus contained in the closure $\overline{\mathbb{S}}$ of \mathbb{S} (in \mathbb{C}^{q+1}).

Recall that by definition the connected component \mathbb{S} of $\text{reg}\mathbb{A}$ is a manifold and since \underline{v}_0 is in $\mathbb{V} \cap \text{reg}\mathbb{A}$, we have that \mathbb{S} contains the manifold $\mathbb{V} \cap \mathbb{U}$ for some neighborhood \mathbb{U} of \underline{v}_0 (in \mathbb{C}^{q+1}). The connected manifold $\mathbb{V} \cap \mathbb{U}$ has an accumulation point in both \mathbb{V} and \mathbb{S} , so all three have the same dimension, that is, $\dim\mathbb{S} = \dim\mathbb{V} = 1$ (see [5, Section 1.2.2]). As shown in [5, Theorem, Section 1.5.4], the irreducible component of \mathbb{A} containing \mathbb{S} is its closure $\overline{\mathbb{S}}$, which by the definition of an irreducible component is an analytic subset of \mathbb{C}^{q+1} and further by [5, Theorem, Section 1.5.1] $\dim\overline{\mathbb{S}} = \dim\mathbb{S} = 1$.

We claim that if $\underline{V}(u)$ is uniformly bounded on $\mathcal{O} \cap U$ for some neighborhood U (in \mathbb{C}) of a boundary point $x \in \overline{\mathcal{O}}$ where \mathcal{O} is locally connected, then the existence of

the one-dimensional irreducible analytic subset $\overline{\mathbb{S}}$ of \mathbb{C}^{q+1} insures that $\underline{V}(\cdot)$ extends continuously at x such that $\underline{F}(x, \underline{V}(x)) = \underline{0}$. Indeed, since \underline{V} is uniformly bounded on $\mathcal{O} \cap U$, by the continuity of $\underline{V}(\cdot)$ on \mathcal{O} the cluster set $Cl(x)$ of all limit points of $\{\underline{V}(u) : u \in \mathcal{O}\}$ as $u \rightarrow x$, is a non-empty, compact, *connected* subset of \mathbb{C}^q (see the proof given in [6, Theorem 1.1] for $q = 1$). Clearly, $\{x\} \times Cl(x)$ is contained in the analytic subset $\mathbb{A}(x) = \{(u, \underline{y}) \in \mathbb{A} : u = x\}$ of $\mathbb{C} \times \mathbb{C}^q$ as well as in the closure $\overline{\mathbb{V}}$ of \mathbb{V} (in \mathbb{C}^{q+1}). With $\overline{\mathbb{V}} \subseteq \overline{\mathbb{S}}$, we thus deduce that $\{x\} \times Cl(x) \subseteq \mathbb{A}(x) \cap \overline{\mathbb{S}}$.

Recall that \mathbb{S} is a one-dimensional, irreducible analytic subset of \mathbb{C}^{q+1} . Since $\overline{\mathbb{S}} \not\subseteq \mathbb{A}(x)$ (as $\underline{v}_0 \in \mathbb{V}$ and u_0 is not a boundary point of \mathcal{O}), by [5, Corollary 1, Section 1.5.3] we have that $\dim \mathbb{A}(x) \cap \overline{\mathbb{S}} = 0$. Thus, $\mathbb{A}(x) \cap \overline{\mathbb{S}}$ is a discrete (analytic) set, so its connected subset $\{x\} \times Cl(x)$ must be a single point, i.e. \underline{V} extends continuously at x . Moreover, \mathbb{A} is a closed subset of $\mathbb{C} \times \mathbb{C}^q$ (by continuity of \underline{F}), hence the extension $\underline{V}(x)$ of \underline{V} satisfies $\underline{F}(x, \underline{V}(x)) = \underline{0}$, as claimed. \square

3.3. Limiting spectral measures: proof of Theorem 1.3. Fixing $z \in \mathbb{C}^+$ let $(\mu_s^z, 1 \leq s \leq q)$ denote some limit point of the compactly supported $(\mathbb{E}[L_{N,s}^{z,\kappa}], 1 \leq s \leq q)$. Then, on the corresponding subsequence, $X_{N,r}(z)$ converges for $r = 1, \dots, q$ to $\int x^{\frac{\alpha}{2}} d\mu_r^z$ which by Propositions 3.5 and 3.7 thus coincides with the unique analytic solution $X_r(z)$ of (3.10) bounded by $(\Im(z))^{-\frac{\alpha}{2}}$. By (3.7) (for $f(x) = x$ bounded and continuous on $\mathbb{K}(z)$), the identity $z^{-1} = -i \int_0^\infty e^{itz} dt$, and Fubini's theorem, we deduce that for each $r \in \{1, \dots, q\}$,

$$\begin{aligned} \int x d\mu_r^z &= -i \int_0^\infty e^{itz} \prod_{s=1}^q \left(\int \exp\{-it\sigma_{rs}^2 \Delta_s^{\frac{2}{\alpha}} x_s\} dP^{\mu_s^z}(x_s) \right) dt \\ (3.25) \quad &= -i \int_0^\infty e^{itz} \exp\{-(it)^{\frac{\alpha}{2}} \widehat{X}_r(z)\} dt = z^{-1} g_{\alpha,2}(Y_r(z)), \end{aligned}$$

where we get the latter equality from (3.13) and the definition (3.11) of $\widehat{X}_r(z)$, followed by the application of (3.17) with $\beta = 2$. In particular, by Proposition 3.7 $\int x d\mu_r^z$ are uniquely determined (for all r and $z \in \mathbb{C}^+$), hence $\mathbb{E}[L_{N,s}^{z,\kappa}(x)]$ converges as $N \rightarrow \infty$ to the right side of (3.25).

Next, by Lemma 2.1 the sequence $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{\kappa,\sigma}}]$ is tight for the topology of weak convergence. Further, recall that for any $z \in \mathbb{C}^+$ and all N ,

$$\int \frac{1}{z-x} d\hat{\mu}_{\mathbf{A}_N^{\kappa,\sigma}}(x) = \sum_{s=1}^q \Delta_{N,s} L_{N,s}^{z,\kappa}(x).$$

Hence, any limit point μ^σ of $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{\kappa,\sigma}}]$ is such that for each $z \in \mathbb{C}^+$,

$$(3.26) \quad \int \frac{1}{z-x} d\mu^\sigma(x) = \sum_{s=1}^q \Delta_s \int x d\mu_s^z(x).$$

Recall that $g_{\alpha,2} = h_\alpha$, so combining (3.25) and (3.26) we thus arrive at the stated formula (1.10) for the values of the Cauchy-Stieltjes transform $G_{\alpha,\sigma}(z)$ of the probability measure μ^σ on the real line, at all $z \in \mathbb{C}^+$. Since h_α is uniformly bounded on the closed set \mathcal{K}_α (see Lemma 3.6), and $Y_s(z) \in \mathcal{K}_\alpha$ for all $z \in \mathbb{C}^+$ and $1 \leq s \leq q$, we deduce from (1.10) that $G_{\alpha,\sigma}(z)$ is uniformly bounded on $\mathbb{C}^+ \cap \mathbb{B}(0, \delta)^c$ for each $\delta > 0$. By the Stieltjes-Perron inversion formula, it follows that the density ρ^σ of the probability measure μ^σ with respect to Lebesgue measure on $\mathbb{R} \setminus \{0\}$ is bounded on $(-\delta, \delta)^c$ for any $\delta > 0$.

With $G_{\alpha,\sigma}(z)$ uniquely determined, we conclude that so is the weak limit μ^σ of $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{\kappa,\sigma}}]$. Further, applying Lemma 3.4 for $f(x) = x$ and considering the union bound over $1 \leq s \leq q$, we find that, with $\epsilon = 1 - \kappa(2 - \alpha) > 0$, for some $c(z)$ finite on \mathbb{C}^+ , any $z \in \mathbb{C}^+$, $\delta > 0$ and $N \in \mathbb{N}$,

$$\mathbb{P}\left(\left|\int \frac{1}{z-x} d\hat{\mu}_{\mathbf{A}_N^{\kappa,\sigma}}(x) - \mathbb{E}\left[\int \frac{1}{z-x} d\hat{\mu}_{\mathbf{A}_N^{\kappa,\sigma}}(x)\right]\right| \geq \delta\right) \leq \frac{qc(z)}{\delta^2} N^{-\epsilon}.$$

Consequently, setting $\phi(n) = [n^\gamma]$ for $\gamma = 2/\epsilon$, by the Borel-Cantelli lemma, with probability one, as $n \rightarrow \infty$,

$$G_n(z) := \int \frac{1}{z-x} d\hat{\mu}_{\mathbf{A}_{\phi(n)}^{\kappa,\sigma}}(x) \rightarrow G_{\alpha,\sigma}(z).$$

Since $G_n(z) \leq (\Im(z))^{-1}$ for all n and $z \in \mathbb{C}^+$, applying this for a countable collection z_k with a cluster point in \mathbb{C}^+ we deduce by Vitali's convergence theorem that with probability one, $G_n(z) \rightarrow G_{\alpha,\sigma}(z)$ for all $z \in \mathbb{C}^+$. Such convergence of the Cauchy-Stieltjes transforms implies of course that $\hat{\mu}_{\mathbf{A}_{\phi(n)}^{\kappa,\sigma}}$ converges weakly to μ^σ and by (2.2) we deduce after yet another application of the Borel-Cantelli lemma, that with probability one $\hat{\mu}_{\mathbf{A}_{\phi(n)}^\sigma}$ converges weakly to μ^σ . Finally, since $\phi(n-1)/\phi(n) \rightarrow 1$ we have from Lemma 2.4 that the same weak convergence to μ^σ holds for $\hat{\mu}_{\mathbf{A}_N^\sigma}$.

With $h_\alpha(\bar{y}) = \overline{h_\alpha(y)}$, combining the identities $Y_r(-\bar{z}) = \overline{Y_r(z)}$ of Proposition 3.7 with the formula (1.10) for the Cauchy-Stieltjes transform $G_{\alpha,\sigma}$ of the probability measure μ^σ on \mathbb{R} we find that $G_{\alpha,\sigma}(-\bar{z}) = -\overline{G_{\alpha,\sigma}(z)} = -G_{\alpha,\sigma}(\bar{z})$ for all $z \in \mathbb{C}^+$, hence necessarily $\mu^\sigma(\cdot) = \mu^\sigma(-\cdot)$ is symmetric about zero. Further, as shown in Proposition 3.7, $z^\alpha \underline{Y}(z)$ is uniformly bounded and extends analytically through the subset (R, ∞) , where $\underline{Y}(z) = \underline{V}(z^{-\alpha}) \in (\mathcal{K}_\alpha)^q$ is the unique analytic solution of (1.11) on $z \in \mathbb{C}^+$ that tend to zero as $|z| \rightarrow \infty$ (and as shown in Lemma 3.9 $z \mapsto \underline{Y}(z)$ is injective when $\sigma \neq 0$). If $\sigma \equiv 0$ then $\underline{V}(u) = \underline{0}$ is analytic on \mathbb{C} . Turning to $\sigma \neq 0$, in view of Lemma 3.9 the function \underline{V} is uniformly bounded on $\mathcal{E}_{\alpha,\epsilon} \cap \mathbb{K}$ for any compact $\mathbb{K} \subseteq \mathbb{C}$. Thus, combining Lemma 3.9 with Proposition 3.8 we find that $\underline{V}(u)$ has a continuous, algebraic extension to $(0, \infty)$. As $Y_r(-\bar{z}) = \overline{Y_r(z)}$, this yields the continuous, algebraic extension of $\underline{Y}(z)$ to $\mathbb{R} \setminus \{0\}$, analytic on (R, ∞) , from which we get by (1.10) and the analyticity of $h_\alpha(\cdot)$ the corresponding continuous/algebraic/analytic extension of $G_{\alpha,\sigma}(z)$. Recall Plemelj formula, that for $x \neq 0$, the limit as $\epsilon \downarrow 0$ of $-\pi^{-1} \Im(G_{\alpha,\sigma}(x+i\epsilon))$ is then precisely the continuous density $\rho^\sigma(x)$ of μ^σ with respect to Lebesgue measure on $\mathbb{R} \setminus \{0\}$, and $\rho^\sigma(x)$ is real-analytic on (R, ∞) .

4. PROOF OF THEOREM 1.7

We start with the following consequence of Proposition 2.2 and Theorem 1.3.

Corollary 4.1. *For any $\sigma \in \mathcal{F}_\alpha$, the probability measures $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\sigma}]$ converge weakly towards some symmetric probability measure μ^σ .*

Proof. We approximate σ in $L^2_\star([0,1]^2)$ by a sequence of piecewise constant functions σ_p . Applying Theorem 1.3 for $\sigma = \sigma_p$ we deduce that hypothesis (2.6) holds. Hence, by Proposition 2.2 $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{\sigma_p}}]$ converges weakly towards the limit μ^{σ_p} of the corresponding measures μ^{σ_p} . We have seen already that μ^{σ_p} are symmetric measures, hence so is their limit μ^σ . \square

Fixing $\sigma \in \mathcal{F}_\alpha$ we proceed to characterize the limiting measure μ^σ . To this end, recall that $k_\sigma := \|\sigma\|^\alpha$ is finite and fix a sequence $\sigma_p \in \mathcal{C}_*$ that converges to σ in L_*^2 , satisfying (1.12) and such that $\sup_{p \in \mathbb{N}} \|\sigma_p\|^\alpha \leq 2k_\sigma$. For each $p \in \mathbb{N}$ let $0 = b_0^p < b_1^p < \dots < b_{q(\sigma_p)}^p = 1$ denote the finite partition of $[0, 1]$ induced by σ_p and per $z \in \mathbb{C}^+$ consider the piecewise constant function $Y^{\sigma_p}(z) : (0, 1] \rightarrow \mathcal{K}_\alpha$ such that

$$Y_x^{\sigma_p}(z) = Y_s(z) \text{ for } x \in (b_{s-1}^p, b_s^p] \text{ and } s = 1, \dots, q(\sigma_p),$$

where $Y_s(z) \in \mathcal{K}_\alpha$ is the unique collection of (analytic) functions of $z \in \mathbb{C}^+$ that satisfy (1.11) for the $q \times q$ matrix of entries $\sigma_{rs} := \sigma_p(b_r^p, b_s^p)$, as in Theorem 1.3. This way (1.14) holds for $\sigma = \sigma_p$ and each $p \in \mathbb{N}$ (being precisely (1.11)).

We next show the existence of $R = R(\sigma)$ finite such that if $|z| \geq R$ then $(Y^{\sigma_p}(z), p \in \mathbb{N})$ is a Cauchy sequence for the L^∞ -norm. To this end, it is convenient to view (1.14) (at each $z \in \mathbb{C}^+$) as the fixed point equation in $L^\infty((0, 1]; \mathcal{K}_\alpha)$

$$(4.1) \quad Y^\sigma = F_z(\sigma, Y^\sigma), \quad F_z(\sigma, Y) := C_\alpha z^{-\alpha} \int_0^1 |\sigma(\cdot, v)|^\alpha g_\alpha(Y_v) dv.$$

Then, with $\|g_\alpha\|_{\mathcal{K}_\alpha} := \sup\{|g_\alpha(y)| : y \in \mathcal{K}_\alpha\}$ finite by Lemma 3.6, bounding the L^∞ -norm of $F_z(\sigma, Y)$ for $Y \in \mathcal{K}_\alpha$ we deduce from (4.1) that for any $\epsilon > 0$

$$(4.2) \quad \sup_{|z| \geq \epsilon} \sup_{p \in \mathbb{N}} \|Y^{\sigma_p}\|_\infty \leq 2k_\sigma \epsilon^{-\alpha} C_\alpha \|g_\alpha\|_{\mathcal{K}_\alpha} =: r_\sigma$$

is finite. Note that for $\|\tilde{Y}\|_\infty \leq r$, $\|\hat{Y}\|_\infty \leq r$ and measurable $\tilde{\sigma}(\cdot, \cdot)$, $\hat{\sigma}(\cdot, \cdot)$,

$$(4.3) \quad \|F_z(\tilde{\sigma}, \tilde{Y}) - F_z(\hat{\sigma}, \hat{Y})\|_\infty \leq |z|^{-\alpha} \|g_\alpha\|_r \left[\|\tilde{\sigma}^\alpha - \hat{\sigma}^\alpha\| + \|\tilde{\sigma}^\alpha\| \|\tilde{Y} - \hat{Y}\|_\infty \right],$$

where $\|g_\alpha\|_r$ is the sum of the supremum and Lipschitz norms of $y \mapsto C_\alpha g_\alpha(y)$ on the ball $\{y \in \mathbb{C} : |y| \leq r\}$. Suppressing hereafter the dependence of $Y_x^{\sigma_p}(z)$ on z , since (σ_p, Y^{σ_p}) , $p \in \mathbb{N}$, satisfy (4.1), from (4.2) and (4.3) we have that for any $p, q \in \mathbb{N}$ and $|z| \geq \epsilon$,

$$\|Y^{\sigma_q} - Y^{\sigma_p}\|_\infty \leq |z|^{-\alpha} \|g_\alpha\|_{r_\sigma} \left[\|\sigma_q^\alpha - \sigma_p^\alpha\| + 2k_\sigma \|Y^{\sigma_q} - Y^{\sigma_p}\|_\infty \right].$$

Taking $R = R(\sigma) \geq \epsilon$ finite such that $R^{-\alpha} \|g_\alpha\|_{r_\sigma} k_\sigma \leq 1/3$, this implies that for $|z| \geq R$

$$\|Y^{\sigma_q} - Y^{\sigma_p}\|_\infty \leq 3|z|^{-\alpha} \|g_\alpha\|_{r_\sigma} \|\sigma_q^\alpha - \sigma_p^\alpha\|.$$

In view of (1.12), we conclude that $(Y^{\sigma_p}, p \in \mathbb{N})$ is for each $|z| \geq R$ a Cauchy sequence in $L^\infty(0, 1]$, which thus converges in this space to a bounded measurable function Y^σ from $(0, 1]$ to the closed set \mathcal{K}_α . Further, then $\|Y^\sigma\|_\infty \leq r_\sigma$ (see (4.2)), so from (4.3) and (1.12) we deduce that

$$\|F_z(\sigma, Y^\sigma) - F_z(\sigma_p, Y^{\sigma_p})\|_\infty \leq \epsilon^{-\alpha} \|g_\alpha\|_{r_\sigma} \left[\|\sigma^\alpha - \sigma_p^\alpha\| + k_\sigma \|Y^\sigma - Y^{\sigma_p}\|_\infty \right] \rightarrow 0,$$

as $p \rightarrow \infty$. With (4.1) holding for the pairs (σ_p, Y^{σ_p}) , $p \in \mathbb{N}$, it follows that the same applies for (σ, Y^σ) , thus establishing (1.14).

Turning to show the uniqueness of the solution to (1.14), suppose $Y_j = F_z(\sigma, Y_j)$ for $\sigma(\cdot, \cdot)$ such that $k_\sigma = \|\sigma\|^\alpha$ is finite, some $|z| \geq R(\sigma)$ and measurable $Y_j : (0, 1] \rightarrow \mathcal{K}_\alpha$, $j = 1, 2$. Then, as in the derivation of (4.2) we have that $\|Y_j\|_\infty \leq r_\sigma$ for $j = 1, 2$. So, applying (4.3) once more,

$$\|Y_1 - Y_2\|_\infty = \|F_z(\sigma, Y_1) - F_z(\sigma, Y_2)\|_\infty \leq |z|^{-\alpha} \|g_\alpha\|_{r_\sigma} k_\sigma \|Y_1 - Y_2\|_\infty \leq \frac{1}{3} \|Y_1 - Y_2\|_\infty$$

and necessarily $Y_1 = Y_2$ almost everywhere on $(0, 1]$.

To recap, the sequence of holomorphic mappings Y^{σ_p} from \mathbb{C}^+ to the closed subset $\mathbb{F} := L^\infty((0, 1]; \mathcal{K}_\alpha)$ of the complex Banach space $L^\infty((0, 1]; \mathbb{C})$ is such that $Y^{\sigma_p}(z) \rightarrow Y^\sigma(z)$ in \mathbb{F} at each point z of the non-empty open subset $\mathbb{B}(0, R)^c \cap \mathbb{C}^+$. Further, in view of (4.2) we have that $(Y^{\sigma_p}, p \in \mathbb{N})$ is locally uniformly bounded on \mathbb{C}^+ , hence by Vitali's convergence theorem for vector-valued holomorphic mappings, it converges at every $z \in \mathbb{C}^+$ to an analytic mapping $Y^\sigma : \mathbb{C}^+ \mapsto \mathbb{F}$ (see [4, Theorem 14.16]). We also characterized $Y^\sigma(z)$ for each $|z| \geq R$ as the unique solution in \mathbb{F} of (1.14), so by the identity theorem for vector-valued holomorphic mappings (see [4, Exercise 9C]), we have thus uniquely determined $Y^\sigma : \mathbb{C}^+ \mapsto \mathbb{F}$.

Next, note that the identity (1.13) holds for $\sigma = \sigma_p \in \mathcal{C}_*$, $p \in \mathbb{N}$, in which case it is merely the formula (1.10). Recall Proposition 2.2 that due to the L_\star^2 -convergence of σ_p to σ , for each $z \in \mathbb{C}^+$ the left hand side of these identities converge as $p \rightarrow \infty$ to $G_{\alpha, \sigma}(z) := \int (z - x)^{-1} d\mu^\sigma(x)$. If in addition $|z| \geq R(\sigma)$ then $\|Y^{\sigma_p} - Y^\sigma\|_\infty \rightarrow 0$ and by dominated convergence the right hand sides of same identities converge to the corresponding expression for $Y^\sigma(z)$. Thus, (1.13) holds also for $\sigma \in \mathcal{F}_\alpha$ and $|z| \geq R(\sigma)$. With μ^σ a probability measure on \mathbb{R} , the left side of (1.13) is obviously an analytic function of $z \in \mathbb{C}^+$. Further, the entire function $h_\alpha(\cdot)$ and its first two derivatives are uniformly bounded on the set \mathcal{K}_α (see Lemma 3.6), in which the analytic mapping $Y^\sigma : \mathbb{C}^+ \mapsto \mathbb{F}$ takes values. Hence, it is not hard to see that $z \mapsto \int_0^1 h_\alpha(Y_v^\sigma(z)) dv$ is also analytic on \mathbb{C}^+ . We thus deduce by the identity theorem that (1.13) holds for all $z \in \mathbb{C}^+$. Consequently, with $\int_0^1 h_\alpha(Y_v^\sigma(z)) dv$ uniformly bounded on \mathbb{C}^+ , the Cauchy-Stieltjes transform of μ^σ is uniformly bounded on $\mathbb{C}^+ \cap \mathbb{B}(0, \delta)^c$. This in turn implies (by the Stieltjes-Perron inversion formula), that the density ρ^σ of μ^σ with respect to Lebesgue measure on $\mathbb{R} \setminus \{0\}$ is bounded outside any neighborhood of zero.

We have seen already that $\|Y^\sigma\|_\infty \leq c(\sigma)|z|^{-\alpha}$ for some $c(\sigma)$ finite and all $|z| \geq R$. Hence, for $z \in \mathbb{C}^+$ such that $|z| \geq R$, we have from (1.13) and (1.14) that

$$\begin{aligned} G_{\alpha, \sigma}(z) &= \frac{1}{z} [h_\alpha(0) + h'_\alpha(0) \int_0^1 Y_x^\sigma(z) dx + O(|z|^{-2\alpha})] \\ &= \frac{1}{z} [h_\alpha(0) + z^{-\alpha} C_\alpha h'_\alpha(0) g_\alpha(0) \int_0^1 \int_0^1 |\sigma(x, v)|^\alpha dx dv + O(|z|^{-2\alpha})]. \end{aligned}$$

Recall Plemelj formula, that $\rho^\sigma(t)$ is the limit of $-\pi^{-1} \Im(G_{\alpha, \sigma}(t + i\epsilon))$ as $\epsilon \downarrow 0$. Thus, as $h_\alpha(0) \in \mathbb{R}$, it follows that $t^{\alpha+1} \rho^\sigma(t) \rightarrow L_\alpha \int |\sigma(x, v)|^\alpha dx dv$ as $t \rightarrow \infty$ and it is not hard to check that $L_\alpha = -\pi^{-1} h'_\alpha(0) g_\alpha(0) \Im(C_\alpha)$ equals $\frac{\alpha}{2}$ (by Euler's reflection formula for the Gamma function).

Turning to verify the last statement of the theorem, note that the equivalence between $\sigma \in \mathcal{F}_\alpha$ and $\tilde{\sigma} \in \mathcal{C}_*$ implies that the piecewise constant $Y^{\tilde{\sigma}}(z) : (0, 1] \mapsto \mathcal{K}_\alpha$ we have constructed before out of $(Y_s(z), 1 \leq s \leq q)$ satisfies (1.14) for any $x \in (0, 1]$ and all $z \in \mathbb{C}^+$. It then follows by the uniqueness of such solution of (1.14) that $Y_x^{\tilde{\sigma}}(z) = Y_x^\sigma(z)$ for all $z \in \mathbb{C}^+$ such that $|z| \geq R(\sigma)$ and almost every $x \in (0, 1]$. In view of (1.13), the Cauchy-Stieltjes transform of μ^σ coincides for such z with the Cauchy-Stieltjes transform $G_{\alpha, \tilde{\sigma}}(z)$ of $\mu^{\tilde{\sigma}}$. As such information uniquely determines the probability measure in question, it follows that $\mu^\sigma = \mu^{\tilde{\sigma}}$.

5. PROOF OF PROPOSITION 1.1 AND THEOREM 1.10

5.1. Convergence to μ_α^γ and its characterization. Consider the $(N + M)$ -dimensional square matrix

$$\mathbf{A}_{N,M} = \begin{pmatrix} 0 & a_{N+M}^{-1} \mathbf{X}_{N,M} \\ a_{N+M}^{-1} \mathbf{X}_{N,M}^t & 0 \end{pmatrix},$$

noting that $\mathbf{B}_{N,M} = \mathbf{A}_{N,M}^2$ is then of the form

$$\mathbf{B}_{N,M} = \begin{pmatrix} a_{N+M}^{-2} \mathbf{X}_{N,M} \mathbf{X}_{N,M}^t & 0 \\ 0 & a_{N+M}^{-2} \mathbf{X}_{N,M}^t \mathbf{X}_{N,M} \end{pmatrix} =: \begin{pmatrix} \mathbf{W}_{N,M} & 0 \\ 0 & \widetilde{\mathbf{W}}_{N,M} \end{pmatrix}$$

and that the eigenvalues of $\mathbf{W}_{N,M}$ consist of the M eigenvalues of $\widetilde{\mathbf{W}}_{N,M}$ augmented by $N - M$ zero eigenvalues. Therefore,

$$(5.1) \quad \hat{\mu}_{\mathbf{B}_{N,M}} = \frac{2N}{N+M} \hat{\mu}_{\mathbf{W}_{N,M}} + \frac{M-N}{N+M} \delta_0.$$

We next show that with probability one $\hat{\mu}_{\mathbf{B}_{N,M}}$ converges weakly. Since $\mathbf{B}_{N,M} = \mathbf{A}_{N,M}^2$, for any $f(\cdot)$ bounded and continuous,

$$(5.2) \quad \int f(x) d\hat{\mu}_{\mathbf{B}_{N,M}} = \int f(x^2) d\hat{\mu}_{\mathbf{A}_{N,M}}$$

so that it is enough to prove the convergence of $\hat{\mu}_{\mathbf{A}_{N,M}}$. To this end, consider \mathbf{A}_{N+M}^σ for

$$(5.3) \quad \sigma(x, y) = \begin{cases} 1 & \text{if } x, y \in (\frac{1}{1+\gamma}, 1] \times (0, \frac{1}{1+\gamma}] \cup (0, \frac{1}{1+\gamma}] \times (\frac{1}{1+\gamma}, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Note that with $M/N \rightarrow \gamma$ and

$$\text{rank}(\mathbf{A}_{N,M} - \mathbf{A}_{N+M}^\sigma) \leq 2 \left\lfloor \left[\frac{N+M}{1+\gamma} \right] - N \right\rfloor,$$

it follows by Lidskii's theorem that $d_1(\hat{\mu}_{\mathbf{A}_{N,M}}, \hat{\mu}_{\mathbf{A}_{N+M}^\sigma}) \rightarrow 0$ as $N \rightarrow \infty$. Therefore, applying Theorem 1.3 we deduce that with probability one $\hat{\mu}_{\mathbf{A}_{N,M}}$ converges weakly to the non-random probability measure μ^σ . By (5.2) and (5.1) this implies that $\hat{\mu}_{\mathbf{B}_{N,M}}$ and $\hat{\mu}_{\mathbf{W}_{N,M}}$ also converge weakly to non-random probability measures,

$$(5.4) \quad \mu_B := \frac{2}{1+\gamma} \mu_\alpha^\gamma + \frac{\gamma-1}{\gamma+1} \delta_0$$

and μ_α^γ , respectively.

We proceed to show that for $z \in \mathbb{C}^+$ the Cauchy-Stieltjes transform of μ_α^γ is

$$(5.5) \quad G_\alpha^\gamma(z) = \frac{1}{z} h_\alpha(Y_1(\sqrt{z})) = \frac{1-\gamma}{z} + \frac{\gamma}{z} h_\alpha(Y_2(\sqrt{z})).$$

Indeed, note that $(Y_1(z), Y_2(z))$ of (1.15) are precisely the solution of (1.11) considered in Proposition 3.7 for $z \in \mathbb{C}^+$ and our special choice of $\sigma(\cdot, \cdot)$. Theorem 1.3 thus asserts that the Cauchy-Stieltjes transform $G_{\alpha,\sigma}$ of μ^σ is then such that, for any $z \in \mathbb{C}^+$,

$$(5.6) \quad z G_{\alpha,\sigma}(z) = \frac{1}{1+\gamma} h_\alpha(Y_1(z)) + \frac{\gamma}{1+\gamma} h_\alpha(Y_2(z)).$$

Moreover, by (5.2) and the symmetry of the law μ^σ (see Corollary 4.1), we have that

$$\int \frac{1}{z-x} d\mu_B(x) = \int \frac{1}{z-x^2} d\mu^\sigma(x) = \frac{1}{\sqrt{z}} \int \frac{1}{\sqrt{z}-x} d\mu^\sigma(x).$$

From this and the formula (5.4) relating μ_B to μ_α^γ , we deduce that

$$(5.7) \quad G_\alpha^\gamma(z) = \frac{1+\gamma}{2\sqrt{z}} G_{\alpha,\sigma}(\sqrt{z}) + \frac{1-\gamma}{2z}.$$

Multiplying the left identity of (1.15) by $Y_2(z)$ and the right identity of (1.15) by $Y_1(z)$ we find that $Y_1(z)g_\alpha(Y_1(z)) = \gamma Y_2(z)g_\alpha(Y_2(z))$ and hence

$$(5.8) \quad h_\alpha(Y_1(z)) = 1 - \gamma + \gamma h_\alpha(Y_2(z)).$$

Upon combining (5.6), (5.7) and (5.8) we get the formula (5.5).

5.2. Analysis of the limiting measures. In case $\gamma = 1$, the function $\sigma(x, y)$ of (5.3) is equivalent to the constant $\tilde{\sigma} = 2^{-1/\alpha}$, which as in Remark 1.9 implies that μ^σ has the density $\rho^\sigma(t) = 2^{1/\alpha} \rho_\alpha(2^{1/\alpha}t)$. Further, we see from (5.7) that $G_\alpha^1(z) = G_{\alpha,\sigma}(\sqrt{z})/\sqrt{z}$, so the probability measure μ_α^1 on $(0, \infty)$ has the density $\rho^\sigma(\sqrt{t})/\sqrt{t}$, as stated.

Considering hereafter $\gamma \in (0, 1)$, observe that by Theorem 1.3, $Y_1(z)$ and $Y_2(z)$ extend continuously to functions on $(0, \infty)$ that are analytic outside of some bounded set. By the analyticity of $h_\alpha(\cdot)$ and (5.5) we have the corresponding continuous extension of $G_\alpha^\gamma(z)$, whereby Plemelj formula provides the density $\rho_\alpha^\gamma(t) = -\pi^{-1} \Im(G_\alpha^\gamma(t))$ of μ_α^γ with respect to Lebesgue measure, as in (1.16). In particular, $\rho_\alpha^\gamma(t) = \frac{1+\gamma}{2\sqrt{t}} \rho^\sigma(\sqrt{t})$ by (5.7), with $\sigma(\cdot, \cdot)$ of (5.3), so we read the tail behavior of ρ_α^γ out of that of ρ^σ (per Theorem 1.7).

Turning next to the behavior near zero of the probability measure μ_α^γ , recall that $G_\alpha^\gamma(z)$ is analytic outside the support $[0, \infty)$ of μ_α^γ and the non-tangential limit of $zG_\alpha^\gamma(z)$ at the boundary point $z = 0$ (i.e., its limit as $|z| \rightarrow 0$ while $\theta_0 \leq \arg(z) \leq 2\pi - \theta_0$ for some fixed $\theta_0 > 0$), exists and equals to the mass at zero of this measure. Further, the identity (5.5) extends by continuity to $z = -x^2$, $x > 0$ and $\sqrt{z} = ix \in \mathbb{C}^+$, hence

$$(5.9) \quad \mu_\alpha^\gamma(\{0\}) = \lim_{z \searrow 0} zG_\alpha^\gamma(z) = \lim_{x \downarrow 0} h_\alpha(Y_1(ix)) = 1 - \gamma + \gamma \lim_{x \downarrow 0} h_\alpha(Y_2(ix)).$$

Since $Y_s(-\bar{z}) = \overline{Y_s(z)}$ for $s = 1, 2$ and all $z \in \mathbb{C}^+$ (see Proposition 3.7), we have in particular that $Y_1(ix)$ and $Y_2(ix)$ are real-valued for all $x > 0$. As $g_\alpha(y) > 0$ for $y \in \mathbb{R}$, it further follows from (1.15) that $Y_s(i\mathbb{R}^+) \subseteq \mathbb{R}^+$ for $s = 1, 2$. With $h_\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ monotone decreasing and $h_\alpha(y) \rightarrow 0$ as $\Re(y) \rightarrow \infty$, it thus follows from (5.8) that $h_\alpha(Y_1(ix)) \geq 1 - \gamma$ for all $x > 0$ and consequently, that $(Y_1(ix), x > 0)$ is uniformly bounded. This of course implies that $(ix)^\alpha Y_1(ix) \rightarrow 0$ as $x \downarrow 0$ which in view of (1.15) requires that $g_\alpha(Y_2(ix)) \rightarrow 0$ as well. As $g_\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is bounded away from zero on compacts, we deduce that $Y_2(ix) \rightarrow \infty$ as $x \downarrow 0$, hence $h_\alpha(Y_2(ix)) \rightarrow 0$ and

$$\mu_\alpha^\gamma(\{0\}) = \lim_{x \downarrow 0} h_\alpha(Y_1(ix)) = 1 - \gamma,$$

as claimed. Moreover, from the preceding $Y_1(ix) \rightarrow h_\alpha^{-1}(1 - \gamma) := b \in \mathbb{R}^+$ as $x \downarrow 0$. Since $Y_1(z)$ is a \mathcal{K}_α -valued continuous function of $z \in \mathbb{C}^+$, its cluster set $Cl(0)$ at the boundary point $z = 0$ of \mathbb{C}^+ is a closed, connected subset of \mathcal{K}_α (see [6, Theorem 1.1]). Further, $Cl(0)$ contains $b \in \mathbb{R}^+$, so its boundary $\partial Cl(0)$ must intersect $[0, \infty)$. We have seen that $Y_1(z)$ extends continuously on $(0, \infty)$ which due to the relation $Y_1(-\bar{z}) = \overline{Y_1(z)}$ implies that it also extends continuously on $(-\infty, 0)$ with $Y_1(-t) = \overline{Y_1(t)}$ for all $t > 0$. In particular, since the cluster set of $Y_1(t)$ for

non-zero, real-valued $t \rightarrow 0$ contains $\partial Cl(0)$ (see [6, Theorem 5.2.1]), necessarily the cluster set of $Y_1(\sqrt{t})$ at the boundary point $t = 0$ of \mathbb{R}^+ also intersects $[0, \infty)$.

Using the bound $\sin(\zeta)/\zeta \geq 1 - \zeta^2/6$, we deduce from (1.5) that if $\Im(h_\alpha(x+iy)) = 0$ for $y \neq 0$ then $y^2 \geq 6h'_\alpha(x)/h''_\alpha(x)$, and direct calculation shows that this function of x is positive and monotone non-decreasing. Thus, with $Y_1(\sqrt{t}) \in \mathcal{K}_\alpha$ there exists $\delta > 0$ such that if $\Im(h_\alpha(Y_1(\sqrt{t}))) = 0$ then either $Y_1(\sqrt{t}) \geq 0$ is real-valued, or $|\Im(Y_1(\sqrt{t}))| \geq \delta$. By (1.16), the latter property applies whenever $t > 0$ is such that $\rho_\alpha^\gamma(t) = 0$. Moreover, by the continuity of $Y_1(\cdot)$ on $(0, \infty)$, if the density ρ_α^γ vanishes on an open interval \mathbb{I} , then either $Y_1(\sqrt{t}) \geq 0$ for all $t \in \mathbb{I}$ or $\inf_{t \in \mathbb{I}} |\Im(Y_1(\sqrt{t}))| \geq \delta$. For $\mathbb{I} = (0, \epsilon)$ we have already seen that the cluster set of $Y_1(\sqrt{t})$ as $t \downarrow 0$ intersects $[0, \infty)$, so necessarily $Y_1(\sqrt{t}) \in [0, \infty)$ for all $t \in \mathbb{I}$. Since (1.15) extends to $z \in \mathbb{R}^+$ and $g_\alpha(\mathbb{R}) \subseteq \mathbb{R}^+$ this in turn implies that $Y_2(\sqrt{t}) = i^\alpha r(t)$ for some continuous function $r : \mathbb{I} \mapsto \mathbb{R}^+$ such that $r(t) \rightarrow \infty$ as $t \downarrow 0$. The entire function $f_{\alpha, \theta}(z) := \frac{1}{2i}[h_\alpha(e^{i\theta}z) - h_\alpha(e^{-i\theta}z)]$ is then by (5.5) such that

$$f_{\alpha, \theta}(r(t)) = \Im(h_\alpha(e^{i\theta}r(t))) = \Im(h_\alpha(Y_2(\sqrt{t}))) = 0$$

for $\theta = \pi\alpha/2$ and all $t \in \mathbb{I}$, which with $f'_{\alpha, \theta}(0) = \sin(\theta)h'_\alpha(0) \neq 0$ contradicts the identity theorem. We thus conclude that ρ_α^γ does not vanish on any non-empty interval $(0, \epsilon)$.

5.3. Properties of μ_α .

Proof of Proposition 1.1. Taking $\sigma \equiv 1$ we deduce from Theorem 1.3 that $Y(z)$ of (1.7) is in \mathcal{K}_α hence uniformly bounded on $\mathbb{C}^+ \setminus \{z : |z| < \delta\}$. Similarly to the argument of Section 5.2, if $y \in Cl(t)$ at $t > 0$ then $y \in \mathcal{K}_\alpha$ and $F(t, y) := t^\alpha y - C_\alpha g_\alpha(y) = 0$, so from the analyticity of $y \mapsto F(t, y)$ and uniform boundedness of $g_\alpha(\cdot)$ on \mathcal{K}_α we deduce by the identity theorem that $Y(z)$ extends continuously to a function $Y(t)$ on $(0, \infty)$. Moreover, $t \mapsto Y(t)$ is real-analytic on $(0, \infty)$ outside the set of those $t > 0$ where both $\partial_y F(t, y) = 0$ and $F(t, y) = 0$ at $y = Y(t)$. The latter set is clearly contained in the set \mathcal{D}_α^+ of $t > 0$ such that $t^\alpha = C_\alpha g'_\alpha(y) > 0$ for some $y \in \mathcal{K}_\alpha$ at which $yg'_\alpha(y) - g_\alpha(y) = 0$. Note that the set \mathcal{D}_α^+ is discrete since $yg'_\alpha(y) - g_\alpha(y)$ is an entire function of y . Further, \mathcal{D}_α^+ is a bounded set (by the uniform boundedness of $g'_\alpha(\cdot)$ on \mathcal{K}_α , see Lemma 3.6). Consequently, \mathcal{D}_α^+ is a finite set. We already saw that $Y(-\bar{z}) = \bar{Y}(z)$ for all $z \in \mathbb{C}^+$, so $Y(-\bar{z})$ extends continuously to $Y(-t) = \bar{Y}(t)$ for any $t > 0$ at which $Y(\cdot)$ extends continuously. We thus deduce that the exceptional set where $t \mapsto Y(t)$ may be non-analytic is contained in the finite set $\{0, \pm t : t \in \mathcal{D}_\alpha^+\}$, as claimed. With h_α an entire function, it then follows that $G_\alpha(\cdot)$ extends continuously to $\mathbb{R} \setminus \{0\}$ with the formula (1.8) for the symmetric density $\rho_\alpha(t)$ on $\mathbb{R} \setminus \{0\}$ that is real-analytic outside \mathcal{D}_α (to verify the right-most expression in (1.8) note that $h_\alpha(Y(z)) = 1 - \frac{\alpha}{2C_\alpha} z^\alpha Y(z)^2$ by (1.5) and (1.7)).

If the symmetric density ρ_α vanishes on an open interval, then it also vanishes on some open interval $\mathbb{I} \subseteq \mathbb{R}^+$ where the continuous function $t \mapsto Y(t)$ is the limit of $Y(z)$ as $\arg(z) \downarrow 0$, hence $\arg(Y(t)) \in [0, \frac{\alpha\pi}{2}]$ (see (3.16)). Further, the right-most expression in (1.8) tells us that $\sin(2\arg(Y(t)) - \frac{\alpha\pi}{2}) = 0$ for all $t \in \mathbb{I}$, so necessarily $Y(t) = e^{i\theta}r(t)$ for $\theta = \frac{\alpha\pi}{4}$ and the continuous $r : \mathbb{I} \mapsto [0, \infty)$. Since (1.7) extends to $t \in \mathbb{I}$ and $g_\alpha(0) \neq 0$, we see that $Y(t) \neq 0$ is injective on \mathbb{I} , so $r(\mathbb{I})$ contains an accumulation point. Finally, as argued at the end of Section 5.2, from (1.8) we also have that $f_{\alpha, \theta}(r(t)) = \Im(h_\alpha(Y(t))) = 0$ for the entire function $f_{\alpha, \theta}(\cdot)$ and all $t \in \mathbb{I}$,

yielding a contradiction. Consequently, the density ρ_α does not vanish on any open interval, as claimed.

It remains to show that μ_α has a uniformly bounded density. We get this by proving the stronger statement that $G_\alpha(z)$ is uniformly bounded on the connected set $\mathbb{C}_*^+ := \mathbb{C}^+ \cup \mathbb{R}^+$. To this end, let $Cl_*(0)$ denote the cluster set of the continuous function $Y(z)$ at the boundary point $z = 0$ of \mathbb{C}_*^+ . If $y \in \mathbb{C}$ is in $Cl_*(0)$ then there exists $z_n \in \mathbb{C}_*^+$ such that $z_n \rightarrow 0$ and $Y(z_n) \rightarrow y$, hence $g_\alpha(y) = 0$ by (1.7). Whereas $Cl_*(0)$ is a closed connected subset of $\mathbb{C} \cup \{\infty\}$ (by [6, Theorem 1.1]), the set of zeros of the entire function $g_\alpha(\cdot)$ is discrete, so necessarily $Cl_*(0)$ is a single point. Taking $z = ix$, $x > 0$ we have that $Y(ix) \in \mathbb{R}^+$, hence $Y(ix) \rightarrow \infty$ by (1.7) and the boundedness of $g_\alpha(\mathbb{R}^+)$, from which we deduce that $Cl_*(0) = \{\infty\}$. Considering (3.18) for $\beta = 2$ we note that $|h_\alpha(y)| \leq c_0 h_\alpha(\xi|y|)$ for some $\xi = \xi(\alpha) > 0$, $c_0 = c_0(\alpha)$ finite and all $y \in \mathcal{K}_\alpha$. In particular, for $z \in \mathbb{C}_*^+$ such that $|z| \rightarrow 0$ we already know that $Y(z) \in \mathcal{K}_\alpha$ and $|Y(z)| \rightarrow \infty$, hence by the preceding bound and the decay to zero of $h_\alpha(r)$ as $r \in \mathbb{R}^+$ goes to infinity, we have that $h_\alpha(Y(z)) \rightarrow 0$. That is, $Y(z)g_\alpha(Y(z)) \rightarrow 2/\alpha$ (see (1.5)). Next, observing that $h_\alpha(r) \leq c_1 r^{-2/\alpha}$ for some positive, finite c_1 and all $r \in \mathbb{R}^+$, we deduce from (1.6) and (1.7) that for some finite constants $c_i = c_i(\alpha)$ and all $z \in \mathbb{C}_*^+$,

$$(5.10) \quad \begin{aligned} |G_\alpha(z)| &= |z|^{-1} |h_\alpha(Y(z))| \leq c_0 |z|^{-1} h_\alpha(\xi|Y(z)|) \\ &\leq c_2 (|z^\alpha Y(z)^2|)^{-1/\alpha} = c_3 |Y(z)g_\alpha(Y(z))|^{-1/\alpha}. \end{aligned}$$

For any $\delta > 0$ we have the uniform boundedness of $G_\alpha(z)$ on $\mathbb{C}_*^+ \cap \mathbb{B}(0, \delta)^c$ (from the uniform boundedness of h_α on \mathcal{K}_α). Further, for $z \in \mathbb{C}_*^+$ converging to zero the right side of (5.10) remains bounded (by $c_3(2/\alpha)^{-1/\alpha}$), hence $G_\alpha(z)$ is uniformly bounded on \mathbb{C}_*^+ , as stated.

Remark 5.1. *We saw that $Y(ix) \in \mathbb{R}^+$ and $x^\alpha Y(ix)^2 = |C_\alpha| Y(ix) g_\alpha(Y(ix)) \rightarrow 2|C_\alpha|/\alpha$ as $x \downarrow 0$. With $\zeta = \sqrt{2|C_\alpha|/\alpha}$, it then follows by dominated convergence that $\pi^{-1} x^{-1} h_\alpha(Y(ix)) \rightarrow \pi^{-1} \int_0^\infty \exp(-\zeta u^{\alpha/2}) du$ finite and positive. This is of course the value of $\rho_\alpha(0)$, provided ρ_α is continuous at $t = 0$.*

Lemma 5.2. *The measures μ_α converge weakly to μ_2 when $\alpha \uparrow 2$.*

Proof. Applying the method of moments, as developed by Zakharevich [9], it is shown in [1, Theorem 1.8] that for any $B < \infty$ fixed, $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^B}]$ converges to some non-random μ_α^B as $N \rightarrow \infty$ (for instance, when x_{ij} are stable variables of index α). Examining the dependence of $C(B)$ of [1, equation (13)] on α , we see that (2.1) applies for some $\delta(\epsilon, B) > 0$, all $B > B(\epsilon, \alpha_0)$ and any $\alpha \in (\alpha_0, 2)$. For such B and α we thus have, in view of the almost sure convergence of $\hat{\mu}_{\mathbf{A}_N}$ to μ_α , that $\mathbb{P}(d_1(\mu_\alpha, \hat{\mu}_{\mathbf{A}_N^B}) \geq 3\epsilon) \rightarrow 0$ as $N \rightarrow \infty$, from which we deduce by the boundedness and convexity of d_1 that

$$d_1(\mu_\alpha, \mu_\alpha^B) = \lim_{N \rightarrow \infty} d_1(\mu_\alpha, \mathbb{E}[\hat{\mu}_{\mathbf{A}_N^B}]) \leq \limsup_{N \rightarrow \infty} \mathbb{E}[d_1(\mu_\alpha, \hat{\mu}_{\mathbf{A}_N^B})] \leq 3\epsilon.$$

Fixing $B < \infty$ it further follows from [1, Lemmas 9.1 and 9.2] that μ_α^B converges weakly to the semi-circle μ_2 when $\alpha \rightarrow 2$. Hence, fixing $\alpha_0 > 0$, $\epsilon > 0$ and $B > B(\epsilon, \alpha_0)$, by the triangle inequality

$$d_1(\mu_\alpha, \mu_2) \leq d_1(\mu_\alpha, \mu_\alpha^B) + d_1(\mu_\alpha^B, \mu_2) \leq 3\epsilon + d_1(\mu_\alpha^B, \mu_2) \rightarrow 3\epsilon$$

as $\alpha \uparrow 2$. Taking $\epsilon \downarrow 0$ we thus conclude that $\mu_\alpha \rightarrow \mu_2$ when $\alpha \uparrow 2$. \square

6. DIAGONAL PERTURBATION: PROOF OF THEOREM 1.12

6.1. The extension of Theorem 1.3. We shall prove the convergence of the expected spectral measures $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N + \mathbf{D}_N}]$ and characterize their limit in case $\sigma \in \mathcal{C}_*$ is given as in Section 3 by (3.1) for some $q \in \mathbb{N}$, $0 = b_0 < b_1 < \dots < b_q = 1$ and $\sigma_{rs} = \sigma_{sr}$ with the corresponding random matrix $\mathbf{A}_N = \mathbf{A}_N^\sigma$ and the $N \times N$ piecewise constant matrix σ^N . To this end, recall that \mathbf{D}_N is a diagonal $N \times N$ matrix, whose entries $\{D_N(k, k), 1 \leq k \leq N\}$ are real valued, independent of the random variables $(x_{ij}, 1 \leq i \leq j < \infty)$ and identically distributed, of law $\mu^{\mathbf{D}}$ having a finite second moment. In view of the assumed finite second moment of $\mu^{\mathbf{D}}$, the proof of (2.2) and Lemma 2.1 also show that the sequences $(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N + \mathbf{D}_N}]; N \in \mathbb{N})$, $(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{\beta} + \mathbf{D}_N}]; N \in \mathbb{N})$ and $(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{\kappa} + \mathbf{D}_N}]; N \in \mathbb{N})$ are tight for the topology of weak convergence on $\mathcal{P}(\mathbb{R})$, and that $(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^{\kappa} + \mathbf{D}_N}]; N \in \mathbb{N})$ has the same set of limit points as $(\mathbb{E}[\hat{\mu}_{\mathbf{A}_N + \mathbf{D}_N}]; N \in \mathbb{N})$. Setting now $\mathbf{G}_N(z) = (z\mathbf{I}_N - \mathbf{D}_N - \mathbf{A}_N)^{-1}$ we define for $z \in \mathbb{C}^+$ the probability measures L_N^z and $L_{N,r}^z$ on \mathbb{C} as in (3.2) and (3.3), with $\mathbf{G}_N^{\kappa}(z)$ and $L_{N,r}^{z,\kappa}$ denoting again the corresponding objects when \mathbf{A}_N is replaced by \mathbf{A}_N^{κ} .

For $0 < \kappa < \frac{1}{2(2-\alpha)}$ any $1 \leq r \leq q$ and bounded Lipschitz function f we then have similarly to Lemma 3.1 that as $N \rightarrow \infty$

$$(6.1) \quad \left| \mathbb{E}[L_{N,r}^{z,\kappa}(f)] - \mathbb{E}\left[f\left((z - D_N(0,0) - \sum_{k=1}^N \tilde{\mathbf{A}}_N^{\kappa}([Nb_r], k)^2 \mathbf{G}_N^{\kappa}(z)_{kk})^{-1}\right)\right] \right| \rightarrow 0,$$

where $\tilde{\mathbf{A}}_N^{\kappa}$ denotes an independent copy of \mathbf{A}_N^{κ} which is also independent of \mathbf{D}_N while $D_N(0,0)$ of law $\mu^{\mathbf{D}}$ is independent of all other variables. Indeed, focusing w.l.o.g. on $r = 1$ and taking $\tilde{\mathbf{G}}_{N+1}^{\kappa}(z) = (z\mathbf{I}_{N+1} - \tilde{\mathbf{D}}_{N+1} - \tilde{\mathbf{A}}_{N+1}^{\kappa})^{-1}$ (with $\tilde{\mathbf{D}}_{N+1}$ denoting the diagonal matrix of entries $D_N(k, k)$, $k = 0, \dots, N$), we get (3.4) by the invariance of the law of $\tilde{\mathbf{D}}_{N+1} + \tilde{\mathbf{A}}_{N+1}^{\kappa}$ to symmetric permutations of its first $[Nb_1] + 1$ rows and columns. Schur's complement formula then leads to the identity (3.5) with $D_N(0,0)$ added to $\tilde{\mathbf{A}}_N^{\kappa}(0,0)$ on its right side. All eigenvalues (and diagonal terms) of $\mathbf{G}_N^{\kappa}(z)$ are in the compact set $\mathbb{K}(z)$, regardless of the value of \mathbf{D}_N , and the centered entries of $\tilde{\mathbf{A}}_N^{\kappa}$ are independent of both $\mathbf{G}_N^{\kappa}(z)$ and $D_N(0,0)$. Thus, as in the proof of Lemma 3.1 we can neglect both $\tilde{\mathbf{A}}_N^{\kappa}(0,0)$ and $\sum_{k \neq l} \tilde{\mathbf{A}}_N^{\kappa}(0,k) \tilde{\mathbf{A}}_N^{\kappa}(l,0) \mathbf{G}_N^{\kappa}(z)_{kl}$ in (3.5) and get (3.6) except for changing here z to $z - D_N(0,0)$ in its right side. Equipped with the latter version of (3.6), fixing $0 < \kappa < \frac{1}{2(2-\alpha)}$ we arrive at (6.1) upon adapting [1, Lemma 4.1] and its proof to our matrices $\tilde{\mathbf{G}}_{N+1}^{\kappa}$ and \mathbf{G}_N^{κ} (while taking there the corresponding matrices $\tilde{\mathbf{G}}_N^{\kappa} = (z\mathbf{I}_{N+1} - \tilde{\mathbf{D}}_{N+1} - \tilde{\mathbf{A}}_N^{\kappa})^{-1}$).

The concentration result of Lemma 3.4 holds in the presence of the diagonal matrix \mathbf{D}_N of i.i.d. entries. Indeed, its proof is easily adapted to the current setting by considering for f continuously differentiable $L_{N,s}^{z,\kappa}(f) := F_N(D_N(l,l), A_N^{\kappa}(k,l), 1 \leq k \leq l \leq N)$, and noting that for $1 \leq l \leq N$,

$$\partial_{D(l,l)} F_N = \frac{1}{N} [\mathbf{G}_N^{\kappa}(z) \mathbf{D}_s(f') \mathbf{G}_N^{\kappa}(z)] u.$$

The spectral radius of $\mathbf{G}_N^{\kappa}(z) \mathbf{D}_s(f') \mathbf{G}_N^{\kappa}(z)$ is again bounded by $\|f'\|_{\infty} / |\Im(z)|^2$, so $\sup_l \|\partial_{D(l,l)} F_N\|_{\infty} \leq \|f\|_{\mathbf{BL}} (N |\Im(z)|^2)^{-1}$. There are only N such variables $\{D_N(l,l)\}$ to consider, each having the same finite second moment, so using the

same martingale bound as in (3.8), their total effect on $\mathbb{E}[(F_N - \mathbb{E}[F_N])^2]$ is taken care off by enlarging the finite constant c_0 .

Equipped with this concentration result and replacing Lemma 3.1 with (6.1), we follow the proof of Proposition 3.3 to deduce that in our current setting, for $r \in \{1, \dots, q\}$ and every bounded continuous function f on $\mathbb{K}(z)$,

$$(6.2) \quad \int f d\mu_r^z = \int f \left((z - \lambda - \sum_{s=1}^q \sigma_{rs}^2 \Delta_s^{\frac{2}{\alpha}} x_s)^{-1} \right) \prod_{s=1}^q dP^{\mu_s^z}(x_s) d\mu^{\mathcal{D}}(\lambda).$$

Following the proof of Proposition 3.5 we find that this in turn implies that any subsequence of the functions $X_{N,r}(z) = \mathbb{E}[L_{N,r}^{z,\kappa}(x^{\alpha/2})]$ has at least one limit point $(X_r(z), 1 \leq r \leq q)$ composed of analytic functions on \mathbb{C}^+ that are bounded by $(\Im(z))^{-\alpha/2}$ and satisfy the following generalization of (3.10)

$$X_r(z) = C(\alpha) \int_0^\infty t^{-1} (it)^{\frac{\alpha}{2}} e^{it(z-\lambda)} \exp\{- (it)^{\frac{\alpha}{2}} \widehat{X}_r(z)\} dt d\mu^{\mathcal{D}}(\lambda),$$

for the analytic functions $\widehat{X}_r : \mathbb{C}^+ \mapsto \widehat{\mathcal{K}}_\alpha$ of (3.11).

We proceed to extend Proposition 3.7 to the setting of $\mathbf{A}_N^\sigma + \mathbf{D}_N$. Indeed, fixing $z \in \mathbb{C}^+$, upon applying per $\lambda \in \mathbb{R}$ the identity (3.17) for $\beta = \alpha$, $y = (\lambda - z)^{-\alpha/2} \widehat{X}_r(z)$ and with $z - \lambda \in \mathbb{C}^+$ replacing z , we see that the preceding generalization of (3.10) is equivalent to

$$X_r(z) = C(\alpha) \int (\lambda - z)^{-\frac{\alpha}{2}} g_{\alpha,\alpha}((\lambda - z)^{-\frac{\alpha}{2}} \widehat{X}_r(z)) d\mu^{\mathcal{D}}(\lambda).$$

By (3.11) we thus deduce that $(\widehat{X}_r(z), 1 \leq r \leq q)$ satisfy (1.19). Namely, it is a solution of $\widehat{\underline{x}} = \underline{F}_z(\widehat{\underline{x}})$ composed of analytic functions from \mathbb{C}^+ to $\widehat{\mathcal{K}}_\alpha$, where $\underline{F}_z(\cdot) = (F_{z,r}(\cdot), 1 \leq r \leq q)$ and

$$F_{z,r}(\widehat{\underline{x}}) := \overline{C}_\alpha \sum_{s=1}^q \widehat{a}_{rs} \int (\lambda - z)^{-\frac{\alpha}{2}} g_\alpha((\lambda - z)^{-\frac{\alpha}{2}} \widehat{x}_s) d\mu^{\mathcal{D}}(\lambda),$$

for $\widehat{a}_{rs} = |\sigma_{rs}|^\alpha \Delta_s$. Note that if $\widehat{x}_s \in \widehat{\mathcal{K}}_\alpha$ then $(\lambda - z)^{-\frac{\alpha}{2}} \widehat{x}_s$ is in \mathcal{K}_α so such solutions must have $|\widehat{x}_r| \leq c (\Im(z))^{-\frac{\alpha}{2}}$ for $c := |C_\alpha| \|g_\alpha\|_{\mathcal{K}_\alpha} \max_r \sum_{s=1}^q |\widehat{a}_{rs}|$ finite and all $z \in \mathbb{C}^+$. Consequently, if $\Im(z) \geq 1$ then $\max_r |\widehat{x}_r| \leq c$. Thus, for such z , any $1 \leq r \leq q$ and any two fixed points $\widehat{\underline{x}}$ and $\widehat{\underline{y}}$ of $\underline{F}_z(\cdot)$ in $(\widehat{\mathcal{K}}_\alpha)^q$,

$$|F_{z,r}(\widehat{\underline{x}}) - F_{z,r}(\widehat{\underline{y}})| \leq \max_{r,s} \{|\widehat{a}_{rs}|\} \|g_\alpha\|_c (\Im(z))^{-\alpha} \|\widehat{\underline{x}} - \widehat{\underline{y}}\|_1$$

(where $\|g_\alpha\|_c$ and $\|g_\alpha\|_{\mathcal{K}_\alpha}$ are as in the proof of Theorem 1.7 and $\|\widehat{\underline{x}}\|_1 := \sum_{s=1}^q |\widehat{x}_s|$). Thus, for some k_0 finite, if $\Im(z) \geq k_0$ then $\|\underline{F}_z(\widehat{\underline{x}}) - \underline{F}_z(\widehat{\underline{y}})\|_1 \leq \frac{1}{2} \|\widehat{\underline{x}} - \widehat{\underline{y}}\|_1$ resulting with uniqueness of the fixed point of $\underline{F}_z(\cdot)$ in $(\widehat{\mathcal{K}}_\alpha)^q$. This in turn implies the stated uniqueness of such fixed point composed of analytic functions $z \mapsto \widehat{x}_s$ from \mathbb{C}^+ to $\widehat{\mathcal{K}}_\alpha$.

To complete the proof of Theorem 1.12 in case $\sigma \in \mathcal{C}_*$, we adapt our proof of Theorem 1.3, where instead of (3.25), combining (6.2) for $f(x) = x$ with (3.13), here the limit points μ_s^z of $(\mathbb{E}[L_{N,s}^{z,\kappa}], 1 \leq s \leq q)$ are such that for each $r \in \{1, \dots, q\}$,

$$(6.3) \quad \int x d\mu_r^z(x) = -i \int d\mu^{\mathcal{D}}(\lambda) \int_0^\infty e^{it(z-\lambda)} \exp\{- (it)^{\frac{\alpha}{2}} \widehat{X}_r(z)\} dt.$$

In particular, since $\int x d\mu_r^z$ is uniquely determined by $\widehat{X}_r(z)$ we deduce that the sequence $\mathbb{E}[L_{N,r}^{z,\kappa}(x)]$ converges as $N \rightarrow \infty$ to the right side of (6.3). So, with

$$\mathbb{E}\left[\int \frac{1}{z-x} d\hat{\mu}_{\mathbf{A}_N^\kappa + \mathbf{D}_N}(x)\right] = \sum_{s=1}^q \Delta_{N,s} \mathbb{E}[L_{N,s}^{z,\kappa}(x)]$$

for any $z \in \mathbb{C}^+$, it follows that

$$\int \frac{1}{z-x} d\mu^{\sigma, \mathbf{D}}(x) = -i \sum_{s=1}^q \Delta_s \int d\mu^{\mathbf{D}}(\lambda) \int_0^\infty e^{it(z-\lambda)} \exp\left\{-it \frac{\alpha}{2} \widehat{X}_s(z)\right\} dt,$$

for any limit point $\mu^{\sigma, \mathbf{D}}$ of $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\kappa + \mathbf{D}_N}]$. With the Cauchy-Stieltjes transform $G_{\alpha, \sigma}^{\mathbf{D}}$ of $\mu^{\sigma, \mathbf{D}} \in \mathcal{P}(\mathbb{R})$ uniquely determined, we deduce that $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N^\kappa + \mathbf{D}_N}]$ converges to $\mu^{\sigma, \mathbf{D}}$, hence so does $\mathbb{E}[\hat{\mu}_{\mathbf{A}_N + \mathbf{D}_N}]$. Finally, for $z \in \mathbb{C}^+$ we arrive at the formula

$$(6.4) \quad G_{\alpha, \sigma}^{\mathbf{D}}(z) = \int \frac{1}{z-\lambda} \sum_{s=1}^q \Delta_s h_\alpha((\lambda-z)^{-\frac{\alpha}{2}} \widehat{X}_s(z)) d\mu^{\mathbf{D}}(\lambda),$$

by applying (3.17) with $\beta = 2$, $y = (\lambda-z)^{-\frac{\alpha}{2}} \widehat{X}_s(z)$ and $z-\lambda$ instead of z .

6.2. The extension of Theorem 1.7. Setting $\sigma \in \mathcal{F}_\alpha$ we adapt the proof of Theorem 1.7 to the current setting. Indeed, using the same approximating sequence $\sigma_p \in \mathcal{C}_\star$ of $\sigma \in \mathcal{F}_\alpha$ as in the proof of Theorem 1.7, we have shown already that (1.18) holds for each of the piecewise constant functions $\widehat{X}^{\sigma_p}(z) : (0, 1] \rightarrow \widehat{\mathcal{K}}_\alpha$, $p \in \mathbb{N}$, where

$$\widehat{X}_x^{\sigma_p}(z) = \widehat{X}_s(z) \text{ for } x \in (b_{s-1}^p, b_s^p] \text{ and } s = 1, \dots, q(\sigma_p),$$

and $\widehat{X}_s(z) \in \widehat{\mathcal{K}}_\alpha$ are the unique collections of (analytic) functions of $z \in \mathbb{C}^+$ we have constructed in Section 6.1.

Similarly to the proof of Theorem 1.7, we get the existence of a bounded measurable solution $\widehat{X}^\sigma(z) : (0, 1] \mapsto \widehat{\mathcal{K}}_\alpha$ of (1.18) whenever $\Im(z) \geq R = R(\sigma)$ by showing that for such z the fixed points $(\widehat{X}^{\sigma_p}(z), p \in \mathbb{N})$ of the mappings

$$F_z(\sigma, \widehat{X}) := \overline{\mathcal{C}}_\alpha \int_0^1 |\sigma(\cdot, v)|^\alpha \int (\lambda-z)^{-\frac{\alpha}{2}} g_\alpha((\lambda-z)^{-\frac{\alpha}{2}} \widehat{X}_v) d\mu^{\mathbf{D}}(\lambda) dv,$$

at $\sigma = \sigma_p$ form a Cauchy sequence in $L^\infty((0, 1])$. To this end, recall that $\|g_\alpha\|_{\mathcal{K}_\alpha}$ is finite (by Lemma 3.6), so fixing $\epsilon \in (0, 1)$ and bounding the L^∞ -norm of $F_z(\sigma, \widehat{X})$ for $\widehat{X} \in \widehat{\mathcal{K}}_\alpha$ we deduce that $\|\widehat{X}^{\sigma_p}\|_\infty \leq r_\sigma$ of (4.2) for all $p \in \mathbb{N}$, whenever $\Im(z) \geq \epsilon$. It is easy to verify that for such z our mapping $F_z(\cdot, \cdot)$ satisfies the inequality (4.3) except for replacing there $|z|^{-\alpha}$ by $(\Im(z))^{-\alpha/2}$. Consequently, with \widehat{X}^{σ_p} fixed points of this mapping, our uniform bound on $\|\widehat{X}^{\sigma_p}\|_\infty$ implies that

$$\|\widehat{X}^{\sigma_q} - \widehat{X}^{\sigma_p}\|_\infty \leq (\Im(z))^{-\alpha/2} \|g_\alpha\|_{r_\sigma} \left[\left| |\sigma_q|^\alpha - |\sigma_p|^\alpha \right| + 2k_\sigma \|\widehat{X}^{\sigma_q} - \widehat{X}^{\sigma_p}\|_\infty \right],$$

for any $p, q \in \mathbb{N}$ and $\Im(z) \geq \epsilon$. Thus, setting $R \geq \epsilon$ such that $R^{-\alpha/2} \|g_\alpha\|_{r_\sigma} k_\sigma \leq 1/3$, we conclude in view of (1.12) that $(\widehat{X}^{\sigma_p}, p \in \mathbb{N})$ is a Cauchy sequence in $L^\infty(0, 1; \mathbb{C})$ whenever z is in $\mathbb{C}_R^+ := \{z : \Im(z) > R\}$. As in the proof of Theorem 1.7, the L^∞ -norm of its limit \widehat{X}^σ is at most r_σ so by (1.12) and the modified inequality (4.3) $\widehat{X}^\sigma(z)$ must be a fixed point of $F_z(\sigma, \cdot)$. Further, equipped with the latter inequality, the uniqueness (almost everywhere) of such a solution to (1.18) is obtained by a re-run of the relevant argument from the proof of Theorem 1.7. We have seen that

the holomorphic mappings \widehat{X}^{σ_p} from \mathbb{C}^+ to the closed subset $\mathbb{F} := L^\infty((0, 1]; \widehat{\mathcal{K}}_\alpha)$ of $L^\infty((0, 1]; \mathbb{C})$ are locally uniformly bounded. Hence, their L^∞ -convergence to \widehat{X}^σ extends by Vitali's convergence theorem from the non-empty open subset \mathbb{C}_R^+ to all of \mathbb{C}^+ , with $\widehat{X}^\sigma : \mathbb{C}^+ \mapsto \mathbb{F}$ an analytic mapping which is uniquely determined by the uniqueness of the solution in \mathbb{F} of (1.18) for each $z \in \mathbb{C}_R^+$ (and the identity theorem).

Next, with the same proof as in Proposition 2.2 we have from the L_\star^2 -convergence of σ_p to σ that $G_{\alpha, \sigma_p}^D(z) \rightarrow G_{\alpha, \sigma}^D(z)$ as $p \rightarrow \infty$, for each $z \in \mathbb{C}^+$. If $z \in \mathbb{C}_R^+$ then also $\|\widehat{X}^{\sigma_p} - \widehat{X}^\sigma\|_\infty \rightarrow 0$. As the identity (1.17) holds for $\sigma = \sigma_p \in \mathcal{C}_\star$, $p \in \mathbb{N}$ (being then merely the formula (6.4)), taking $p \rightarrow \infty$ we deduce by dominated convergence that (1.17) holds for $\sigma \in \mathcal{F}_\alpha$ and $z \in \mathbb{C}_R^+$. For all $x \in (0, 1]$, $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}^+$ the argument $(\lambda - z)^{-\alpha/2} \widehat{X}_x^\sigma(z)$ of the entire function h_α is in the set \mathcal{K}_α where h_α and its derivatives are uniformly bounded. Further, for such λ the mapping $z \mapsto (\lambda - z)^{-\alpha/2} \widehat{X}^\sigma(z)$ from \mathbb{C}^+ to $L^\infty((0, 1]; \mathbb{C})$ is analytic, out of which one can verify that the right side of (1.17) is analytic on \mathbb{C}^+ . With $G_{\alpha, \sigma}^D$ also analytic on \mathbb{C}^+ the validity of (1.17) extends from \mathbb{C}_R^+ to \mathbb{C}^+ (by the identity theorem).

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