

SPECTRAL METHOD ON QUADRILATERALS

GUO BEN-YU AND JIA HONG-LI

ABSTRACT. In this paper, we investigate the spectral method on quadrilaterals. We introduce an orthogonal family of functions induced by Legendre polynomials, and establish some results on the corresponding orthogonal approximation. These results play important roles in the spectral method for partial differential equations defined on quadrilaterals. As examples of applications, we provide spectral schemes for two model problems and prove their spectral accuracy in Jacobi weighted Sobolev space. Numerical results coincide well with the analysis. We also investigate the spectral method on convex polygons whose solutions possess spectral accuracy. The approximation results of this paper are also applicable to other problems.

1. INTRODUCTION

During the past three decades, spectral method has gained increasing popularity in scientific computations; see [1]–[7] and [9]–[11] and the references therein. The standard spectral method is traditionally confined to periodic problems and problems defined on rectangular domains. However, many practical problems are set on complex domains. We usually use finite element methods for such problems. For obtaining accurate numerical results, it is also interesting to consider spectral methods and other high order methods for non-rectangular domains. Some authors proposed spectral methods for triangles, quadrilaterals and unbounded domains, see, e.g., [1], [2], [4], [14]–[17]. In particular, pseudospectral methods for polygons were developed with precise analysis; see [2, 5, 17] and the references therein.

In this paper, we investigate the spectral method on convex quadrilaterals. This work is motivated by several facts. For instance, we consider numerical solutions of partial differential equations defined on a polygon. In this case, we may divide the polygon into several convex quadrilaterals, and then use a spectral method on each quadrilateral. Next, for exterior problems of partial differential equations with a polygon obstacle, we could use mixed Laguerre-Legendre approximation outside a rectangle containing the obstacle, and use a spectral method for quadrilaterals on the remaining subdomain. Clearly, as the first step of those algorithms, we need to study spectral methods on quadrilaterals.

The paper is organized as follows. The next section is for preliminaries. In Section 3, we introduce an orthogonal system on a convex quadrilateral, induced by

Received by the editor July 15, 2008 and, in revised form, April 30, 2009 and June 21, 2009.

2010 *Mathematics Subject Classification.* Primary 65N35, 41A30, 35J05.

Key words and phrases. Orthogonal approximation on quadrilaterals, spectral method.

The work of this author is supported in part by NSF of China N.10871131, Science and Technology Commission of Shanghai Municipality, Grant N.075105118, Shanghai Leading Academic Discipline Project N.S30405, and Fund for E-institute of Shanghai Universities N.E03004.

©2010 American Mathematical Society
Reverts to public domain 28 years from publication

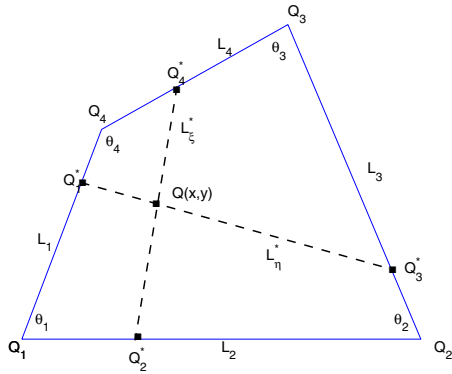


FIGURE 1. Quadrilateral Ω .

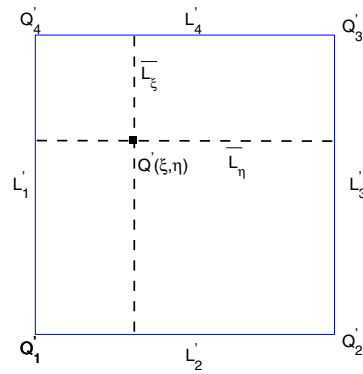


FIGURE 2. Square S .

the Legendre polynomials. Then we establish the basic results on the corresponding orthogonal approximation, which possesses spectral accuracy. Moreover, it still keeps high accuracy, even if the approximated function possesses certain singularities at the edges or the vertices of a quadrilateral. These results play important roles in the spectral method for partial differential equations defined on quadrilaterals. As examples of applications, we provide the spectral schemes for two model problems in Section 4, with the analysis of convergence. We describe their numerical implementation in Section 5, and present some numerical results in Section 6. In Section 7, we consider the spectral method for convex polygons and prove the spectral accuracy of their numerical solutions. The final section is for concluding remarks.

2. PRELIMINARIES

Let Ω be a convex quadrilateral with the edges L_j , the vertices $Q_j = (x_j, y_j)$ and the angles θ_j , $1 \leq j \leq 4$; see Figure 1. The length of L_j is denoted by l_j . For any $Q = (x, y) \in \Omega$, we set

$$(2.1) \quad \begin{aligned} \sigma_1(\xi, \eta) &= \frac{1}{4}(1 - \xi)(1 - \eta), & \sigma_2(\xi, \eta) &= \frac{1}{4}(1 + \xi)(1 - \eta), \\ \sigma_3(\xi, \eta) &= \frac{1}{4}(1 + \xi)(1 + \eta), & \sigma_4(\xi, \eta) &= \frac{1}{4}(1 - \xi)(1 + \eta). \end{aligned}$$

We make the variable transformation (cf. [2, 5, 17, 19]):

$$(2.2) \quad x = x(\xi, \eta) = \sum_{j=1}^4 x_j \sigma_j(\xi, \eta), \quad y = y(\xi, \eta) = \sum_{j=1}^4 y_j \sigma_j(\xi, \eta).$$

More precisely,

$$(2.3) \quad x = a_0 + a_1\xi + a_2\eta + a_3\xi\eta, \quad y = b_0 + b_1\xi + b_2\eta + b_3\xi\eta$$

where

$$(2.4) \quad \begin{aligned} a_0 &= \frac{1}{4}(x_1 + x_2 + x_3 + x_4), & b_0 &= \frac{1}{4}(y_1 + y_2 + y_3 + y_4), \\ a_1 &= \frac{1}{4}(-x_1 + x_2 + x_3 - x_4), & b_1 &= \frac{1}{4}(-y_1 + y_2 + y_3 - y_4), \\ a_2 &= \frac{1}{4}(-x_1 - x_2 + x_3 + x_4), & b_2 &= \frac{1}{4}(-y_1 - y_2 + y_3 + y_4), \\ a_3 &= \frac{1}{4}(x_1 - x_2 + x_3 - x_4), & b_3 &= \frac{1}{4}(y_1 - y_2 + y_3 - y_4). \end{aligned}$$

By the transformation (2.2), the quadrilateral Ω is transformed to the reference square $S = \{(\xi, \eta) \mid -1 < \xi, \eta < 1\}$, with the edges L'_j and the vertices $Q'_j = (\xi_j, \eta_j)$, $1 \leq j \leq 4$; see Figure 2. In fact, $\xi_1 = \xi_4 = \eta_1 = \eta_2 = -1$ and $\xi_2 = \xi_3 = \eta_3 = \eta_4 = 1$.

It is noted that if Ω is a parallelogram, then $a_3 = b_3 = 0$. In this case, the transformation (2.2) is an affine mapping. In particular, $a_2 = a_3 = b_1 = b_3 = 0$ for any rectangle Ω .

For simplicity, we denote $\frac{\partial x}{\partial \xi}$ by $\partial_\xi x$, etc. The Jacobi matrix of transformation (2.2) is

$$M_\Omega = \begin{pmatrix} \partial_\xi x & \partial_\xi y \\ \partial_\eta x & \partial_\eta y \end{pmatrix} = \begin{pmatrix} a_1 + a_3\eta & b_1 + b_3\eta \\ a_2 + a_3\xi & b_2 + b_3\xi \end{pmatrix}.$$

Its Jacobian determinant is

$$(2.5) \quad J_\Omega(\xi, \eta) = \begin{vmatrix} a_1 + a_3\eta & b_1 + b_3\eta \\ a_2 + a_3\xi & b_2 + b_3\xi \end{vmatrix} = d_0 + d_1\xi + d_2\eta$$

where

$$d_0 = a_1b_2 - a_2b_1, \quad d_1 = a_1b_3 - a_3b_1, \quad d_2 = a_3b_2 - a_2b_3.$$

By virtue of (2.4), we get

$$(2.6) \quad \begin{aligned} d_0 &= \frac{1}{8}(x_4 - x_2)(y_1 - y_3) + \frac{1}{8}(x_1 - x_3)(y_2 - y_4), \\ d_1 &= \frac{1}{8}(x_4 - x_3)(y_2 - y_1) + \frac{1}{8}(x_2 - x_1)(y_3 - y_4), \\ d_2 &= \frac{1}{8}(x_3 - x_2)(y_4 - y_1) + \frac{1}{8}(x_1 - x_4)(y_3 - y_2). \end{aligned}$$

Obviously, $J_\Omega(\xi, \eta)$ is a linear function of variables ξ and η , and so reaches its extremum at the four vertices. A direct calculation shows that

$$\begin{aligned} J_\Omega(-1, -1) &= \frac{1}{4}(x_4 - x_2)y_1 + \frac{1}{4}(x_1 - x_4)y_2 + \frac{1}{4}(x_2 - x_1)y_4, \\ J_\Omega(1, -1) &= \frac{1}{4}(x_3 - x_2)y_1 + \frac{1}{4}(x_1 - x_3)y_2 + \frac{1}{4}(x_2 - x_1)y_3, \\ J_\Omega(1, 1) &= \frac{1}{4}(x_4 - x_3)y_2 + \frac{1}{4}(x_2 - x_4)y_3 + \frac{1}{4}(x_3 - x_2)y_4, \\ J_\Omega(-1, 1) &= \frac{1}{4}(x_4 - x_3)y_1 + \frac{1}{4}(x_1 - x_4)y_3 + \frac{1}{4}(x_3 - x_1)y_4, \end{aligned}$$

or equivalently,

$$\begin{aligned} J_\Omega(-1, -1) &= \frac{1}{4}(x_2 - x_1)(y_4 - y_1) - \frac{1}{4}(x_4 - x_1)(y_2 - y_1), \\ J_\Omega(1, -1) &= \frac{1}{4}(x_3 - x_2)(y_1 - y_2) - \frac{1}{4}(x_1 - x_2)(y_3 - y_2), \\ J_\Omega(1, 1) &= \frac{1}{4}(x_4 - x_3)(y_2 - y_3) - \frac{1}{4}(x_2 - x_3)(y_4 - y_3), \\ J_\Omega(-1, 1) &= \frac{1}{4}(x_1 - x_4)(y_3 - y_4) - \frac{1}{4}(x_3 - x_4)(y_1 - y_4). \end{aligned}$$

As pointed out by Strang and Fix in Section 3.3 of [18], we have

$$\begin{aligned} J_\Omega(-1, -1) &= l_1l_2 \sin \theta_1, & J_\Omega(1, -1) &= l_2l_3 \sin \theta_2, \\ J_\Omega(1, 1) &= l_3l_4 \sin \theta_3, & J_\Omega(-1, 1) &= l_4l_1 \sin \theta_4. \end{aligned}$$

Due to the convexity of Ω , the above four quantities are positive. Thereby, there exist positive constants δ_Ω and δ_Ω^* , such that

$$(2.7) \quad \begin{aligned} 0 < \delta_\Omega &= \min\{J_\Omega(-1, -1), J_\Omega(1, -1), J_\Omega(1, 1), J_\Omega(-1, 1)\} \\ &\leq J_\Omega(\xi, \eta) \leq \max\{J_\Omega(-1, -1), J_\Omega(1, -1), J_\Omega(1, 1), J_\Omega(-1, 1)\} = \delta_\Omega^*. \end{aligned}$$

The inverse of transformation (2.2) is

$$\xi = \xi(x, y), \quad \eta = \eta(x, y).$$

The explicit expressions of $\xi(x, y)$ and $\eta(x, y)$ are given in Appendix A of this paper. The Jacobi matrix of this inverse transformation is

$$(2.8) \quad M_S = M_\Omega^{-1} = \begin{pmatrix} \partial_x \xi & \partial_x \eta \\ \partial_y \xi & \partial_y \eta \end{pmatrix} = \frac{1}{J_\Omega(\xi, \eta)} \begin{pmatrix} b_2 + b_3 \xi & -b_1 - b_3 \eta \\ -a_2 - a_3 \xi & a_1 + a_3 \eta \end{pmatrix}.$$

Thanks to (2.7), its Jacobian determinant $J_S(x, y)$ satisfies

$$(2.9) \quad 0 < \frac{1}{\delta_\Omega^*} \leq J_S(x, y) = J_\Omega^{-1}(\xi, \eta) \leq \frac{1}{\delta_\Omega}.$$

We now derive several relations, which will be used in forthcoming discussions. For any point $Q = (x, y) \in \Omega$, there exists the corresponding point $Q' = (\xi(x, y), \eta(x, y)) \in S$. Denote by \bar{L}_η the line in S , which is parallel to the ξ -axis and passes by Q' . Clearly, all points on this line have the same coordinate η . The mapping of the line \bar{L}_η , denoted by L_η^* , is not parallel to the x -axis usually. But it intersects the edges L_1 and L_3 at the points $Q_1^* = (x_1^*(x, y), y_1^*(x, y))$ and $Q_3^* = (x_3^*(x, y), y_3^*(x, y))$, respectively; see Figures 1 and 2. Similarly, all points on the line \bar{L}_ξ , which are parallel to the η -axis and pass by Q' , have the same coordinate ξ . The mapping of the line \bar{L}_ξ , denoted by L_ξ^* , is not parallel to the y -axis usually; but it intersects the edges L_2 and L_4 at the points $Q_2^* = (x_2^*(x, y), y_2^*(x, y))$ and $Q_4^* = (x_4^*(x, y), y_4^*(x, y))$, respectively.

We now calculate $x_j^*(x, y)$ and $y_j^*(x, y)$, $1 \leq j \leq 4$. Indeed, (2.2) implies

$$(2.10) \quad x_1^* = \sum_{j=1}^4 x_j \sigma_j(-1, \eta), \quad y_1^* = \sum_{j=1}^4 y_j \sigma_j(-1, \eta),$$

whence

$$(2.11) \quad x_1^* = \frac{1}{2}(x_1 + x_4) + \frac{1}{2}(x_4 - x_1)\eta, \quad y_1^* = \frac{1}{2}(y_1 + y_4) + \frac{1}{2}(y_4 - y_1)\eta.$$

Similarly,

$$(2.12) \quad x_2^* = \frac{1}{2}(x_1 + x_2) + \frac{1}{2}(x_2 - x_1)\xi, \quad y_2^* = \frac{1}{2}(y_1 + y_2) + \frac{1}{2}(y_2 - y_1)\xi,$$

$$(2.13) \quad x_3^* = \frac{1}{2}(x_2 + x_3) + \frac{1}{2}(x_3 - x_2)\eta, \quad y_3^* = \frac{1}{2}(y_2 + y_3) + \frac{1}{2}(y_3 - y_2)\eta,$$

$$(2.14) \quad x_4^* = \frac{1}{2}(x_3 + x_4) + \frac{1}{2}(x_3 - x_4)\xi, \quad y_4^* = \frac{1}{2}(y_3 + y_4) + \frac{1}{2}(y_3 - y_4)\xi.$$

Subtracting the first formula of (2.10) from the first formula of (2.2), we obtain

$$x - x_1^* = \frac{1}{4}(x_2 - x_1)(1 + \xi)(1 - \eta) + \frac{1}{4}(x_3 - x_4)(1 + \xi)(1 + \eta).$$

Similarly,

$$y - y_1^* = \frac{1}{4}(y_2 - y_1)(1 + \xi)(1 - \eta) + \frac{1}{4}(y_3 - y_4)(1 + \xi)(1 + \eta).$$

The above two equalities, together with (2.4), lead to

$$(2.15) \quad x - x_1^* = (1 + \xi)(a_1 + a_3\eta), \quad y - y_1^* = (1 + \xi)(b_1 + b_3\eta).$$

Similarly,

$$(2.16) \quad x - x_2^* = (1 + \eta)(a_2 + a_3\xi), \quad y - y_2^* = (1 + \eta)(b_2 + b_3\xi),$$

$$(2.17) \quad x - x_3^* = -(1 - \xi)(a_1 + a_3\eta), \quad y - y_3^* = -(1 - \xi)(b_1 + b_3\eta),$$

$$(2.18) \quad x - x_4^* = -(1 - \eta)(a_2 + a_3\xi), \quad y - y_4^* = -(1 - \eta)(b_2 + b_3\xi).$$

3. ORTHOGONAL APPROXIMATION ON A QUADRILATERAL

In this section, we establish the basic results on the orthogonal approximation on a convex quadrilateral.

3.1. Legendre orthogonal approximation. We first recall some recent results on the Legendre orthogonal approximation. Let $\Lambda_\xi = \{\xi \mid |\xi| < 1\}$ and $\chi^{(\alpha,\beta)}(\xi) = (1-\xi)^\alpha(1+\xi)^\beta$, $\alpha, \beta > -1$. We define the weighted space $L^2_{\chi^{(\alpha,\beta)}}(\Lambda_\xi)$ in the usual way, with the following inner product and norm,

$$(u, v)_{\chi^{(\alpha,\beta)}, \Lambda_\xi} = \int_{\Lambda} u(\xi)v(\xi)\chi^{(\alpha,\beta)}(\xi)d\xi, \quad \|v\|_{\chi^{(\alpha,\beta)}, \Lambda_\xi} = (v, v)_{\chi^{(\alpha,\beta)}, \Lambda_\xi}^{\frac{1}{2}},$$

$$\forall u, v \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda_\xi).$$

We omit the subscript $\chi^{(\alpha,\beta)}$ in notation whenever $\alpha = \beta = 0$.

The Legendre polynomial of degree l is defined by

$$L_l(\xi) = \frac{(-1)^l}{2^l l!} \partial_\xi^l (1 - \xi^2)^l.$$

The set of Legendre polynomials is a complete $L^2(\Lambda_\xi)$ -orthogonal system. Moreover,

$$(3.1) \quad \|L_l\|_{\Lambda_\xi}^2 = \left(l + \frac{1}{2}\right)^{-1}.$$

Let N be any positive integer. Denote by $\mathcal{P}_N(\Lambda_\xi)$ the set of all polynomials of degree at most N . Moreover, $\mathcal{P}_N^0(\Lambda_\xi) = \mathcal{P}_N(\Lambda_\xi) \cap H_0^1(\Lambda_\xi)$. The $L^2(\Lambda_\xi)$ -orthogonal projection $P_{N, \Lambda_\xi} : L^2(\Lambda_\xi) \rightarrow \mathcal{P}_N(\Lambda_\xi)$, is defined by

$$(3.2) \quad (P_{N, \Lambda_\xi} v - v, \phi)_{\Lambda_\xi} = 0, \quad \forall \phi \in \mathcal{P}_N(\Lambda_\xi).$$

Throughout this paper, we denote by c a generic positive constant independent of N and any function. According to Theorem 2.1 of [12], we know that if $v \in L^2(\Lambda_\xi)$, $\partial_\xi^r v \in L^2_{\chi^{(r,r)}}(\Lambda_\xi)$, integers $r \geq 0$ and $r \leq N + 1$, then

$$(3.3) \quad \|P_{N, \Lambda} v - v\|_{\Lambda_\xi} \leq cN^{-r} \|\partial_\xi^r v\|_{\chi^{(r,r)}, \Lambda_\xi}.$$

Next, the orthogonal projection $P_{N, \Lambda_\xi}^{1,0} : H_0^1(\Lambda_\xi) \rightarrow \mathcal{P}_N^0(\Lambda_\xi)$, is defined by

$$(3.4) \quad (\partial_\xi (P_{N, \Lambda_\xi}^{1,0} v - v), \partial_\xi \phi)_{\Lambda_\xi} = 0, \quad \forall \phi \in \mathcal{P}_N^0(\Lambda_\xi).$$

As a special case of Theorem 3.4 of [12], we have that if $v \in H_0^1(\Lambda_\xi)$, $\partial_\xi^r v \in L^2_{\chi^{(r-1, r-1)}}(\Lambda_\xi)$ and integers $1 \leq r \leq N + 1$, then

$$(3.5) \quad \|\partial_\xi^\mu (P_{N, \Lambda_\xi}^{1,0} v - v)\|_{\Lambda_\xi} \leq cN^{\mu-r} \|\partial_\xi^r v\|_{\chi^{(r-1, r-1)}, \Lambda_\xi}, \quad \mu = 0, 1.$$

3.2. $L^2(\Omega)$ -orthogonal approximation. We now consider the $L^2(\Omega)$ -orthogonal approximation on the quadrilateral Ω .

Let $\Lambda_\eta = \{\eta \mid |\eta| < 1\}$. Clearly, the square $S = \Lambda_\xi \times \Lambda_\eta$. We denote the inner product and the norm of $L^2(\Omega)$ by $(u, v)_\Omega$ and $\|v\|_\Omega$, respectively.

We introduce the functions

$$(3.6) \quad \begin{aligned} \psi_{l,m}(x, y) &= L_l(\xi(x, y))L_m(\eta(x, y))J_S^{\frac{1}{2}}(x, y) \\ &= L_l(\xi(x, y))L_m(\eta(x, y))J_\Omega^{-\frac{1}{2}}(\xi(x, y), \eta(x, y)), \quad l, m \geq 0. \end{aligned}$$

By virtue of (3.1), we get

$$\begin{aligned} \int \int_{\Omega} \psi_{l,m}(x, y)\psi_{l',m'}(x, y)dxdy &= \int_{\Lambda_{\xi}} L_l(\xi)L_{l'}(\xi)d\xi \int_{\Lambda_{\eta}} L_m(\eta)L_{m'}(\eta)d\eta \\ &= (l + \frac{1}{2})^{-1}(m + \frac{1}{2})^{-1}\delta_{l,l'}\delta_{m,m'}, \end{aligned}$$

where $\delta_{l,l'}$ is the Kronecker symbol. Moreover, the set of all $\psi_{l,m}$ is complete in the space $L^2(\Omega)$. Thus, for any $v \in L^2(\Omega)$, we have

$$(3.7) \quad v(x, y) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \hat{v}_{l,m}\psi_{l,m}(x, y),$$

with

$$(3.8) \quad \hat{v}_{l,m} = (l + \frac{1}{2})(m + \frac{1}{2}) \int \int_{\Omega} v(x, y)\psi_{l,m}(x, y)dxdy.$$

Furthermore, let

$$V_N(\Omega) = span\{\psi_{l,m}(x, y) \mid 0 \leq l, m \leq N\}.$$

The $L^2(\Omega)$ -orthogonal projection $P_N : L^2(\Omega) \rightarrow V_N(\Omega)$, is defined by

$$(3.9) \quad (P_N v - v, \phi)_{\Omega} = 0, \quad \forall \phi \in V_N(\Omega).$$

For description of approximation errors, we shall use the following notation:

$$\begin{aligned} A_{r,\Omega}(v) &= \sum_{k=0}^r \sum_{j=0}^k \delta_{\Omega}^{-r+k} (|d_1|^{r-k} \|(1 - \xi^2)^{\frac{r}{2}}(a_1 + a_3 \eta)^j (b_1 + b_3 \eta)^{k-j} \partial_x^j \partial_y^{k-j} v\|_{\Omega} \\ &\quad + |d_2|^{r-k} \|(1 - \eta^2)^{\frac{r}{2}}(a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j} \partial_x^j \partial_y^{k-j} v\|_{\Omega}). \end{aligned}$$

Theorem 3.1. *If $v \in L^2(\Omega)$, $A_{r,\Omega}(v)$ is finite for integer $r \geq 0$, and $r \leq N + 1$, then*

$$(3.10) \quad \|P_N v - v\|_{\Omega} \leq cN^{-r} A_{r,\Omega}(v).$$

Proof. By projection theorem,

$$\|P_N v - v\|_{\Omega}^2 \leq \|\phi - v\|_{\Omega}^2, \quad \forall \phi \in V_N(\Omega).$$

Let

$$(3.11) \quad \begin{aligned} u(\xi, \eta) &= v(x(\xi, \eta), y(\xi, \eta)) J_{\Omega}^{\frac{1}{2}}(\xi, \eta), \\ \phi(x, y) &= (P_{N,\Lambda_{\xi}} \bullet P_{N,\Lambda_{\eta}} u)(\xi(x, y), \eta(x, y)) J_{\Omega}^{-\frac{1}{2}}(\xi(x, y), \eta(x, y)) \in V_N(\Omega). \end{aligned}$$

Due to $J_S(x, y) = J_{\Omega}^{-1}(\xi, \eta)$, we use (3.3) to obtain

$$(3.12) \quad \begin{aligned} \|v - \phi\|_{\Omega}^2 &= \int \int_S (u - P_{N,\Lambda_{\xi}} \bullet P_{N,\Lambda_{\eta}} u)^2 d\xi d\eta \\ &\leq 2 \int \int_S (u - P_{N,\Lambda_{\xi}} u)^2 d\xi d\eta + 2 \int \int_S (P_{N,\Lambda_{\xi}} \bullet (u - P_{N,\Lambda_{\eta}} u))^2 d\xi d\eta \\ &\leq cN^{-2r} \|\partial_{\xi}^r u\|_{L_{\chi}^2(r,r)(\Lambda_{\xi}, L^2(\Lambda_{\eta}))}^2 + c\|P_{N,\Lambda_{\eta}} u - u\|_{L^2(\Lambda_{\xi}, L^2(\Lambda_{\eta}))}^2 \\ &\leq cN^{-2r} (\|\partial_{\xi}^r u\|_{L_{\chi}^2(r,r)(\Lambda_{\xi}, L^2(\Lambda_{\eta}))}^2 + \|\partial_{\eta}^r u\|_{L^2(\Lambda_{\xi}, L_{\chi}^2(r,r)(\Lambda_{\eta}))}^2). \end{aligned}$$

Thus, it remains to estimate the right side of (3.12).

Let $\sigma(q) = (-1)^{q-1}2^{-q}(2q-3)!!$. Thanks to (2.5), we have

$$\partial_\xi^q J_\Omega^{\frac{1}{2}} = \sigma(q)d_1^q J_\Omega^{\frac{1}{2}-q}, \quad \partial_\eta^q J_\Omega^{\frac{1}{2}} = \sigma(q)d_2^q J_\Omega^{\frac{1}{2}-q}.$$

Hence, we have from (3.11) that

$$(3.13) \quad \partial_\xi^r u = \partial_\xi^r (v J_\Omega^{\frac{1}{2}}) = J_\Omega^{\frac{1}{2}} \sum_{k=0}^r C_r^k \sigma(r-k) d_1^{r-k} J_\Omega^{-r+k} \partial_\xi^k v,$$

$$(3.14) \quad \partial_\eta^r u = \partial_\eta^r (v J_\Omega^{\frac{1}{2}}) = J_\Omega^{\frac{1}{2}} \sum_{k=0}^r C_r^k \sigma(r-k) d_2^{r-k} J_\Omega^{-r+k} \partial_\eta^k v.$$

Furthermore, thanks to (2.3), we have

$$(3.15) \quad \partial_\xi v = \partial_x v \partial_\xi x + \partial_y v \partial_\xi y = (a_1 + a_3 \eta) \partial_x v + (b_1 + b_3 \eta) \partial_y v,$$

$$(3.16) \quad \partial_\eta v = \partial_x v \partial_\eta x + \partial_y v \partial_\eta y = (a_2 + a_3 \xi) \partial_x v + (b_2 + b_3 \xi) \partial_y v.$$

Thus, we derive inductively that

$$(3.17) \quad \partial_\xi^k v = \sum_{j=0}^k C_k^j (a_1 + a_3 \eta)^j (b_1 + b_3 \eta)^{k-j} \partial_x^j \partial_y^{k-j} v,$$

$$(3.18) \quad \partial_\eta^k v = \sum_{j=0}^k C_k^j (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j} \partial_x^j \partial_y^{k-j} v.$$

Inserting (3.17) into (3.13) and inserting (3.18) into (3.14), respectively, we find that

$$(3.19) \quad \partial_\xi^r u = J_\Omega^{\frac{1}{2}} \sum_{k=0}^r \sum_{j=0}^k C_r^k C_k^j \sigma(r-k) d_1^{r-k} J_\Omega^{-r+k} (a_1 + a_3 \eta)^j (b_1 + b_3 \eta)^{k-j} \partial_x^j \partial_y^{k-j} v,$$

$$(3.20) \quad \partial_\eta^r u = J_\Omega^{\frac{1}{2}} \sum_{k=0}^r \sum_{j=0}^k C_r^k C_k^j \sigma(r-k) d_2^{r-k} J_\Omega^{-r+k} (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j} \partial_x^j \partial_y^{k-j} v.$$

Therefore,

$$(3.21) \quad \begin{aligned} & \|\partial_\xi^r u\|_{L^2_{\chi^{(r,r)}}(\Lambda_\xi, L^2(\Lambda_\eta))} \\ & \leq c \sum_{k=0}^r \sum_{j=0}^k |d_1|^{r-k} \delta_\Omega^{-r+k} \|(1-\xi^2)^{\frac{r}{2}} (a_1 + a_3 \eta)^j (b_1 + b_3 \eta)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega. \end{aligned}$$

Similarly,

$$(3.22) \quad \begin{aligned} & \|\partial_\eta^r u\|_{L^2(\Lambda_\xi, L^2_{\chi^{(r,r)}}(\Lambda_\eta))} \\ & \leq c \sum_{k=0}^r \sum_{j=0}^k |d_2|^{r-k} \delta_\Omega^{-r+k} \|(1-\eta^2)^{\frac{r}{2}} (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega. \end{aligned}$$

Finally, the desired result follows from a combination of (3.12), (3.21) and (3.22). \square

Remark 3.1. By virtue of (2.15)–(2.18),

$$(3.23) \quad \begin{aligned} a_1 + a_3\eta &= \frac{x-x_1^*}{1+\xi} = -\frac{x-x_3^*}{1-\xi}, & b_1 + b_3\eta &= \frac{y-y_1^*}{1+\xi} = -\frac{y-y_3^*}{1-\xi}, \\ a_2 + a_3\xi &= \frac{x-x_2^*}{1+\eta} = -\frac{x-x_4^*}{1-\eta}, & b_2 + b_3\xi &= \frac{y-y_2^*}{1+\eta} = -\frac{y-y_4^*}{1-\eta}. \end{aligned}$$

Consequently,

$$(3.24) \quad \begin{aligned} |a_1 + a_3\eta| &= \sqrt{-\frac{(x-x_1^*)(x-x_3^*)}{1-\xi^2}}, & |b_1 + b_3\eta| &= \sqrt{-\frac{(y-y_1^*)(y-y_3^*)}{1-\xi^2}}, \\ |a_2 + a_3\xi| &= \sqrt{-\frac{(x-x_2^*)(x-x_4^*)}{1-\eta^2}}, & |b_2 + b_3\xi| &= \sqrt{-\frac{(y-y_2^*)(y-y_4^*)}{1-\eta^2}}. \end{aligned}$$

Therefore, (3.10) implies

$$(3.25) \quad \|P_N v - v\|_\Omega \leq cN^{-r} A_{r,\Omega}^*(v)$$

where

$$\begin{aligned} A_{r,\Omega}^*(v) &= \sum_{k=0}^r \sum_{j=0}^k \delta_\Omega^{-r+k} (|d_1|^{r-k} \|((x-x_1^*)(x-x_3^*))^{\frac{j}{2}} ((y-y_1^*)(y-y_3^*))^{\frac{k-j}{2}} \partial_x^j \partial_y^{k-j} v\|_\Omega \\ &\quad + |d_2|^{r-k} \|((x-x_2^*)(x-x_4^*))^{\frac{j}{2}} ((y-y_2^*)(y-y_4^*))^{\frac{k-j}{2}} \partial_x^j \partial_y^{k-j} v\|_\Omega). \end{aligned}$$

On the other hand, let $x_5 = x_1, y_5 = y_1$, and

$$\sigma_\Omega = \max_{(\xi,\eta) \in S} (|b_2 + b_3\xi|, |b_1 + b_3\eta|, |a_2 + a_3\xi|, |a_1 + a_3\eta|).$$

It can be checked that

$$(3.26) \quad \sigma_\Omega = \frac{1}{2} \max_{1 \leq j \leq 4} (|x_j - x_{j+1}|, |y_j - y_{j+1}|).$$

Accordingly, by (3.10),

$$(3.27) \quad \|P_N v - v\|_\Omega \leq cN^{-r} A_{r,\Omega}^{**}(v)$$

where

$$A_{r,\Omega}^{**}(v) = \sum_{k=0}^r \sum_{j=0}^k \delta_\Omega^{-r+k} \sigma_\Omega^k (|d_1|^{r-k} \|(1-\xi^2)^{\frac{j}{2}} \partial_x^j \partial_y^{k-j} v\|_\Omega + |d_2|^{r-k} \|(1-\eta^2)^{\frac{j}{2}} \partial_x^j \partial_y^{k-j} v\|_\Omega).$$

Remark 3.2. We see from the definition of $A_{r,\Omega}(v)$ and (3.24) that in the norm of derivative $\partial_x^j \partial_y^{k-j} v$, there are the weight functions

$$\begin{aligned} f_{j,k} &= |(1-\xi^2)^{r-k} ((x-x_1^*)(x-x_3^*))^j ((y-y_1^*)(y-y_3^*))^{k-j}|, \quad 0 \leq j \leq k \leq r, \\ g_{j,k} &= |(1-\eta^2)^{r-k} ((x-x_2^*)(x-x_4^*))^j ((y-y_2^*)(y-y_4^*))^{k-j}|, \quad 0 \leq j \leq k \leq r. \end{aligned}$$

Let $|Q - Q_\nu^*|$ be the distance between $Q(x, y)$ and $Q_\nu^*(x, y)$. Clearly,

$$\begin{aligned} f_{j,k} &\leq c(1-\xi^2)^{r-k} |Q - Q_\nu^*|^{2k}, & 0 \leq k \leq r, \nu &= 1, 3, \\ g_{j,k} &\leq c(1-\eta^2)^{r-k} |Q - Q_\nu^*|^{2k}, & 0 \leq k \leq r, \nu &= 2, 4. \end{aligned}$$

Thus, all $f_{j,k}$ tend to zero as $Q(x, y)$ goes to L_1 or L_3 , while all $g_{j,k}$ tend to zero as $Q(x, y)$ goes to L_2 or L_4 . Moreover, the higher the order k of derivative, the smaller the weight functions $f_{j,k}$ and $g_{j,k}$. Next, let $Q_5(x, y) = Q_1(x, y)$. We find, especially from (2.11)–(2.14), that the points $Q_\nu^*(x, y)$ and $Q_{\nu+1}^*(x, y)$ tend to the same vertex Q_ν , as the point $Q(x, y)$ goes to Q_ν , $1 \leq \nu \leq 4$. Therefore, the corresponding weight functions tend to zero simultaneously. As a result, $\|P_N v - v\|_\Omega$ still keeps the order N^{-r} , even if the approximated function has the singularity

at the vertices, such as $|\partial_x^j \partial_y^{k-j} v| = o((1 - \xi^2)^{-\frac{r+k}{2}} |Q - Q_\nu^*|^{-\frac{2k-1}{2}})$, $\nu = 1, 3$ and $|\partial_x^j \partial_y^{k-j} v| = o((1 - \eta^2)^{-\frac{r+k}{2}} |Q - Q_\nu^*|^{-\frac{2k-1}{2}})$, $\nu = 2, 4$.

Remark 3.3. If $\Omega = \{(x, y) \mid |x| < a, |y| < b, a, b, > 0\}$, then $a_1 = a, b_2 = b, a_2 = a_3 = b_1 = b_3 = 0$. In this case, $\partial_\xi v = a \partial_x v, \partial_\eta v = b \partial_y v, J_\Omega = ab, x_1^*(x, y) = -a, x_2^*(x, y) = x_4^*(x, y) = x, x_3^*(x, y) = a, y_1^*(x, y) = y_3^*(x, y) = y, y_2^*(x, y) = -b, y_4^*(x, y) = b$, and $\xi(x, y) = \frac{x}{a}, \eta(x, y) = \frac{y}{b}$. Therefore,

$$\|P_N v - v\|_\Omega \leq cN^{-r} (\|(a^2 - x^2)^{\frac{r}{2}} \partial_x^r v\|_\Omega + \|(b^2 - y^2)^{\frac{r}{2}} \partial_y^r v\|_\Omega).$$

It is noted that in this special case, the $L^2(\Omega)$ -orthogonal approximation keeps the same spectral accuracy, even if the considered function possesses certain singularities at the edges of the quadrilateral. If, in addition, $a = b = 1$, then the above estimate turns out to be the same result as in [13].

3.3. $H_0^1(\Omega)$ -orthogonal approximation. We now turn to the $H_0^1(\Omega)$ -orthogonal approximation. According to the Poincaré inequality, there exists a positive constant c_Ω such that

$$(3.28) \quad \|w\|_\Omega \leq c_\Omega \|\nabla w\|_\Omega, \quad \forall w \in H_0^1(\Omega).$$

For simplicity of statements, let $(a_1 + a_3 \eta)_\dagger^\nu = (a_1 + a_3 \eta)^\nu$ for $\nu \geq 0$, and $(a_1 + a_3 \eta)_\dagger^\nu = 0$ otherwise. The meanings of $(b_1 + b_3 \eta)_\dagger^\nu$, $(a_2 + a_3 \xi)_\dagger^\nu$ and $(b_2 + b_3 \xi)_\dagger^\nu$ are similar. Let $\gamma_\Omega = \max(|a_3|, |b_3|)$. Clearly, $\gamma_\Omega \leq \sigma_\Omega$. In order to describe the approximation error, we also introduce the following notation:

$$\begin{aligned} B_{r,\Omega}(v) = & \sigma_\Omega ((|d_1| + |d_2|) \delta_\Omega^{-1} + 1) \\ & \cdot \left(\sum_{k=0}^r \sum_{j=0}^k \delta_\Omega^{-r+k-1} \{ |d_1|^{r-k} \|(1 - \xi^2)^{\frac{r-1}{2}} (a_1 + a_3 \eta)^j (b_1 + b_3 \eta)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega \right. \\ & + |d_2|^{r-k} \|(1 - \eta^2)^{\frac{r-1}{2}} (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega \} \\ & + \sum_{k=0}^{r-1} \sum_{j=0}^k \delta_\Omega^{-r+k} \{ |d_1|^{r-k-1} [|d_2| \delta_\Omega^{-1} \|(1 - \xi^2)^{\frac{r-2}{2}} (a_1 + a_3 \eta)^j (b_1 + b_3 \eta)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega \\ & + \gamma_\Omega \|(1 - \xi^2)^{\frac{r-2}{2}} (a_1 + a_3 \eta)_\dagger^{j-1} (b_1 + b_3 \eta)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega \\ & + \gamma_\Omega \|(1 - \xi^2)^{\frac{r-2}{2}} (a_1 + a_3 \eta)^j (b_1 + b_3 \eta)_\dagger^{k-j-1} \partial_x^j \partial_y^{k-j} v\|_\Omega \\ & + \sigma_\Omega \|(1 - \xi^2)^{\frac{r-2}{2}} (a_1 + a_3 \eta)^j (b_1 + b_3 \eta)^{k-j} \partial_x^{j+1} \partial_y^{k-j} v\|_\Omega \\ & + \sigma_\Omega \|(1 - \xi^2)^{\frac{r-2}{2}} (a_1 + a_3 \eta)^j (b_1 + b_3 \eta)^{k-j} \partial_x^j \partial_y^{k-j+1} v\|_\Omega \\ & + |d_2|^{r-k-1} [|d_1| \delta_\Omega^{-1} \|(1 - \eta^2)^{\frac{r-2}{2}} (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega \\ & + \gamma_\Omega \|(1 - \eta^2)^{\frac{r-2}{2}} (a_2 + a_3 \xi)_\dagger^{j-1} (b_2 + b_3 \xi)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega \\ & + \gamma_\Omega \|(1 - \eta^2)^{\frac{r-2}{2}} (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)_\dagger^{k-j-1} \partial_x^j \partial_y^{k-j} v\|_\Omega \\ & + \sigma_\Omega \|(1 - \eta^2)^{\frac{r-2}{2}} (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j} \partial_x^{j+1} \partial_y^{k-j} v\|_\Omega \\ & + \sigma_\Omega \|(1 - \eta^2)^{\frac{r-2}{2}} (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j} \partial_x^j \partial_y^{k-j+1} v\|_\Omega \} \Big). \end{aligned}$$

Let $V_N^0(\Omega) = H_0^1(\Omega) \cap V_N(\Omega)$. The orthogonal projection $P_N^{1,0} : H_0^1(\Omega) \rightarrow V_N^0(\Omega)$ is defined by

$$(3.29) \quad (\nabla(P_N^{1,0} v - v), \nabla \phi)_\Omega = 0, \quad \forall \phi \in V_N^0(\Omega).$$

Theorem 3.2. *If $v \in H_0^1(\Omega)$ and $B_{r,\Omega}(v)$ is finite for integers $2 \leq r \leq N + 1$, then*

$$(3.30) \quad \|\nabla(P_N^{1,0}v - v)\|_\Omega \leq cN^{1-r}B_{r,\Omega}(v), \quad \|P_N^{1,0}v - v\|_\Omega \leq cc_\Omega^*N^{-r}B_{r,\Omega}(v),$$

where c_Ω^* is a positive constant depending on Ω .

Proof. By the projection theorem,

$$(3.31) \quad \|\nabla(P_N^{1,0}v - v)\|_\Omega \leq \|\nabla(\phi - v)\|_\Omega, \quad \forall \phi \in V_N^0(\Omega).$$

Let $u(\xi, \eta)$ be the same as in (3.11), and $\psi(\xi, \eta) = (P_{N,\Lambda_\xi}^{1,0} \bullet P_{N,\Lambda_\eta}^{1,0} u)(\xi, \eta)$. Take

$$\phi(x, y) = \psi(\xi(x, y), \eta(x, y))J_\Omega^{-\frac{1}{2}}(\xi(x, y), \eta(x, y)) \in V_N^0(\Omega).$$

We denote by $\|w\|_S$ the norm of the space $L^2(S)$, and $\nabla_S w = (\partial_\xi w, \partial_\eta w)^T$. It can be shown that $\nabla_S w = M_\Omega \nabla w$. Thus by (2.8), $\nabla w = M_S \nabla_S w$. Hence, we use (2.8) and (2.5) successively to obtain

$$\begin{aligned} \partial_x(\phi - v) &= (b_2 + b_3\xi)J_\Omega^{-1}\partial_\xi((\psi - u)J_\Omega^{-\frac{1}{2}}) - (b_1 + b_3\eta)J_\Omega^{-1}\partial_\eta((\psi - u)J_\Omega^{-\frac{1}{2}}) \\ &= (b_2 + b_3\xi)J_\Omega^{-\frac{3}{2}}(\partial_\xi(\psi - u) - \frac{1}{2}d_1J_\Omega^{-1}(\psi - u)) \\ &\quad - (b_1 + b_3\eta)J_\Omega^{-\frac{3}{2}}(\partial_\eta(\psi - u) - \frac{1}{2}d_2J_\Omega^{-1}(\psi - u)). \end{aligned}$$

With the aid of (2.9), (3.26) and the Poincaré inequality on S , a direct calculation gives

$$\begin{aligned} \|\partial_x(\phi - v)\|_\Omega &\leq c\sigma_\Omega\delta_\Omega^{-1}(\|\nabla(\psi - u)\|_S + (|d_1| + |d_2|)\delta_\Omega^{-1}\|\psi - u\|_S) \\ &\leq c\sigma_\Omega\delta_\Omega^{-1}((|d_1| + |d_2|)\delta_\Omega^{-1} + 1)\|\nabla_S(\psi - u)\|_S. \end{aligned}$$

We can estimate $\|\partial_y(\phi - v)\|_\Omega$ in the same manner. Accordingly,

$$(3.32) \quad \|\nabla(\phi - v)\|_\Omega \leq c\sigma_\Omega\delta_\Omega^{-1}((|d_1| + |d_2|)\delta_\Omega^{-1} + 1)\|\nabla_S(\psi - u)\|_S.$$

We now estimate $\|\nabla_S(\psi - u)\|_S$. Clearly,

$$\begin{aligned} \|\partial_\xi(\psi - u)\|_S^2 &= \|\partial_\xi(P_{N,\Lambda_\xi}^{1,0} \bullet P_{N,\Lambda_\eta}^{1,0} u - u)\|_S^2 \\ &\leq \|\partial_\xi(P_{N,\Lambda_\xi}^{1,0} u - u)\|_S^2 + \|\partial_\xi P_{N,\Lambda_\xi}^{1,0}(P_{N,\Lambda_\eta}^{1,0} u - u)\|_S^2. \end{aligned}$$

Using (3.5) with $\mu = 1$ gives

$$\|\partial_\xi(P_{N,\Lambda_\xi}^{1,0} u - u)\|_S^2 \leq cN^{2-2r}\|\partial_\xi^r u\|_{L^2_{\chi^{(r-1,r-1)}}(\Lambda_\xi, L^2(\Lambda_\eta))}^2.$$

Using (3.5) with $r = \mu = 1$ gives

$$\|\partial_\xi P_{N,\Lambda_\xi}^{1,0}(P_{N,\Lambda_\eta}^{1,0} u - u)\|_S^2 \leq \|\partial_\xi(P_{N,\Lambda_\eta}^{1,0} u - u)\|_{L^2(\Lambda_\xi, L^2(\Lambda_\eta))}^2.$$

Thereby, using (3.5) with $\mu = 0$ again yields

$$\|\partial_\xi P_{N,\Lambda_\xi}^{1,0}(P_{N,\Lambda_\eta}^{1,0} u - u)\|_S^2 \leq cN^{2-2r}\|\partial_\xi\partial_\eta^{r-1}u\|_{L^2(\Lambda_\xi, L^2_{\chi^{(r-2,r-2)}}(\Lambda_\eta))}^2.$$

We can estimate $\|\partial_\eta(\psi - u)\|_S^2$ similarly. Consequently, we get that

$$(3.33) \quad \begin{aligned} \|\nabla_S(\psi - u)\|_S^2 &\leq cN^{2-2r}(\|\partial_\xi^r u\|_{L^2_{\chi^{(r-1,r-1)}}(\Lambda_\xi, L^2(\Lambda_\eta))}^2 + \|\partial_\eta^r u\|_{L^2(\Lambda_\xi, L^2_{\chi^{(r-1,r-1)}}(\Lambda_\eta))}^2 \\ &\quad + \|\partial_\xi^{r-1}\partial_\eta u\|_{L^2_{\chi^{(r-2,r-2)}}(\Lambda_\xi, L^2(\Lambda_\eta))}^2 + \|\partial_\xi\partial_\eta^{r-1}u\|_{L^2(\Lambda_\xi, L^2_{\chi^{(r-2,r-2)}}(\Lambda_\eta))}^2). \end{aligned}$$

We next estimate the right side of (3.33). By (3.19), (3.20) and the same argument as in the derivations of (3.21) and (3.22), we verify that

$$(3.34) \quad \begin{aligned} & \|\partial_\xi^r u\|_{L^2_{\chi^{(r-1, r-1)}}(\Lambda_\xi, L^2(\Lambda_\eta))} \\ & \leq c \sum_{k=0}^r \sum_{j=0}^k |d_1|^{r-k} \delta_\Omega^{-r+k} \|(1-\xi^2)^{\frac{r-1}{2}} (a_1 + a_3\eta)^j (b_1 + b_3\eta)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega, \end{aligned}$$

$$(3.35) \quad \begin{aligned} & \|\partial_\eta^r u\|_{L^2(\Lambda_\xi, L^2_{\chi^{(r-1, r-1)}}(\Lambda_\eta))} \\ & \leq c \sum_{k=0}^r \sum_{j=0}^k |d_2|^{r-k} \delta_\Omega^{-r+k} \|(1-\eta^2)^{\frac{r-1}{2}} (a_2 + a_3\xi)^j (b_2 + b_3\xi)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega. \end{aligned}$$

Furthermore, thanks to (2.3) and (2.5), we have $\partial_\xi J_\Omega = d_1$ and $\partial_\eta J_\Omega = d_2$. Moreover,

$$\begin{aligned} \partial_\xi(\partial_x^j \partial_y^{k-j} v) &= (a_1 + a_3\eta) \partial_x^{j+1} \partial_y^{k-j} v + (b_1 + b_3\eta) \partial_x^j \partial_y^{k-j+1} v, \\ \partial_\eta(\partial_x^j \partial_y^{k-j} v) &= (a_2 + a_3\xi) \partial_x^{j+1} \partial_y^{k-j} v + (b_2 + b_3\xi) \partial_x^j \partial_y^{k-j+1} v. \end{aligned}$$

Therefore, by differentiating (3.19) with respect to η , we obtain

$$(3.36) \quad \begin{aligned} \partial_\xi^{r-1} \partial_\eta u &= \sum_{k=0}^{r-1} \sum_{j=0}^k C_{r-1}^k C_k^j \sigma(r-k-1) d_1^{r-k-1} J_\Omega^{-r+k+\frac{1}{2}} \\ & \cdot [(d_2(-r+k+\frac{3}{2})(a_1 + a_3\eta)^j (b_1 + b_3\eta)^{k-j} \\ & + J_\Omega(a_3(a_1 + a_3\eta)_\dagger^{j-1} (b_1 + b_3\eta)^{k-j} + b_3(a_1 + a_3\eta)^j (b_1 + b_3\eta)_\dagger^{k-j-1}) \partial_x^j \partial_y^{k-j} v \\ & + J_\Omega(a_1 + a_3\eta)^j (b_1 + b_3\eta)^{k-j} ((a_2 + a_3\xi) \partial_x^{j+1} \partial_y^{k-j} v + (b_2 + b_3\xi) \partial_x^j \partial_y^{k-j+1} v)]. \end{aligned}$$

Similarly, by differentiating (3.20) with respect to ξ , we obtain

$$(3.37) \quad \begin{aligned} \partial_\xi \partial_\eta^{r-1} u &= \sum_{k=0}^{r-1} \sum_{j=0}^k C_{r-1}^k C_k^j \sigma(r-k-1) d_2^{r-k-1} J_\Omega^{-r+k+\frac{1}{2}} \\ & \cdot [(d_1(-r+k+\frac{3}{2})(a_2 + a_3\xi)^j (b_2 + b_3\xi)^{k-j} \\ & + J_\Omega(a_3(a_2 + a_3\xi)_\dagger^{j-1} (b_2 + b_3\xi)^{k-j} + b_3(a_2 + a_3\xi)^j (b_2 + b_3\xi)_\dagger^{k-j-1}) \partial_x^j \partial_y^{k-j} v \\ & + J_\Omega(a_2 + a_3\xi)^j (b_2 + b_3\xi)^{k-j} ((a_1 + a_3\eta) \partial_x^{j+1} \partial_y^{k-j} v + (b_1 + b_3\eta) \partial_x^j \partial_y^{k-j+1} v)]. \end{aligned}$$

Then, following the same line as in the derivations of (3.21) and (3.22), we obtain

$$(3.38) \quad \begin{aligned} \|\partial_\xi^{r-1} \partial_\eta u\|_{L^2_{\chi^{(r-2, r-2)}}(\Lambda_\xi, L^2(\Lambda_\eta))} & \leq c \sum_{k=0}^{r-1} \sum_{j=0}^k |d_1|^{r-k-1} \delta_\Omega^{-r+k+1} \\ & \cdot [|d_2| \delta_\Omega^{-1} \|(1-\xi^2)^{\frac{r-2}{2}} (a_1 + a_3\eta)^j (b_1 + b_3\eta)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega \\ & + \gamma_\Omega \|(1-\xi^2)^{\frac{r-2}{2}} (a_1 + a_3\eta)_\dagger^{j-1} (b_1 + b_3\eta)^{k-j} \partial_x^j \partial_y^{k-j} v\|_\Omega \\ & + \gamma_\Omega \|(1-\xi^2)^{\frac{r-2}{2}} (a_1 + a_3\eta)^j (b_1 + b_3\eta)_\dagger^{k-j-1} \partial_x^j \partial_y^{k-j} v\|_\Omega \\ & + \sigma_\Omega \|(1-\xi^2)^{\frac{r-2}{2}} (a_1 + a_3\eta)^j (b_1 + b_3\eta)^{k-j} \partial_x^{j+1} \partial_y^{k-j} v\|_\Omega \\ & + \sigma_\Omega \|(1-\xi^2)^{\frac{r-2}{2}} (a_1 + a_3\eta)^j (b_1 + b_3\eta)^{k-j} \partial_x^j \partial_y^{k-j+1} v\|_\Omega] \end{aligned}$$

and

$$\begin{aligned}
 \|\partial_\xi \partial_\eta^{r-1} u\|_{L^2(\Lambda_\xi, L^2_{\chi(r-2, r-2)}(\Lambda_\eta))} &\leq c \sum_{k=0}^{r-1} \sum_{j=0}^k |d_2|^{r-k-1} \delta_\Omega^{-r+k+1} \\
 &\cdot [|d_1| \delta_\Omega^{-1} \|(1-\eta^2)^{\frac{r-2}{2}} (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j} \partial_x^j \partial_y^{k-j} v \|_\Omega \\
 &+ \gamma_\Omega \|(1-\eta^2)^{\frac{r-2}{2}} (a_2 + a_3 \xi)^{j-1} (b_2 + b_3 \xi)^{k-j} \partial_x^j \partial_y^{k-j} v \|_\Omega \\
 &+ \gamma_\Omega \|(1-\eta^2)^{\frac{r-2}{2}} (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j-1} \partial_x^j \partial_y^{k-j} v \|_\Omega \\
 &+ \sigma_\Omega \|(1-\eta^2)^{\frac{r-2}{2}} (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j} \partial_x^{j+1} \partial_y^{k-j} v \|_\Omega \\
 &+ \sigma_\Omega \|(1-\eta^2)^{\frac{r-2}{2}} (a_2 + a_3 \xi)^j (b_2 + b_3 \xi)^{k-j} \partial_x^j \partial_y^{k-j+1} v \|_\Omega].
 \end{aligned}
 \tag{3.39}$$

Then, the first result of (3.30) comes from a combination of (3.31)–(3.35), (3.38) and (3.39).

We now prove the second result of (3.30). Let $g \in L^2(\Omega)$ and consider an auxiliary problem which is to find $w \in H_0^1(\Omega)$ such that

$$(\nabla w, \nabla z)_\Omega = (g, z)_\Omega, \quad \forall z \in H_0^1(\Omega).
 \tag{3.40}$$

Taking $z = w$ in (3.40) and using (3.28), we obtain $\|\nabla w\|_\Omega \leq c_\Omega \|g\|_\Omega$. Moreover, by the property of the elliptic equation with the homogeneous boundary condition, there exists a positive constant \bar{c}_Ω such that

$$\begin{aligned}
 \|w\|_{H^2(\Omega)} &\leq \bar{c}_\Omega (\|w\|_\Omega + \|g\|_\Omega) \leq \bar{c}_\Omega (c_\Omega \|\nabla w\|_\Omega + \|g\|_\Omega) \\
 &\leq \bar{c}_\Omega (c_\Omega^2 + 1) \|g\|_\Omega.
 \end{aligned}
 \tag{3.41}$$

Taking $z = P_N^{1,0} v - v$ in (3.40). Then we use (3.29) and the first result of (3.30) to obtain

$$\begin{aligned}
 |(P_N^{1,0} v - v, g)_\Omega| &= |(\nabla w, \nabla(P_N^{1,0} v - v))_\Omega| = |(\nabla(P_N^{1,0} w - w), \nabla(P_N^{1,0} v - v))_\Omega| \\
 &\leq \|\nabla(P_N^{1,0} w - w)\|_\Omega \|\nabla(P_N^{1,0} v - v)\|_\Omega \leq cN^{-r} B_{r,\Omega}(v) B_{2,\Omega}(w).
 \end{aligned}
 \tag{3.42}$$

Since $r \geq 2$, we use (3.26) to assert that $B_{2,\Omega}(w) \leq \beta_\Omega \|w\|_{H^2(\Omega)}$ where

$$\begin{aligned}
 \beta_\Omega &= ((|d_1| + |d_2|)\delta_\Omega^{-1} + 1) \left(\sum_{k=0}^r \delta_\Omega^{k-3} \sigma_\Omega^k (k+1) (|d_1|^{2-k} + |d_2|^{2-k}) \right. \\
 &+ \sum_{k=0}^{r-1} \delta_\Omega^{k-2} \sigma_\Omega^k (k+1) [|d_1|^{1-k} (|d_2| \delta_\Omega^{-1} + \gamma_\Omega \sigma_\Omega + \sigma_\Omega) \\
 &\quad \left. + |d_2|^{1-k} (|d_1| \delta_\Omega^{-1} + \gamma_\Omega \sigma_\Omega + \sigma_\Omega)] \right).
 \end{aligned}$$

Finally, we have from (3.41) and (3.42) that

$$\begin{aligned}
 \|P_N^{1,0} v - v\|_\Omega &= \sup_{g \in L^2(\Omega), g \neq 0} \frac{|(P_N^{1,0} v - v, g)_\Omega|}{\|g\|_\Omega} \leq cN^{-r} \frac{B_{r,\Omega}(v) B_{2,\Omega}(w)}{\|g\|_\Omega} \\
 &\leq c\beta_\Omega \frac{B_{r,\Omega}(v) \|w\|_{H^2(\Omega)}}{\|g\|_\Omega} \leq c\bar{c}_\Omega \beta_\Omega N^{-r} B_{r,\Omega}(v).
 \end{aligned}$$

This ends the proof. □

Remark 3.4. By the definition of $B_{r,\Omega}(v)$ and (3.24), there are the weight functions $|(1-\xi^2)^{r-k-2} ((x-x_1^*)(x-x_3^*))^j ((y-y_1^*)(y-y_3^*))^{k-j}|$ and $|(1-\eta^2)^{r-k-2} ((x-x_2^*)(x-x_4^*))^j ((y-y_2^*)(y-y_4^*))^{k-j}|$, in the norm of derivative $\partial_x^j \partial_y^{k-j} v$, etc. Since $r \geq \max(k, 2)$, they decay to zero as the point $Q(x, y)$ goes to the corners of the quadrilateral. Therefore, the error of $H_0^1(\Omega)$ -orthogonal approximation keeps the order N^{1-r} , even if the considered function possesses certain singularities at

the vertices of the quadrilateral. For instance, $|\partial_x^j \partial_y^{k-j} v| = o((1 - \xi^2)^{-\frac{r+k+2}{2}} |Q - Q_\nu^*|^{-\frac{2k-1}{2}})$ as $Q(x, y)$ tends to the edges L_1 or L_3 , etc.

Remark 3.5. If $\Omega = \{(x, y) \mid |x| < a, |y| < b, a, b, > 0\}$, then $\sigma_\Omega = \max(a, b)$, $\delta_\Omega = ab$ and $d_1 = d_2 = \gamma_\Omega = 0$. Moreover, $a_1 = a, b_2 = b$ and $a_2 = a_3 = b_1 = b_3 = 0$. Thus, we have from (3.36) and (3.37) that

$$\partial_\xi^{r-1} \partial_\eta u = a^{r-\frac{1}{2}} b^{\frac{3}{2}} \partial_x^{r-1} \partial_y v, \quad \partial_\xi \partial_\eta^{r-1} u = a^{\frac{3}{2}} b^{r-\frac{1}{2}} a^{\frac{3}{2}} \partial_x \partial_y^{r-1} v.$$

Accordingly, the quantity $B_{r,\Omega}(v)$ is reduced to

$$B_{r,\Omega}(v) = \max(a, b) b^{-1} \|(a^2 - x^2)^{\frac{r-1}{2}} \partial_x^r v\|_\Omega + a^{-1} \|(b^2 - y^2)^{\frac{r-1}{2}} \partial_y^r v\|_\Omega \\ + ab \max(a, b) (\|(a^2 - x^2)^{\frac{r-2}{2}} \partial_x^{r-1} \partial_y v\|_\Omega + \|(b^2 - y^2)^{\frac{r-2}{2}} \partial_x \partial_y^{r-1} v\|_\Omega).$$

Therefore, the error of the $H_0^1(\Omega)$ -orthogonal approximation still keeps the order N^{1-r} , even if the considered function possesses certain singularities at the edges of the quadrilateral. If, in addition, $a = b = 1$, then the above estimate turns out to be the same result as in [13].

4. SPECTRAL METHOD FOR QUADRILATERALS

In this section, we propose the spectral method for quadrilaterals.

4.1. A steady problem. As an example, we consider the following simple problem:

$$(4.1) \quad \begin{cases} -\Delta U(x, y) = f(x, y), & (x, y) \in \Omega, \\ U(x, y) = g(x, y), & (x, y) \in \partial\Omega. \end{cases}$$

We first consider the case $g(x, y) \equiv 0$. Then, the weak formulation of (4.1) is to seek solution $U \in H_0^1(\Omega)$, such that

$$(4.2) \quad (\nabla U, \nabla v)_\Omega = (f, v)_\Omega, \quad \forall v \in H_0^1(\Omega).$$

The spectral scheme for (4.2) is to find $u_N \in V_N^0(\Omega)$, such that

$$(4.3) \quad (\nabla u_N, \nabla \phi)_\Omega = (f, \phi)_\Omega, \quad \forall \phi \in V_N^0(\Omega).$$

For the convergence analysis, we set $U_N = P_N^{1,0} U$. We have from (4.2) with (3.29) that

$$(4.4) \quad (\nabla U_N, \nabla \phi)_\Omega = (f, \phi)_\Omega, \quad \forall \phi \in V_N^0(\Omega).$$

Let $\tilde{U}_N = u_N - U_N$. Subtracting (4.4) from (4.3), we obtain

$$(4.5) \quad (\nabla \tilde{U}_N, \nabla \phi)_\Omega = 0, \quad \forall \phi \in V_N^0(\Omega).$$

Taking $\phi = \tilde{U}_N$ in the above, we get $\|\nabla \tilde{U}_N\|_\Omega = 0$. This fact with the Poincaré inequality on Ω implies $\tilde{U}_N = 0$, i.e., $u_N = P_N^{1,0} U$. Finally, by (3.30),

$$(4.6) \quad \|\nabla(u_N - U)\|_\Omega \leq cN^{1-r} B_{r,\Omega}(U), \quad \|u_N - U\|_\Omega \leq cc_\Omega^* N^{-r} B_{r,\Omega}(U).$$

Remark 4.1. In practice, $U(x, y)$ might not vanish on the boundary. In this case, we construct a function $W(x, y)$ which equals $U(x, y)$ on $\partial\Omega$; see (7.3) of this paper. Then, we solve the corresponding problem and obtain the numerical solution of original problem.

Remark 4.2. The spectral scheme (4.4) is similar to scheme (1.15) of [2]. But scheme (1.15) of [2] is a pseudospectral method. Our method is more similar to scheme (5.3.11) of [5] for a single domain, as well as the related method in [17]. Whereas they are not exactly the same. First, one uses the base functions $L_l(\xi(x, y))L_m(\eta(x, y))$ in [2, 5, 17], which are not mutually orthogonal in the space $L^2(\Omega)$. However, we use the base functions $L_l(\xi(x, y))L_m(\eta(x, y))J_S^{\frac{1}{2}}(x, y)$, which are mutually orthogonal in the space $L^2(\Omega)$. It is more natural. As a result, we could use the recent result on the Jacobi orthogonal approximation to derive better error estimate (3.30) of numerical solution, with the Jacobi weights, which cover certain singularities on the vertices of Ω .

4.2. An unsteady problem. Let β be a constant. We consider the problem (4.7)

$$\begin{cases} \partial_t U(x, y, t) = \Delta U(x, y, t) + \beta U(x, y, t) + f(x, y, t), & (x, y) \in \Omega, 0 < t \leq T, \\ U(x, y, t) = g(x, y, t), & (x, y) \in \partial\Omega, 0 < t \leq T, \\ U(x, y, 0) = U_0(x, y), & (x, y) \in \Omega \cup \partial\Omega. \end{cases}$$

We first consider the case with $g(x, y, t) = 0$. Then, the weak formulation of (4.7) is to find $U \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, such that

$$(4.8) \quad \begin{cases} (\partial_t U(t), v)_\Omega + (\nabla U(t), \nabla v)_\Omega = \beta(U(t), v)_\Omega + (f(t), v)_\Omega, \\ \quad \forall v \in H_0^1(\Omega), 0 < t \leq T, \\ U(0) = U_0. \end{cases}$$

The spectral scheme for (4.8) is to find $u_N \in V_N^0(\Omega)$ for all $0 < t \leq T$, such that

$$(4.9) \quad \begin{cases} (\partial_t u_N(t), \phi)_\Omega + (\nabla u_N(t), \nabla \phi)_\Omega = \beta(u_N(t), v)_\Omega + (f(t), \phi)_\Omega, \\ \quad \forall \phi \in V_N^0(\Omega), 0 < t \leq T, \\ u_N(0) = P_N U_0. \end{cases}$$

We now analyze the convergence. Letting $U_N = P_N^{1,0}U$, we obtain from (4.8) that

$$(4.10) \quad \begin{cases} (\partial_t U_N(t), \phi)_\Omega + (\nabla U_N(t), \nabla \phi)_\Omega = \beta(U_N(t), v)_\Omega \\ \quad + (f(t), \phi)_\Omega + G_1(t, \phi) + G_2(t, \phi), \quad \forall \phi \in V_N^0(\Omega), 0 < t \leq T, \\ U_N(0) = P_N^{1,0}U_0, \end{cases}$$

with

$$G_1(t, \phi) = (\partial_t(U_N(t) - U(t)), \phi)_\Omega, \quad G_2(t, \phi) = \beta(U(t) - U_N(t), \phi)_\Omega.$$

Letting $\tilde{U}_N = u_N - U_N$, and subtracting (4.10) from (4.9), we obtain

$$(4.11) \quad \begin{cases} (\partial_t \tilde{U}_N, \phi)_\Omega + (\nabla \tilde{U}_N, \nabla \phi)_\Omega = \beta(\tilde{U}_N(t), v)_\Omega - G_1(t, \phi) - G_2(t, \phi), \\ \quad \forall \phi \in V_N^0(\Omega), 0 < t \leq T, \\ \tilde{U}_N(0) = P_N U_0 - P_N^{1,0}U_0. \end{cases}$$

Take $\phi = 2\tilde{U}_N$ in (4.11). Then,

$$(4.12) \quad \partial_t \|\tilde{U}_N(t)\|_\Omega^2 + 2\|\nabla \tilde{U}_N(t)\|_\Omega^2 \leq 2\beta\|\tilde{U}_N(t)\|_\Omega^2 + 2|G_1(t, \tilde{U}_N)| + 2|G_2(t, \tilde{U}_N)|.$$

By virtue of (3.28) and (3.30), we have

$$(4.13) \quad \begin{aligned} 2|G_1(t, \tilde{U}_N)| &\leq \frac{1}{2c_\Omega^2} \|\tilde{U}_N(t)\|_\Omega^2 + 2c_\Omega^2 \|\partial_t(U_N(t) - U(t))\|_\Omega^2 \\ &\leq \frac{1}{2} \|\nabla \tilde{U}_N(t)\|_\Omega^2 + cc_\Omega^2 (c_\Omega^*)^2 N^{-2r} B_{r,\Omega}^2 (\partial_t U(t)). \end{aligned}$$

Similarly,

$$(4.14) \quad 2|G_2(t, \tilde{U}_N)| \leq \frac{1}{2} \|\nabla \tilde{U}_N\|_{\Omega}^2 + c\beta^2 c_{\Omega}^2 (c_{\Omega}^*)^2 N^{-2r} B_{r,\Omega}^2(U(t)).$$

For simplicity of statements, let

$$E(v(t)) = \|v(t)\|_{\Omega}^2 + \int_0^t \|\nabla v(\xi)\|_{\Omega}^2 d\xi.$$

Substituting (4.13) and (4.14) into (4.12), we obtain

$$(4.15) \quad \partial_t(E(\tilde{U}_N(t))e^{-2\beta t}) \leq ce^{-2\beta t}(\beta^2 + 1)c_{\Omega}^2 (c_{\Omega}^*)^2 N^{-2r} (B_{r,\Omega}^2(\partial_t U(t)) + B_{r,\Omega}^2(U(t))).$$

On the other hand, we use (3.10) and the second result of (3.30) to obtain

$$(4.16) \quad \begin{aligned} \|\tilde{U}_N(0)\|_{\Omega}^2 &\leq 2\|P_N U_0 - U_0\|_{\Omega}^2 + 2\|U_0 - P_N^{1,0} U_0\|_{\Omega}^2 \\ &\leq cN^{-2r}(A_{r,\Omega}^2(U_0) + (c_{\Omega}^*)^2 B_{r,\Omega}^2(U_0)). \end{aligned}$$

Integrating (4.15) with respect to t and using (4.16), we obtain

$$(4.17) \quad E(\tilde{U}_N(t)) \leq cN^{-2r} e^{2\beta t} R_{r,\Omega,\beta}(U, t)$$

where

$$\begin{aligned} R_{r,\Omega,\beta}(U, t) &= (\beta^2 + 1)c_{\Omega}^2 (c_{\Omega}^*)^2 \int_0^t e^{-2\beta\xi} (B_{r,\Omega}^2(\partial_{\xi} U(\xi)) + B_{r,\Omega}^2(U(\xi))) d\xi \\ &\quad + A_{r,\Omega}^2(U_0) + (c_{\Omega}^*)^2 B_{r,\Omega}^2(U_0). \end{aligned}$$

Finally, a combination of (3.30) and (4.17) leads to

$$(4.18) \quad \begin{aligned} \int_0^t \|\nabla(u_N(\xi) - U(\xi))\|_{\Omega}^2 d\xi &\leq cN^{-2r} e^{2\beta t} R_{r,\Omega,\beta}(U, t) \\ &\quad + cN^{2-2r} \int_0^t B_{r,\Omega}^2(U(\xi)) d\xi, \end{aligned}$$

$$(4.19) \quad \|u_N(t) - U(t)\|_{\Omega}^2 \leq cN^{-2r} (e^{2\beta t} R_{r,\Omega,\beta}(U, t) + (c_{\Omega}^*)^2 B_{r,\Omega}^2(U(t))).$$

5. NUMERICAL IMPLEMENTATION

In this section, we describe numerical implementation. Let

$$(5.1) \quad \sigma_l(\zeta) = c_l(L_l(\zeta) - L_{l+2}(\zeta)), \quad c_l = (4l + 6)^{-\frac{1}{2}}.$$

Clearly, $\sigma_l(\pm 1) = 0$. Moreover, a calculation shows

$$(5.2) \quad \partial_{\zeta} \sigma_l(\zeta) = d_l L_l(\zeta), \quad d_l = 2^{-\frac{1}{2}}(l+1)^{-1}(2l+3)^{\frac{3}{2}}.$$

Further, we set

$$(5.3) \quad \psi_{l,m}(x, y) = \sigma_l(\xi(x, y))\sigma_m(\eta(x, y))J_{\frac{1}{2}}^{\frac{1}{2}}(x, y), \quad 0 \leq l, m \leq N-2,$$

which form a basis of $V_N^0(\Omega)$.

We first consider the scheme (4.3). We expand the numerical solution $u_N(x, y)$ as

$$(5.4) \quad u_N(x, y) = \sum_{k,j=0}^{N-2} u_{N,k,j} \psi_{k,j}(x, y).$$

Substituting (5.4) into (4.3) and putting $\phi(x, y) = \psi_{l,m}(x, y)$ in the resulting equation, we derive a linear system with the unknown coefficients $u_{N,k,j}, 0 \leq k, j \leq N - 2$. This system can be rewritten as a compact matrix form. To do this, let

$$\mathbf{u} = (u_{N,0,0}, u_{N,0,1}, \dots, u_{N,0,N-2}, \dots, u_{N,N-2,0}, \dots, u_{N,N-2,N-2})^T, \\ \mathbf{F} = (f_{0,0}, f_{0,1}, \dots, f_{0,N-2}, \dots, f_{N-2,0}, \dots, f_{N-2,N-2})^T, \quad f_{l,m} = (f, \psi_{l,m})_\Omega.$$

Furthermore, let $a_{l,m,k,j} = (\partial_x \psi_{k,j}, \partial_x \psi_{l,m})_\Omega + (\partial_y \psi_{k,j}, \partial_y \psi_{l,m})_\Omega$. The matrix

$$A = \begin{pmatrix} a_{0,0,0,0} & a_{0,0,0,1} & \cdots & a_{0,0,N-2,0} & \cdots & a_{0,0,N-2,N-2} \\ a_{0,1,0,0} & a_{0,1,0,1} & \cdots & a_{0,1,N-2,0} & \cdots & a_{0,1,N-2,N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{N-2,N-2,0,0} & a_{N-2,N-2,0,1} & \cdots & a_{N-2,N-2,N-2,0} & \cdots & a_{N-2,N-2,N-2,N-2} \end{pmatrix}.$$

Consequently, we have

$$(5.5) \quad A\mathbf{u} = \mathbf{F}.$$

The matrix A is full. But its condition number (labeled by Cond.) is acceptable; see Table 5.1.

TABLE 5.1. Condition number of matrix A .

N	5	10	15	20	25
Cond.	34.3	410.5	1873.6	5624.1	13316.2

We now turn to (4.8). We approximate the term $\partial_t u_N(t)$ by the Crank-Nicholson discretization with the mesh size τ . It is of the form

$$(5.6) \quad \frac{1}{\tau}(u_N(t + \tau) - u_N(t), \psi_{l,m})_\Omega + \frac{1}{2}(\partial_x(u_N(t + \tau)) + \partial_x u_N(t), \partial_x \psi_{l,m})_\Omega \\ + \frac{1}{2}(\partial_y(u_N(t + \tau)) + \partial_y u_N(t), \partial_y \psi_{l,m})_\Omega \\ = \frac{1}{2}\beta(u_N(t + \tau) + u_N(t), \psi_{l,m})_\Omega + f_{l,m}(t + \frac{\tau}{2}), \quad 0 \leq l, m \leq N - 2,$$

where $f_{l,m}(t + \frac{\tau}{2}) = (f(t + \frac{\tau}{2}), \psi_{l,m})_\Omega$. We expand the numerical solution as

$$(5.7) \quad u_N(x, y, t) = \sum_{k,j=0}^{N-2} u_{N,k,j}(t) \psi_{k,j}(x, y).$$

Inserting (5.7) into (5.6), we obtain a scheme for the unknown coefficients $u_{N,k,j}(t), t = \tau, 2\tau, \dots, [\frac{T}{\tau}]$. It can be also written as a compact matrix form. To do this, let

$$\mathbf{u}(t) = (u_{N,0,0}(t), u_{N,0,1}(t), \dots, u_{N,N-2,N-2}(t))^T, \\ \mathbf{F}(t + \frac{\tau}{2}) = (f_{0,0}(t + \frac{\tau}{2}), f_{0,1}(t + \frac{\tau}{2}), \dots, f_{N-2,N-2}(t + \frac{\tau}{2}))^T.$$

Let A be the same matrix as in (5.5). The matrix B is similar to A , with the entries $b_{l,m,k,j} = (\psi_{k,j}, \psi_{l,m})_\Omega$. Then, the matrix form of (5.6) is

$$(5.8) \quad (\tau A + (2 - \tau\beta)B)\mathbf{u}(t + \tau) = (-\tau A + (2 + \tau\beta)B)\mathbf{u}(t) + 2\tau\mathbf{F}(t + \frac{\tau}{2}).$$

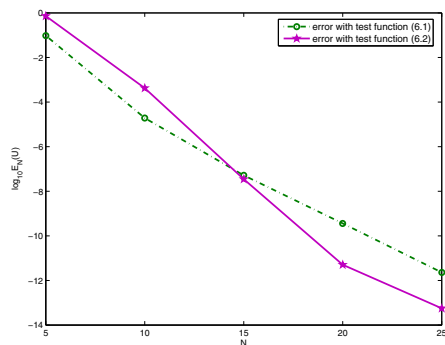


FIGURE 3. Numerical errors of (4.3).

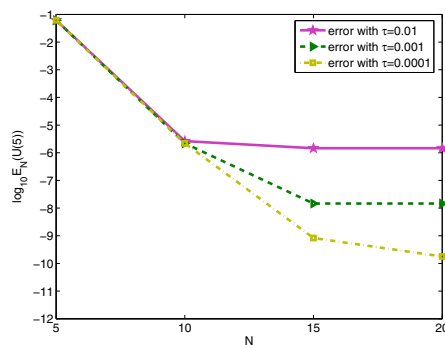


FIGURE 4. Numerical errors of (5.6).

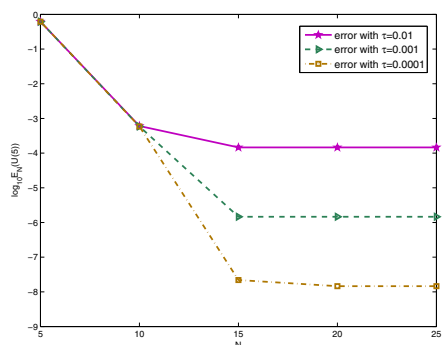


FIGURE 5. Numerical errors of (5.6).

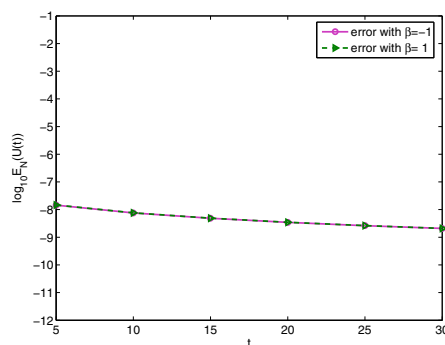


FIGURE 6. Stability of (5.6).

6. NUMERICAL RESULTS

We first use (4.3) to solve (4.2) with the test function

$$(6.1) \quad U(x, y) = \left(y - \frac{(x-x_1)(y_4-y_1)}{(x_4-x_1)} - y_1\right) \left(y - \frac{(x-x_2)(y_2-y_1)}{(x_2-x_1)} - y_2\right) \left(y - \frac{(x-x_3)(y_3-y_2)}{(x_3-x_2)} - y_3\right) \left(y - \frac{(x-x_4)(y_4-y_3)}{(x_4-x_3)} - y_4\right) \sqrt{x^2 + y^2 + 1},$$

where x_j and y_j , $1 \leq j \leq 4$ are the coordinates of the vertices of Ω . We shall also use the test function

$$(6.2) \quad U(x, y) = 4 \sin(2x + 3y),$$

which does not vanish on the boundary. In actual calculation, we take $x_1 = 0, x_2 = 3, x_3 = 2, x_4 = 1, y_1 = 0, y_2 = 0, y_3 = 3$ and $y_4 = 2$. For description of numerical

errors, let $\zeta_{M,i}$ and $\omega_{M,i}$ be the nodes and the weights of one-dimensional Legendre-Gauss-Lobatto quadrature. We measure the numerical errors by

$$\begin{aligned}
 E_N(U) &= \left(\sum_{i,j=1}^M (U(x(\zeta_{M,i}, \zeta_{M,j}), y(\zeta_{M,i}, \zeta_{M,j})) \right. \\
 &\quad \left. - u_N(x(\zeta_{M,i}, \zeta_{M,j}), y(\zeta_{M,i}, \zeta_{M,j})))^2 (d_0 + d_1 \zeta_{M,i} + d_2 \zeta_{M,j}) \omega_{M,i} \omega_{M,j} \right)^{\frac{1}{2}} \\
 &\approx \left(\int \int_S (U(x(\xi, \eta), y(\xi, \eta)) - u_N(x(\xi, \eta), y(\xi, \eta)))^2 J_\Omega(\xi, \eta) d\xi d\eta \right)^{\frac{1}{2}} = \|U - u_N\|_\Omega.
 \end{aligned}$$

In Figure 3, we plot the values of $\log_{10} E_N(U)$ vs. the mode N , with the test functions (6.1) and (6.2). They demonstrate the high accuracy of scheme (4.3).

We next use (4.9) to solve (4.8), with the test function

$$\begin{aligned}
 (6.3) \quad U(x, y, t) &= \left(y - \frac{(x-x_1)(y_4-y_1)}{(x_4-x_1)} - y_1 \right) \left(y - \frac{(x-x_2)(y_2-y_1)}{(x_2-x_1)} - y_2 \right) \left(y - \frac{(x-x_3)(y_3-y_2)}{(x_3-x_2)} - y_3 \right) \\
 &\quad \left(y - \frac{(x-x_4)(y_4-y_3)}{(x_4-x_3)} - y_4 \right) \sqrt{x^2 + y^2 + t + 1}.
 \end{aligned}$$

For describing the numerical errors at a different time t , we use the notation $E_N(U(t))$. In Figure 4, we plot the values of $E_N(U(t))$ with $\beta = 1, t = 5, N = 5, 10, 15, 20$, and $\tau = 0.01, 0.001, 0.0001$. Clearly, the error decays rapidly as N increases and τ decreases. It also shows the high accuracy in space. We also use (4.12) to solve (4.11), with the test function

$$(6.4) \quad U(x, y) = 4 \sin(2x + 3y + t).$$

In Figure 5, we plot the corresponding values of $E_N(U(t))$ with $\beta = 1$. They indicate again the high accuracy in space.

In Figure 6, we plot the values of $E_N(U(t))$ with the test function (6.3), $\beta = \pm 1, N = 20$ and $\tau = 0.001$. They demonstrate the stability of (5.6).

7. THE SPECTRAL METHOD FOR POLYGONS

In this section, we investigate the spectral method for polygons.

7.1. Some preparations. We first consider several quasi-Legendre orthogonal approximations on the convex quadrilateral Ω . To do this, we put $\hat{v}(\xi, \eta) = v(x(\xi, \eta), y(\xi, \eta))$, and

$$\begin{aligned}
 (7.1) \quad \hat{v}_{b,L_1}(\eta) &= \frac{1}{2}(\hat{v}(-1, -1)(1 - \eta) + \hat{v}(-1, 1)(1 + \eta)), \\
 \hat{v}_{b,L_2}(\xi) &= \frac{1}{2}(\hat{v}(-1, -1)(1 - \xi) + \hat{v}(1, -1)(1 + \xi)), \\
 \hat{v}_{b,L_3}(\eta) &= \frac{1}{2}(\hat{v}(1, -1)(1 - \eta) + \hat{v}(1, 1)(1 + \eta)), \\
 \hat{v}_{b,L_4}(\xi) &= \frac{1}{2}(\hat{v}(-1, 1)(1 - \xi) + \hat{v}(1, 1)(1 + \xi)).
 \end{aligned}$$

Next, we set

$$\begin{aligned}
 (7.2) \quad \hat{v}_{b,L_1}^0(\eta) &= \hat{v}(-1, \eta) - \hat{v}_{b,L_1}(\eta), & \hat{v}_{b,L_2}^0(\xi) &= \hat{v}(\xi, -1) - \hat{v}_{b,L_2}(\xi), \\
 \hat{v}_{b,L_3}^0(\eta) &= \hat{v}(1, \eta) - \hat{v}_{b,L_3}(\eta), & \hat{v}_{b,L_4}^0(\xi) &= \hat{v}(\xi, 1) - \hat{v}_{b,L_4}(\xi).
 \end{aligned}$$

The above four functions vanish at the endpoints of Λ_ξ or Λ_η , respectively. We also set

$$(7.3) \quad \begin{aligned} \hat{v}_{b,\partial\Omega}(\xi, \eta) &= \hat{v}_{b,\partial\Omega}^{(1)}(\xi, \eta) + \hat{v}_{b,\partial\Omega}^{(2)}(\xi, \eta), \\ \hat{v}_{b,\partial\Omega}^{(1)}(\xi, \eta) &= \frac{1}{2}((1-\xi)\hat{v}(-1, \eta) + (1-\eta)\hat{v}(\xi, -1) + (1+\xi)\hat{v}(1, \eta) + (1+\eta)\hat{v}(\xi, 1)), \\ \hat{v}_{b,\partial\Omega}^{(2)}(\xi, \eta) &= -\frac{1}{4}((1-\xi)(1-\eta)\hat{v}(-1, -1) + (1+\xi)(1-\eta)\hat{v}(1, -1) \\ &\quad + (1+\xi)(1+\eta)\hat{v}(1, 1) + (1-\xi)(1+\eta)\hat{v}(-1, 1)), \end{aligned}$$

or equivalently,

$$\begin{aligned} \hat{v}_{b,\partial\Omega}(\xi, \eta) &= \frac{1}{2}((1-\xi)\hat{v}_{b,L_1}^0(\eta) + (1-\eta)\hat{v}_{b,L_2}^0(\xi) + (1+\xi)\hat{v}_{b,L_3}^0(\eta) + (1+\eta)\hat{v}_{b,L_4}^0(\xi)) \\ &\quad + \frac{1}{4}((1-\xi)(1-\eta)\hat{v}(-1, -1) + (1+\xi)(1-\eta)\hat{v}(1, -1) \\ &\quad + (1+\xi)(1+\eta)\hat{v}(1, 1) + (1-\xi)(1+\eta)\hat{v}(-1, 1)). \end{aligned}$$

Let $\hat{v}_\Omega^0(\xi, \eta) = \hat{v}(\xi, \eta) - \hat{v}_{b,\partial\Omega}(\xi, \eta)$. Obviously, $\hat{v}_\Omega^0(\xi, \eta)$ vanishes on $\partial\Omega$.

For the spectral method for a union of quadrilaterals, it is simpler to take the base functions $\psi_{l,m}(x, y) = L_l(\xi(x, y))L_m(\eta(x, y))$, as in [2, 5, 17]. Let $V_N(\Omega) = \{\psi_{l,m}(x, y), 0 \leq l, m \leq N\}$ and $V_N^0(\Omega) = H_0^1(\Omega) \cap V_N(\Omega)$. The orthogonal projection $P_{N,\Omega}^{1,0} : H_0^1(\Omega) \rightarrow V_N^0(\Omega)$ is defined in the usual way. Furthermore, let N_b be any positive integer. We define the projection on $\partial\Omega$, as

$$\begin{aligned} * \hat{P}_{N_b,\partial\Omega}^1 \hat{v}_{b,\partial\Omega}(\xi, \eta) &= \frac{1}{2}((1-\xi)P_{N_b,\Lambda_\eta}^{1,0} \hat{v}_{b,L_1}^0(\eta) + (1-\eta)P_{N_b,\Lambda_\xi}^{1,0} \hat{v}_{b,L_2}^0(\xi) \\ &\quad + (1+\xi)P_{N_b,\Lambda_\eta}^{1,0} \hat{v}_{b,L_3}^0(\eta) + (1+\eta)P_{N_b,\Lambda_\xi}^{1,0} \hat{v}_{b,L_4}^0(\xi)) \\ &\quad + \frac{1}{4}((1-\xi)(1-\eta)\hat{v}(-1, -1) + (1+\xi)(1-\eta)\hat{v}(1, -1) \\ &\quad + (1+\xi)(1+\eta)\hat{v}(1, 1) + (1-\xi)(1+\eta)\hat{v}(-1, 1)). \end{aligned}$$

Furthermore,

$$\begin{aligned} v_\Omega^0(x, y) &= \hat{v}_\Omega^0(\xi, \eta)|_{\xi=\xi(x,y), \eta=\eta(x,y)}, \quad v_{b,\partial\Omega}(x, y) = \hat{v}_{b,\partial\Omega}(\xi, \eta)|_{\xi=\xi(x,y), \eta=\eta(x,y)}, \\ * P_{N_b,\partial\Omega}^1 v_{b,\partial\Omega}(x, y) &= * \hat{P}_{N_b,\partial\Omega}^1 \hat{v}_{b,\partial\Omega}(\xi, \eta)|_{\xi=\xi(x,y), \eta=\eta(x,y)}. \end{aligned}$$

Then, the first quasi-Legendre orthogonal projection $*P_{N,N_b,\Omega}^1 v$ is defined by

$$(7.4) \quad *P_{N,N_b,\Omega}^1 v(x, y) = P_{N,\Omega}^{1,0} v_\Omega^0(x, y) + *P_{N_b,\partial\Omega}^1 v_{b,\partial\Omega}(x, y).$$

It can be checked that $*P_{N,N_b,\Omega}^1 v(x, y) = v(x, y)$ at the four vertices of Ω . Since

$$*P_{N,N_b,\Omega}^1 v - v = P_{N,\Omega}^{1,0} v_\Omega^0(x, y) - v_\Omega^0(x, y) + *P_{N_b,\partial\Omega}^1 v_{b,\partial\Omega}(x, y) - v_{b,\partial\Omega}(x, y),$$

we have

$$(7.5) \quad \|\nabla(*P_{N,N_b,\Omega}^1 v - v)\|_\Omega \leq \|\nabla(P_{N,\Omega}^{1,0} v_\Omega^0 - v_\Omega^0)\|_\Omega + \|\nabla(*P_{N_b,\partial\Omega}^1 v_{b,\partial\Omega} - v_{b,\partial\Omega})\|_\Omega.$$

For notational convenience, let

$$\begin{aligned} D_{r,S}(\hat{w}) &= (\|\partial_\xi^r \hat{w}\|_{L_\chi^{2(r-1,r-1)}(\Lambda_\xi, L^2(\Lambda_\eta))}^2 + \|\partial_\eta^r \hat{w}\|_{L^2(\Lambda_\xi, L_\chi^{2(r-1,r-1)}(\Lambda_\eta))}^2 \\ &\quad + \|\partial_\xi^{r-1} \partial_\eta \hat{w}\|_{L_\chi^{2(r-2,r-2)}(\Lambda_\xi, L^2(\Lambda_\eta))}^2 + \|\partial_\xi \partial_\eta^{r-1} \hat{w}\|_{L^2(\Lambda_\xi, L_\chi^{2(r-2,r-2)}(\Lambda_\eta))}^2)^{\frac{1}{2}}. \end{aligned}$$

By (2.5), we have $\|\nabla w\|_\Omega \leq c\sigma_\Omega \delta_\Omega^{-\frac{1}{2}} \|\nabla_S \hat{w}\|_S$. Now, let

$$\phi(x, y) = P_{N,\Lambda_\xi}^{1,0} \bullet P_{N,\Lambda_\eta}^{1,0} \hat{v}_\Omega^0|_{\xi=\xi(x,y), \eta=\eta(x,y)}.$$

Then, following the same line as the derivations of (3.32) and (3.33), we deduce that

$$(7.6) \quad \begin{aligned} \|\nabla(P_{N,\Omega}^{1,0}v_\Omega^0 - v_\Omega^0)\|_\Omega &\leq \|\nabla(\phi - v_\Omega^0)\|_\Omega \leq c\sigma_\Omega\delta_\Omega^{-\frac{1}{2}}\|\nabla_S(P_{N,\Lambda_\xi}^{1,0} \bullet P_{N,\Lambda_\eta}^{1,0}\hat{v}_\Omega^0 - \hat{v}_\Omega^0)\|_S \\ &\leq c\sigma_\Omega\delta_\Omega^{-\frac{1}{2}}N^{1-r}D_{r,S}(\hat{v}_\Omega^0) \leq c\sigma_\Omega\delta_\Omega^{-\frac{1}{2}}N^{1-r}(D_{r,S}(\hat{v}) + D_{r,S}(\hat{v}_{b,\partial\Omega}^{(1)}) + D_{r,S}(\hat{v}_{b,\partial\Omega}^{(2)})). \end{aligned}$$

We are going to estimate the upper bound of the right side of (7.6). First, an argument similar to the derivation of (3.30), leads to

$$(7.7) \quad D_{r,S}(\hat{v}) \leq c\delta_\Omega^{-\frac{1}{2}}E_{r,\Omega}(v)$$

where $E_{r,\Omega}(v) = E_{r,\Omega}^{(1)}(v) + E_{r,\Omega}^{(2)}(v) + E_{r,\Omega}^{(3)}(v)$, with

$$\begin{aligned} E_{r,\Omega}^{(1)}(v) &= \sum_{k=0}^r (\|(1 - \xi^2)^{\frac{r-1}{2}}(a_1 + a_3\eta)^k(b_1 + b_3\eta)^{r-k}\partial_x^k\partial_y^{r-k}v\|_\Omega \\ &\quad + \|(1 - \eta^2)^{\frac{r-1}{2}}(a_2 + a_3\xi)^k(b_2 + b_3\xi)^{r-k}\partial_x^k\partial_y^{r-k}v\|_\Omega), \\ E_{r,\Omega}^{(2)}(v) &= \sum_{k=0}^{r-1} (\|(1 - \xi^2)^{\frac{r-2}{2}}(a_1 + a_3\eta)^k(b_1 + b_3\eta)^{r-k-1}(a_2 + a_3\xi)\partial_x^{k+1}\partial_y^{r-k-1}v\|_\Omega \\ &\quad + \|(1 - \eta^2)^{\frac{r-2}{2}}(a_2 + a_3\xi)^k(b_2 + b_3\xi)^{r-k-1}(a_1 + a_3\eta)\partial_x^{k+1}\partial_y^{r-k-1}v\|_\Omega \\ &\quad + \|(1 - \xi^2)^{\frac{r-2}{2}}(a_1 + a_3\eta)^k(b_1 + b_3\eta)^{r-k-1}(b_2 + b_3\xi)\partial_x^k\partial_y^{r-k}v\|_\Omega \\ &\quad + \|(1 - \eta^2)^{\frac{r-2}{2}}(a_2 + a_3\xi)^k(b_2 + b_3\xi)^{r-k-1}(b_1 + b_3\eta)\partial_x^k\partial_y^{r-k}v\|_\Omega), \\ E_{r,\Omega}^{(3)}(v) &= \sum_{k=0}^{r-2} (\|(1 - \xi^2)^{\frac{r-2}{2}}a_3(a_1 + a_3\eta)^k(b_1 + b_3\eta)^{r-k-2}\partial_x^{k+1}\partial_y^{r-k-2}v\|_\Omega \\ &\quad + \|(1 - \eta^2)^{\frac{r-2}{2}}a_3(a_2 + a_3\xi)^k(b_2 + b_3\xi)^{r-k-2}\partial_x^{k+1}\partial_y^{r-k-2}v\|_\Omega \\ &\quad + \|(1 - \xi^2)^{\frac{r-2}{2}}b_3(a_1 + a_3\eta)^k(b_1 + b_3\eta)^{r-k-2}\partial_x^k\partial_y^{r-k-1}v\|_\Omega \\ &\quad + \|(1 - \eta^2)^{\frac{r-2}{2}}b_3(a_2 + a_3\xi)^k(b_2 + b_3\xi)^{r-k-2}\partial_x^k\partial_y^{r-k-1}v\|_\Omega). \end{aligned}$$

We now estimate $D_{r,S}(\hat{v}_{b,\partial\Omega}^{(1)})$. We have from (7.3) that for $r \geq 2$,

$$\begin{aligned} \partial_\xi^r\hat{v}_{b,\partial\Omega}^{(1)}(\xi, \eta) &= \frac{1}{2}\partial_\xi^r((1 - \eta)\hat{v}(\xi, -1) + (1 + \eta)\hat{v}(\xi, 1)), \\ \partial_\eta^r\hat{v}_{b,\partial\Omega}^{(1)}(\xi, \eta) &= \frac{1}{2}\partial_\eta^r((1 - \xi)\hat{v}(-1, \eta) + (1 + \xi)\hat{v}(1, \eta)). \end{aligned}$$

On the other hand, for $r \geq 3$,

$$\begin{aligned} \partial_\xi^{r-1}\partial_\eta\hat{v}_{b,\partial\Omega}^{(1)}(\xi, \eta) &= \frac{1}{2}\partial_\xi^{r-1}(-\hat{v}(\xi, -1) + \hat{v}(\xi, 1)) = \frac{1}{2}\int_{\Lambda_\eta}\partial_\xi^{r-1}\partial_\eta\hat{v}(\xi, \eta)d\eta, \\ \partial_\xi\partial_\eta^{r-1}\hat{v}_{b,\partial\Omega}^{(1)}(\xi, \eta) &= \frac{1}{2}\partial_\eta^{r-1}(-\hat{v}(-1, \eta) + \hat{v}(1, \eta)) = \frac{1}{2}\int_{\Lambda_\xi}\partial_\xi\partial_\eta^{r-1}\hat{v}(\xi, \eta)d\xi, \end{aligned}$$

while

$$\begin{aligned} \partial_\xi\partial_\eta\hat{v}_{b,\partial\Omega}^{(1)}(\xi, \eta) &= \frac{1}{2}(-\partial_\eta\hat{v}(-1, \eta) - \partial_\xi\hat{v}(\xi, -1) + \partial_\eta\hat{v}(1, \eta) + \partial_\xi\hat{v}(\xi, 1)) \\ &= \frac{1}{2}\int_{\Lambda_\eta}\partial_\xi\partial_\eta\hat{v}(\xi, \eta)d\eta + \frac{1}{2}\int_{\Lambda_\xi}\partial_\xi\partial_\eta\hat{v}(\xi, \eta)d\xi. \end{aligned}$$

With the aid of the previous equalities, a direct calculation yields

$$(7.8) \quad D_{r,S}(\hat{v}_{b,\partial\Omega}^{(1)}) \leq c\delta_\Omega^{-\frac{1}{2}}F_{r,\Omega}(v)$$

where $F_{r,\Omega}(v) = F_{r,\Omega}^{(1)}(v) + E_{r,\Omega}^{(2)}(v) + E_{r,\Omega}^{(3)}(v)$, with

$$F_{r,\Omega}^{(1)}(v) = \sum_{k=0}^r \left(\sum_{\nu=2,4} \|(1-\xi^2)^{\frac{r-1}{2}}(a_1+a_3\eta)^k(b_1+b_3\eta)^{r-k} \partial_x^k \partial_y^{r-k} v\|_{L^2(L_\nu)} \right. \\ \left. + \sum_{\nu=1,3} \|(1-\eta^2)^{\frac{r-1}{2}}(a_2+a_3\xi)^k(b_2+b_3\xi)^{r-k} \partial_x^k \partial_y^{r-k} v\|_{L^2(L_\nu)} \right).$$

We next estimate $D_{r,S}(\hat{v}_{b,\partial\Omega}^{(2)})$. In fact, $\partial_\xi^r \hat{v}_{b,\partial\Omega}^{(2)}(\xi, \eta) = \partial_\eta^r \hat{v}_{b,\partial\Omega}^{(2)}(\xi, \eta) = 0$ for $r \geq 2$, and $\partial_\xi^{r-1} \partial_\eta \hat{v}_{b,\partial\Omega}^{(2)}(\xi, \eta) = \partial_\xi \partial_\eta^{r-1} \hat{v}_{b,\partial\Omega}^{(2)}(\xi, \eta) = 0$ for $r \geq 3$. Moreover,

$$\partial_\xi \partial_\eta \hat{v}_{b,\partial\Omega}^{(2)}(\xi, \eta) = -\frac{1}{4}(\hat{v}(-1, -1) - \hat{v}(1, -1) + \hat{v}(1, 1) - \hat{v}(-1, 1)) \\ = -\frac{1}{4} \int_S \partial_\xi \partial_\eta \hat{v}(\xi, \eta) d\xi d\eta.$$

As a result, $D_{r,S}(\hat{v}_{b,\partial\Omega}^{(2)}) = 0$ for $r \geq 3$, and

(7.9)

$$D_{2,S}(\hat{v}_{b,\partial\Omega}^{(2)}) \leq \|(a_1+a_3\eta)(a_2+a_3\xi)\partial_x^2 v\|_\Omega + \|(b_1+b_3\eta)(b_2+b_3\xi)\partial_y^2 v\|_\Omega \\ + \|(a_1+a_3\eta)(b_2+b_3\xi)\partial_x \partial_y v\|_\Omega + \|(b_1+b_3\eta)(a_2+a_3\xi)\partial_x \partial_y v\|_\Omega \\ + \|a_3 \partial_x v\|_\Omega + \|b_3 \partial_y v\|_\Omega \leq c\delta_\Omega^{-\frac{1}{2}}(E_{r,\Omega}^{(2)}(v) + E_{r,\Omega}^{(3)}(v)).$$

Furthermore, we estimate $\|\nabla(*P_{N_b,\partial\Omega}^1 v_{b,\partial\Omega} - v_{b,\partial\Omega})\|_\Omega$. By virtue of (7.3), we have

$$* \hat{P}_{N_b,\partial\Omega}^1 \hat{v}_{b,\partial\Omega} - \hat{v}_{b,\partial\Omega} = \frac{1}{2}((1-\xi)(P_{N_b,\Lambda_\eta}^{1,0} \hat{v}_{b,L_1}^0(\eta) - \hat{v}_{b,L_1}^0(\eta)) \\ + (1-\eta)(P_{N_b,\Lambda_\xi}^{1,0} \hat{v}_{b,L_2}^0(\xi) - \hat{v}_{b,L_2}^0(\xi)) \\ + (1+\xi)(P_{N_b,\Lambda_\eta}^{1,0} \hat{v}_{b,L_3}^0(\eta) - \hat{v}_{b,L_3}^0(\eta)) + (1+\eta)(P_{N_b,\Lambda_\xi}^{1,0} \hat{v}_{b,L_4}^0(\xi) - \hat{v}_{b,L_4}^0(\xi))).$$

With the aid of (3.5), we verify that for $r_b \geq 2$,

$$\|\nabla_S(* \hat{P}_{N_b,\partial\Omega}^1 \hat{v}_{b,\partial\Omega} - \hat{v}_{b,\partial\Omega})\|_S \\ \leq cN_b^{1-r_b} (\|(1-\eta^2)^{\frac{r_b-1}{2}} \partial_\eta^{r_b} \hat{v}_{b,L_1}^0\|_S + \|(1-\xi^2)^{\frac{r_b-1}{2}} \partial_\xi^{r_b} \hat{v}_{b,L_2}^0\|_S \\ + \|(1-\eta^2)^{\frac{r_b-1}{2}} \partial_\eta^{r_b} \hat{v}_{b,L_3}^0\|_S + \|(1-\xi^2)^{\frac{r_b-1}{2}} \partial_\xi^{r_b} \hat{v}_{b,L_4}^0\|_S) \\ = cN_b^{1-r_b} (\|(1-\eta^2)^{\frac{r_b-1}{2}} \partial_\eta^{r_b} \hat{v}(-1, \eta)\|_S + \|(1-\xi^2)^{\frac{r_b-1}{2}} \partial_\xi^{r_b} \hat{v}(\xi, -1)\|_S \\ + \|(1-\eta^2)^{\frac{r_b-1}{2}} \partial_\eta^{r_b} \hat{v}(1, \eta)\|_S + \|(1-\xi^2)^{\frac{r_b-1}{2}} \partial_\xi^{r_b} \hat{v}(\xi, 1)\|_S).$$

The above inequality with (2.9) implies

$$(7.10) \quad \|\nabla(*P_{N_b,\partial\Omega}^1 v_{b,\partial\Omega} - v_{b,\partial\Omega})\|_\Omega \leq c\sigma_\Omega \delta_\Omega^{-1} N_b^{1-r_b} G_{r_b,\Omega}(v)$$

where

$$G_{r_b,\Omega}(v) = \sum_{k=0}^{r_b} \left(\sum_{\nu=1,3} \|(1-\eta^2)^{\frac{r_b-1}{2}}(a_2+a_3\xi)^k(b_2+b_3\xi)^{r_b-k} \partial_x^k \partial_y^{r_b-k} v\|_{L^2(L_\nu)} \right. \\ \left. + \sum_{\nu=2,4} \|(1-\xi^2)^{\frac{r_b-1}{2}}(a_1+a_3\eta)^k(b_1+b_3\eta)^{r_b-k} \partial_x^k \partial_y^{r_b-k} v\|_{L^2(L_\nu)} \right).$$

A combination of (7.5)–(7.10) gives

$$(7.11) \quad \|\nabla(*P_{N_b,\Omega}^1 v - v)\|_\Omega \\ \leq c\sigma_\Omega \delta_\Omega^{-1} (N^{1-r}(E_{r,\Omega}(v) + F_{r,\Omega}(v)) + N_b^{1-r_b} G_{r_b,\Omega}(v)).$$

We now turn to the quasi-Legendre orthogonal approximations corresponding to Neumann boundary conditions imposed on certain edges of Ω . To do this, we need some preparation. For fixedness, we assume that the Neumann boundary condition is given on L_1 ; see Figure 1. Let

$${}^0H^1(\Lambda_\xi) = \{ v \in H^1(\Lambda_\xi) \mid v(1) = 0 \}, \quad {}^0\mathcal{P}_N(\Lambda_\xi) = \{ \phi \in \mathcal{P}_N(\Lambda_\xi) \mid \phi(1) = 0 \}.$$

The orthogonal projection ${}^0P_{N,\Lambda_\xi}^1 : {}^0H^1(\Lambda_\xi) \rightarrow {}^0\mathcal{P}_N(\Lambda_\xi)$, is defined by

$$(\partial_\xi({}^0P_{N,\Lambda_\xi}^1 v - v), \partial_\xi \phi)_{\Lambda_\xi} = 0, \quad \forall \phi \in {}^0\mathcal{P}_N(\Lambda_\xi).$$

Take

$$\phi(\xi) = - \int_\xi^1 P_{N-1,\Lambda_\eta} \partial_\eta v(\eta) d\eta \in {}^0\mathcal{P}_N(\Lambda_\xi).$$

If $v \in {}^0H(\Lambda_\xi)$ and $\partial_\xi^r v \in L^2_{\chi^{(r-1,r-1)}}(\Lambda_\xi)$ for integers $1 \leq r \leq N + 1$, then by virtue of the projection theorem and (3.3), we deduce that

$$(7.12) \quad \begin{aligned} \|\partial_\xi({}^0P_{N,\Lambda_\xi}^1 v - v)\|_{\Lambda_\xi} &\leq \|\partial_\xi(\phi - v)\|_{\Lambda_\xi} \\ &= \|P_{N-1,\Lambda_\xi} \partial_\xi v - \partial_\xi v\|_{\Lambda_\xi} \leq cN^{1-r} \|\partial_\xi^r v\|_{\chi^{(r-1,r-1)},\Lambda_\xi}. \end{aligned}$$

Moreover, a standard duality argument yields

$$(7.13) \quad \|{}^0P_{N,\Lambda_\xi}^1 v - v\|_{\Lambda_\xi} \leq cN^{-r} \|\partial_\xi^r v\|_{\chi^{(r-1,r-1)},\Lambda_\xi}.$$

Now, we set

$${}^0H^1(\Omega) = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } L_2, L_3 \text{ and } L_4 \}, \quad {}^0V_N(\Omega) = {}^0H^1(\Omega) \cap V_N(\Omega).$$

The orthogonal projection ${}^0P_{N,\Omega}^1 : {}^0H^1(\Omega) \rightarrow {}^0V_N(\Omega)$ is defined by

$$(7.14) \quad (\nabla({}^0P_{N,\Omega}^1 v - v), \nabla \phi)_\Omega = 0, \quad \forall \phi \in {}^0V_N(\Omega).$$

Next, let $\hat{v}_{b,L_2}(\xi) = \frac{1}{2}\hat{v}(1, -1)(1 + \xi)$, $\hat{v}_{b,L_4}(\xi) = \frac{1}{2}\hat{v}(1, 1)(1 + \xi)$, and let $\hat{v}_{b,L_3}(\eta)$ be the same as in (7.1). We also define $\hat{v}_{b,L_2}^0(\xi)$, $\hat{v}_{b,L_3}^0(\eta)$ and $\hat{v}_{b,L_4}^0(\xi)$ by (7.2) with the above new $\hat{v}_{b,L_2}(\xi)$ and $\hat{v}_{b,L_4}(\xi)$. Evidently, $\hat{v}_{b,L_2}^0(1) = \hat{v}_{b,L_3}^0(\pm 1) = \hat{v}_{b,L_4}^0(1) = 0$. Furthermore, we set $\hat{v}_\Omega^0(\xi, \eta) = \hat{v}(\xi, \eta) - \hat{v}_{b,\partial\Omega}(\xi, \eta)$, with

$$(7.15) \quad \begin{aligned} \hat{v}_{b,\partial\Omega}(\xi, \eta) &= \frac{1}{2}((1 - \eta)\hat{v}_{b,L_2}^0(\xi) + (1 + \xi)\hat{v}_{b,L_3}^0(\eta) + (1 + \eta)\hat{v}_{b,L_4}^0(\xi)) \\ &\quad + \frac{1}{4}((1 + \xi)(1 - \eta)\hat{v}(1, -1) + (1 + \xi)(1 + \eta)\hat{v}(1, 1)). \end{aligned}$$

The function $\hat{v}_\Omega^0(\xi, \eta)$ vanishes on $L_j, j = 2, 3, 4$. We also introduce the projection on $\partial\Omega$, as

$$\begin{aligned} * \hat{P}_{N_b,\partial\Omega}^1 \hat{v}_{b,\partial\Omega}(\xi, \eta) &= \frac{1}{2}((1 - \eta)^0 P_{N_b,\Lambda_\xi}^1 \hat{v}_{b,L_2}^0(\xi) + (1 + \xi) P_{N_b,\Lambda_\eta}^{1,0} \hat{v}_{b,L_3}^0(\eta) \\ &\quad + (1 + \eta)^0 P_{N_b,\Lambda_\xi}^1 \hat{v}_{b,L_4}^0(\xi)) \\ &\quad + \frac{1}{4}((1 + \xi)(1 - \eta)\hat{v}(1, -1) + (1 + \xi)(1 + \eta)\hat{v}(1, 1)). \end{aligned}$$

Finally, we define the quasi-Legendre orthogonal projection $*P_{N,N_b,\Omega}^1 v$ as

$$(7.16) \quad *P_{N,N_b,\Omega}^1 v(x, y) = {}^0P_{N,\Omega}^1 v_\Omega^0(x, y) + *P_{N_b,\partial\Omega}^1 v_{b,\partial\Omega}(x, y),$$

where the meanings of $v_\Omega^0(x, y)$, $v_{b,\partial\Omega}(x, y)$ and $*P_{N_b,\partial\Omega}^1 v_{b,\partial\Omega}(x, y)$ are similar to those in (7.4), with the new functions $\hat{v}_\Omega^0(\xi, \eta)$ and $\hat{v}_{b,\partial\Omega}(\xi, \eta)$, and the new projection $*\hat{P}_{N_b,\partial\Omega}^1 \hat{v}_{b,\partial\Omega}(\xi, \eta)$. It can be checked that $*P_{N,N_b,\Omega}^1 v(x, y) = v(x, y)$ at the vertices $(x, y) = (1, -1)$ or $(1, 1)$.

In the same manner, we can define the quasi-Legendre orthogonal projections $*P_{N,N_b,\Omega}^1 v$ corresponding to the Neumann boundary condition imposed on the edges L_2, L_3 and L_4 , respectively. Moreover, by using (3.5), (7.13) and an argument like the derivation of (7.11), we can derive the error estimates for these quasi-Legendre orthogonal projections, which are similar to (7.11).

7.2. Composite quasi-orthogonal approximation on polygons. We are now in a position to study the quasi-Legendre orthogonal projection on a polygon with the boundary $\partial\Omega$ and $\partial^*\Omega \subseteq \partial\Omega$. We divide Ω into convex quadrilaterals $\Omega_i, 1 \leq i \leq n$, with the boundary $\partial\Omega_i$, the edges $L_{i,\nu}$, the vertices $Q_{i,\nu}$ and the angles $\theta_{i,\nu}, 1 \leq \nu \leq 4$. Therefore, $\partial^*\Omega_i = \partial^*\Omega \cap \partial\Omega_i$. The local variable transformation are denoted by $\xi_i = \xi_i(x, y)$ and $\eta_i = \eta_i(x, y), 1 \leq i \leq n$. The corresponding quantities $\sigma_{\Omega_i}, \gamma_{\Omega_i}, \delta_{\Omega_i}, d_{i,1}, d_{i,2}, a_{i,1}, a_{i,2}, a_{i,3}, b_{i,1}, b_{i,2}$ and $b_{i,3}$ are defined in the same way as for the single quadrilateral; see Section 3. Let $h_i = \text{diam } \Omega_i$. Assume that the partition of Ω satisfies the following hypotheses:

- (H₁) $\Omega = \bigcup_{i=1}^n \Omega_i$ and $\Omega_i \cap \Omega_k = \emptyset$ if $i \neq k$,
- (H₂) each vertex of Ω_i is also one of the vertices of adjacent quadrilaterals,
- (H₃) if $\Omega_i \cap \partial\Omega \neq \emptyset$, then Ω_i has at most one edge $\partial^*\Omega_i$,
- (H₄) there are positive constants β_0 and β_1 such that $0 < \beta_0 \leq \theta_{i,\nu} \leq \beta_1 < \pi, 1 \leq \nu \leq 4, 1 \leq i \leq n$, and so $0 < \delta_0 h_i^2 \leq \delta_{\Omega_i} \leq \delta_1 h_i^2, 1 \leq i \leq n$.

Let $N = (N_1, N_2, \dots, N_n)$ and $r = (r_1, r_2, \dots, r_n)$. We define the quasi-Legendre orthogonal projection $*P_{N,N_b,\Omega}^1 v$ as $*P_{N,N_b,\Omega}^1 v|_{\Omega_i} = *P_{N_i,N_b,\Omega_i}^1 v, 1 \leq i \leq n$, where the projections $*P_{N_i,N_b,\Omega_i}^1 v$ are constructed in such a way that

- (A) if $\bar{\Omega}_i \cap \partial\Omega = \emptyset$, then $*P_{N_i,N_b,\Omega_i}^1 v$ is given by (7.4),
- (B) if $\partial^*\Omega_i \neq \emptyset$, say $\partial^*\Omega_i = L_{i,1}$, then $*P_{N_i,N_b,\Omega_i}^1 v$ is similar to (7.16).

Clearly, if $L_{i,\nu}$ and $L_{k,\nu}$ are the same segment, say $L_{i,3} = L_{k,1}$, then the coefficients in the expansions of $P_{N_b,\Lambda_{\eta_i}}^{1,0} \hat{v}_{b,L_{i,3}}^0(\eta_i)$ and $P_{N_b,\Lambda_{\eta_k}}^{1,0} \hat{v}_{b,L_{k,1}}^0(\eta_k)$ are the same. It can be checked that $*P_{N,N_b,\Omega}^1 v \in C(\bar{\Omega})$.

For a description of the approximation error, we introduce the notation $E_{r_i,\Omega_i}(v), F_{r_i,\Omega_i}(v)$ and $G_{r_b,\Omega_i}(v)$, with the quantities $\xi_i, \eta_i, \sigma_{\Omega_i}, \gamma_{\Omega_i}, \delta_{\Omega_i}, d_{i,1}, d_{i,2}, a_{i,1}, a_{i,2}, a_{i,3}, b_{i,1}, b_{i,2}$ and $b_{i,3}$, respectively. According to the previous statements and an standard argument as in [2, 5, 7], we observe that if $v \in H^1(\Omega_i) \cap C(\bar{\Omega}_i)$ and $E_{r_i,\Omega_i}(v), F_{r_i,\Omega_i}(v)$ and $G_{r_b,\Omega_i}(v)$ are finite for integers $2 \leq r_i \leq N_i + 1(1 \leq i \leq n)$ and $2 \leq r_b \leq N_b + 1$, then

$$(7.17) \quad \begin{aligned} & \|\nabla(*P_{N,N_b,\Omega}^1 v - v)\|_{\Omega} \\ & \leq c\sigma_{\Omega}\delta_{\Omega}^{-1} \sum_{i=1}^n (N_i^{1-r_i}(E_{r_i,\Omega_i}(v) + F_{r_i,\Omega_i}(v)) + N_b^{1-r_b}G_{r_b,\Omega_i}(v)). \end{aligned}$$

7.3. Spectral element method for polygons. We now consider the following problem,

$$(7.18) \quad \begin{cases} -\Delta U(x, y) = f(x, y), & (x, y) \in \Omega, \\ \partial_n U(x, y) = g_2(x, y), & (x, y) \in \partial^*\Omega, \\ U(x, y) = g_1(x, y), & (x, y) \in \partial\Omega \setminus \partial^*\Omega. \end{cases}$$

If $\partial^*\Omega = \partial\Omega$, we require, in addition, $(f, 1)_\Omega = 0$ and $(U, 1)_\Omega = 0$ for fixing the solution. In what follows, we suppose $\partial^*\Omega \neq \partial\Omega$ for simplicity. Let

$$V_{g_1}(\Omega) = \{ v \in H^1(\Omega) \cap C(\bar{\Omega}) \mid v = g_1 \text{ on } \partial\Omega \setminus \partial^*\Omega \},$$

$$\bar{V}(\Omega) = \{ v \in H^1(\Omega) \cap C(\bar{\Omega}) \mid v = 0 \text{ on } \partial\Omega \setminus \partial^*\Omega \}.$$

The weak formulation of (7.18) is to find $U \in V_{g_1}(\Omega)$ such that

$$(7.19) \quad (\nabla U, \nabla v)_\Omega + \int_{\partial^*\Omega} g_2(x, y)v(x, y)ds = (f, v)_\Omega, \quad \forall v \in \bar{V}(\Omega).$$

For the numerical solution of (7.19), we need three kinds of base functions. Let $L_l(\xi)$ be the Legendre polynomial of degree l as usual. The base functions corresponding to Ω_i , are given by

$$\psi_{\Omega_i, l_i, m_i}^0(x, y) = \begin{cases} \frac{1}{\sqrt{(4l_i + 6)(4m_i + 6)}} (L_{l_i}(\xi_i) - L_{l_i+2}(\xi_i))(L_{m_i}(\eta_i) - L_{m_i+2}(\eta_i))|_{\xi_i=\xi_i(x,y), \eta_i=\eta_i(x,y)}, & \text{on } \bar{\Omega}_i, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we define the base functions corresponding to the edges of quadrilaterals. For instance, if $L_{i,k} = L_{i,1} = L_{k,3}$, then the corresponding base function

$$\psi_{L_{i,k}, l_{i,k}}^0(x, y) = \begin{cases} \frac{1}{2\sqrt{4l_{i,k} + 6}}(1 - \xi_i)(L_{l_{i,k}}(\eta_i) - L_{l_{i,k}+2}(\eta_i))|_{\xi_i=\xi_i(x,y), \eta_i=\eta_i(x,y)}, & \text{on } \bar{\Omega}_i, \\ \frac{1}{2\sqrt{4l_{i,k} + 6}}(1 + \xi_k)(L_{l_{i,k}}(\eta_k) - L_{l_{i,k}+2}(\eta_k))|_{\xi_k=\xi_k(x,y), \eta_k=\eta_k(x,y)} & \text{on } \bar{\Omega}_k, \\ 0, & \text{otherwise,} \end{cases}$$

The third kind of base functions correspond to the vertices of quadrilaterals. For example, if $Q_{i_1,1} = Q_{i_2,2} = Q_{i_3,3} = Q_{i_4,4}$, then the the corresponding base function

$$\psi_{Q_{i_1, i_2, i_3, i_4}}(x, y) = \begin{cases} \frac{1}{4}(1 - \xi_{i_1})(1 - \eta_{i_1})|_{\xi_{i_1}=\xi_{i_1}(x,y), \eta_{i_1}=\eta_{i_1}(x,y)}, & \text{on } \bar{\Omega}_{i_1}, \\ \frac{1}{4}(1 + \xi_{i_2})(1 - \eta_{i_2})|_{\xi_{i_2}=\xi_{i_2}(x,y), \eta_{i_2}=\eta_{i_2}(x,y)}, & \text{on } \bar{\Omega}_{i_2}, \\ \frac{1}{4}(1 + \xi_{i_3})(1 + \eta_{i_3})|_{\xi_{i_3}=\xi_{i_3}(x,y), \eta_{i_3}=\eta_{i_3}(x,y)}, & \text{on } \bar{\Omega}_{i_3}, \\ \frac{1}{4}(1 - \xi_{i_4})(1 + \eta_{i_4})|_{\xi_{i_4}=\xi_{i_4}(x,y), \eta_{i_4}=\eta_{i_4}(x,y)}, & \text{on } \bar{\Omega}_{i_4}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $W_{N, N_b}(\Omega)$ be the set spanned by $\psi_{\Omega_i, l_i, m_i}^0(x, y), 0 \leq l_i, m_i \leq N_i - 2$, all $\psi_{L_{i,k}, l_{i,k}}^0(x, y), 0 \leq l_{i,k} \leq N_b$ and all $\psi_{Q_{i_1, i_2, i_3, i_4}}(x, y)$. Clearly, $W_{N, N_b}(\Omega) \subset H^1(\Omega) \cap C(\bar{\Omega})$. Furtherore, we let

$$V_{N, N_b}(\Omega) = \{ \phi \in W_{N, N_b}(\Omega) \mid \phi = P_{N_b, \partial\Omega}^1 g_1 \text{ on } \partial\Omega \setminus \partial^*\Omega \},$$

where, for example, if the edge $L_{i,1} \subseteq \partial\Omega \setminus \partial^*\Omega$, then we take $\hat{g}_1(-1, \eta_i) = g_1(x(-1, \eta_i), y(-1, \eta_i))$ and

$$\begin{aligned} *P_{N_b, \partial\Omega}^1 g_1(x, y) &= P_{N_b, \Lambda_{\eta_i}}^{1,0}(\hat{g}_1(-1, \eta_i) - \frac{1}{2}\hat{g}_1(-1, -1)(1 - \eta_i) - \frac{1}{2}\hat{g}_1(-1, 1)(1 + \eta_i)) \\ &\quad + \frac{1}{2}\hat{g}_1(-1, -1)(1 - \eta_i) + \frac{1}{2}\hat{g}_1(-1, 1)(1 + \eta_i)|_{\xi_i=\xi_i(x,y), \eta_i=\eta_i(x,y)}. \end{aligned}$$

The set $V_{\mathbf{N},N_b}^*(\Omega) = \{ \phi \in W_{\mathbf{N},N_b}(\Omega) \mid \phi = 0 \text{ on } \partial\Omega \setminus \partial^*\Omega \}$.

We now construct the spectral method for (7.19). We first consider an auxiliary problem (cf. [8]), which is to find the solution $W \in V_{*P_{N_b, \partial\Omega}^1}(\Omega)$ such that

$$(7.20) \quad (\nabla W, \nabla v)_\Omega + \int_{\partial^*\Omega} g_2(x, y)v(x, y)ds = (f, v)_\Omega, \quad \forall v \in \bar{V}(\Omega).$$

The corresponding spectral method is to find $w_{\mathbf{N},N_b} \in V_{\mathbf{N},N_b}(\Omega)$ such that

$$(7.21) \quad (\nabla w_{\mathbf{N},N_b}, \nabla \phi)_\Omega + \int_{\partial^*\Omega} g_2(x, y)\phi(x, y)ds = (f, \phi)_\Omega, \quad \forall \phi \in V_{\mathbf{N},N_b}^*(\Omega).$$

For derivation of error estimate of numerical solution, we introduce the orthogonal projection $P_{\mathbf{N},N_b,\Omega}^1 v : H^1(\Omega) \cap C(\bar{\Omega}) \rightarrow V_{\mathbf{N},N_b}(\Omega)$ such that

$$(\nabla(P_{\mathbf{N},N_b,\Omega}^1 v - v), \nabla \phi)_\Omega = 0, \quad \forall \phi \in V_{\mathbf{N},N_b}^*(\Omega).$$

This, along with (7.20), leads to

$$(7.22) \quad (\nabla P_{\mathbf{N},N_b,\Omega}^1 W, \nabla \phi)_\Omega + \int_{\partial^*\Omega} g_2(x, y)\phi(x, y)ds = (f, \phi)_\Omega, \quad \forall \phi \in V_{\mathbf{N},N_b}^*(\Omega).$$

Subtracting (7.22) from (7.21) yields

$$(\nabla(P_{\mathbf{N},N_b,\Omega}^1 W - w_{\mathbf{N},N_b}), \nabla \phi)_\Omega = 0, \quad \forall \phi \in V_{\mathbf{N},N_b}^*(\Omega).$$

This implies $w_{\mathbf{N},N_b} = P_{\mathbf{N},N_b,\Omega}^1 W$. Consequently (cf. [8]),

$$(7.23) \quad \|\nabla(w_{\mathbf{N},N_b} - W)\|_\Omega = \|\nabla(P_{\mathbf{N},N_b,\Omega}^1 W - W)\|_\Omega \leq \|\nabla(*P_{\mathbf{N},N_b,\Omega}^1 U - W)\|_\Omega \\ \leq \|\nabla(*P_{\mathbf{N},N_b,\Omega}^1 U - U)\|_\Omega + \|\nabla(U - W)\|_\Omega.$$

Furthermore, we have from (7.18) and (7.20) that

$$\begin{cases} -\Delta(U(x, y) - W(x, y)) = 0, & (x, y) \in \Omega, \\ \partial_n(U(x, y) - W(x, y)) = 0, & (x, y) \in \partial^*\Omega, \\ U(x, y) - W(x, y) = g_1(x, y) - *P_{N_b, \partial\Omega}^1 g_1(x, y), & (x, y) \in \partial\Omega \setminus \partial^*\Omega. \end{cases}$$

According to the properties of elliptic equation (cf. [8]) and the quasi-Legendre orthogonal approximation, we obtain

$$(7.24) \quad \|U - W\|_{H^1(\Omega)} \\ \leq c \|g_1 - *P_{N_b, \partial\Omega}^1 g_1\|_{H^1(\partial\Omega \setminus \partial^*\Omega)} \leq c\sigma_\Omega \delta_\Omega^{-1} N_b^{1-r_b} \sum_{i=1}^n K_{r_b, \Omega_i}(g_1)$$

where $K_{r_b, \Omega_i}(g_1) = 0$ if $\bar{\Omega}_i \cap \partial\Omega = \emptyset$, and otherwise,

$$K_{r_b, \Omega_i}(g_1) = \begin{cases} \sum_{k=0}^{r_b} \|(1 - \eta^2)^{\frac{r_b-1}{2}} (a_2 + a_3\xi)^k (b_2 + b_3\xi)^{r_b-k} \partial_x^k \partial_y^{r_b-k} g_1\|_{L^2(L_{i,1})}, & \text{if } L_{i,1} \subseteq \partial\Omega \setminus \partial^*\Omega, \\ \sum_{k=0}^{r_b} \|(1 - \xi^2)^{\frac{r_b-1}{2}} (a_1 + a_3\eta)^k (b_1 + b_3\eta)^{r_b-k} \partial_x^k \partial_y^{r_b-k} g_1\|_{L^2(L_{i,2})}, & \text{if } L_{i,2} \subseteq \partial\Omega \setminus \partial^*\Omega, \\ \sum_{k=0}^{r_b} \|(1 - \eta^2)^{\frac{r_b-1}{2}} (a_2 + a_3\xi)^k (b_2 + b_3\xi)^{r_b-k} \partial_x^k \partial_y^{r_b-k} g_1\|_{L^2(L_{i,3})}, & \text{if } L_{i,3} \subseteq \partial\Omega \setminus \partial^*\Omega, \\ \sum_{k=0}^{r_b} \|(1 - \xi^2)^{\frac{r_b-1}{2}} (a_1 + a_3\eta)^k (b_1 + b_3\eta)^{r_b-k} \partial_x^k \partial_y^{r_b-k} g_1\|_{L^2(L_{i,4})}, & \text{if } L_{i,4} \subseteq \partial\Omega \setminus \partial^*\Omega. \end{cases}$$

A combination of (7.17), (7.23) and (7.24) leads to the following conclusion.

Theorem 7.1. *If the hypotheses (H₁)–(H₄) hold, $U \in H^1(\Omega) \cap C(\bar{\Omega})$, and all $E_{r_i, \Omega_i}(U)$, $F_{r_i, \Omega_i}(U)$, $G_{r_b, \Omega_i}(U)$ and $K_{r_b, \Omega_i}(g_1)$ are finite for integers $2 \leq r_i \leq N_i + 1$, $1 \leq i \leq n$ and $2 \leq r_b \leq N_b + 1$, then*

$$\begin{aligned} & \|U - w_{\mathbf{N}, N_b}\|_{H^1(\Omega)} \\ & \leq c\sigma_\Omega \delta_\Omega^{-1} \sum_{i=1}^n (N_i^{1-r_i} (E_{r_i, \Omega_i}(U) + F_{r_i, \Omega_i}(U)) + N_b^{1-r_b} (G_{r_b, \Omega_i}(U) + K_{r_b, \Omega_i}(g_1))). \end{aligned}$$

Remark 7.1. By (3.26), we have $\max(|a_{i,3}|, |b_{i,3}|, |\sigma_{\Omega_i}|) \leq ch_i$. Therefore,

$$\begin{aligned} \|U - W_{\mathbf{N}, N_b}\|_{H^1(\Omega)} & \leq c \sum_{i=1}^n N_i^{1-r_i} \left(\sum_{k=0}^{r_i} h_i^{r_i-1} \|\partial_x^k \partial_y^{r_i-k} U\|_\Omega + \sum_{k=0}^{r_i-1} h_i^{r_i-2} \|\partial_x^k \partial_y^{r_i-k} U\|_\Omega \right. \\ & \quad + \sum_{\nu=1}^4 \sum_{k=0}^{r_i} h_i^{r_i-1} \|\partial_x^k \partial_y^{r_i-k} U\|_{L^2(L_{i,\nu})} \\ & \quad + c \sum_{i=1}^n N_b^{1-r_b} h_i^{r_b-1} \sum_{\nu=1}^4 \sum_{k=0}^{r_b} (\|\partial_x^k \partial_y^{r_b-k} U\|_{L^2(L_{i,\nu})} \\ & \quad \left. + \|\partial_x^k \partial_y^{r_b-k} g_1\|_{L^2(L_{i,\nu} \cap (\partial\Omega - \partial^*\Omega))} \right). \end{aligned}$$

This is similar to the result (5.4.16) of [5] and the corresponding result of [17] for multidomain pseudospectral method.

Remark 7.2. In actual computation, we evaluate the terms $\int_{\partial^*\Omega} g_2(x, y)\phi(x, y)ds$ and $(f, \phi)_\Omega$ approximately. Thus, in general, there exist two additional errors depending on the accuracy of numerical quadratures and the smoothness of f and g_2 .

8. CONCLUDING REMARKS

In this paper, we first established the basic results on the orthogonal approximation for arbitrary convex quadrilaterals, which conform the mathematical foundation of the related spectral method possessing the spectral accuracy. Moreover,

the Jacobi weighted norms appearing in the upper bounds of approximation errors, cover certain singularities of considered functions at the edges or vertices of quadrilaterals.

As examples of applications, the spectral schemes were proposed for two model problems, with the optimal error estimates of numerical solutions. The numerical results demonstrated their high accuracy, and coincide with the analysis well. We also developed a spectral method for convex polygons. Our method could be applied to exterior problems with non-rectangular obstacles.

It is noted that the super-convergence of finite element method was obtained for analytical solutions; see [19]. But our results are valid for solutions in certain weighted Sobolev space, which is much more practical.

APPENDIX A

We have from (2.3) that

$$(a_1b_3 - a_3b_1)\xi^2 - (b_3(x - a_0) - a_3(y - b_0) + a_2b_1 - a_1b_2)\xi - b_2(x - a_0) + a_2(y - b_0) = 0.$$

If $a_1b_3 - a_3b_1 \neq 0$, then

$$\xi(x, y) = \frac{1}{2(a_1b_3 - a_3b_1)}(b_3(x - a_0) - a_3(y - b_0) + a_2b_1 - a_1b_2 + \sigma\sqrt{(b_3(x - a_0) - a_3(y - b_0) + a_2b_1 - a_1b_2)^2 + 4(a_1b_3 - a_3b_1)(b_2(x - a_0) - a_2(y - b_0))}),$$

where $\sigma = 1$, if the points Q_j are numbered counterclockwise as in Section 2, and $\sigma = -1$ otherwise. If $a_1b_3 - a_3b_1 = 0$, then

$$\xi(x, y) = -\frac{b_2(x - a_0) - a_2(y - b_0)}{b_3(x - a_0) - a_3(y - b_0) + a_2b_1 - a_1b_2}.$$

Similarly, we have

$$(a_2b_3 - a_3b_2)\eta^2 - (b_3(x - a_0) - a_3(y - b_0) + a_1b_2 - a_2b_1)\eta - b_1(x - a_0) + a_1(y - b_0) = 0.$$

If $a_2b_3 - a_3b_2 \neq 0$, then

$$\eta(x, y) = \frac{1}{2(a_2b_3 - a_3b_2)}(b_3(x - a_0) - a_3(y - b_0) + a_1b_2 - a_2b_1 + \lambda\sqrt{(b_3(x - a_0) - a_3(y - b_0) + a_1b_2 - a_2b_1)^2 + 4(a_2b_3 - a_3b_2)(b_1(x - a_0) - a_1(y - b_0))}),$$

where $\lambda = -1$, if the points Q_j are numbered clockwise as in Section 2, and $\lambda = 1$ otherwise. If $a_2b_3 - a_3b_2 = 0$, then

$$\eta(x, y) = -\frac{b_1(x - a_0) - a_1(y - b_0)}{b_3(x - a_0) - a_3(y - b_0) + a_1b_2 - a_2b_1}.$$

REFERENCES

- [1] Bernardi C. and Maday Y., Spectral methods, in *Handbook of Numerical Analysis*, 209-486, ed. by Ciarlet P. G. and Lions J. L., Elsevier, Amsterdam, 1997. MR1470226
- [2] Bernardi C., Maday Y. and Rapetti F., *Discretisations Variationnelles de Problemes aux Limites Elliptique, Collection: Mathematique et Applications, Vol. 45*, Springer-Verlag, Berlin, 2004. MR2068204 (2005g:65004)
- [3] Boyd J. P., *Chebyshev and Fourier Spectral Methods, Second edition*, Dover Publication Inc., Mineda, New York, 2001. MR1874071 (2002k:65160)
- [4] Canuto C., Hussaini M. Y., Quarteroni A. and Zang T. A., *Spectral Methods: Fundamentals in Single Domains*, Springer-Verlag, Berlin, 2006. MR2223552 (2007c:65001)
- [5] Canuto C., Hussaini M. Y., Quarteroni A. and Zang T. A., *Spectral Methods: Evolution to complex Geometries and Applications to Fluid Dynamics*, Springer-Verlag, Berlin, 2007. MR2340254 (2009d:76084)

- [6] Funaro D., *Polynomial Approximations of Differential Equations*, Springer-Verlag, Berlin, 1992. MR1176949 (94c:65078)
- [7] Gottlieb D. and Orszag S. A., *Numerical Analysis of Spectral Methods: Theory and Applications*, SIAM-CBMS, Philadelphia, 1977. MR0520152 (58:24983)
- [8] Guo Benqi, Approximation theory for the p -version of the finite element method in three dimensions, Part II: Convergence of the p -version of the finite element method, *SIAM J. on Numer. Anal.* **47** (2009), no. 4, 2578–2611. MR2217381 (2007b:65121)
- [9] Guo Ben-yu, *Spectral Methods and Their Applications*, World Scientific, Singapore, 1998. MR1641586 (2000b:65194)
- [10] Guo Ben-yu, Jacobi approximations in certain Hilbert spaces and their applications to singular differential equations, *J. Math. Anal. Appl.*, **243** (2000), 373-408. MR1741531 (2001b:65082)
- [11] Guo Ben-yu, Shen Jie and Wang Li-lian, Optical spectral-Galerkin methods using generalized Jacobi polynomials, *J. Sci. Comp.*, **27** (2006), 305-322. MR2285783 (2008f:65233)
- [12] Guo Ben-yu and Wang Li-lian, Jacobi approximations in non-uniformly Jacobi-Weighted Sobolev spaces, *J. Appr. Theo.*, **128** (2004), 1-41. MR2063010 (2005h:41010)
- [13] Guo Ben-yu and Wang Li-lian, Non-isotropic Jacobi spectral method, *Cont. Math.*, **329** (2003), 157-164. MR2022641 (2004k:65123)
- [14] Guo Ben-yu and Wang Li-lian, Error analysis of spectral method on a triangle, *Adv. in Comp. Math.*, **128** (2005), 1-24. MR2291668 (2007k:41008)
- [15] Guo Ben-yu and Wang Tian-jun, Composite generalized Laguerre-Legendre spectral method with domain decomposition and its application to Fokker-Planck equation in an finite channel, *Math. Comp.*, **78**(2009), 129-151. MR2448700
- [16] Guo Ben-yu and Wang Tian-jun, Composite Laguerre-Legendre spectral method for exterior problems, *Adv. in Comp. Math.*, DOI: 10.1017/s.10444-008-9112-5.
- [17] Karniadakis G. E. and Sherwin S. J., *Spectral/hp Element Methods for CFD, The Second Edition*, Oxford Univ. Press, Oxford, 2005. MR2165335 (2006j:65001)
- [18] Strang G. and Fix G. J., *An Analysis of the Finite Element Method*, Prentice-Hall, 1973. MR0443377 (56:1747)
- [19] Zhang Zhi-ming, Polynomial preserving gradient recovery and a posteriori estimate for bilinear element on irregular quadrilaterals, *Int. J. Numer. Anal. Model.*, **1** (2004), 1-24. MR2052728 (2005b:65119)

DEPARTMENT OF MATHEMATICS, SHANGHAI NORMAL UNIVERSITY, 200234, SHANGHAI, PEOPLE'S REPUBLIC OF CHINA

DEPARTMENT OF MATHEMATICS, DONGHUA UNIVERSITY, 200065, SHANGHAI, PEOPLE'S REPUBLIC OF CHINA