

SPECTRAL METHODS FOR ONE-DIMENSIONAL KINETIC MODELS OF GRANULAR FLOWS AND NUMERICAL QUASI ELASTIC LIMIT

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Abstract. In this paper we introduce numerical schemes for a one-dimensional kinetic model of the Boltzmann equation with dissipative collisions and variable coefficient of restitution. In particular, we study the numerical passage of the Boltzmann equation with singular kernel to nonlinear friction equations in the so-called quasi elastic limit. To this aim we introduce a Fourier spectral method for the Boltzmann equation [25, 26] and show that the kernel modes that define the spectral method have the correct quasi elastic limit providing a consistent spectral method for the limiting nonlinear friction equation.

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1. INTRODUCTION

This paper is devoted to the development of numerical schemes for the accurate computation of the solution of some kinetic models which arise in the study of driven granular flows [9, 30].

The kinetic equations are considered in a thermal bath

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = Q_{\gamma}^{(i)}(f, f)(x, v, t) + \varepsilon \frac{\partial^2 f}{\partial v^2}, \quad (1)$$

where $f(x, v, t)$ is a nonnegative function that represents the density of particles in position $x \in \mathbb{R}$ at time $t \in \mathbb{R}^+$ with velocity $v \in \mathbb{R}$ and ε is a small diffusion coefficient. These models describe inelastic particle collisions driven by random accelerations induced by a Brownian dynamics [4, 9]. Since particles loose energy in time the inclusion of an energy input is essential to achieve steady states with non zero energy.

In (1) $Q_{\gamma}^{(i)}$, $i = 1, 2$ are the so-called granular collision operators. $Q_{\gamma}^{(1)}$ describes a model Boltzmann equation for rigid spheres [30], with dissipative collisions and variable coefficient of restitution. It is a quadratic integral

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operator describing the change in the density function due to creation and annihilation of particles in binary collisions

$$Q_\gamma^{(1)}(f, f) = \int_{\mathbb{R}} \int_{\mathbb{R}^+} \beta(\vartheta) \left\{ |v^* - w^*| \frac{1}{J} f(v^*) f(w^*) - |v - w| f(v) f(w) \right\} dw d\vartheta. \quad (2)$$

In (2), $v^*(v, w)$ and $w^*(v, w)$ are the pre-collisional velocities corresponding to v, w . On the contrary to what happens for perfectly elastic collisions, in this case we have to distinguish between pre- and post-collisional velocities. If we denote the post-collisional velocities with v' and w' , these are related to v and w by

$$v' = \frac{1}{2}(v + w) + \frac{1}{2}(v - w)h; \quad w' = \frac{1}{2}(v + w) - \frac{1}{2}(v - w)h \quad (3)$$

where $h \in [0, 1]$ represents the coefficient of restitution, *i.e.* $v' - w' = h(v - w)$. It is assumed that this coefficient depends on the relative velocity,

$$h = h(|v - w|, \vartheta) = \frac{1}{1 + \vartheta|v - w|^{\gamma-1}}, \quad (4)$$

while $\beta(\vartheta)|v - w|$ is the associate collision kernel. Finally, in (2) J is the Jacobian of the transformation $(v, w) \rightarrow (v', w')$

$$J = J(\vartheta|v - w|^{\gamma-1}) = h(\vartheta|v - w|^{\gamma-1}) (1 - (\gamma - 1)\vartheta|v - w|^{\gamma-1} h(\vartheta|v - w|^{\gamma-1})) \quad (5)$$

$J(\vartheta|v - w|^{\gamma-1})$ is nonnegative for all $\gamma < 2$.

$Q_\gamma^{(2)}$ describes a nonlinear friction operator [30]. This operator reads

$$Q_\gamma^{(2)}(f, f) = \frac{\lambda}{2} \frac{\partial}{\partial v} \left[f(v, t) \int_{\mathbb{R}} |v - w|^\gamma (v - w) f(w, t) dw \right], \quad (6)$$

where λ is a relaxation parameter, and $-1 \leq \gamma < 2$.

The Boltzmann operator (2) has been derived in [30] under the hypothesis that, when collections of grains are forced to flow, they may be in a regime in which they interact *via* near-instantaneous collisions [17]. This regime is known as *rapid granular flows* and is reminiscent of the dynamics of molecular gases. For systems composed by a large number of particles whose size is of a few microns, the kinetic description seems to be the natural one (see [1, 2, 15, 22] and the references therein). All the Boltzmann-like equations introduced so far, describe grains as partially inelastic rigid spheres. This choice relies in the physical hypothesis that the grains must be cohesionless, which implies the hard-sphere interaction only, and no long-range forces of any kind.

A fundamental role in dissipative collisions is played by the coefficient of restitution. Experimental works show that the coefficient of restitution depends on the relative velocity. The grains are close to be elastic for binary collisions with a small relative velocity, while they exhibit a certain degree of inelasticity when the relative velocity in the binary collision is high. This dependence has been made explicit in [30] by assuming (4), where the exponent γ characterizes the asymptotics of the restitution coefficient with respect to the relative velocity. This choice is the analogous of the classical case of viscoelastic spheres obtained by the Hertz equation [7, 28].

The variable $\vartheta \in \mathbb{R}^+$ furnishes a measure of the degree of inelasticity of the collision. Purely elastic collisions are obtained for $\vartheta = 0$, while perfectly inelastic collisions correspond to $\vartheta = +\infty$. For any fixed value of the inelasticity parameter ϑ , $\gamma > 1$ corresponds to grains that are close to be elastic for small relative velocity. Of course, $\gamma < 1$ gives the opposite phenomenon, namely the grains are close to be elastic for large relative velocities. We will refer to this case as the case of “anomalous” granular materials. To any collision the associate kernel is $\beta(\vartheta)|v - w|$, which takes into account both the rate function of the rigid spheres interactions, and the probability that collisions with a degree of inelasticity ϑ occur.

For particular choices both of the function β and of the exponent γ , the collision operator (2) reduces to models which are present in the literature. If $\gamma = 1$, and $\beta(\vartheta)$ equals the Dirac delta function $\delta(\vartheta - (1 - q)/q)$, where $q < 1$ is a positive constant, we obtain the Boltzmann equation introduced in [1, 15, 22]. This equation considers the grains like rigid spheres, and has constant coefficient of restitution q .

In [22] McNamara and Young introduced and studied the nonlinear friction operator $Q_1^{(2)}$. The same operator was derived independently some year later in [1] in a suitable scaling limit from a one-dimensional system of N particles colliding inelastically.

The Boltzmann operator and the nonlinear friction operator are connected each other, through the so-called quasi elastic limit procedure [22, 30]. For a sequence of (singular) kernels $\beta(\vartheta)$ concentrating on $\vartheta = 0$, the solution to the Boltzmann equation with restitution coefficient of exponent $\gamma - 1$, converges to the solution of nonlinear friction equation (6), with exponent γ .

Collisions of type (3) determine a loss of kinetic energy, given by

$$v'^2 + w'^2 - (v^2 + w^2) = -\frac{1}{2}(1 - h^2)(v - w)^2. \quad (7)$$

For real systems, we require that the total amount of energy transfer is finite. Since

$$\frac{1}{2}(1 - h^2)(v - w)^2 \leq \frac{\vartheta}{2}|v - w|^{1+\gamma}, \quad (8)$$

this corresponds to impose the condition

$$\int_{\mathbb{R}^+} \beta(\vartheta)\vartheta \, d\vartheta < \infty. \quad (9)$$

Condition (9) is satisfied even if the kernel $\beta(\vartheta)$ has a singularity in $\vartheta = 0$. This singularity is taken in correspondence to the elastic collisions and when these collisions prevail (in a way we shall make precise in the next section) solutions to the Boltzmann equation converge towards solutions of the nonlinear friction equation. This asymptotic problem has many similarities with the grazing collision asymptotics of the Boltzmann equation, studied in recent years. There, a noticeable amount of theoretical results [12, 13, 18, 33, 34] concerned with the limiting process involved, has been obtained.

As in the grazing collision asymptotics, the usual numerical resolution methods for the Boltzmann equation, either of probabilistic type [3, 23] as well as of deterministic type [8, 29], can not be used in this context. These methods are in fact based on *cutting off* the small values of the elasticity coefficient so that β becomes integrable. As the cut-off parameter becomes smaller, the computational cost of these methods increases dramatically so that they are unable to treat in practice the non cut-off problem. Only recently, some numerical efforts to solve non cut-off situations for Boltzmann-like equations have been investigated numerically using Monte Carlo methods in [6, 14] and using finite element methods in [21].

A different approach, based on a Fourier spectral method, has been introduced in [25, 26]. The advantage of this method is that no cut-off assumptions are necessary and that the whole structure of the collisional operator (in practice all the information characterizing the kinetic equation) is contained into a series of kernel modes. The numerical passage to the Landau equation in the grazing collision asymptotics has been studied in this way in [27].

Here, we consider this spectral method in the case of the granular singular one-dimensional Boltzmann collisional operator (2), by showing that the corresponding kernel modes are well-defined for any kernel with singularity satisfying (9). The main result here is that these kernel modes converge (in the quasi elastic limit) to the corresponding kernel modes of the nonlinear friction equation and that the spectral method is uniformly accurate with respect to the parameter describing the quasi elastic limit.

We remark here two interesting characteristics of our analysis. First, on the contrary to what happens for the classical Boltzmann equation, where the spectral method is applicable only in more than one-dimension

of the velocity variable, the method is applicable in dimension one. Second, since dissipative models like (1) with $\varepsilon = 0$ are such that any solution corresponding to an initial distribution which is compactly supported remains compactly supported, the spectral method does not introduce additional approximations on the tails of the distribution.

We briefly discuss the organization of the paper. Section 2 is devoted to recall some mathematical and physical properties of the equations and to present the precise formulation of both the fluid and quasi elastic limits. Section 3 deals with the derivation of the spectral projection of the singular dissipative Boltzmann equation. The quasi elastic limit of the kernel modes is contained in Section 4. Finally numerical experiments and some final considerations are contained in Sections 5 and 6.

2. MATHEMATICAL AND PHYSICAL PRELIMINARIES

In this section, we briefly recall the mathematical and physical background both of the dissipative Boltzmann and nonlinear friction equations. The main part of our analysis will be devoted to the space homogeneous equation

$$\frac{\partial f}{\partial t} = Q_\gamma^{(i)}(f, f)(x, v, t) + \varepsilon \frac{\partial^2 f}{\partial v^2}. \quad (10)$$

It is well-known, in fact, that by the standard splitting algorithm we may consider separately the transport

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0, \quad (11)$$

and the relaxation given by (10).

The long-time behavior of (10) with $i = 2$ and $\gamma > 0$ has been deeply investigated in [10]. In this paper, by means of suitable generalization of logarithmic Sobolev inequalities and mass transportation inequalities, exact rates of convergence to equilibrium have been derived. The case $i = 2$, with $\varepsilon = 0$ has been recently studied in [20] for the whole range of the parameter γ . The pure nonlinear friction case (with $\varepsilon = 0$) is of independent interest since in this case there exist similarity solutions (homogeneous cooling states). The analysis of [20] showed that these equations split naturally into two classes, depending whether their similarity solutions (homogeneous cooling states) extinguish or not in finite time. For both classes, uniqueness of the solution has been shown by proving decay to zero in the Vasershtein metric [19, 32] of any two solutions with the same mass and mean velocity. Furthermore, if the similarity solution extinguishes in finite time (which is the case corresponding to $\gamma < 0$), it has been proven that any other solution with initially bounded support extinguishes in finite time, and explicit upper bounds for the life-time of the solution in terms of the length of the support have been computed. These results are applicable in our context, since they allow to establish rigorous relationships between the case $\varepsilon = 0$ and the case $\varepsilon > 0$. In the sequel, we will briefly recall the notion of weak solution to equation (10), as well as the main results of [20, 30], which constitute the starting point of our approximation method.

Denote by \mathcal{M}_0 the space of all probability measures in \mathbb{R} and by

$$\mathcal{M}_{2p} = \left\{ \mu \in \mathcal{M}_0 : \int_{\mathbb{R}} v^{2p} \mu(dv) < +\infty, p \geq 0 \right\}, \quad (12)$$

the space of all Borel probability measures of finite momentum of order $2p$, equipped with the topology of the weak convergence of the measures. Let us define with $\langle \cdot, \cdot \rangle$ the inner product in $L_1(\mathbb{R})$. By a weak solution of the initial value problem for equation (10), case $i = 1$, corresponding to the initial distribution $\mu_0(v) \in \mathcal{M}_2$ we

shall mean any distribution function $\mu \in C^1(\mathbb{R}_t^+, \mathcal{M}_2)$ satisfying

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \varphi(v) \mu(dv, t) &= \langle Q_\gamma^{(1)}(\mu), \varphi \rangle \\ &= \frac{1}{2} \int_{\mathbb{R}^+} d\vartheta \beta(\vartheta) \int_{\mathbb{R}^2} |v-w| [\varphi(v') + \varphi(w') - \varphi(v) - \varphi(w)] \mu(dv, t) \mu(dw, t) \\ &\quad + \varepsilon \int_{\mathbb{R}} \varphi''(v) \mu(dv, t) \end{aligned} \quad (13)$$

for $t > 0$ and all $\varphi \in C^2(\mathbb{R})$, and such that for all $\varphi \in C^2(\mathbb{R})$

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}} \varphi(v) \mu(dv, t) = \int_{\mathbb{R}} \varphi(v) \mu_0(dv). \quad (14)$$

Analogously, by a weak solution of the initial value problem for equation (10), case $i = 2$, corresponding to the initial distribution $\mu_0(v) \in \mathcal{M}_2$ we shall mean any distribution function $\mu \in C^1(\mathbb{R}_t^+, \mathcal{M}_2)$ satisfying

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \varphi(v) \mu(dv, t) &= -\frac{\lambda}{2} \int_{\mathbb{R}^2} |v-w|^\gamma (v-w) \varphi'(v) \mu(dv, t) \mu(dw, t) + \varepsilon \int_{\mathbb{R}} \varphi''(v) \mu(dv, t) \\ &\quad - \frac{\lambda}{4} \int_{\mathbb{R}^2} |v-w|^\gamma (v-w) [\varphi'(v) - \varphi'(w)] \mu(dv, t) \mu(dw, t) + \varepsilon \int_{\mathbb{R}} \varphi''(v) \mu(dv, t) \end{aligned} \quad (15)$$

for $t > 0$ and all $\varphi \in C^2(\mathbb{R})$, and such that for all $\varphi \in C^2(\mathbb{R})$ (14) is satisfied.

Then, choosing $\varphi(v) = 1, v$ into (13), (15) shows that for $i = 1, 2$ both the mass

$$\rho(x, t) = \int_{\mathbb{R}} f(x, v, t) dv,$$

and the momentum

$$u(x, t) = \int_{\mathbb{R}} v f(x, v, t) dv,$$

are conserved in time. Choosing now $\varphi(v) = v^2$ and $\varepsilon = 0$ shows that the energy

$$E(x, t) = \int_{\mathbb{R}} v^2 f(x, v, t) dv,$$

is dissipated by the purely granular models. Different functionals which decrease along the solution to (10), $i = 2$, were considered in [10]. The steady state when $\varepsilon = 0$ equals the Dirac delta function $\delta(v)$, and corresponds to concentration, whereas when $\varepsilon > 0$ is a smooth function that exhibits strong deviations from classical Gaussian distributions (see [9] and the references therein for more details). On the contrary to the Boltzmann equation [5], the nonlinear friction equation (with $\varepsilon = 0$) has a similarity solution of given mass for all values of the parameter γ [30]. This solution is the combination of two Dirac masses located symmetrically with respect to the origin

$$g_s(v, t) = \frac{1}{2} \delta\left(\frac{v}{\alpha(t)} + \left(\frac{1}{\lambda 2^{\gamma-1}}\right)^{1/\gamma}\right) + \frac{1}{2} \delta\left(\frac{v}{\alpha(t)} - \left(\frac{1}{\lambda 2^{\gamma-1}}\right)^{1/\gamma}\right), \quad (16)$$

where $\alpha(t)$ is given by

$$\alpha(t) = \left(\frac{1}{1 + \gamma t} \right)_+^{\frac{1}{\gamma}}, \quad -1 \leq \gamma \leq 2, \quad \gamma \neq 0.$$

We denoted by f_+ the positive part of the function f . This solution is also called a homogeneous cooling state.

The *quasi elastic asymptotic* of the Boltzmann equation consists in letting the rate function $\beta(\vartheta)$ of the restitution coefficient in (2) concentrate on the singularity in such a way that total amount of energy transfer stays bounded. To be precise, we give the following:

Definition 2.1. Let $(\beta_n(\vartheta))_{n>1}, \vartheta \in \mathbb{R}^+$, be a sequence of kernel functions. We say that (β_n) concentrate on elastic collisions if

- (i) For all n , $\beta_n(\vartheta)$ have a non integrable singularity at the point $\vartheta = 0$.
- (ii) $\beta_n(\vartheta)\vartheta^2 \in L^1(\mathbb{R}^+)$.
- (iii)

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^+} \beta_n(\vartheta)\vartheta \, d\vartheta = \lambda < \infty. \quad (17)$$

The main results of [30] can then be stated as follows:

Theorem 2.2. *Let the nonnegative initial measure $\mu_0 \in \mathcal{M}_p$, where $p > 1 + 2\gamma$ if $\gamma > 1$, and $p > 3$ if $\gamma < 1$. Then, if $-1 \leq \gamma < 2$, and β satisfies condition (9), the initial value problem for (10), $i = 1$ and $\varepsilon = 0$, has a solution that satisfies (13) and (14). This solution conserves mass and momentum, while the energy is non increasing with time. Moreover, let $(\beta_n(\vartheta))_{n>1}, \vartheta \in \mathbb{R}^+$, be a sequence of rate functions concentrating to zero in the sense of Definition 2.1. Then, for all $t > 0$, the solutions $(\mu_n(dv, t))$ to the dissipative Boltzmann equation converge, up to extraction of a subsequence, to a measure $\mu(dv, t)$ that is a weak solution of the nonlinear friction equation (10), $i = 2$ and $\varepsilon = 0$. Finally, the support of the initial measure μ_0 is non increasing with time.*

A direct consequence of the previous theorem is that the energy decays to zero as time goes to infinity. This implies, at least when the energy is finite, that the unique possible steady state is the concentration. For recent results about the steady state with $\varepsilon \neq 0$ we refer to [9].

The recent analysis of [20] shows that the weak solution of (10), $i = 2$, is unique. This analysis can be easily extended to prove that the solution to (15) with $\varepsilon \ll 1$ is close to the solution to (15) with $\varepsilon = 0$, uniformly in time. To this aim, let us recall that on $\mathcal{M}_p, p \geq 2$ one can consider several types of metrics. One of these metrics is known as the Vasershtein distance [19, 32]. Given any measure $\mu \in \mathcal{M}_p$, let us define, for $v \in \mathbb{R}$, $F(v) = \int_{w \leq v} \mu(dw)$. Let $F^{-1}(w) = \inf\{v : F(v) > w\}$ denote the pseudo inverse function of the distribution function $F(v)$. The Vasershtein distance between μ and ν can be rewritten as the L^2 -distance of the pseudo inverse functions F^{-1} and G^{-1} , where $G(v) = \int_{w \leq v} \nu(dw)$

$$d(\mu, \nu) = d(F, G) = \left(\int_0^1 [F^{-1}(\rho) - G^{-1}(\rho)]^2 \, d\rho \right)^{1/2}. \quad (18)$$

The result of [20] can be resumed as follows:

Theorem 2.3. *Let $\mu(dv, t), \nu(dv, t) \in C^1(\mathbb{R}_t^+, \mathcal{M}_2)$ be two solutions to the initial value problem for equation (10), $i = 2$ and $\varepsilon = 0$, corresponding to the initial distributions $\mu_0(dv), \nu_0(dv) \in \mathcal{M}_2$, respectively. Then, if $0 < \gamma < 2$, the Vasershtein distance of $\mu(dv, t)$ and $\nu(dv, t)$ is monotonically decreasing with time, and the following decay holds*

$$\frac{d}{dt} d(\mu(t), \nu(t)) \leq -\frac{\lambda}{2^\gamma} d(\mu(t), \nu(t))^{1+\gamma}. \quad (19)$$

Moreover, if $-1 < \gamma < 0$ and the initial distribution $\mu_0(dv)$ has bounded support, $\text{Supp}(\mu_0) = L < \infty$, the support of the solution decays to zero in finite time, and the following bound holds

$$\text{Supp}(\mu(dv, t)) \leq \left[\text{Supp}(\mu_0)^{|\gamma|} - \frac{|\gamma|}{2|\gamma|} t \right]_+^{\frac{1}{|\gamma|}}. \quad (20)$$

Furthermore, if both the initial distributions $\mu_0(dv), \nu_0(dv) \in \mathcal{M}_2$, have bounded supports, the Vasershtein distance of $\mu(dv, t)$ and $\nu(dv, t)$ decays to zero at finite time, and the following time-decay holds

$$d(\mu(t), \nu(t)) \leq d(\mu_0, \nu_0) \left[1 - \frac{|\gamma|}{(2L)^{|\gamma|}} t \right]_+^{\frac{1}{2}(\frac{1}{|\gamma|} - 1)}, \quad (21)$$

where L denotes the maximum of the supports.

Let us denote with $\mu_\varepsilon(dv, t)$ the solution to (15), and let $\mu(dv, t)$ denote the solution to (15) corresponding to $\varepsilon = 0$. Making use of the bound (19), we obtain:

Corollary 2.1. *Let $\mu_\varepsilon(dv, t), \mu(dv, t) \in C^1(\mathbb{R}_t^+, \mathcal{M}_2)$ be the solutions to the initial value problem for equation (10), $i = 2$, corresponding to the same initial distribution $\mu_0(dv) \in \mathcal{M}_2$. Then, if $0 < \gamma < 2$, the Vasershtein distance of $\mu_\varepsilon(dv, t)$ and $\mu(dv, t)$ is uniformly bounded in time, and the following bound holds*

$$d(\mu_\varepsilon(t), \mu(t)) \leq \left(\frac{2^\gamma}{\lambda} \varepsilon \right)^{2/(2+\gamma)}. \quad (22)$$

Proof. Let us recall that, if $f(v, t)$ satisfies (10), $i = 2$, through a simple computation involving differential calculus the pseudo inverse function $F^{-1}(\rho)$ satisfies

$$\begin{aligned} \frac{\partial F^{-1}(\rho, t)}{\partial t} &= - \frac{1}{f(v, t)} \frac{\partial F^{-1}(v, t)}{\partial t} \Big|_{v=F^{-1}(\rho, t)} \\ &= -\lambda \int_0^1 |F^{-1}(\rho) - F^{-1}(p)|^\gamma (F^{-1}(\rho) - F^{-1}(p)) \, dp - \varepsilon \frac{\partial}{\partial \rho} \left(\frac{1}{\frac{\partial F^{-1}(\rho)}{\partial \rho}} \right). \end{aligned} \quad (23)$$

Hence, if $g(v, t)$ satisfies (10), $i = 2$, with $\varepsilon = 0$, we obtain from (19)

$$\begin{aligned} \frac{d}{dt} d(F, G)^2 &= \frac{d}{dt} \int_0^1 [F^{-1}(\rho) - G^{-1}(\rho)]^2 \, d\rho \\ &= 2 \int_0^1 [F^{-1}(\rho) - G^{-1}(\rho)] \left(\frac{\partial}{\partial t} F^{-1}(\rho) - \frac{\partial}{\partial t} G^{-1}(\rho) \right) \, d\rho \\ &\leq -\frac{\lambda}{2\gamma-1} d(F(t), G(t))^{(2+\gamma)} - 2\varepsilon \int_0^1 [F^{-1}(\rho) - G^{-1}(\rho)] \frac{\partial}{\partial \rho} \left(\frac{1}{\frac{\partial F^{-1}(\rho)}{\partial \rho}} \right) \, d\rho. \end{aligned} \quad (24)$$

Integrating by parts the last integral we get

$$\begin{aligned} - \int_0^1 [F^{-1}(\rho) - G^{-1}(\rho)] \frac{\partial}{\partial \rho} \left(\frac{1}{\frac{\partial F^{-1}(\rho)}{\partial \rho}} \right) \, d\rho &= \int_0^1 \frac{\partial}{\partial \rho} ([F^{-1}(\rho) - G^{-1}(\rho)]) \frac{1}{\frac{\partial F^{-1}(\rho)}{\partial \rho}} \, d\rho \\ &= \int_0^1 \left(1 - \frac{\frac{\partial G^{-1}(\rho)}{\partial \rho}}{\frac{\partial F^{-1}(\rho)}{\partial \rho}} \right) \, d\rho \leq 1. \end{aligned} \quad (25)$$

Indeed, the boundary term is equal to zero, due to the fact that

$$\left(\frac{\partial F^{-1}(\rho)}{\partial \rho}\right)^{-1} = f(F^{-1}(\rho)) = 0 \quad \text{if } \rho = 0, 1. \quad (26)$$

Moreover we deduce from (26) the non negativity of the derivatives of $F^{-1}(\rho)$ and $G^{-1}(\rho)$. Thus we showed that the distance $d(F, G)$ satisfies the differential inequality

$$\frac{d}{dt}d(F, G)^2 \leq -\frac{\lambda}{2^{\gamma-1}}d(F(t), G(t))^{2+\gamma} + 2\varepsilon. \quad (27)$$

Let D solve

$$-\frac{\lambda}{2^{\gamma-1}}D^{2+\gamma} + 2\varepsilon = 0.$$

Then, if $d(F_0, G_0) \leq D$, we conclude from (27) that the maximum value for $d(F, G)$ can not overcome D . This concludes the proof. \square

At present, we are not able to prove analogous result for the Boltzmann equation, even if we have numerical evidence of it. Corollary 2.1 is at the basis of the spectral scheme we will introduce in the next section. It guarantees that we can treat numerically convergence to equilibrium (which in this case is the Dirac delta function), simply by adding a small diffusion. Grace to the previous result, we have a control of the error (in Vasershtein metric) uniformly in time. Using previous results on the equivalence of this metric with other more usual metrics [31], we can easily obtain a control of the error in terms of Fourier transforms.

3. THE SPECTRAL SCHEME

In this section we derive the Fourier spectral method for the non cut-off Boltzmann equation following [25, 26]. To simplify notations, we will consider here only absolutely continuous measures μ , so that we can write, for $f \in L_1^+(\mathbb{R})$, $\mu(dv, t) = f(v, t)dv$. The general case can be treated in the same way.

Let us consider the weak form of the equation (10)

$$\begin{aligned} \langle \partial_t f, \varphi \rangle &= \langle Q_\gamma^{(1)}(f, f), \varphi \rangle + \varepsilon \langle \partial_{vv} f, \varphi \rangle \\ &= \int_{\mathbb{R}^+} \beta(\vartheta) \int_{\mathbb{R}^2} |v - w| [\varphi(v') - \varphi(v)] f(v, t) f(w, t) dw d\vartheta dv + \varepsilon \int_{\mathbb{R}} \varphi''(v) f(v) dv \end{aligned} \quad (28)$$

for $t > 0$ and all test functions φ .

A simple change of variables permits to write $\langle Q_\gamma^{(1)}(f, f), \varphi \rangle$ in the form

$$\langle Q_\gamma^{(1)}(f, f), \varphi \rangle = \int_{\mathbb{R}^+} \beta(\vartheta) \int_{\mathbb{R}^2} |q| [\varphi(v + q^+) - \varphi(v)] f(v, t) f(v + q, t) dq d\vartheta dv \quad (29)$$

where $q = w - v$ is the relative velocity, and the vectors q^+ and q^- that define the collisional velocities are given by

$$q^+ = \frac{q}{2} \left(1 + \frac{1}{1 + \vartheta |q|^{\gamma-1}} \right), \quad q^- = \frac{q}{2} \left(1 - \frac{1}{1 + \vartheta |q|^{\gamma-1}} \right). \quad (30)$$

Note that the possibility to integrate the collision operator over the relative velocity is essential in the derivation of the method. We consider now an initial density function $f_0(v)$ with compact support $\text{Supp}(f_0(v)) \subset [-R, R]$.

Then for $\varepsilon = 0$ the solution to (10) has compact support for all later times. In fact by (3), if $|v|, |w| \leq R$

$$|v'| \leq \frac{1}{2}|v|(1+h) + \frac{1}{2}|w|(1-h) \leq R,$$

and similarly $|w'| \leq R$. In addition $|q|, |q^+|, |q^-| \leq 2R$, thus we have the following:

Lemma 3.1. *If the function $f(v, t)$ is such that $\text{Supp}(f(v, t)) \subset [-R, R]$ then*

- (i) $\text{Supp}(Q_\gamma^{(1)}(f, f)(v, t)) \subset [-R, R]$,
- (ii)

$$\langle Q_\gamma^{(1)}(f, f), \varphi \rangle = \int_{\mathbb{R}^+} \beta(\vartheta) \int_{|v| \leq R} \int_{|q| \leq 2R} |q| [\varphi(v + q^+) - \varphi(v)] f(v, t) f(v + q, t) dq d\vartheta dv \quad (31)$$

with $v + q^+, v + q \in [-3R, 3R]$.

Remark 3.2. The previous results shows that for compactly supported functions f in order to evaluate $Q_\gamma^{(1)}(f, f)$ by a spectral method without aliasing error we can consider the density function f restricted on the interval $[-2R, 2R]$, and extend it by periodicity to a periodic function on $[-2R, 2R]$. Clearly whenever $\varepsilon \neq 0$ aliasing errors are unavoidable for $t > 0$.

To simplify the notation let us take $2R = \pi$. The approximate function f_N is represented as the truncated Fourier series

$$f_N(v) = \sum_{k=-N}^N \hat{f}_k e^{ikv}, \quad (32)$$

$$\hat{f}_k = \frac{1}{(2\pi)} \int_{|v| \leq \pi} f(v) e^{-ikv} dv. \quad (33)$$

Hence, taking $f = f_N$ and $\varphi = e^{-ikv}$ for $k = -N, \dots, N$ we have

$$\int_{|v| \leq \pi} [\partial_t f_N - Q_\gamma(f_N, f_N) - \varepsilon \partial_{vv} f_N] e^{-ikv} dv = 0. \quad (34)$$

By substituting expression (32) in (34) we get a set of ordinary differential equations for the Fourier coefficients [11]

$$\partial_t \hat{f}_k = \sum_{\substack{l, m=-N \\ l+m=k}}^N \hat{f}_l \hat{f}_m \hat{\beta}(l, m) - \varepsilon k^2 \hat{f}_k, \quad k = -N, \dots, N, \quad (35)$$

where the *Boltzmann kernel modes* $\hat{\beta}(l, m)$ are given by

$$\hat{\beta}(l, m) = \int_{\mathbb{R}^+} d\vartheta \beta(\vartheta) \int_{|q| \leq \pi} |q| [\cos(lq^+ - mq^-) - \cos(lq)] dq. \quad (36)$$

In fact, by evaluating (31) for $\varphi = e^{-ikv}$ and $f = f_N$, one obtains

$$\hat{\beta}(l, m) = \int_{\mathbb{R}^+} d\vartheta \beta(\vartheta) \int_{|q| \leq \pi} |q| [e^{-ikq^+} - 1] e^{ilq} dq \quad (37)$$

and (36) follows by using the parities of the trigonometric functions. Note that (36) is a real quantity completely independent on the argument v , depending just on the particular kernel structure.

In practice all the informations characterizing the kinetic equation are now contained in the kernel modes. Clearly, these quantities can be computed in advance and then stored in a two dimensional matrix of size $2N$.

In a similar way we can derive a symmetrized form for the kernel modes

$$\hat{\beta}(l, m) = \frac{1}{2} \int_{\mathbb{R}^+} d\vartheta \beta(\vartheta) \int_{|q| \leq \pi} |q| [\cos(lq^+ - mq^-) + \cos(mq^+ - lq^-) - \cos(mq) - \cos(lq)]. \quad (38)$$

Thanks to (38) the effective number of kernel modes that need to be computed and stored for the implementation of the method is reduced in practice since

$$\hat{\beta}(l, m) = \hat{\beta}(m, l) = \hat{\beta}(-l, -m). \quad (39)$$

Note that in the case $\gamma = 1$, expression (36) for $\hat{\beta}(l, m)$ simplifies as

$$\hat{\beta}(l, m) = \pi^2 \int_{\mathbb{R}^+} d\vartheta \beta(\vartheta) \left[2 \operatorname{Sinc}(p\pi) - \operatorname{Sinc}\left(\frac{p\pi}{2}\right)^2 - 2 \operatorname{Sinc}(l\pi) + \operatorname{Sinc}\left(\frac{l\pi}{2}\right)^2 \right],$$

where $p = ((l - m) + (l + m)h)/2$ and $\operatorname{Sinc}(x) = \sin(x)/x$.

Finally we can rewrite scheme (35) as

$$\partial_t \hat{f}_k = \sum_{m=-N}^N \hat{f}_{k-m} \hat{f}_m \hat{\beta}(l, m) - \varepsilon k^2 \hat{f}_k, \quad k = -N, \dots, N. \quad (40)$$

In the previous expression we assume that the Fourier coefficients are extended to zero for $|k| > N$. The evaluation of (40) requires exactly $O(N^2)$ operations which is smaller than the cost of a standard method based on N parameters for f in the velocity space since we gain the integration over the variable ϑ . Thus the straightforward evaluation of (40) is slightly less expensive than a usual discrete-velocity algorithm.

Remark 3.3. By construction the spectral method preserves the mass, whereas momentum and energy are approximated with spectral accuracy if the solution is smooth. We refer to [26] for a more detailed discussion on these topics.

4. NUMERICAL QUASI-ELASTIC LIMIT

By Definition 2.1, the quasi elastic limit is obtained by letting the collisions to be elastic, that is $h = 1$. To this end, consider that, by Taylor expansion

$$\cos(lq^+ - mq^-) - \cos(lq) = (l + m)q^- \sin(lq) - \frac{(l + m)^2}{2} (q^-)^2 \cos(lq) - \frac{(l + m)^3}{6} (q^-)^3 \sin[lq + \mu(l + m)q^-] \quad (41)$$

for some $0 \leq \mu \leq 1$. Analogous expression for the other difference in (38). Next, by expanding q^- in powers of ϑ , we get,

$$q^-(\vartheta) = \frac{q}{2} \left[\vartheta |q|^{\gamma-1} - (\vartheta |q|^{\gamma-1})^2 + (\vartheta |q|^{\gamma-1})^3 \right] + O(\vartheta^4). \quad (42)$$

Hence we can write

$$\hat{\beta}(l, m) = \hat{\beta}_0(l, m) \int_{\mathbb{R}^+} d\vartheta \vartheta \beta(\vartheta) + \hat{\beta}_1(l, m) \int_{\mathbb{R}^+} d\vartheta \beta(\vartheta) \vartheta^2 + \hat{\beta}_2(l, m) \int_{\mathbb{R}^+} d\vartheta \beta(\vartheta) \vartheta^3 + \dots, \quad (43)$$

where we defined

$$\hat{\beta}_0(l, m) = \frac{l+m}{4} \int_{|q| \leq \pi} |q|^\gamma q [\sin(lq) + \sin(mq)] dq, \quad (44)$$

$$\begin{aligned} \hat{\beta}_1(l, m) &= -\frac{l+m}{4} \int_{|q| \leq \pi} |q|^{2\gamma-1} q [\sin(lq) + \sin(mq)] dq \\ &\quad - \frac{(l+m)^2}{16} \int_{|q| \leq \pi} |q|^{2\gamma+1} [\cos(lq) + \cos(mq)] dq, \end{aligned} \quad (45)$$

$$\begin{aligned} \hat{\beta}_2(l, m) &= \frac{l+m}{4} \int_{|q| \leq \pi} |q|^{3\gamma-2} q [\sin(lq) + \sin(mq)] dq \\ &\quad + \frac{(l+m)^2}{8} \int_{|q| \leq \pi} |q|^{3\gamma} [\cos(lq) + \cos(mq)] dq \\ &\quad - \frac{(l+m)^3}{96} \int_{|q| \leq \pi} |q|^{3\gamma} q [\sin(lq) + \sin(mq)] dq, \end{aligned} \quad (46)$$

and in a similar way we can compute the higher order terms. Let us remark here that the coefficients $\hat{\beta}_1(l, m)$ represent, in the quasi elastic limit procedure, the (second order) diffusion approximation to the Boltzmann equation.

Let us now apply the same procedure to the nonlinear friction equation (10), $i = 2$. Changing variable into (15) we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} \varphi(v) f(v, t) dv = \frac{\lambda}{4} \int_{\mathbb{R}^2} |q|^\gamma q [\varphi'(v) - \varphi'(v+q)] f(v, t) f(v+q, t) dv dq. \quad (47)$$

As before, if the initial density f_0 is such that $\text{Supp}(f_0(v)) \subset [-\pi, \pi]$, (47) reduces to

$$\frac{d}{dt} \int_{\mathbb{R}} \varphi(v) f(v, t) dv = \frac{\lambda}{4} \int_{|v| \leq \pi} dv \int_{|q| \leq \pi} |q|^\gamma q [\varphi'(v) - \varphi'(v+q)] f(v, t) f(v+q, t) dq, \quad (48)$$

with $v, v+q \in [-\pi, \pi]$. Evaluating (48) for $\varphi = e^{-ikv}$ and $f = f_N$, we get

$$\begin{aligned} \int_{|v| \leq \pi} dv \int_{|q| \leq \pi} |q|^\gamma q \left[\frac{d}{dv} e^{-ikv} - \frac{d}{dv} e^{-ik(v+q)} \right] f_N(v, t) f_N(v+q, t) dq = \\ -ik \int_{|v| \leq \pi} e^{-ikv} dv \int_{|q| \leq \pi} |q|^\gamma q [1 - e^{-ikq}] f_N(v, t) f_N(v+q, t) dq. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \partial_t \hat{f}_k &= -i \frac{k\lambda}{4} \sum_{\substack{l+m=k \\ l, m=-N}}^N \hat{f}_l \hat{f}_m \int_{|q| \leq \pi} |q|^\gamma q [e^{imq} - e^{-ilq}] dq \\ &= \frac{k\lambda}{4} \sum_{\substack{l+m=k \\ l, m=-N}}^N \hat{f}_l \hat{f}_m \int_{|q| \leq \pi} |q|^\gamma q [\sin(lq) + \sin(mq)] dq. \end{aligned} \quad (49)$$

Finally, by substituting expression (44) into (49), from (48) we get for the nonlinear friction equation the set of ordinary differential equations for the Fourier coefficients

$$\partial_t \hat{f}_k = \sum_{\substack{l+m=k \\ l,m=-N}}^N \hat{f}_l \hat{f}_m \hat{\beta}_\infty(l, m), \quad k = -N, \dots, N \quad (50)$$

where, in (50)

$$\hat{\beta}_\infty(l, m) = \lambda \hat{\beta}_0(l, m). \quad (51)$$

Let us consider a sequence $(\beta_n(\vartheta))_{n>1}$, of rate functions concentrating to zero in the sense of Definition 2.1, and let us define $\hat{\beta}_n(l, m)$ to be the kernel modes of the Boltzmann equation with kernel β_n , that is

$$\hat{\beta}_n(l, m) = \hat{\beta}_0(l, m) \int_{\mathbb{R}^+} d\vartheta \vartheta \beta_n(\vartheta) + \hat{\beta}_1(l, m) \int_{\mathbb{R}^+} d\vartheta \vartheta^2 \beta_n(\vartheta) + \hat{\beta}_2(l, m) \int_{\mathbb{R}^+} d\vartheta \vartheta^3 \beta_n(\vartheta) + \dots \quad (52)$$

Passing to the limit in (52), we obtain:

Theorem 4.1. *Let $\hat{\beta}_n(l, m)$ be the kernel modes of the Boltzmann equation with kernel $\beta_n(\vartheta)$. If $(\beta_n(\vartheta))_{n>1}$ is a sequence of kernel functions concentrating to zero in the sense of Definition 2.1 we have*

$$\lim_{n \rightarrow \infty} \hat{\beta}_n(l, m) = \hat{\beta}_\infty(l, m),$$

where $\hat{\beta}_\infty(l, m)$ are the kernel modes of the nonlinear friction equation.

4.1. Fast algorithms

The final expression (50) is much simpler than the starting expression (35). It is possible to show [24, 27] that this simplification allows to compute the resulting final algorithm in only $O(N \log N)$ instead of $O(N^2)$ operations. Similarly, also the approximation defined by (52), that can be used to study numerically the behavior of the non cut-off Boltzmann equation when the collisions become inelastic, can be computed with only $O(N \log N)$ operations.

We describe the method for the approximation defined by (52) which includes as a particular case the limiting friction equation.

To this aim we observe that expression (52) admits the decomposition

$$\hat{\beta}(l, m) = \sum_{h=0}^M \int_{\mathbb{R}^+} d\vartheta \vartheta^{h+1} \beta_n(\vartheta) \left\{ \sum_{j=0}^h a_j(l+m) [b_j(l) + b_j(m)] \right\}, \quad (53)$$

where $M = 0, 1, 2, \dots$

Whenever this is possible the scheme can be evaluated through $(M+1)(M+2)/2$ convolution sums for $k = -N, \dots, N$

$$\partial_t \hat{f}_k = \sum_{h=0}^M \int_{\mathbb{R}^+} d\vartheta \vartheta^{h+1} \beta_n(\vartheta) \left\{ \sum_{j=0}^h a_j(k) \sum_{m=-N}^N [\hat{f}_{k-m} b_j(k-m) \hat{f}_m + \hat{f}_m b_j(m) \hat{f}_{k-m}] \right\},$$

and by standard transform methods each of these sums requires only $O(N \log_2 N)$ operations. We refer the reader to [11] for further details.

Remark 4.2. In the original variables the expansion characterized by (52) corresponds to introduce an expansion of $Q_\gamma^{(1)}(f, f)$ that involves derivatives of increasing order in the v variable (which in the spectral representation are given by powers of $k = l + m$). Thus truncating expression (52) may originate unstable PDEs approximations of the Boltzmann equation. Numerical experiments indicate that instabilities occur for $M = 2$. Thus in all our numerical tests we will consider only the case $M = 1$. A different kind of expansion with better stability property has been studied in [24].

5. NUMERICAL RESULTS

In this section we present some numerical results in the space homogeneous case which confirm the previous analysis. Since we are interested only in checking the accuracy of our schemes with respect to the variable v the error in t has been neglected. This can be achieved either using very small time steps or a suitably high order time discretization. In all our computations we used a fourth order Runge-Kutta method. Note that we do not advocate the use of such time integrator to solve these equations in practical applications. We address the interested reader to [16] and the references therein for some recent advances on this topic.

We have considered the sequence of kernel functions

$$\beta_n(\vartheta) = \frac{ne^{-n\vartheta}}{\vartheta}, \quad \vartheta \in [0, \infty]. \quad (54)$$

Thus we obtain in (52)

$$\int_{\mathbb{R}^+} \beta_n(\vartheta) \vartheta^k d\vartheta = \frac{(k-1)!}{n^{k-1}}, \quad k = 1, 2, \dots$$

and hence $\lambda = 1$.

The initial data is the sum of two Gaussian distribution

$$f(v, 0) = \exp(-(2v + 2)^2) + \exp(-(2v - 2)^2),$$

with $v \in [-\pi, \pi]$. The final integration time is $t_f = 8$.

We denote with *SM* the spectral method given by (35) and (36) taking $\beta(\vartheta) = \beta_1(\vartheta)$ in (54), and with *FM*₀ and *FM*₁ the fast spectral methods characterized by (35) and (52) truncated to the term $\hat{\beta}_0$ and $\hat{\beta}_1$ respectively. Clearly *FM*₁ \rightarrow *FM*₀ in the grazing collision limit $n \rightarrow \infty$, where *FM*₀ does not depend on n and corresponds to scheme (50) for the limiting friction equation. Since the solution is symmetric in v all the spectral methods are conservative in both mass and mean velocity.

5.1. Test 1

In the first test case we report the numerical results obtained for $\gamma = 1$ (*i.e.* the restitution coefficient does not depend on the relative velocity). This test is used to check the spectral accuracy of the method and to study the different behavior with respect to ε .

In Figure 1 we report the time evolution of the distribution function for $\varepsilon = 0$ and $\varepsilon = 0.1$ with $N = 64$ modes.

The decay of the energy in time together with the final solution at $t_f = 8$ is reported in Figure 2. The different concentration of the distribution function is evident. For larger times in the case $\varepsilon = 0$ the spectral method starts to oscillate since the equilibrium state is $\delta(v)$ and a very fine grid is required to approximate the spike at $v = 0$. At variance, one should choose $\varepsilon > 0$ as a numerical viscosity in order to eliminate the spurious oscillations and to approximate the spike through a smooth function.

In Table 1 we report the relative error norms of the error at time $t = 0.5$ obtained with the spectral method for $N = 8$, $N = 16$ and $N = 32$. The ‘exact’ reference solution has been computed with $N = 64$. The

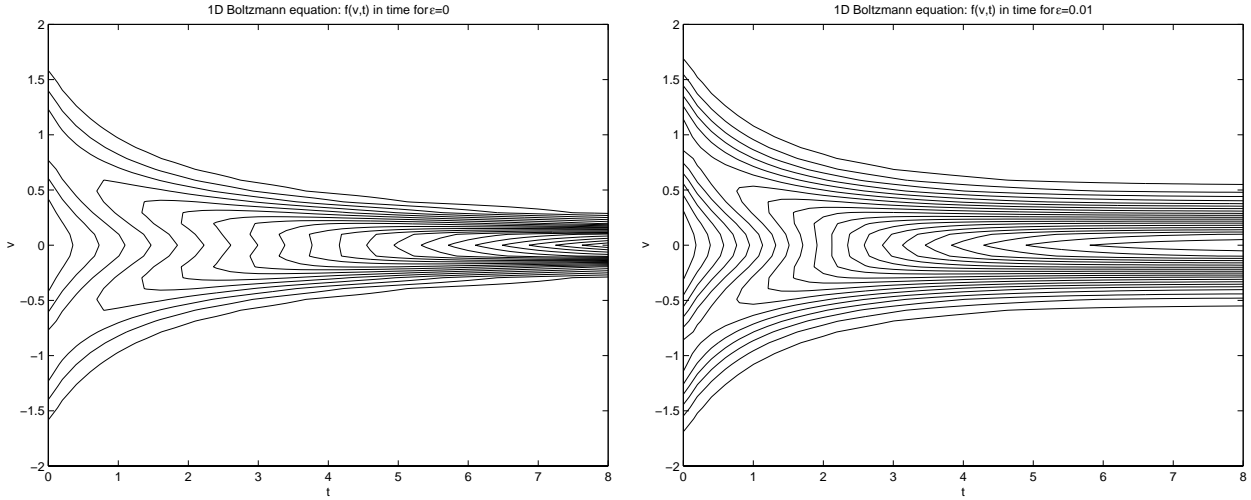


FIGURE 1. Boltzmann equation: time evolution of the distribution function for $N = 64$ and $\varepsilon = 0$ (left), $\varepsilon = 0.01$ (right).

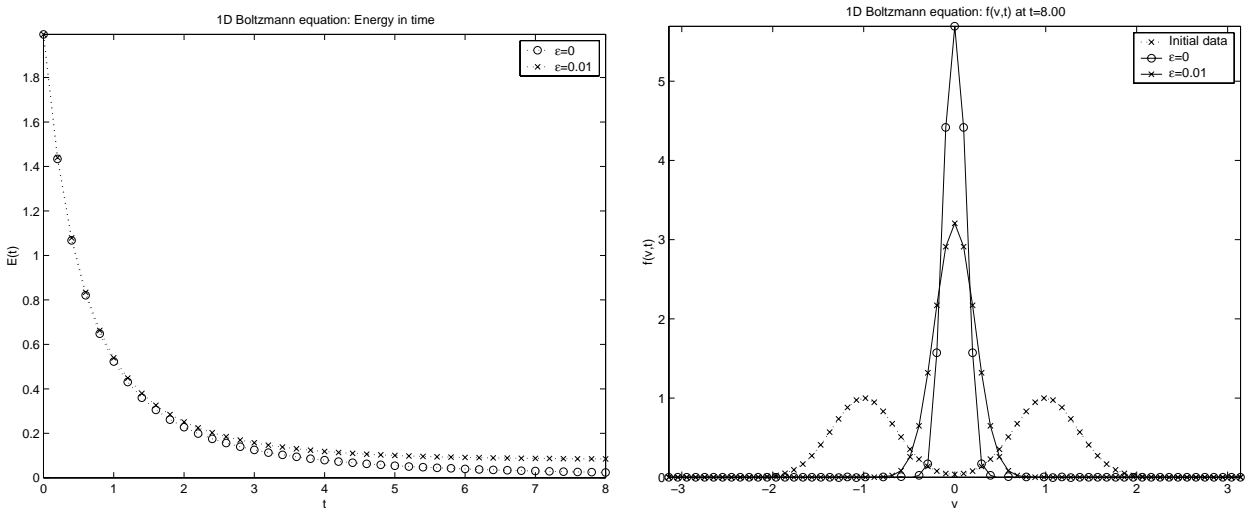


FIGURE 2. Boltzmann equation: time evolution of the energy (left) and distribution function at $t = 8$ (right) for $N = 64$ and $\varepsilon = 0$ (\times), $\varepsilon = 0.01$ (\circ).

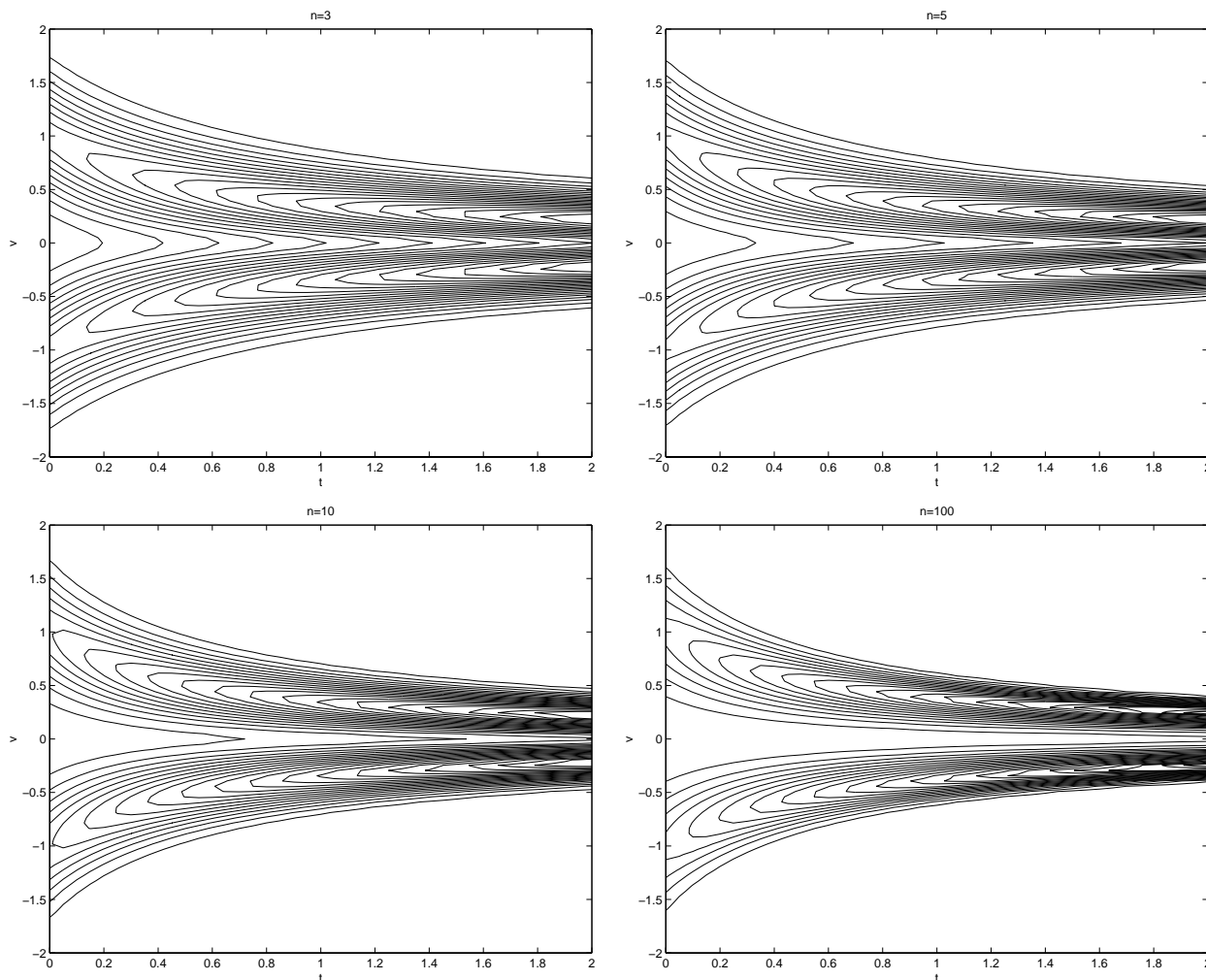
convergence rate in the L_1 -norm is about 11.7 when we pass from 8 to 16 modes. The error with $N = 32$ modes is of order 10^{-8} which was the tolerance used in the computation of the kernel modes. In the last column the total CPU time in seconds for a single evaluation of the collision integral is reported. Similar convergence rates are obtained also for $\varepsilon \neq 0$.

5.2. Test 2

Next we consider the variable restitution coefficient case taking $\gamma = 6/5$ that corresponds to the variable coefficient restitution case. This exponent comes out considering in the Boltzmann equation a granular gas composed by viscoelastic spheres [28]. To this exponent corresponds a time-decay of the temperature of order $t^{-5/3}$. This test is used to study the numerical grazing collision limit passage.

TABLE 1. Relative error norms at time $t = 0.5$ and CPU-time for scheme SM with $\gamma = 1$.

SM	L_1	L_∞	L_2	CPU time
$N = 8$	3.988e-003	2.559e-003	3.017e-003	0.012 s
$N = 16$	1.153e-006	4.678e-007	7.922e-007	0.027 s
$N = 32$	8.217e-008	8.547e-008	7.745e-008	0.084 s

FIGURE 3. Inelastic collision limit: time evolution of the distribution function for $n = 3, 5, 10, 100$ with $N = 64$ and $\gamma = 6/5$ obtained with scheme SM .

To this aim we consider scheme SM and scheme FM_1 for different values of n . Note however that the coefficients for scheme SM need to be recomputed for each value of n .

The sequence of images in Figures 3 and 4 shows the grazing limit process for $n = 3, 5, 10, 100$ at time $t = 2$ and with $N = 64$. During the simulations the fast solver has been approximately 26 times faster than the standard spectral scheme.

The different concentration behavior for various n appears clearly. As expected good agreement between the two schemes is observed for large values of n ($n = 10, 100$) whereas for small n ($n = 3, 5$) deviations between

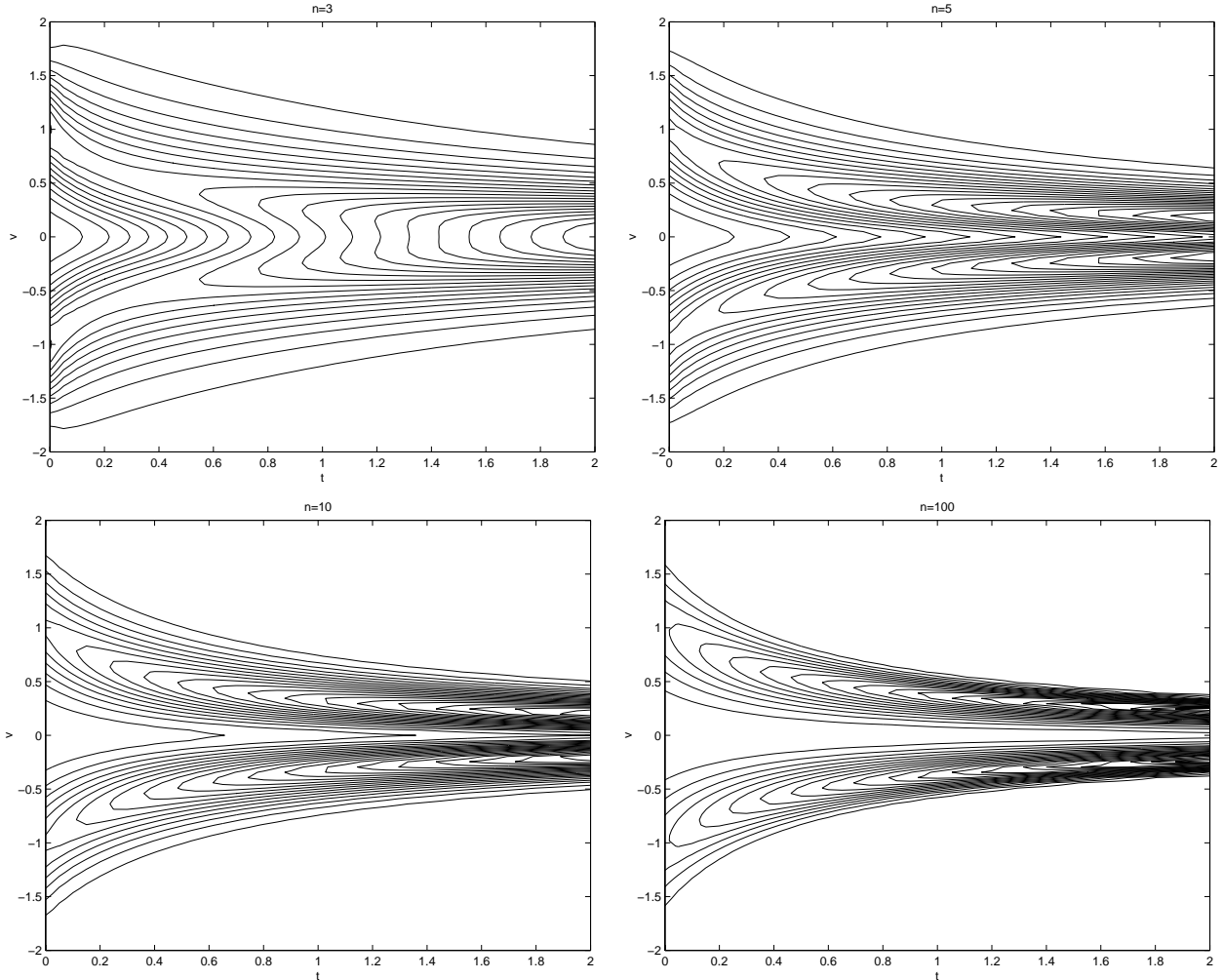


FIGURE 4. Inelastic collision limit: time evolution of the distribution function for $n = 3, 5, 10, 100$ with $N = 64$ and $\gamma = 6/5$ obtained with the fast scheme FM_1 .

SM and FM_1 are observed. This confirms the validity of scheme FM_1 for computation regimes close to the quasi-elastic limit.

In Table 2 we report the relative L_1 error norms and convergence rates at $t = 0.25$ for scheme FM_1 . We omit for brevity the accuracy table of scheme SM which qualitatively confirms the results of Table 1. Spectral accuracy independently of n can be observed, although there is a slight deterioration of the error for large values of n . Aliasing occur for $n = 3$ and $n = 5$ due to the diffusive nature of the approximation and this is the main reason of error when we pass from $N = 32$ to $N = 64$ modes. Here the coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$ have been computed with a tolerance of 10^{-10} . Note that for $N = 8$ the method is about 4 times faster than the corresponding spectral scheme. The gain increases to a factor 7 for $N = 16$ and a factor 14 for $N = 32$.

5.3. Test 3

The last test case consider the limiting friction equation with $\gamma = 1, -1$. Finite time extinction of the solution has been proven in the case $\gamma < 0$. Thus we consider the long time behavior for scheme FM_0 using a fine grid with $N = 1024$ and a small numerical viscosity $\varepsilon = 10^{-3}$. The solution at time $t = 2$ together with the energy

TABLE 2. Relative L_1 error norms at time $t = 0.5$ and CPU-time for scheme FM_1 with $\gamma = 6/5$.

FM_1	$n = 3$	$n = 5$	$n = 10$	$n = 100$	CPU time
$N = 8$	1.656e-002	6.611e-002	8.368e-002	2.663e-001	0.003 s
$N = 16$	6.331e-006	1.098e-003	1.276e-003	1.785e-002	0.004 s
$N = 32$	3.733e-009	3.614e-008	9.062e-007	3.516e-005	0.006 s
$N = 64$	3.349e-009	3.633e-008	1.674e-010	2.602e-010	0.012 s

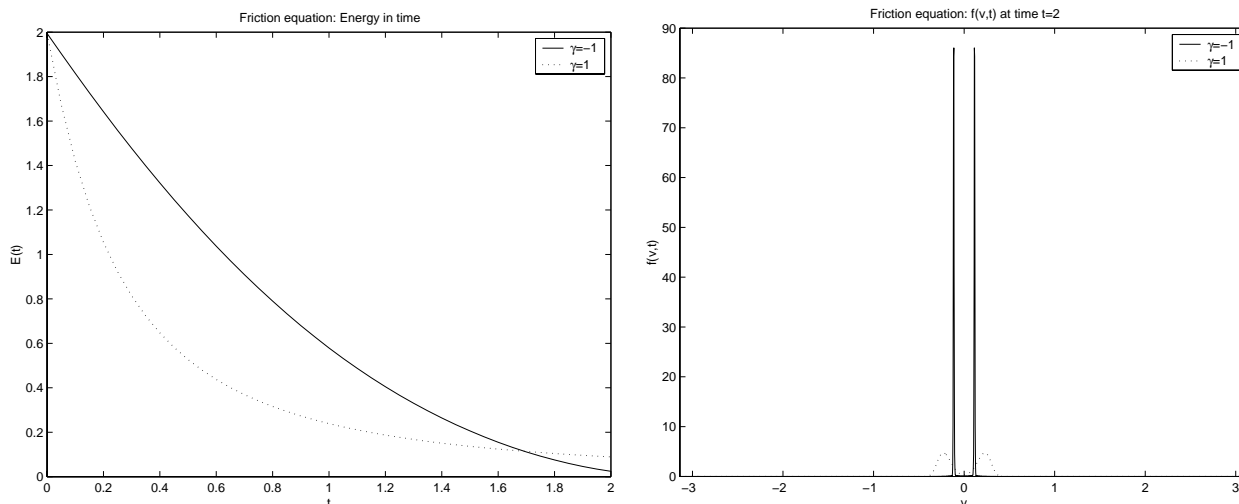


FIGURE 5. Friction equation: time evolution of the energy (left) and distribution function for the friction equation at $t = 2$ (right) for $N = 1024$ and $\varepsilon = 10^{-3}$ with $\gamma = 1$ (dotted line), $\gamma = -1$ (continuous line).

plot in the same time interval is shown in Figure 5. The final energy is of order 10^{-1} for $\gamma = 1$ and 10^{-2} for $\gamma = -1$.

6. CONCLUSIONS

In this paper we have studied Fourier spectral methods that allow a numerical passage from a Boltzmann equation for driven granular media to the friction equation in the quasi-elastic collision limit. This permits to show that in the small quasi-elastic limit the methods provide a consistent discretization of the limiting friction equation. Approximate formulas that give intermediate asymptotics of the Boltzmann kernel modes and that can be evaluated with fast algorithms have been also derived. Finally, uniform spectral accuracy of the methods with respect to the grazing collision parameter has been given numerically and numerical evidence of the finite time extinction phenomena has been obtained.

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