

# Spectral Methods of Automorphic Forms

## Second Edition

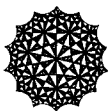
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Graduate Studies  
in Mathematics

Volume 53



American Mathematical Society  
Providence, Rhode Island



Revista Matemática Iberoamericana

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