SPECTRAL NORM OF CIRCULANT TYPE MATRICES

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Abstract

We first discuss the convergence in probability and in distribution of the spectral norm of scaled Toeplitz, circulant, reverse circulant, symmetric circulant and a class of k-circulant matrices when the input sequence is independent and identically distributed with finite moments of suitable order and the dimension of the matrix tends to ∞.

When the input sequence is a stationary two sided moving average process of infinite order, it is difficult to derive the limiting distribution of the spectral norm but if the eigenvalues are scaled by the spectral density then the limits of the maximum of modulus of these scaled eigenvalues can be derived in most of the cases.

Keywords Large dimensional random matrix, eigenvalues, Toeplitz matrix, Hankel matrix, circulant matrix, symmetric circulant matrix, reverse circulant matrix, k circulant matrix, spectral norm, moving average process, spectral density, normal approximation.

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1 Introduction

Matrices with suitable patterned random inputs where the dimension tends to infinity, are known as large dimensional random matrices. The sequence $\{x_i\}$ which is used to build these matrices will be called the *input sequence*. Such patterned matrices have been objects of great interest and many different types of results are known for them. In this article we focus on the (symmetric) Toeplitz, (symmetric) Hankel, circulant, reverse circulant and symmetric circulant matrices.

Nonrandom Toeplitz matrices and the corresponding Toeplitz operators are of course well studied objects in mathematics. Circulant matrices play a crucial role in the study of large dimensional Toeplitz matrices with nonrandom input. See, for example, Grenander and Szegő (1984). The k-circulant matrix and its block versions arise in many contexts and have been considered in many works in mathematics, statistics and related areas. As examples, we mention the book by Davis (1979) and the articles by Pollock (2002) and Zhou (1996). Here is a quick description of the above matrices. Let $\{x_0, x_1, \ldots\}$ be a sequence of real random variables. Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{Z}_{\geq 0}$ the set of all nonnegative integers.

1. Toeplitz matrix. The $n \times n$ random (symmetric) Toeplitz matrix T_n with input $\{x_i\}$ is the matrix whose (i, j)-th entry is $x_{|i-j|}$.

2. Hankel matrix. Similarly, the (symmetric) Hankel matrix H_n with input $\{x_i\}$ is the matrix whose (i, j)-th entry is x_{i+j-2}.

3. Reverse circulant matrix. This is also a symmetric matrix (denoted by RC_n) where the (i, j)-th element of the matrix is $x_{(i+j-2) mod\ n}$.

4. Circulant matrix. The $n \times n$ circulant matrix C_n with input $\{x_i\}$ is the matrix whose (i, j)-th entry is $x_{(j-i+n) mod\ n}$. This is not a symmetric matrix.

5. Symmetric circulant matrix. The symmetric version of the usual circulant matrix (denoted by SC_n) may be defined with (i, j)-th element of the matrix given by $x_{n/2-|n/2-|i-j||}$.

6. k-Circulant matrix. For positive integers k and n, define the $n \times n$ square matrix

$$A_{k,n} = \begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_{n-k} & x_{n-k+1} & x_1 & \ldots & x_{n-k-2} & x_{n-k-1} \\ x_{n-2k} & x_{n-2k+1} & x_0 & \ldots & x_{n-2k-2} & x_{n-2k-1} \\ & & & \vdots & & \end{bmatrix}_{n \times n}.$$

We emphasize that all subscripts appearing in the entries above are calculated modulo n. The first row of $A_{k,n}$ is $(x_0, x_1, x_2, ..., x_{n-1})$ and for $1 \leq j < n - 1$, its $(j + 1)$-th row is obtained by giving its j-th row a right circular shift by k positions (equivalently, k mod n positions). This is a generalization of the usual circulant matrix when $k = 1$. It may be noted that the reverse circulant is a special case of the k-circulant when we let $k = n - 1$.

Recent focus has been to understand the behaviour of the eigenvalues when the input sequence is random and the dimension of the matrix tends to ∞. For example, the limiting spectral distributions of such matrices has been dealt with in Bryc, Dembo and Jiang (2006), Hammond and Miller (2005), Bose and Sen (2008) and Bose, Hazra and Saha (2009). A few results are also available for the spectral norm. See for example Silverstein (1996), Adamczak (2008), Bose and Sen (2007), Meckes (2007) and Bryc and Sethuraman (2009).

The spectral norm $\|A\|$ of a matrix A with complex entries is the square root of the largest eigenvalue of the positive semidefinite matrix A^*A:

$$\|A\| = \sqrt{\lambda_{max}(A^*A)}$$

where A^* denotes the conjugate transpose of A. Therefore if A is an $n \times n$ real symmetric matrix or A is a normal matrix, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$\|A\| = \max_{1\le i\le n} |\lambda_i|.$$

In this article we study the spectral norm and related objects for the above matrices. In particular, we study the distributional convergence of the spectral norm and of the maximum and minimum eigenvalues when the input sequence is independent and identically distributed (i.i.d.). We also study an appropriately modified version of the spectral norm when the input sequence is a linear process and establish a few interesting results.

The outline of the paper is as follows. In Section 2 we review some known results and state a few new results on the spectral norm of random matrices with i.i.d. inputs. In Theorems 3 and 5 we show that for RC_n and SC_n respectively, the limit distribution is Normal or Gumbel according as the mean μ is nonzero or zero. In Theorem 4 we show that the maximum and minimum of the eigenvalues of the symmetric circulant matrices jointly converge after scaling and centering. In Section 3 we take the input sequence $\{x_n\}$ to be an infinite order moving average process, $x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}$, where $\sum_n |a_n| < \infty$, are nonrandom and $\{\epsilon_i; i \in \mathbb{Z}\}$ are i.i.d. with $E(\epsilon_i) = 0$ and $V(\epsilon_i) = 1$. It seems to be a nontrivial problem to derive properties of the spectral norm in this case. We resort to scaling each eigenvalue by the spectral density at the appropriate ordinate as described below and then consider their maximum. This scaling has the effect of equalizing the variance of the eigenvalues. Similar scaling has been used in the study of periodograms (see Walker (1965), Davis and Mikosch (1999), Lin and Liu (2009)). For any of the above mentioned matrix A_n we define $\mathrm{M}(A_n, f) = \max_{1\le k\le n} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$ where f is the spectral density corresponding to $\{x_n\}$ and $\{\lambda_k\}$ are eigenvalues of A_n. We show in Theorem 7 and Theorem 11 that $\mathrm{M}(n^{-1/2}RC_n, f)$ and $\mathrm{M}(n^{-1/2}A_{k,n}, f)$ converge to the Gumbel distribution after proper centering and scaling. For the symmetric circulant, in Theorem 8 we show that $\mathrm{M}(n^{-1/2}SC_n, f)$ converges to the same limit as above when we impose the extra condition $a_j = a_{-j}$ for all j. Without this condition it is difficult to conclude the distributional convergence even if ϵ_i's are i.i.d $N(0,1)$. The convergence in probability of $\mathrm{M}(n^{-1/2}SC_n, f)$ is discussed in Lemma 7 and Theorem 10. In Section 4 we provide some concluding remarks and point out some interesting problems which arise from the results.

A bit of notation. By $f(t) \sim g(t)$, we shall mean $\frac{f(t)}{g(t)} \to 1$ as $t \to \infty$. By $|x|$ we denote the Euclidean norm of $x \in \mathbb{R}^d$ and also the modulus if x is a complex number. Throughout, C will denote a generic constant and Λ will denote the standard Gumbel distribution

$$\Lambda(x) = e^{-e^{-x}}.$$

The following normalizing constants, well known in the context of maxima of i.i.d. normal variables, will be repeatedly used in the statements of our results.

$$a_n = (2\ln n)^{-1/2} \text{ and } b_n = (2\ln n)^{1/2} - \frac{\ln\ln n + \ln 4\pi}{2(2\ln n)^{1/2}}. \tag{1.1}$$

2 Results for i.i.d. input

2.1 Toeplitz matrix

First we state a result which is known for Toeplitz and Hankel matrices. Let $u_n = n^{-1/2}(1, 1, \ldots, 1)^T$.

Theorem 1. *(Bose and Sen (2007)) Let $\{x_i\}$ be i.i.d. with $E(x_0) = \mu > 0$ and $Var(x_0) = 1$ and let T_n be the symmetric Toeplitz matrix $((x_{|i-j|}))$. Let $T_n^0 = T_n - \mu n u_n u_n^T$. Then*

(i)

$$\frac{\|T_n\|}{n} \to \mu \quad \textit{almost surely and} \; \|\frac{T_n^0}{\|T_n\|}\| \to 0 \quad \textit{almost surely.}$$

(ii) If $E(x_0^4) < \infty$, then for $M_n = \|T_n\|$ or $M_n = \lambda_n(T_n)$, the maximum eigenvalue of T_n,

$$\frac{M_n - \mu n}{\sqrt{n}} \to N(0, 4/3) \; \textit{in distribution.}$$

(iii) If T_n and T_n^0 are replaced by the corresponding symmetric Hankel matrices H_n and H_n^0, then (i) holds. Further, (ii) holds with the limiting variance being changed from $4/3$ to $2/3$.

Remark 1. *When $\{x_i\}$ are centered random variables some results are known for Toeplitz matrix. Meckes (2007) showed that if $\{x_i\}$'s are centered uniformly subgaussian then $E\|T_n\| \sim \sqrt{n \ln n}$ and the same holds for $\|T_n\|$ with probability 1 provided $\{x_i\}$'s have some concentration of measures property. These results were further improved in Adamczak (2008), where it was shown that for $\{x_i\}$ i.i.d. mean zero and finite variance,*

$$\lim_{n\to\infty} \frac{\|T_n\|}{E\|T_n\|} = 1 \; a.s.$$

Further,

$$\limsup \frac{\|T_n\|}{\sqrt{n \ln n}} < \infty \; a.s. \; \textit{if and only if} \; Ex_0 = 0 \; \textit{and} \; Ex_0^2 < \infty.$$

2.2 Circulant and Reverse Circulant matrix

Similar results can be established for reverse circulant, symmetric circulant and circulant matrices. In fact we shall show that the spectral norm converges in distribution when centered and scaled appropriately. Observe that since C_n is normal, the eigenvalues of $n^{-1}C_nC_n^T$ are same as square of eigenvalues of reverse circulant matrix. So $\|n^{-1/2}C_n\| = \|n^{-1/2}RC_n\|$. Hence the spectral norm for these two matrices do not have to be dealt with separately. Some results about the maximum of the singular values of circulant matrices with standard complex normal entries is known from the form of the eigenvalues. See for example Corollary 5 of Meckes (2009).

We start with a result on the reverse circulant which follows easily from the existing literature.

Theorem 2. *Suppose $\{x_i\}$ is i.i.d. with $E(x_0) = \mu$ and $Var(x_0) = 1$. Suppose RC_n is the reverse circulant matrix formed by the $\{x_i\}$. Let $RC_n^0 = RC_n - \mu n u_n u_n^T$. If $\mu > 0$, then*

$$\frac{\|RC_n\|}{n} \to \mu \quad \textit{almost surely and} \; \|\frac{RC_n^0}{\|RC_n\|}\| \to 0 \quad \textit{almost surely.}$$

Similar results hold for C_n also.

Proof. The proof follows from arguments for Toeplitz and Hankel matrices given in Theorem 3 and Lemma 1(i) of Bose and Sen (2007). $\square$

Remark 2. *If we assume $E(x_0^4) < \infty$, then the distributional convergence when $\mu > 0$ can also be proved following the proof of Bose and Sen (2007). However, below we establish the distributional convergence under the assumption $E|x_0|^{2+\delta} < \infty$.*

Theorem 3. *Suppose $\{x_i\}_{i\geq 0}$ is i.i.d. with mean μ and $E|x_i|^{2+\delta} < \infty$ for some $\delta > 0$. Consider the reverse circulant (RC_n) and circulant (C_n) matrices with the input $\{x_i\}$.*

(i) If $\mu \neq 0$ then,

$$\frac{\|RC_n\| - |\mu|n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1).$$

(ii) If $\mu = 0$ then,

$$\frac{\|\frac{1}{\sqrt{n}}RC_n\| - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

where

$$q = q(n) = [\frac{n-1}{2}], \quad d_q = \sqrt{\ln q} \quad \textit{and} \quad c_q = \frac{1}{2\sqrt{\ln q}}\ .$$

The above conclusions continue to hold for C_n also.

Proof. As pointed out earlier, it is enough to deal with only RC_n. Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of $n^{-1/2}RC_n$. These eigenvalues are given by (see Bose and Mitra (2002)):

$$\begin{cases} \lambda_0 & = n^{-1/2}\sum_{t=0}^{n-1} x_t \\ \lambda_{n/2} & = n^{-1/2}\sum_{t=0}^{n-1}(-1)^t x_t, \quad \text{if } n \quad \text{is even} \\ \lambda_k = -\lambda_{n-k} & = \sqrt{I_{n,x}(\omega_k)}, \ 1 \leq k \leq [\frac{n-1}{2}]. \end{cases} \tag{2.1}$$

where

$$I_{n,x}(\omega_k) = \frac{1}{n}|\sum_{t=0}^{n-1} x_t e^{-it\omega_k}|^2 \text{ and } \omega_k = \frac{2\pi k}{n}.$$

Note that $\{|\lambda_k|^2;\ 1 \leq k < n/2\}$ is the periodogram of $\{x_i\}$ at the frequencies $\{\frac{2\pi k}{n};\ 1 \leq k < n/2\}$. If $\mu = 0$ then under the given conditions Davis and Mikosch (1999) have shown that

$$\max_{1\leq k<\frac{n}{2}} I_{n,x}(\omega_k) - \ln q \xrightarrow{\mathcal{D}} \Lambda.$$

Therefore

$$\max_{1\leq k<n/2} |\lambda_k|^2 - \ln q \xrightarrow{\mathcal{D}} \Lambda. \tag{2.2}$$

Define $g(x) = \sqrt{x}$. Then by mean value theorem,

$$g(\max_{1\leq k<n/2} |\lambda_k|^2) - g(\ln q) = g'(\xi_n)(\max_{1\leq k<n/2} |\lambda_k|^2 - \ln q)$$

where ξ_n lies between $\max_{1\leq k<n/2} |\lambda_k|^2$ and $\ln q$. From (2.2) we have

$$\frac{\max_{1\leq k<n/2} |\lambda_k|^2}{\ln q} \xrightarrow{\mathcal{P}} 1.$$

Therefore $\frac{\xi_n}{\ln q} \xrightarrow{\mathcal{P}} 1$. Now

$$\frac{g'(\xi_n)}{g'(\ln q)} = \left(\frac{\ln q}{\xi_n}\right)^{1/2} \xrightarrow{\mathcal{P}} 1$$

and therefore

$$\frac{g(\max_{1\le k<n/2}|\lambda_k|^2) - g(\ln q)}{g'(\ln q)} = \frac{g'(\xi_n)}{g'(\ln q)}(\max_{1\le k<n/2}|\lambda_k|^2 - \ln q) \xrightarrow{\mathcal{D}} \Lambda.$$

So if $\{x_i\}$ are i.i.d. with mean zero, variance 1 and $E|x_i|^{2+\delta} < \infty$, then

$$\frac{\max_{1\le k<\frac{n}{2}}|\lambda_k| - \sqrt{\ln q}}{\frac{1}{2\sqrt{\ln q}}} \xrightarrow{\mathcal{D}} \Lambda. \tag{2.3}$$

Observe that we have left out λ_0 and $\lambda_{n/2}$ (if n is even) where

$$\lambda_0 = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1} x_t \quad \text{and} \quad \lambda_{n/2} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}(-1)^t x_t.$$

Now suppose that mean of $\{x_i\}$ is $\mu > 0$. For $1 \le k < n/2$,

$$|\lambda_k| = \frac{1}{\sqrt{n}}\Big|\sum_{t=0}^{n-1} x_t e^{it\omega_k}\Big| = \frac{1}{\sqrt{n}}\Big|\sum_{t=0}^{n-1}(x_t - \mu)e^{it\omega_k}\Big|,$$

and $(x_t - \mu)$ has mean zero and variance 1. Therefore even when $E(x_i) > 0$, (2.3) holds. Note that by CLT

$$\frac{\sqrt{n}\lambda_0 - \mu n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1). \tag{2.4}$$

(2.4) implies $\lambda_0 \xrightarrow{\mathcal{P}} \infty$ and hence

$$|\lambda_0| - \mu\sqrt{n} \xrightarrow{\mathcal{D}} N(0,1).$$

Let $A_n = \max_{1\le k<q}|\lambda_k|$. From (2.3) and (2.4)

$$\frac{A_n}{\sqrt{\ln q}} \xrightarrow{\mathcal{P}} 1 \quad \text{and} \quad \frac{\lambda_0}{\mu\sqrt{n}} \xrightarrow{\mathcal{P}} 1$$

and so it follows that

$$\mathbb{P}\big[\max(A_n, |\lambda_0|) - \mu\sqrt{n} > x\big] \to \mathbb{P}\big[N(0,1) > x\big],$$

proving (i) for odd n.

Since for even n,

$$\lambda_{n/2} = n^{-1/2}\sum_{t=0}^{n-1}(-1)^t x_t \xrightarrow{\mathcal{D}} N(0,1),$$

this can also be neglected as before, and hence (i) holds also for even n. Similar proof works when $\mu < 0$. This proves (i) completely.

(ii) Now assume $\mu = 0$. In contrast to the previous case, here A_n dominates $|\lambda_0|$, since $|\lambda_0|$ is tight and

$$\frac{|\lambda_0| - \sqrt{\ln q}}{(\ln q)^{-1/2}} \xrightarrow{\mathcal{P}} -\infty.$$

Hence in this case

$$\frac{\|\frac{1}{\sqrt{n}} RC_n\| - \sqrt{\ln q}}{\frac{1}{2\sqrt{\ln q}}} \xrightarrow{\mathcal{D}} \Lambda.$$

□

2.3 Symmetric circulant matrix

The spectral norm of the symmetric circulant matrices behaves quite similar to reverse circulant matrices but the normalizing constants change. We need the following Lemmata which are well known and hence we omit their proofs.

Lemma 1. *The eigenvalues of $\frac{1}{\sqrt{n}} SC_n$ are given by:*

(i) for n odd:

$$\lambda_0 = \frac{1}{\sqrt{n}} \big[x_0 + 2 \sum_{j=1}^{[n/2]} x_j\big]$$
$$\lambda_k = \frac{1}{\sqrt{n}} \big[x_0 + 2 \sum_{j=1}^{[n/2]} x_j \cos \frac{2\pi kj}{n}\big], \quad 1 \le k \le [n/2]$$

(ii) for n even:

$$\lambda_0 = \frac{1}{\sqrt{n}} \big[x_0 + 2 \sum_{j=1}^{\frac{n}{2}-1} x_j + x_{n/2}\big]$$
$$\lambda_k = \frac{1}{\sqrt{n}} \big[x_0 + 2 \sum_{j=1}^{\frac{n}{2}-1} x_j \cos \frac{2\pi kj}{n} + (-1)^k x_{n/2}\big], \quad 1 \le k \le \frac{n}{2}$$

with $\lambda_{n-k} = \lambda_k$ in both the cases.

The next Lemma is on the joint behaviour of maxima and minima of i.i.d normal random variables.

Lemma 2. *Let $\{N_i\}$ be i.i.d. $N(0,1)$. If $m_n = \min_{1 \le i \le n} N_i$ and $M_n = \max_{1 \le i \le n} N_i$, then with a_n and b_n as in (1.1),*

$$\Big(\frac{-m_n - b_n}{a_n}, \frac{M_n - b_n}{a_n}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda,$$

where $\Lambda \otimes \Lambda$ denotes joint distribution of two independent standard Gumbel random variables.

The statement of Lemma 3 is taken from Einmahl (1989) Corollary 1(b), page 31, in combination with Remark on page 32.

Lemma 3. *Let $\{\psi_i\}$ be independent random vectors with mean zero and values in $\mathbb{R}^d$. Assume that the moment generating functions of ψ_i, $1 \le i \le n$, exist in a neighborhood of the origin and that*

$$Cov(\psi_1 + \psi_2 + ... + \psi_n) = B_n I_d,$$

where $B_n > 0$ and I_d denotes the d-dimensional identity matrix. Let η_k be independent $N(0, \sigma^2 Cov(\psi_k))$ random vectors, $k = 1, 2, ...n$, independent of $\{\psi_k\}$ and $\sigma^2 \in (0, 1]$. Let $\psi_k^* = \psi_k + \eta_k$, $k = 1, 2, ...n$ and write p_n^* for the density of $B_n^{-1/2} \sum_{k=1}^n \psi_k^*$. Choose $\alpha \in (0, \frac{1}{2})$ such that

$$\alpha \sum_{k=1}^{n} E|\psi_k|^3 \exp(\alpha|\psi_k|) \le B_n,$$

where $|x|$ denotes the Euclidean norm in $\mathbb{R}^d$. Let

$$\beta_n = \beta_n(\alpha) = B_n^{-3/2} \sum_{k=1}^{n} E|\psi_k|^3 \exp(\alpha|\psi_k|).$$

If $|x| \le c_1 \alpha B_n^{1/2}$, $\sigma^2 \ge -c_2 \beta_n^2 \ln \beta_n$ and $B_n \ge c_3 \alpha^{-2}$, where c_1, c_2, c_3 are constants depending only on d, then

$$p_n^* = \phi_{(1+\sigma^2)I_d}(x) \exp(\bar{T}_n(x)) \text{ with } |\bar{T}_n(x)| \le c_4 \beta_n (|x|^3 + 1),$$

where ϕ_c is the density of a d-dimensional centered Gaussian vector with covariance matrix c and c_4 is a constant depending on d.

We shall use the above Lemma now to derive a normal approximation result which shall be used in the proof of Theorem 4. Define

$$\bar{x}_t = x_t \mathbf{I}(|x_t| \le (1 + 2j)^{1/s}) - E[x_t \mathbf{I}(|x_t| \le (1 + 2j)^{1/s})]. \tag{2.5}$$

For $1 \le i_1 < i_2 < ... < i_d < j$ let

$$v_d(0) = \sqrt{2}(1, 1, ..., 1), \quad v_d(t) = 2\left(\cos\frac{2\pi i_1 t}{2j+1}, \cos\frac{2\pi i_2 t}{2j+1}, ..., \cos\frac{2\pi i_d t}{2j+1}\right) \text{ for } 1 \le t \le j.$$

Lemma 4. *Let $n = 1 + 2j$ and $\sigma_j^2 = (1 + 2j)^{-c}$ for some $c > 0$ and let $\{x_t\}$ be i.i.d mean zero with $Ex_0^2 = 1$ and $Ex_0^s < \infty$ for some $s > 2$. If we denote $\tilde{p}_j(x)$ to be the density of*

$$\frac{1}{\sqrt{1 + 2j}} \sum_{t=0}^{j} (\bar{x}_t + \sigma_j N_t) v_d(t),$$

where N_t's are i.i.d. $N(0, 1)$ random variables independent of $\{x_t\}$ then for any measurable subset E of $\mathbb{R}^d$,

$$|\int_E \tilde{p}_j(x)dx - \int_E \phi_{(1+\sigma_j^2)I_d}(x)dx| \le \epsilon_j \int_E \phi_{(1+\sigma_j^2)I_d}(x)dx + O(\exp(-(1 + 2j)^\eta))$$

where $\epsilon_j \to 0$, $\eta > 0$ and the above holds uniformly over d-tuples $1 \le i_1 < i_2 < ... < i_d < j$.

Proof. Let $S_{j,\bar{x}} = \sum_{t=0}^{j} \bar{x}_t v_d(t)$ and let $s = 2 + \delta$. Then observe $Cov(S_{j,\bar{x}}) = B_j I_d$ where, $B_j = (2j+1)Var(\bar{x}_t)$ and I_d is the $d \times d$ identity matrix. Since $\{\bar{x}_t v_d(t)\}_{0 \le t \le j}$ is an independent collection of mean zero random vectors in $\mathbb{R}^d$, we shall use Lemma 3. By choosing $\alpha = \frac{c_5(1+2j)^{-\frac{1}{s}}}{2\sqrt{d}}$, it can be easily shown that,

$$\alpha \sum_{t=0}^{j} E|\bar{x}_t v_d(t)|^3 \exp(\alpha |x_t v_d(t)|) < B_j.$$

If we define $\tilde{\beta}_j = B_j^{-3/2} \sum_{t=0}^{j} E|\bar{x}_t v_d(t)|^3 \exp(\alpha |\bar{x}_t v_d(t)|)$, then it follows that

$$\tilde{\beta}_j \le C(1+2j)^{-(\frac{1}{2} - \frac{1-\delta}{s})}.$$

Let $c = \frac{1}{2} - \frac{1-\delta}{s} > 0$. Now choose $|x| \le c_1 \alpha B_j^{1/2} \sim c_2 (1+2j)^{\frac{1}{2} - \frac{1}{s}}$ and σ_j^2 satisfying,

$$1 \ge \sigma_j^2 \ge c_3 (\ln(2j+1))(2j+1)^{-2c}.$$

Clearly $B_j \ge c_4 \alpha^{-2}$ and $B_j \sim (1+2j)$. We mention here that c_1, c_2, c_3, c_4 are constants depending only on d. Then Lemma 3 implies that,

$$\tilde{p}_j(x) = \phi_{(1+\sigma_j^2)I_d}(x) \exp(|T_j(x)|)$$

with $|T_j(x)| \le c_5 \tilde{\beta}_j (|x|^3 + 1)$. Note that, $|T_j(x)| \to 0$ uniformly for $|x|^3 = o\{\min((1+2j)^{-c}, (1+2j)^{\frac{1}{2} - \frac{1}{s}})\}$. For the choice of $\sigma_j^2 = (1+2j)^{-c}$ the above condition can be seen to be satisfied. Now it follows from Corollary 1 of Bose, Mitra and Sen (2008) that for any measurable subset E of $\mathbb{R}^d$,

$$\Big| \int_E \tilde{p}_j(x)dx - \int_E \phi_{(1+\sigma_j^2)I_d}(x)dx \Big| \quad \le \quad \epsilon_j \int_E \phi_{(1+\sigma_j^2)I_d}(x)dx + O(\exp(-(1+2j)^\eta))$$

where $\epsilon_j \to 0$. ☐

For the reverse circulant, leaving out the eigenvalues λ_0 and $\lambda_{n/2}$, the maximum and minimum eigenvalues are equal in magnitude. This is not the case for symmetric circulant. Hence we now look at the joint behaviour of the maximum and minimum of the eigenvalues. While preparing the manuscript we came to know that the limiting distribution of the maximum of eigenvalues of symmetric circulant matrices has been worked out in Bryc and Sethuraman (2009). It is easy to see that Theorem 1 of their paper can be derived from the following result.

Theorem 4. *Let $q = [n/2]$ and $M_{q,x} = \max_{1 \le k \le q} \lambda_k$ and $m_{q,x} = \min_{1 \le k \le q} \lambda_k$. If $\{x_i\}$ are i.i.d. with $Ex_0 = 0$, $Ex_0^2 = 1$ and $E|x_0|^s < \infty$ for some $s > 2$ then we have,*

$$\Big(\frac{-m_{q,x} - b_q}{a_q}, \frac{M_{q,x} - b_q}{a_q}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda,$$

where a_q and b_q are given by (1.1). The same limit continues to hold if the eigenvalue λ_0 is included in the definition of max *and* min *above.*

Proof. First assume $n = 2j+1$, odd and let $s = 2 + \delta$. The proof may be broken down into two major steps–truncation and application of Bonferroni's inequality.

Step 1: Truncation. Let $\bar{x}_t$ be as in (2.5) and

$$\tilde{x}_t = x_t \mathbf{I}(|x_t| \le (1+2j)^{1/s}).$$

If $\bar\lambda_k$ and $\tilde\lambda_k$ denote the eigenvalues of symmetric circulant matrices with entries $\bar x_t$ and $\tilde x_t$ respectively, then $\bar\lambda_k = \tilde\lambda_k$. By Borel-Cantelli lemma, $\sum_{t=1}^{\infty} |x_t| \mathbf{I}(|x_t| > (1+2j)^{1/s})$ is bounded with probability 1 and consists of only a finite number of nonzero terms. Thus there exists a positive integer $N(\omega)$ such that

$$\begin{aligned}\sum_{t=0}^{j} |x_t - \tilde x_t| &= \sum_{t=0}^{j} |x_t| \mathbf{I}(|x_t| > (1+2j)^{1/s}) \\ &\leq \sum_{t=0}^{\infty} |x_t| \mathbf{I}(|x_t| > (1+2j)^{1/s}) \\ &= \sum_{t=0}^{N(\omega)} |x_t| \mathbf{I}(|x_t| > (1+2j)^{1/s}).\end{aligned}$$

It follows that for $2j+1 \geq \{N(\omega), |x_1|^s, \ldots, |x_{N(\omega)}|^s\}$ the left side is zero. Consequently, for all j sufficiently large, $\tilde\lambda_k = \lambda_k$ a.s. for all k. Therefore

$$\Big(\frac{-m_{j,x} - b_j}{a_j}, \frac{M_{j,x} - b_j}{a_j}\Big) \stackrel{\mathcal{D}}{=} \Big(\frac{-m_{j,\bar x} - b_j}{a_j}, \frac{M_{j,\bar x} - b_j}{a_j}\Big) \tag{2.6}$$

where $m_{j,\bar x} = \min_{1\leq k\leq j} \bar\lambda_k$ and $M_{j,\bar x} = \max_{1\leq k\leq j} \bar\lambda_k$.

Step 2: Bonferroni Inequalities. Define for $1 \leq k \leq j$,

$$\begin{aligned}\bar\lambda_k' &= \frac{1}{\sqrt{2j+1}}\Big(\sqrt{2}\bar x_0 + 2\sum_{t=1}^{j} \bar x_t \cos\frac{2\pi k t}{2j+1}\Big), \\ \bar{\bar\lambda}_k' &= \bar\lambda_k' + \frac{\sigma_j}{\sqrt{1+2j}}\big(\sqrt{2}N_0 + 2\sum_{t=1}^{j} N_t \cos\frac{2\pi k t}{n}\big) \\ &= \bar\lambda_k' + \sigma_j N_{j,k}'.\end{aligned}$$

Observe $N_{j,k}'$ are i.i.d. $N(0,1)$ for $k = 1, 2, \cdots j$. Define

$$M_{j,\bar x+\sigma N} = \max_{1\leq k\leq j} \bar{\bar\lambda}_k' \quad \text{and} \quad m_{j,\bar x+\sigma N} = \min_{1\leq k\leq j} \bar{\bar\lambda}_k'.$$

$$\begin{aligned}\mathbb{P}\big(\frac{-m_{j,\bar x+\sigma N} - b_j}{a_j} > x, \frac{M_{j,\bar x+\sigma N} - b_j}{a_j} > y\big) &= \mathbb{P}\big(m_{j,\bar x+\sigma N} < -a_j x - b_j, M_{j,\bar x+\sigma N} > a_j y + b_j\big) \\ &= \mathbb{P}\big(\cup_{k=1}^{j} \{\bar{\bar\lambda}_k' < -a_j x - b_j\} \cap \cup_{k=1}^{j} \{\bar{\bar\lambda}_k' > a_j y + b_j\}\big) \\ &= \mathbb{P}\big(\cup_{k=1}^{j} \{\bar{\bar\lambda}_k' \in I_{x,y}^j\}\big) = \mathbb{P}\big(\cup_{k=1}^{j} A_{k,j}\big)\end{aligned}$$

where, $I_{x,y}^j = (a_j y + b_j, -a_j x - b_j)$ and $A_{k,j} = \{\bar{\bar\lambda}_k' \in I_{x,y}^j\}$.
Now by Bonferroni's inequality,

$$\sum_{t=1}^{2k} (-1)^{t-1} \tilde A_{t,j} \leq \mathbb{P}(A) \leq \sum_{t=1}^{2k-1} (-1)^{t-1} \tilde A_{t,j} \tag{2.7}$$

where

$$A = \big(\frac{-m_{j,\bar x+\sigma N} - b_j}{a_j} > x, \frac{M_{j,\bar x+\sigma N} - b_j}{a_j} > y\big) \text{ and } \tilde A_{t,j} = \sum_{1\leq i_1 < i_2 < \cdots < i_t \leq j} \mathbb{P}\big(A_{i_1,j} \cap .. \cap A_{i_t,j}\big).$$

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}\Big(\frac{-\min_{1\le k\le j}(1+\sigma_j^2)N_k - b_j}{a_j} > x, \frac{\max_{1\le k\le j}(1+\sigma_j^2)N_k - b_j}{a_j} > y\Big) \\ &= \mathbb{P}\big(\cup_{k=1}^{j}\{(1+\sigma_j^2)^{1/2}N_k \in I_{x,y}^j\}\big) = \mathbb{P}\big(\cup_{k=1}^{j} B_{k,j}\big)\end{aligned}$$

where $B_{k,j} = \{(1+\sigma_j^2)^{1/2}N_k \in I_{x,y}^j\}$. By Bonferroni's inequality,

$$\sum_{t=1}^{2k}(-1)^{t-1}\tilde{B}_{t,j} \le \mathbb{P}(B) \le \sum_{t=1}^{2k-1}(-1)^{t-1}\tilde{B}_{t,j} \tag{2.8}$$

where,

$$\tilde{B}_{t,j} = \sum_{1\le i_1<i_2<\cdots<i_t\le j} \mathbb{P}\big(B_{i_1,j}\cap B_{i_2,j}\cap ..\cap B_{i_t,j}\big).$$

From (2.7) and (2.8) we get

$$\sum_{t=1}^{2k}(-1)^{t-1}(\tilde{A}_{t,j}-\tilde{B}_{t,j}) - \tilde{B}_{2k+1,j} \le \mathbb{P}(A)-\mathbb{P}(B) \le \sum_{t=1}^{2k-1}(-1)^{t-1}(\tilde{A}_{t,j}-\tilde{B}_{t,j}) + \tilde{B}_{2k,j}. \tag{2.9}$$

Now note that,

$$\begin{aligned}\tilde{B}_{t,j} &= \sum_{1\le i_1<i_2\cdots<i_t\le j} \mathbb{P}\big(B_{i_1,j}\cap B_{i_2,j}\cap\ldots\cap B_{i_t,j}\big) \\ &= \sum_{1\le i_1<i_2\cdots<i_t\le j} \mathbb{P}\big((1+\sigma_j^2)^{1/2}N_{i_l}\in I_{x,y}^j; l=1,2,..,t\big) \\ &= \sum_{1\le i_1<i_2\cdots<i_t\le j} \mathbb{P}^t\big((1+\sigma_j^2)^{1/2}N_{i_l}\in I_{x,y}^j\big).\end{aligned}$$

Note here that

$$\begin{aligned}\mathbb{P}\big((1+\sigma_j^2)^{1/2}N_1 \in (a_jy+b_j, -a_jx-b_j)\big) &\le \mathbb{P}\big((1+\sigma_j^2)^{1/2}N_1 > a_jy+b_j\big) \\ &= \mathbb{P}\big(N_1 > (a_jy+b_j)(1+\sigma_j^2)^{-1/2}\big) \\ &\le \mathbb{P}\big(N_1 > (a_jy+b_j)(1-\frac{1}{2}\sigma_j^2)\big).\end{aligned}$$

Now $(a_jy+b_j)(1-\frac{\sigma_j^2}{2}) \sim b_j + o(1)$ and $\mathbb{P}(N_1 > b_j) \sim \frac{1}{j}$. Therefore

$$\mathbb{P}\big(N_1 > (1-\frac{1}{2}\sigma_j^2)(a_jy+b_j)\big) \le \frac{K}{j}$$

and hence

$$\tilde{B}_{t,j} \le \binom{j}{t}\frac{K^t}{j^t} \le \frac{K^t}{t!}.$$

Thus

$$\lim_{t\to\infty}\overline{\lim_{j\to\infty}}\,\tilde{B}_{t,j} = 0.$$

On the other hand, fixing $t\ge 1$ we get,

$$\mathbb{P}(A_{i_1,j}\cap A_{i_2,j}\cap\ldots\cap A_{i_t,j}) = \mathbb{P}\Big(\frac{1}{\sqrt{1+2j}}\sum_{t=0}^{j}(\bar{x}_t+\sigma_jN_t)v_d(t)\in E_t\Big),$$

where $E_t = \{(x_1, x_2, ..., x_t) : x_i \in I^j_{x,y}\}$. So by Lemma 4 we have that uniformly over all d-tuples $1 \le i_1 < i_2 < ... < i_d \le j$,

$$\Big|\mathbb{P}(\frac{1}{\sqrt{1+2j}}\sum_{t=0}^{j}(\bar{x}_t + \sigma_j N_t)v_d(t) \in E_t) - \mathbb{P}((1+\sigma_j^2)^{1/2}N_{i_l} \in I^j_{x,y}, 1 \le l \le t)\Big|$$
$$\le \epsilon_j \mathbb{P}((1+\sigma_j^2)^{1/2}N_{i_l} > a_j y + b_j, 1 \le l \le t) + O(\exp(-(1+2j)^\eta)).$$

So as $j \to \infty$ we get,

$$|\tilde{A}_{t,j} - \tilde{B}_{t,j}| \le \epsilon_j \tilde{B}_{t,j} + \binom{j}{t} O(\exp(-(1+2j)^\eta)) \to 0.$$

Therefore,

$$\overline{\lim_{j\to\infty}}\, |\mathbb{P}(A) - \mathbb{P}(B)| \le \overline{\lim_{j\to\infty}}\, \tilde{B}_{2k+1,j} + \overline{\lim_{j\to\infty}}\, \tilde{B}_{2k,j}$$

and letting $k \to \infty$ we get,

$$\lim_{j\to\infty} [\mathbb{P}(A) - \mathbb{P}(B)] = 0.$$

As $\max_{1\le k\le j} N_k = O_p((\ln j)^{1/2})$, it follows that,

$$\Big|\frac{(1+\sigma_j^2)^{1/2}\max_{1\le k\le j} N_k - b_j}{a_j} - \frac{\max_{1\le k\le j} N_k - b_j}{a_j}\Big| \le \frac{\sigma_j \max_{1\le k\le j}|N_k|}{a_j} \xrightarrow{\mathcal{P}} 0.$$

Therefore

$$\frac{(1+\sigma_j^2)^{1/2}\max_{1\le k\le j} N_k - b_j}{a_j} \xrightarrow{\mathcal{D}} \Lambda.$$

Since $-\min_{1\le k\le j}(1+\sigma_j^2)^{1/2}N_k = \max_{1\le k\le j}\big(-(1+\sigma_j^2)^{1/2}N_k\big)$ and $-(1+\sigma_j^2)^{1/2}N_k \overset{\mathcal{D}}{=} (1+\sigma_j^2)^{1/2}N_k$ we get

$$\frac{\min_{1\le k\le j} -(1+\sigma_j^2)^{1/2}N_k - b_j}{a_j} \xrightarrow{\mathcal{D}} \Lambda.$$

Since $(1+\sigma_j^2)^{1/2}N_i$ are i.i.d. symmetric distributions, by Resnick (1987) Exercise 5.5.2

$$\Big(\frac{\min_{1\le k\le j} -(1+\sigma_j^2)^{1/2}N_k - b_j}{a_j}, \frac{\max_{1\le k\le j}(1+\sigma_j^2)^{1/2}N_k - b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda.$$

Therefore combining the previous steps we get,

$$\Big(\frac{-m_{j,\bar{x}+\sigma N} - b_j}{a_j}, \frac{M_{j,\bar{x}+\sigma N} - b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda.$$

Now, to complete the proof it remains to show the result for truncated eigenvalues by (2.6). Now

$$\Big|\frac{\max(\bar{\bar{\lambda}}'_k)}{a_j} - \frac{\max(\bar{\lambda}'_k)}{a_j}\Big| \le \frac{\sigma_j}{a_j}\max|N'_{j,k}| \xrightarrow{\mathcal{P}} 0.$$

Similarly $-\bar{\bar{\lambda}}'_k = -\bar{\lambda}'_k - \sigma N'_{j,k}$ and

$$\Big|\frac{\max(-\bar{\bar{\lambda}}'_k)}{a_j} - \frac{\max(-\bar{\lambda}'_k)}{a_j}\Big| \le \frac{\sigma_j}{a_j}\max|N'_{j,k}| \xrightarrow{\mathcal{P}} 0.$$

Now if we denote $m'_{j,\bar{x}} = \min_{1\leq k\leq j} \bar{\lambda}'_k$ and $M'_{j,\bar{x}} = \max_{1\leq k\leq j} \bar{\lambda}'_k$ then,

$$\begin{aligned}
&\Big|\Big(\frac{-m_{j,\bar{x}+\sigma N} - b_j}{a_j}, \frac{M_{j,\bar{x}+\sigma N} - b_j}{a_j}\Big) - \Big(\frac{-m'_{j,\bar{x}} - b_j}{a_j}, \frac{M'_{j,\bar{x}} - b_j}{a_j}\Big)\Big| \\
\leq\ & C\Big[\Big|\frac{-m_{j,\bar{x}+\sigma N} - (-m'_{j,\bar{x}})}{a_j}\Big| + \Big|\frac{M_{j,\bar{x}+\sigma N} - M'_{j,\bar{x}}}{a_j}\Big|\Big] \\
\leq\ & C\Big[\Big|\frac{\max(-\bar{\bar{\lambda}}'_k) - \max(-\bar{\lambda}'_k)}{a_j}\Big| + \Big|\frac{\max(\bar{\bar{\lambda}}'_k) - \max(\bar{\lambda}'_k)}{a_j}\Big|\Big] \xrightarrow{\mathcal{P}} 0.
\end{aligned}$$

Therefore

$$\Big(\frac{-m'_{j,\bar{x}} - b_j}{a_j}, \frac{M'_{j,\bar{x}} - b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda.$$

Again $\bar{\lambda}_k = \bar{\lambda}'_k + \frac{(1-\sqrt{2})}{\sqrt{2j+1}}\bar{x}_0$, therefore

$$\Big|\frac{M'_{j,\bar{x}} - b_j}{a_j} - \frac{M_{j,\bar{x}} - b_j}{a_j}\Big| \xrightarrow{\mathcal{P}} 0,$$

and

$$\Big|\frac{-m_{j,\bar{x}} - b_j}{a_j} - \frac{-m'_{j,\bar{x}} - b_j}{a_j}\Big| \xrightarrow{\mathcal{P}} 0.$$

Hence

$$\Big(\frac{-m_{j,\bar{x}} - b_j}{a_j}, \frac{M_{j,\bar{x}} - b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda$$

Now for large j we know that $\bar{\lambda}_k = \lambda_k$ a.s. So, it follows that

$$\Big(\frac{-m_{j,x} - b_j}{a_j}, \frac{M_{j,x} - b_j}{a_j}\Big) \xrightarrow{\mathcal{D}} \Lambda \otimes \Lambda.$$

This proves the theorem when n is odd. For the even case say $n = 2j$ it should be noted that if we work with $\lambda'_k = \sqrt{2}x_0 + \sqrt{2}(-1)^k x_j + 2\sum_{t=1}^{j-1} x_t \cos\frac{2\pi kt}{2j}$ then similar normal approximations can be done and the subsequent calculations follow after that. We omit the obvious details. This proves the theorem completely. □

The next theorem follows by calculations similar to those used in the proof of Theorem 3.

Theorem 5. *Suppose $\{x_i\}_{i\geq 0}$ is an i.i.d. sequence with mean μ and $E|x_i|^{2+\delta} < \infty$ for some $\delta > 0$. Consider the symmetric circulant matrix (SC_n) with these $\{x_i\}$.*

(i) If $\mu = 0$ then,

$$\frac{\|\frac{1}{\sqrt{n}}SC_n\| - b_q - a_q \ln 2}{a_q} \xrightarrow{\mathcal{D}} \Lambda$$

where $q = q(n) \sim \frac{n}{2}$ and a_q and b_q are as in equation (1.1).

(ii) If $\mu \neq 0$ then,

$$\frac{\|SC_n\| - |\mu|n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, 2).$$

Proof. To prove (i), since mean $\mu = 0$, $\lambda_0 \xrightarrow{\mathcal{D}} N(0,2)$. So we can neglect this as shown in Theorem 3. Therefore, for large n,

$$\|\frac{1}{\sqrt{n}}SC_n\| = \max\{-\min_{1\le i\le[n/2]}\lambda_i, \max_{1\le i\le[n/2]}\lambda_i\}.$$

Hence

$$\begin{aligned}\mathbb{P}(\|\frac{1}{\sqrt{n}}SC_n\| \le a_q x + b_q) &= \mathbb{P}(\frac{-\min\lambda_i - b_q}{a_q} \le x, \frac{\max\lambda_i - b_q}{a_q} \le x) \\ &\xrightarrow{\mathcal{D}} \Lambda(x)\Lambda(x) = \Lambda(x + \ln\frac{1}{2}).\end{aligned}$$

Now by convergence of types

$$\mathbb{P}\Big(\frac{\|\frac{1}{\sqrt{n}}SC_n\| - \tilde{b}_q}{\tilde{a}_q} \le x\Big) \xrightarrow{\mathcal{D}} \Lambda(x)$$

where, $\tilde{a}_q = a_q$ and $\tilde{b}_q = b_q + a_q \ln 2$.
In part (ii), λ_0 dominates and the proof proceeds as in Theorem 3. We omit the details. □

2.4 k-Circulant matrix

For the k- circulant matrices the special case $n = k^2+1$ was considered Bose, Mitra and Sen (2008) who proved the following Theorem. We are investigating the general case.

Theorem 6. *Suppose $\{x_i\}_{i\ge0}$ is an i.i.d. sequence with mean zero and variance 1 and $E|x_i|^{2+\delta} < \infty$ for some $\delta > 2$. If $n = k^2+1$ then*

$$\frac{\|n^{-1/2}A_{k,n}\| - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

as $n \to \infty$ where $q = q(n) = \lfloor \frac{n}{4} \rfloor$ and

$$c_n = (8\ln n)^{-1/2} \quad \text{and} \quad d_n = \frac{(\ln n)^{1/2}}{\sqrt{2}}\left(1 + \frac{1}{4}\frac{\ln\ln n}{\ln n}\right) + \frac{1}{2(8\ln n)^{1/2}}\ln\frac{\pi}{2}. \tag{2.10}$$

3 Results for dependent input

Now let $\{x_n; n \ge 0\}$ be a two sided moving average process,

$$x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i} \tag{3.1}$$

where $\{a_n; n \in \mathbb{Z}\} \in l_1$, that is $\sum_n |a_n| < \infty$, are nonrandom and $\{\epsilon_i; i \in \mathbb{Z}\}$ are i.i.d. with $E(\epsilon_i) = 0$ and $V(\epsilon_i) = 1$. Let $f(\omega)$, $\omega \in [0, 2\pi]$ be the spectral density of $\{x_n\}$. Note that if $\{x_n\}$ is i.i.d. with mean 0 and variance σ^2, then $f \equiv \frac{\sigma^2}{2\pi}$. It appears to be a very difficult problem to establish limit results for the spectral norm when the spectral density is non-constant. One intuitive reason for this is that the variance of each eigenvalue is of the order of the spectral density at the corresponding ordinate. Thus it is meaningful to rescale by the spectral density. This is, for example, the approach taken by Walker (1965), Davis and Mikosch (1999), Lin and Liu (2009) while studying the periodogram. This rescaling by the spectral density makes them approximately same variance and that makes it relatively easy to handle their maxima.

3.1 Reverse Circulant and Circulant Matrix

Define $\mathrm{M}(\cdot, f)$ for the reverse circulant matrix as follows:

$$\mathrm{M}(n^{-1/2}RC_n, f) = \max_{1\le k<\frac{n}{2}} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$$

where λ_k are the eigenvalues of $n^{-1/2}RC_n$ as defined in proof of Theorem 3. Note that $\mathrm{M}(n^{-1/2}C_n, f)$ for the circulant matrix defined similarly satisfies $\mathrm{M}(n^{-1/2}RC_n, f) = \mathrm{M}(n^{-1/2}C_n, f)$.

Theorem 7. *Let $\{x_n\}$ be the two sided moving average process (3.1) where $\{\epsilon_i\}$ are i.i.d. with $E(\epsilon_i) = 0$, $E(\epsilon_i^2) = 1$, $E|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$ and*

$$\sum_{j=-\infty}^{\infty} |a_j||j|^{1/2} < \infty \text{ and } f(\omega) > 0 \text{ for all } \omega \in [0, 2\pi]. \tag{3.2}$$

Then

$$\frac{M(n^{-1/2}RC_n, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda,$$

where $q = q(n) = [\frac{n-1}{2}]$, $d_q = \sqrt{\ln q}$ and $c_q = \frac{1}{2\sqrt{\ln q}}$. Same result continues to hold for $M(n^{-1/2}C_n, f)$.

Proof. Under (3.2) it is known that (see Walker (1965), Theorem 3) for some $\delta' > 0$,

$$\max_{1\le k<\frac{n}{2}} \Big| \frac{I_{x,n}(\omega_k)}{2\pi f(\omega_k)} - I_{\epsilon,n}(\omega_k) \Big| = o_p(n^{-\delta'}) \tag{3.3}$$

where

$$I_{x,n}(\omega_k) = \frac{1}{n}|\sum_{t=0}^{n-1} x_t e^{-it\omega_k}|^2 \text{ and } I_{\epsilon,n}(\omega_k) = \frac{1}{n}|\sum_{t=0}^{n-1} \epsilon_t e^{-it\omega_k}|^2.$$

Combining this with Theorem 2.1 of Davis and Mikosch (1999) we have

$$\max_{1\le k<\frac{n}{2}} \frac{I_{x,n}(\omega_k)}{2\pi f(\omega_k)} - \ln q \xrightarrow{\mathcal{D}} \Lambda.$$

Now proceeding as in the proof of Theorem 3, we can conclude that

$$\frac{\mathrm{M}(n^{-1/2}RC_n, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

□

Remark 3. *If we define $\overline{M}(n^{-1/2}RC_n, f) = \max_{0\le k<n/2} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$ then different limits may appear depending on mean μ' of the process $\{x_n\}$. If mean μ of ϵ_0 is 0 then by Theorem 7.1.2 of Brockwell and Davis (2002) it follows that $\frac{\lambda_0}{\sqrt{2\pi f(0)}} \xrightarrow{\mathcal{D}} N(0,1)$. So by arguments similar to Theorem 3 we have*

$$\frac{\overline{M}(n^{-1/2}RC_n, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

When $\mu \neq 0$ then,

$$\overline{M}(n^{-1/2}RC_n, f) - |\mu|\sqrt{n} \xrightarrow{\mathcal{D}} N(0,1).$$

Remark 4. *It appear that by the results of Lin and Liu (2009), if $\{x_n\}$ is the two sided moving average process (3.1) where $E(\epsilon_0) = 0$, $E(\epsilon_0^2) = 1$, $E[\epsilon_0^2 \mathbb{I}\{|\epsilon| \geq n\}] = o(1/\ln n)$ and*

$$\sum_{|j| \geq n} |a_j| = o(1/\ln n) \text{ and } \min_{\omega \in [0,2\pi]} f(\omega) > 0, \tag{3.4}$$

then also

$$\frac{M(n^{-1/2} RC_n, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda,$$

where c_q, d_q are as in Theorem 7.

3.2 Symmetric Circulant matrix

We now come to the symmetric circulant case. The result of Walker (1965) is not directly applicable but we use his results appropriately.

Lemma 5. *Let $\{x_n\}$ be the two sided moving average process (3.1) where $E(\epsilon_i) = 0$, $E(\epsilon_i^2) = 1$, $E|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$ and*

$$\sum_{j=-\infty}^{\infty} |a_j||j|^{1/2} < \infty \text{ and } f(\omega) > 0 \text{ for all } \omega \in [0, 2\pi]. \tag{3.5}$$

Then we have,

$$\max_{1 \leq k \leq [n/2]} \Big| \frac{\lambda_k}{\sqrt{2\pi f(\omega_k)}} - 2\frac{A_k}{\sqrt{n}} \sum_{t=1}^{[n/2]} \epsilon_t \cos(\frac{2\pi kt}{n}) + 2\frac{B_k}{\sqrt{n}} \sum_{t=1}^{[n/2]} \epsilon_t \sin(\frac{2\pi kt}{n}) \Big| = o_p(n^{-1/4}) \tag{3.6}$$

where

$$\sqrt{2\pi f(\omega_k)} A_k = \sum_{j=-\infty}^{\infty} a_j \cos(\frac{2\pi kj}{n}) \quad \text{and} \quad \sqrt{2\pi f(\omega_k)} B_k = \sum_{j=-\infty}^{\infty} a_j \sin(\frac{2\pi kj}{n}).$$

Proof. First observe that $\min_{\omega \in [0,2\pi]} f(\omega) > \alpha > 0$. Consider $n = 2m + 1$ for simplicity and for $n = 2m$ calculations are similar.

$$\frac{\lambda_k}{\sqrt{2\pi f(\omega_k)}} - 2\frac{A_k}{\sqrt{n}} \sum_{t=1}^{m} \epsilon_t \cos(\frac{2\pi kt}{n}) + 2\frac{B_k}{\sqrt{n}} \sum_{t=1}^{m} \epsilon_t \sin(\frac{2\pi kt}{n}) = Y_{n,k}$$

where

$$Y_{n,k} = \frac{1}{\sqrt{n}\sqrt{2\pi f(\omega_k)}} \sum_{j=-\infty}^{\infty} a_j \big[\cos \frac{2\pi kj}{n} U_{k,j} - \sin \frac{2\pi kj}{n} V_{k,j} \big],$$

$$U_{k,j} = \sum_{t=1}^{m} \big[\epsilon_{t-j} \cos \frac{2\pi k(t-j)}{n} - \epsilon_t \cos \frac{2\pi kt}{n} \big], \ V_{k,j} = \sum_{t=1}^{m} \big[\epsilon_{t-j} \sin \frac{2\pi k(t-j)}{n} - \epsilon_t \sin \frac{2\pi kt}{n} \big].$$

Note that

$$|U_{k,j}| \leq \begin{cases} |\sum_{t=1-j}^{0} \epsilon_t \cos \frac{2\pi kt}{n}| + |\sum_{t=m-j+1}^{m} \epsilon_t \cos \frac{2\pi kt}{n}| & \text{if} \quad |j| < m, j \geq 0 \\ |\sum_{t=1}^{|j|} \epsilon_t \cos \frac{2\pi kt}{n}| + |\sum_{t=m+1}^{m+|j|} \epsilon_t \cos \frac{2\pi kt}{n}| & \text{if} \quad |j| < m, j < 0 \\ |\sum_{t=1-j}^{m-j} \epsilon_t \cos \frac{2\pi kt}{n}| + |\sum_{t=1}^{m} \epsilon_t \cos \frac{2\pi kt}{n}| & \text{if} \quad |j| \geq m, j \geq 0 \\ |\sum_{t=|j|+1}^{|j|+m} \epsilon_t \cos \frac{2\pi kt}{n}| + |\sum_{t=1}^{m} \epsilon_t \cos \frac{2\pi kt}{n}| & \text{if} \quad |j| \geq m, j < 0. \end{cases}$$

Now for any $r > 1$,

$$|\sum_{t=1}^{r} \epsilon_t \cos \frac{2\pi kt}{n}|^2 \leq |\sum_{t=1}^{r} \epsilon_t e^{\frac{i2\pi kt}{n}}|^2 \leq \sum_{s=-r}^{r} |\sum_{t=1}^{r-|s|} \epsilon_t \epsilon_{t+|s|}|.$$

Hence by equation (29) of Walker (1965),

$$\mathbb{E}\{\max_k |\sum_{t=1}^{r} \epsilon_t \cos \frac{2\pi kt}{n}|^2\} \leq K r^{\frac{3}{2}}.$$

Therefore

$$\mathbb{E}\{\max_k U_{k,j}^2\} \leq \begin{cases} 4K|j|^{3/2} & \text{if} \quad |j| < m, \\ 4Km^{3/2} & \text{if} \quad |j| \geq m. \end{cases}$$

Similarly

$$\mathbb{E}\{\max_k V_{k,j}^2\} \leq \begin{cases} 4K|j|^{3/2} & \text{if} \quad |j| < m, \\ 4Km^{3/2} & \text{if} \quad |j| \geq m. \end{cases}$$

Now

$$\begin{aligned} \mathbb{E}\{\max_k |Y_{n,k}|\} &\leq \frac{1}{\sqrt{2\pi\alpha}} \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} |a_j| \left[\mathbb{E}\{\max_k |U_{k,j}|\} + \mathbb{E}\{\max_k |V_{k,j}|\}\right] \\ &\leq \frac{2K^{1/2}}{\sqrt{2\pi\alpha}} \frac{1}{\sqrt{n}} \left[\sum_{|j|<m} |a_j||j|^{3/4} + \sum_{|j|\geq m} |a_j| m^{3/4} \right] \\ &\leq \frac{2K^{1/2}}{\sqrt{2\pi\alpha}} \frac{1}{n^{1/4}} \left[\sum_{|j|<m} |j|^{1/2}|a_j|(j/n)^{1/4} + \sum_{|j|\geq m} j^{1/2}|a_j| \right] \\ &= o(n^{-1/4}) \end{aligned}$$

since the second sum goes to zero as $n \to \infty$ and the first sum is not greater than

$$\sum_{k(n)<|j|<m} |j|^{1/2}|a_j| + \{k(n)/n)\}^{1/4} \sum_{0\leq|j|\leq k(n)} |j|^{1/2}|a_j|,$$

where $k(n)$ is such that $\lim_{n\to\infty}\{k(n)/n\} = 0$ and $\lim_{n\to\infty} k(n) = \infty$. □

Define $\mathrm{M}(\cdot, f)$ for the symmetric circulant matrix as was done for the reverse circulant matrix:

$$\mathrm{M}(n^{-1/2}SC_n, f) = \max_{1\leq k<\frac{n}{2}} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$$

where λ_k are the eigenvalues of $n^{-1/2}SC_n$ as defined in Lemma 1. Under the additional restriction of $a_j = a_{-j}$, for all j, the following result is easy to prove.

Theorem 8. *Let $\{x_n\}$ be the two sided moving average process (3.1) where $E(\epsilon_i) = 0$, $E(\epsilon_i^2) = 1$, $E|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$ and*

$$a_j = a_{-j} \text{ for all } j \text{ and } \sum_{j=-\infty}^{\infty} |a_j||j|^{1/2} < \infty \text{ and } f(\omega) > 0 \text{ for all } \omega \in [0, 2\pi]. \tag{3.7}$$

Then

$$\frac{M(n^{-1/2}SC_n, f) - b_q - a_q \ln 2}{a_q} \xrightarrow{\mathcal{D}} \Lambda$$

where $q = q(n) = [n/2] \sim \frac{n}{2}$ and a_q and b_q are as in equation (1.1).

Proof. Note that if $a_j = a_{-j}$ then in Lemma 5, $B_k = 0$ and hence from the same lemma, it is easy to see that,

$$\max_{1\leq k\leq [n/2]} \left| \frac{\lambda_k}{\sqrt{2\pi f(\omega_k)}} - \lambda_{k,\epsilon} \right| = o_p(n^{-1/4}) \tag{3.8}$$

where $\lambda_{k,\epsilon}$ denote eigenvalue of symmetric circulant matrix with $\{x_i\}$ replaced by $\{\epsilon_i\}$. Combining this with part *(ii)* of Theorem 5 we have

$$\frac{\mathrm{M}(n^{-1/2}SC_n, f) - b_q - a_q \ln 2}{a_q} \xrightarrow{\mathcal{D}} \Lambda.$$

□

Now we focus on the case where a_j is not necessarily equal to a_{-j}. We first define some notation which will be used in the proofs of Lemma 7 and Theorem 9. For $0 < \delta_1 < 1/2$ define $p_n = (1 - \frac{1}{n^{1/2+\delta_1}})$ and denote $L_n = \{k : 1 \leq k \leq [np_n/2]\}$ and $L_n^1 = \{k \in L_n : k \text{ is even}\}$ and $L_n^2 = \{k \in L_n : k \text{ is odd}\}$.
Let

$$\sigma_k^2 = 1 + \frac{A_k B_k}{n} \tan(\frac{\pi k}{n}) \text{ and } \nu_{k,k'} = \frac{D_{k,k'}}{n} \tan\frac{\pi(k+k')}{2n} + \frac{E_{k,k'}}{n} \tan\frac{\pi(k'-k)}{2n} \tag{3.9}$$

where, $D_{k,k'} = A_k B_{k'} + A_{k'} B_k$ and $E_{k,k'} = A_{k'} B_k - A_k B_{k'}$.

The following lemma from Dai and Mukherjea (2001) (Theorem 2.1) is an analogue of Mill's ratio in higher dimension.

Lemma 6. *Let $(X_1, X_2, ...X_n)$ be multivariate normal with zero means and a positive definite covariance matrix Σ. Let $\tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq ... \geq \tilde{\sigma}_n$ denote the variances and let $I(t) = \mathbb{P}(X_i \geq t, 1 \leq i \leq n)$. If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) = \vec{1}\Sigma^{-1}$ where $\vec{1} = (1, 1, ..., 1)$ with $\alpha_i > 0$ then*

$$I(t) \sim \frac{1}{(\sqrt{2\pi})^n \sqrt{|\Sigma|} (\prod_{i=1}^n \alpha_i) t^n} \exp(-\frac{1}{2} t^2 \vec{1}\Sigma^{-1}\vec{1}^T).$$

Now we find the rate of convergence of the maximum of the eigenvalues when $\{\epsilon_i\}$ are standard normal random variables.

Lemma 7. *Let $\{N_i\}$ be i.i.d. $N(0,1)$ and let*

$$\lambda_{k,\Phi} = \frac{\sqrt{2} A_k N_0}{\sqrt{n}} + \frac{1}{\sqrt{n}} \sum_{t=1}^{[n/2]} N_t \left(2A_k \cos(\frac{2\pi kt}{n}) - 2B_k \sin(\frac{2\pi kt}{n}) \right).$$

Then

$$\frac{\max_{k \in L_n^1} \lambda_{k,\Phi} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda \tag{3.10}$$

and

$$\frac{\max_{k \in L_n^2} \lambda_{k,\Phi} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda \tag{3.11}$$

where $q = q_n = [n/4]$ and a_n and b_n are as in (1.1).

In particular,

$$\frac{\max_{1\leq k\leq [n/2]} \lambda_{k,\Phi}}{\sqrt{\ln n}} \xrightarrow{\mathcal{P}} 1. \tag{3.12}$$

Proof. Consider the case $n = 2m+1$. First observe that $Var\lambda_{k,\Phi} = \sigma_k^2$ and for $k' > k$ we have $Cov(\lambda_{k,\Phi}, \lambda_{k',\Phi}) = \nu_{k,k'}$ where σ_k and $\nu_{k,k'}$ is defined in (3.9). Let $x_q = a_q x + b_q \sim \sqrt{2\ln q}$. By Bonferroni inequalities we have for $j > 1$

$$\sum_{d=1}^{2j}(-1)^{d-1}\tilde{B}_d \leq \mathbb{P}(\max_{k\in L_n^1}\lambda_{k,\Phi} > x_q) \leq \sum_{d=1}^{2j-1}(-1)^{d-1}\tilde{B}_d,$$

where

$$\tilde{B}_d = \sum_{i_1,i_2,\ldots,i_d\in L_n^1,\text{ all distinct}} \mathbb{P}(\lambda_{i_1,\Phi} > x_q, \ldots\lambda_{i_d,\Phi} > x_q)$$

Observe by the choice of p_n we have,

$$\frac{1}{n}\tan(\frac{\pi p_n}{2}) \sim \frac{2n^{1/2+\delta_1}}{\pi n} \to 0.$$

Hence for some $\epsilon > 0$, for large n we have $1-\epsilon < \sigma_k^2 < 1+\epsilon$ and for any $k, k' \in L_n^1$ (or L_n^2) we have $|\nu_{k,k'}| \to 0$ as $n \to \infty$. Next we make the following claim:

$$\sum_{i_i,i_2,\ldots,i_d\in L_n^1,\text{ all distinct}} \mathbb{P}(\lambda_{i_1,\Phi} > x_q, \ldots\lambda_{i_d,\Phi} > x_q) \sim \frac{q^d\exp(-\frac{x_q^2 d}{2})}{d!x_q^d(\sqrt{2\pi})^d},\ \text{ for } d \geq 1. \tag{3.13}$$

To avoid notational complications we show the above claim for $d = 1$ and $d = 2$ and indicate what changes are necessary for higher dimension.

d=1: Using the fact that $\frac{\sigma_k^2}{x_q^2} \to 0$ and

$$\left(1-\frac{1}{x^2}\right)\frac{\exp(-x^2/2)}{\sqrt{2\pi}x} \leq \mathbb{P}(N(0,1) > x) \leq \frac{\exp(-x^2/2)}{\sqrt{2\pi}x}$$

it easily follows that,

$$\sum_{k\in L_n^1}\mathbb{P}(N(0,1) > x_q/\sigma_k) \sim \sum_{k\in L_n^1}\frac{\sigma_k}{\sqrt{2\pi}x_q}\exp(-\frac{x_q^2}{2\sigma_k^2}).$$

Observe that

$$\begin{aligned}\frac{\sum_{k\in L_n^1}\frac{\sigma_k}{\sqrt{2\pi}x_q}\exp(-\frac{x_q^2}{2\sigma_k^2})}{\frac{qp_n}{\sqrt{2\pi}x_q}\exp(-\frac{x_q^2}{2})} &= \frac{1}{qp_n}\sum_{k\in L_n^1}\sigma_k\exp(-\frac{x_q^2}{2}(\frac{1}{\sigma_k^2}-1)) \\ &= \frac{1}{qp_n}\sum_{k\in L_n^1}\sigma_k\exp(-\frac{x_q^2}{2\sigma_k^2}\frac{A_kB_k}{n}\tan(\frac{\pi k}{n})).\end{aligned}$$

Now using the facts that $\frac{A_kB_kx_q^2}{n\sigma_k^2}\tan(\frac{\pi p_n}{n}) \to 0$, $\sup_{k\in L_n^1}\sigma_k^2 \to 1$ and $|\{k : k \in L_n^1\}| \sim qp_n$, it is easy to see that the last term above goes to 1. Since $p_n \sim 1$ the claim is proved for $d = 1$.

d=2: We shall use Lemma 6 for this case. Without loss of generality assume that $\sigma_k^2 > \sigma_{k'}^2$. Let $\alpha = (\alpha_1, \alpha_2)$ where $\alpha = \vec{1}V^{-1}$ and

$$V = \begin{bmatrix}\sigma_k^2 & \nu_{k,k'} \\ \nu_{k,k'} & \sigma_{k'}^2\end{bmatrix}.$$

Hence $(\alpha_1, \alpha_2) = \left(\frac{\sigma_{k'}^2 - \nu_{k,k'}}{|V|}, \frac{\sigma_k^2 - \nu_{k,k'}}{|V|}\right)$. For any $0 < \epsilon < 1$ it easily follows that $\alpha_i > \frac{1-\epsilon}{|V|}$ for large n and for $i = 1, 2$. Hence from Lemma 6 it follows that as $n \to \infty$,

$$\sum_{k,k' \in L_n^1} \mathbb{P}(\lambda_{k,\Phi} > x_q, \lambda_{k',\Phi} > x_q) \sim \sum_{k,k' \in L_n^1} \frac{1}{2\pi\sqrt{|V|}} \frac{\exp(-\frac{1}{2}x_q^2 \vec{1} V^{-1} \vec{1}^T)}{\alpha_1 \alpha_2 x_q^2}.$$

Now denote

$$\psi_{k,k'} = \frac{1}{|V|}\Big[-\frac{A_k B_k}{n}\tan(\frac{\pi k}{n}) - \frac{A_{k'} B_{k'}}{n}\tan(\frac{\pi k'}{n}) + \frac{A_k B_k}{n}\tan(\frac{\pi k}{n})\frac{A_{k'} B_{k'}}{n}\tan(\frac{\pi k'}{n}) - 2\nu_{k,k'} + 2\nu_{k,k'}^2\Big]$$

and observe

$$|x_q^2 \psi_{k,k'}| \leq C\frac{x_q^2}{n}\tan(\frac{\pi p_n}{2}) \to 0 \ \text{ as } \ n \to \infty.$$

$$\begin{aligned}
\frac{\sum_{k,k' \in L_n^1} \frac{1}{2\pi\sqrt{|V|}\alpha_1\alpha_2 x_q^2}\exp(-\frac{1}{2}x_q^2 \vec{1} V^{-1}\vec{1}^T)}{\frac{q^2\exp(-x_q^2)}{2! x_q^2 2\pi}} &= \frac{2}{q^2}\sum_{k,k' \in L_n^1} \frac{1}{\sqrt{|V|}\alpha_1\alpha_2}\exp\left(-\frac{1}{2}x_q^2(\alpha_1+\alpha_2) + x_q^2\right) \\
&= \frac{2}{q^2}\sum_{k,k' \in L_n^1} \frac{|V|^{3/2}}{(\sigma_{k'}^2 - \nu_{k,k'})(\sigma_k^2 - \nu_{k,k'})}\exp\left(-\frac{x_q^2}{2}(\alpha_1+\alpha_2-2)\right) \\
&\leq \frac{2}{q^2}\sum_{k,k' \in L_n^1} \frac{|V|^{3/2}}{(1-\epsilon)^2}\exp(-\frac{x_q^2}{2}\psi_{k,k'}) \\
&\to 1 \text{ as } n \to \infty \text{ and as } \epsilon \to 0.
\end{aligned}$$

Similarly the lower bound can be obtained to show that the claim is true for $d = 2$.

$\mathbf{d \geq 2}$: Now the probability inside the sum in claim (3.13) is $\mathbb{P}(N(0, V_n) \in E_n)$ where $E_n = \{(y_1, y_2, ..., y_d) : y_i > x_q, i = 1, 2..., d\}$, and V_n denote covariance matrix $\{V_n(s,t)\}_{s,t=1}^d$ with $V_n(s,s) = \sigma_{i_s}^2$ and for $s \neq t$ we have $V_n(s,t) = \nu_{i_s i_t}$. Without loss of generality assume that $\sigma_{i_1} \geq \sigma_{i_2} \geq ... \geq \sigma_{i_d}$, since we can always permute the original vector to achieve this and the covariance matrix changes accordingly. Note that as $n \to \infty$ we get

$$\|V_n - I_d\|_\infty \to 0,$$

where $\|A\|_\infty = \max |a_{i,j}|$. As $V_n^{-1} = \sum_{j=0}^\infty (I_d - V_n)^j$ we have $\alpha = \vec{1} + \sum_{j=1}^\infty \vec{1}(I_d - V_n)^j$. Now since $\|I_d - V_n\|_\infty \to 0$ so $\|(I_d - V_n)^j\|_\infty \to 0$ and hence elements of $(I_d - V_n)^j$ goes to zero for all j. So we get that $\alpha_i \in (1-\epsilon, 1+\epsilon)$ for $i = 1, 2, ..., d$ and $0 < \epsilon < 1$ and hence we can again apply Lemma 6. For further calculations it is enough to observe that for $|x| \neq 0$,

$$\frac{x V_n x^T}{|x|^2} = 1 + \frac{1}{|x|^2}\sum_{k=1}^d {x_k}^2 A_{i_k} B_{i_k}\frac{1}{n}\tan(\frac{\pi i_k}{n}) + \frac{1}{|x|^2}\sum_{1 \leq k \neq k' \leq d} x_k x_{k'} \nu_{i_k, i_{k'}}$$

Since the last two term goes to zero in their modulus so given any $\epsilon > 0$, we get for large n

$$1 - \epsilon \leq \lambda_{min}(V_n) \leq \lambda_{max}(V_n) \leq 1 + \epsilon,$$

where $\lambda_{min}(V_n)$ and $\lambda_{max}(V_n)$ denote the minimum and maximum eigenvalue of V_n. Rest of the calculation is similar to $d = 2$ case.

Now using the fact that a_n and b_n are normalizing constants for maxima of standard normal it follows that,

$$\frac{q^d \exp(-\frac{x_q^2 d}{2})}{d! x_q^d} \sim \frac{1}{d!} \exp(-dx).$$

So from the Bonferroni inequalities and observing $\exp(-\exp(-x)) = \sum_{d=0}^{\infty} \frac{(-1)^d}{d!} \exp(-dx)$ it follows that

$$\mathbb{P}(\max_{k \in L_n^1} \lambda_{k,\Phi} > x_q) \to \exp(-\exp(-x)),$$

proving (3.10). For (3.11) calculations are similar and we omit the details.
To prove (3.12) we first observe that,

$$\sum_{k=np_n/2}^{n/2} \mathbb{P}(N(0,1) > x_q/\sigma_k) \leq \frac{n}{2}(1-p_n)\mathbb{P}(N(0,1) > \frac{x_q}{\sqrt{2}}),$$

since $\sigma_k^2 \leq 2$ for $k \leq n/2$. Expanding the expressions for a_n and b_n we get,

$$\frac{x_q^2}{4} = \frac{1}{4}(a_q x + b_q)^2 = o(1) + \frac{\ln q}{2} - \frac{1}{4}\ln(4\pi \ln q) + \frac{x}{2}.$$

Now

$$\begin{aligned}
\frac{n(1-p_n)}{2}\mathbb{P}(N(0,2) > x_q) &\leq C\frac{n(1-p_n)}{2}\frac{\exp(-\frac{x_q^2}{4})}{x_q} \\
&\sim Cn^{-1/2}\frac{n(1-p_n)}{2\sqrt{\ln q}} \\
&\sim C\frac{1}{n^{\delta_1}\sqrt{\ln q}} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Breaking up the set $L_1 = \{k : 1 \leq k \leq [n/2]$ and k is even $\}$ into L_n^1 and $\tilde{L}_n^1 = \{k : [np_n/2] < k < [n/2]$ and k is even$\}$ we get,

$$\begin{aligned}
\mathbb{P}(\max_{k \in L_1} \lambda_{k,\Phi} > x_q) &= \mathbb{P}(\max(\max_{k \in L_n^1} \lambda_{k,\Phi}, \max_{k \in \tilde{L}_n^1} \lambda_{k,\Phi}) > x_q) \\
&\leq \mathbb{P}(\max_{k \in L_n^1} \lambda_{k,\Phi} > x_q) + \mathbb{P}(\max_{k \in \tilde{L}_n^1} \lambda_{k,\Phi} > x_q) \\
&\leq \mathbb{P}(\max_{k \in L_n^1} \lambda_{k,\Phi} > x_q) + \sum_{t=[np_n/2]}^{[n/2]} \mathbb{P}(N(0,\sigma_k^2) > x_q) \\
&= \mathbb{P}(\max_{k \in L_n^1} \lambda_{k,\Phi} > x_q) + o(1).
\end{aligned}$$

Hence the upper bound is obtained. The lower bound easily follows from (3.10). Similar calculations for set $L_2 = \{k : 1 \leq k < [n/2]$ and k is odd$\}$ can be done. To complete the proof it is enough to observe that,

$$\mathbb{P}(\frac{\max_{1 \leq k < [n/2]} \lambda_{k,\Phi}}{\sqrt{\ln n}} > 1-\epsilon) \leq \mathbb{P}(\frac{\max_{k \in L_1} \lambda_{k,\Phi}}{\sqrt{\ln n}} > 1-\epsilon) + \mathbb{P}(\frac{\max_{k \in L_2} \lambda_{k,\Phi}}{\sqrt{\ln n}} > 1-\epsilon)$$

and the last two probabilities go to zero. This completes the proof of the Lemma. □

Remark 5. *By calculations similar to above, it can be shown that for* $\sigma^2 = n^{-c}$ *where* $c > 0$*,*

$$\sum_{i_i,i_2,\ldots,i_d \in L_n^1, \text{ all distinct}} \mathbb{P}((1+\sigma^2)^{1/2}\lambda_{i_1,\Phi} > x_q, \ldots, (1+\sigma^2)^{1/2}\lambda_{i_d,\Phi} > x_q) \le \frac{K^d}{d!} \tag{3.14}$$

for some constant $K > 0$*. This will be used in the proof of Theorem 9.*

We now consider the symmetric circulant matrix with the general moving average process $\{x_i\}$.

Theorem 9. *Let* SC_n *be the symmetric circulant matrix with entries from* $\{x_n\}$*, the two sided moving average process (3.1). Let* $E(\epsilon_i) = 0$*,* $E(\epsilon_i^2) = 1$*,* $E|\epsilon_i|^s < \infty$ *for some* $s > 2$ *and*

$$\sum_{j=-\infty}^{\infty} |a_j||j|^{1/2} < \infty \text{ and } f(\omega) > 0 \text{ for all } \omega \in [0, 2\pi]. \tag{3.15}$$

If $\lambda_{k,x}$ *denote the eigenvalues of* $\frac{1}{\sqrt{n}}SC_n$ *with input* $\{x_i\}$ *then*

$$\frac{\max_{k \in L_n^1} \lambda_{k,x} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda \tag{3.16}$$

and

$$\frac{\max_{k \in L_n^2} \lambda_{k,x} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda \tag{3.17}$$

where $q = q_n = [n/4]$ *and* a_n *and* b_n *are as in (1.1).*

Proof. Again for simplicity we assume that $n = 2m + 1$.

Truncation: Define

$$\tilde{\epsilon}_t = \epsilon_t I(|\epsilon_t| \le n^{1/s}), \quad \bar{\epsilon}_t = \tilde{\epsilon}_t - E\tilde{\epsilon}_t, \quad \tilde{x}_t = \sum_{j=-\infty}^{\infty} a_j \tilde{\epsilon}_{t-j}, \quad \bar{x}_t = \sum_{j=-\infty}^{\infty} a_j \bar{\epsilon}_{t-j},$$

$$\lambda_{k,\tilde{x}} = \frac{1}{\sqrt{n}}[\tilde{x}_0 + 2\sum_{t=1}^{m} \tilde{x}_t \cos\frac{2\pi kt}{n}], \quad \lambda_{k,\bar{x}} = \frac{1}{\sqrt{n}}[\bar{x}_0 + 2\sum_{t=1}^{m} \bar{x}_t \cos\frac{2\pi kt}{n}].$$

Note that

$$\begin{aligned}
\sqrt{n}\lambda_{k,\bar{x}} &= \bar{x}_0 + 2\sum_{t=1}^{m} \bar{x}_t \cos\frac{2\pi kt}{n} \\
&= \tilde{x}_0 + 2\sum_{t=1}^{m} \tilde{x}_t \cos\frac{2\pi kt}{n} + \sum_{j=-\infty}^{\infty} a_j E(\tilde{\epsilon}_{-j}) + 2\sum_{t=1}^{m}[\sum_{j=-\infty}^{\infty} a_j E(\tilde{\epsilon}_{t-j})]\cos\frac{2\pi kt}{n} \\
&= \sqrt{n}\lambda_{k,\tilde{x}} + [\sum_{j=-\infty}^{\infty} a_j E(\tilde{\epsilon}_j)][1 + 2\sum_{t=1}^{m} \cos\frac{2\pi kt}{n}] \\
&= \sqrt{n}\lambda_{k,\tilde{x}}.
\end{aligned}$$

Choose δ such that $(\frac{1}{2} - \frac{1}{s} - \delta) > 0$ and observe

$$n^\delta E[\max_{1 \le k \le [n/2]} |\lambda_{k,\bar{x}} - \lambda_{k,x}|] \quad = \quad n^\delta E[\max_{1 \le k \le [n/2]} |\lambda_{k,\tilde{x}} - \lambda_{k,x}|]$$

$$
\begin{aligned}
&\leq \frac{2}{n^{1/2-\delta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| E(|\epsilon_{t-j}| I(|\epsilon_{t-j}| > n^{1/s})) \\
&\leq \frac{2}{n^{1/2-\delta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| \big[n^{1/s} \mathbb{P}(|\epsilon_{t-j}| > n^{1/s}) + \int_{n^{1/s}}^{\infty} \mathbb{P}(|\epsilon_{t-j}| > u) du \big] \\
&= I_1 + I_2, \text{ say,}
\end{aligned}
$$

and

$$
\begin{aligned}
I_1 &= \frac{2}{n^{1/2-\delta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| n^{1/s} \mathbb{P}(|\epsilon_{t-j}| > n^{1/s}) \\
&\leq \frac{2}{n^{1/2-\delta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| n^{1/s} \frac{1}{n} E(|\epsilon_{t-j}|^s) \\
&\leq \frac{E(|\epsilon_0|^s)}{n^{1/2-1/s-\delta}} \sum_{j=-\infty}^{\infty} |a_j|.
\end{aligned}
$$

and right side goes to zero as $n \to \infty$ since $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Similarly

$$
\begin{aligned}
I_2 &= \frac{2}{n^{1/2-\delta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| \int_{n^{1/s}}^{\infty} \mathbb{P}(|\epsilon_{t-j}| > u) du \\
&\leq \frac{2}{n^{1/2-\delta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| \int_{n^{1/s}}^{\infty} \frac{E(|\epsilon_{t-j}|^s)}{u^s} du \\
&\leq \frac{2E(|\epsilon_0|^s)}{(s-1)n^{1/2-\delta}} \sum_{t=0}^{m} \sum_{j=-\infty}^{\infty} |a_j| \frac{1}{n^{1-1/s}} \\
&\leq \frac{E(|\epsilon_0|^s)}{(s-1)n^{1/2-1/s-\delta}} \sum_{j=-\infty}^{\infty} |a_j|
\end{aligned}
$$

and goes to zero as $n \to \infty$ for above choice of δ. Hence $\max_{1 \leq k \leq [n/2]} |\lambda_{k,\bar{x}} - \lambda_{k,x}| = o_p(n^{-\delta})$. By Lemma 5 we get that

$$
\max_{k \in L_n^1} \Big| \frac{\lambda_{k,\bar{x}}}{a_q \sqrt{2\pi f(\omega_k)}} - \frac{2A_k}{\sqrt{n} a_q} \sum_{t=1}^{m} \bar{\epsilon}_t \cos(\frac{2\pi kt}{n}) + \frac{2B_k}{\sqrt{n} a_q} \sum_{t=1}^{m} \bar{\epsilon}_t \sin(\frac{2\pi kt}{n}) \Big| = o_p(\frac{\sqrt{\ln n}}{n^{\delta_1}}).
$$

Similar conclusions can drawn for maximum over L_n^2. So to show the result it is enough to show that,

$$
\frac{\max_{k \in L_n^1} \lambda_{k,\epsilon} - b_q}{a_q} \xrightarrow{\mathcal{D}} \Lambda, \tag{3.18}
$$

where

$$
\lambda_{k,\epsilon} = \frac{\sqrt{2} A_k \bar{\epsilon}_0}{\sqrt{n}} + \frac{2A_k}{\sqrt{n}} \sum_{t=1}^{m} \bar{\epsilon}_t \cos(\frac{2\pi kt}{n}) - \frac{2B_k}{\sqrt{n}} \sum_{t=1}^{m} \bar{\epsilon}_t \sin(\frac{2\pi kt}{n}).
$$

Normal Approximation: This step is similar to the proof of Lemma 4. Now we use Lemma 3 to approximate $\lambda_{k,\epsilon}$ with Gaussian random variables. Let $d \geq 1$ and $i_1, i_2, ... i_d$ be d distinct numbers from L_n^1.

$$
v_d(0) = \sqrt{2}(A_{i_1}, ..., A_{i_d}) \text{ and}
$$

$$v_d(t) = 2\left(A_{i_1}\cos(\frac{2\pi i_1 t}{n}) - B_{i_1}\sin(\frac{2\pi i_1 t}{n}), ..., A_{i_d}\cos(\frac{2\pi i_d t}{n}) - B_{i_d}\sin(\frac{2\pi i_d t}{n})\right)$$

Let $S_n = \sum_{t=0}^{m} \bar{\epsilon}_t v_d(t)$, and observe that $cov(S_n) = V_n$ where V_n is the covariance matrix with diagonal entries $V_n(k,k) = B_n \sigma^2_{i_k}$ and off-diagonal entries $V_n(k,k') = B_n \nu_{i_k,i_{k'}}$ and $B_n = Var(\bar{\epsilon}_t)n \sim n$. We have infact already seen that,

$$\|\frac{V_n}{B_n} - I_d\|_\infty \to 0.$$

To apply Lemma 3 we define

$$\epsilon'_t = {B_n}^{1/2} V_n^{-1/2} \bar{\epsilon}_t v_d(t) \text{ for } 0 \le t \le [n/2] \text{ and } S'_n = \sum_{t=0}^{m} \epsilon'_t.$$

It is easy to see that $Cov(S'_n) = B_n I_d$. Also note the since $\|(\frac{V_n}{B_n})^{-1} - I_d\|_\infty < c'$ for some constant $c' > 0$ and hence for large n we get that $|\epsilon'_t| < 2dCn^{1/s}$ for some constant C . Hence ϵ'_t are sequence of independent, mean zero random vectors with moment generating function finite in a neighborhood of zero. For verification of the other conditions choose $\tilde{\alpha} = \frac{c_1}{n^{1/s}2dC}$, where c_1 is a constant to be chosen later. Hence,

$$\begin{aligned}
\tilde{\alpha}\sum_{t=0}^{m} E|\epsilon'_t|^3 \exp(\tilde{\alpha}|\epsilon'_t|) &\le \tilde{\alpha} B_n^{3/2} |V_n|^{-3/2} (2d)^3 \sum_{t=0}^{m} E|\bar{\epsilon}_t|^3 \exp(c_1) \\
&\le 4c_1 \exp(c_1) C^2 d^2 n^{(1-\frac{1}{s})} E|\bar{\epsilon}_t|^3 \\
&\le 4c_1 \exp(c_1) C^2 d^2 n^{(1-\frac{\delta_2}{s})} E|\epsilon_t|^{2+\delta_2},
\end{aligned}$$

where $\delta_2 \in (0,1)$ such that $E|\epsilon_t|^{2+\delta_2} < \infty$. Now choose c_1 such that the the required condition is satisfied. Similar calculations show that

$$\beta_n = B_n^{-3/2} \sum_{t=0}^{m} E|\epsilon'_t|^3 \exp(\tilde{\alpha}|\epsilon'_t|) \le Cn^{-c_3},$$

where $c_3 = \frac{1}{2} - \frac{1-\delta_2}{s} > 0$. The rest of calculations are similar to the proof of Lemma 4. Let $\bar{\sigma}^2 = n^{-c_3}$ and if N'_t are i.i.d. $N(0, \bar{\sigma}^2 Cov(\epsilon'_t))$ independent of ϵ'_t and $\tilde{p}_n$ be density of $S^*_n = \frac{1}{\sqrt{B_n}} \sum_{t=0}^{m} (\epsilon'_t + N'_t)$, then,

$$\tilde{p}_n(x) = \phi_{(1+\bar{\sigma}^2)I_d}(x)(1 + o(1)),$$

uniformly for all x such that $|x|^3 = o(n^{(\frac{1}{2}-\frac{1}{s})})$. Here ϕ_C denotes the d-dimensional normal density with covariance matrix C.

Let $\sigma^2 = Var(\bar{\epsilon})\bar{\sigma}^2 \sim n^{-c_3}$ and observe that $N'_t \overset{\mathcal{D}}{=} B_n^{1/2} V_n^{-1/2} \sigma N_t v_d(t)$, where N_t are i.i.d. $N(0,1)$ for $t = 0, 1, ..., m$.

Now define the following for $x \in \mathbb{R}^d$, $\|x\|_0 = \min_{1 \le i \le d} x_i$. Recall $|\cdot|$ denotes the Euclidean norm and observe that $\|x + y\|_0 \le \|x\|_0 + |y|$. Let $\overline{S}_n = \frac{1}{\sqrt{n}} \sum_{t=0}^{m} (\bar{\epsilon}_t + N_t) v_d(t)$. Then note that $S^*_n = B_n^{1/2} V_n^{-1/2} \overline{S}_n$.

Let $r_n = o(n^{(\frac{1}{2}-\frac{1}{s})})$ and denote $K_n = \{y \in \mathbb{R}^d : \|B_n^{-1/2} V_n^{1/2} y\|_0 > x_q\}$ and break it into the following two sets $K_{1,n} = \{y \in \mathbb{R}^d : \|B_n^{-1/2} V_n^{1/2} y\|_0 > x_q, |y| > r_n\}$ and $K_{2,n} = \{y \in \mathbb{R}^d : \|B_n^{-1/2} V_n^{1/2} y\|_0 > x_q, |y| \le r_n\}$.

Then

$$
\begin{aligned}
\mathbb{P}(\|\overline{S}_n\|_0 > x_q) &\leq \mathbb{P}(\|B_n^{-1/2}V_n^{1/2}S_n^*\|_0 > x_q) \\
&= \int_{K_n} \tilde{p}_n(y)dy \\
&= \int_{K_{2,n}} \tilde{p}_n(y)dy + \int_{K_{1,n}} \tilde{p}_n(y)dy \\
&= (1+o(1)) \int_{K_{2,n}} \phi_{(1+\sigma^2)I_d}(y)dy + \int_{K_{1,n}} \tilde{p}_n(y)dy \\
&= (1+o(1)) \int_{K_n} \phi_{(1+\sigma^2)I_d}(y)dy - (1+o(1)) \int_{K_{1,n}} \phi_{(1+\sigma^2)I_d}(y)dy + \int_{K_{1,n}} \tilde{p}_n(y)dy \\
&= (1+o(1))\mathbb{P}(\|(1+\sigma^2)^{1/2}\frac{1}{\sqrt{n}}\sum_{t=0}^{m} N_t v_d(t)\|_0 > x_q) \\
&\quad - (1+o(1)) \int_{K_{1,n}} \phi_{(1+\sigma^2)I_d}(y)dy + \int_{K_{1,n}} \tilde{p}_n(y)dy
\end{aligned}
$$

The third integral is less than

$$
\mathbb{P}\left(|\frac{1}{\sqrt{n}}\sum_{t=0}^{m} B_n^{1/2}V_n^{-1/2}(\epsilon_t + \sigma N_t)v_d(t)| > r_n\right).
$$

Now we using the fact that $\|(\frac{V_n}{B_n})^{-1/2}\|_\infty \leq C_5$ for some constant $C_5 > 0$ and calculations similar to Corollary 1 of Bose, Mitra and Sen (2008) we get that the third integral is bounded by $K_1 \exp(-K_2 n^{\delta_3})$ for some constant $K_1, K_2 > 0$ and depending only on d and $\delta_3 > 0$. Similarly the integral in the second term is bounded by,

$$
\int_{|y|>r_n} \phi_{(1+\sigma^2)I_d}(y)dy \leq 2d\exp(-\frac{r_n}{2d}).
$$

From all the above observations it is easy to conclude that, for $\epsilon_n \to 0$ we get uniformly over d distinct tuples $i_1, i_2, ...i_d \in L_n^1$ that

$$
\begin{aligned}
\left|\mathbb{P}(\|\overline{S}_n\|_0 > x_q) - P(\|(1+\sigma^2)\frac{1}{\sqrt{n}}\sum_{t=0}^{m} N_t v_d(t)\|_0 > x_q)\right| \quad &\leq \quad \epsilon_n P(\|(1+\sigma^2)\frac{1}{\sqrt{n}}\sum_{t=0}^{m} N_t v_d(t)\|_0 > x_q) \\
&\quad + K_3\exp(-K_4 n^{\delta_3}), \qquad (3.19)
\end{aligned}
$$

where K_3, K_4 are constants depending on d. Now define,

$$
\lambda_{k,\epsilon+\sigma N} = \frac{\sqrt{2}A_k}{\sqrt{n}}(\overline{\epsilon}_0 + \sigma N_0) + \frac{2A_k}{\sqrt{n}}\sum_{t=1}^{m}(\overline{\epsilon}_t + \sigma N_t)\cos(\frac{2\pi kt}{n}) - \frac{2B_k}{\sqrt{n}}\sum_{t=1}^{m}(\overline{\epsilon}_t + \sigma N_t)\sin(\frac{2\pi kt}{n}).
$$

By arguments similar to Step 2 of Theorem 4 and using (3.14) and (3.19) it follows that,

$$
\left|\mathbb{P}(\max_{k\in L_n^1} \lambda_{k,\epsilon+\sigma N} > x_q) - \mathbb{P}(\max_{k\in L_n^1}(1+\sigma^2)^{1/2}\lambda_{k,\Phi} > x_q)\right| \to 0,
$$

where $\lambda_{k,\Phi}$ is defined in Lemma 7. Now since $\max_{k\in L_n^1}\lambda_{k,\Phi}=O_P(\sqrt{\ln n})$ and $\sigma^2=n^{-c_3}$ we get as $n\to\infty$,

$$\mathbb{P}\left(\max_{k\in L_n^1}(1+\sigma^2)^{1/2}\lambda_{k,\Phi}>x_q\right)\to\Lambda(x).$$

It follows that (3.18) is true. Similar calculations hold for the second part of the Theorem. □

Theorem 10. *If* $\{\lambda_{k,x}\}$ *are the eigenvalues of* $\frac{1}{\sqrt{n}}SC_n$ *then under the assumptions of Theorem 9,*

$$\frac{\max_{1\le k\le[n/2]}\frac{\lambda_{k,x}}{\sqrt{2\pi f(w_k)}}}{\sqrt{\ln n}}\xrightarrow{\mathcal{P}}1 \quad where \quad \omega_k=\frac{2\pi k}{n}.$$

Proof. As before we assume $n=2m+1$. It is now easy to see from the truncation part of Theorem 9 and Lemma 5 that it is enough to show that,

$$\frac{\max_{1\le k\le[n/2]}\lambda_{k,\epsilon}}{\sqrt{\ln n}}\xrightarrow{\mathcal{P}}1,$$

where,

$$\lambda_{k,\epsilon}=\frac{\sqrt{2}A_k\bar{\epsilon}_0}{\sqrt{n}}+\frac{2A_k}{\sqrt{n}}\sum_{t=1}^{m}\bar{\epsilon}_t\cos(\frac{2\pi kt}{n})-\frac{2B_k}{\sqrt{n}}\sum_{t=1}^{m}\bar{\epsilon}_t\sin(\frac{2\pi kt}{n}),$$

and $\bar{\epsilon}_t=\epsilon_tI(|\epsilon_t|\le n^{1/s})-E\epsilon_tI(|\epsilon_t|\le n^{1/s})$. The steps are same as the steps required to prove (3.12) in Lemma 7 and observe from there that to complete the proof it is enough to show,

$$\sum_{k=[np_n/2]+1}^{[n/2]}\mathbb{P}(\lambda_{k,\epsilon}>x_q)\to0 \text{ as } n\to\infty. \tag{3.20}$$

Denote

$$m=[n/2],\ v_1(0)=\sqrt{2}A_k \text{ and } v_1(t)=2A_k\cos(\frac{2\pi kt}{n})-2B_k\sin(\frac{2\pi kt}{n}).$$

Since $\{\bar{\epsilon}_tv_1(t)\}$ is a sequence of bounded independent mean zero random variable, by applying Bernstein's inequality we get

$$\begin{aligned}\mathbb{P}(\frac{1}{\sqrt{m}}\sum_{t=0}^{m}\bar{\epsilon}_tv_1(t)>x_q) &\le \mathbb{P}(|\sum_{t=0}^{m}\bar{\epsilon}_tv_1(t)|>\sqrt{m}x_q)\\ &= \mathbb{P}(|\sum_{t=0}^{m}\bar{\epsilon}_tv_1(t)|>m\frac{x_q}{\sqrt{m}})\\ &\le 2\exp\big(-\frac{mx_q^2}{2\sum_{t=0}^{m}Var(\epsilon_tv_1(t))+\frac{2}{3}Cn^{1/s}m\frac{x_q}{\sqrt{m}}}\big).\end{aligned}$$

Denote by $C_k=A_kB_k$ and observe

$$\begin{aligned}D &:= \frac{mx_q^2}{2\sum_{t=0}^{m}Var(\epsilon_tv_1(t))+\frac{2}{3}Cn^{1/s}m\frac{x_q}{\sqrt{m}}}\\ &\ge \frac{x_q^2}{4\frac{1}{n}\sum_{t=0}^{m}Var(\epsilon_tv_1(t))+\frac{4}{3}Cn^{1/s-1/2}x_q}\end{aligned}$$

$$
\begin{aligned}
&= \frac{x_q^2}{4(1+\frac{C_k}{n}\tan\frac{\pi k}{n}) + \frac{4}{3}\frac{Cx_q}{n^{1/2-1/s}}} \\
&\geq \frac{x_q^2}{4(1+\frac{2}{\pi}) + o(1)} \geq \frac{x_q^2}{8}.
\end{aligned}
$$

Therefore

$$
\mathbb{P}(|\sum_{t=0}^{m} \bar{\epsilon}_t v_1(t)| > \sqrt{m} x_q) \leq 2\exp(-\frac{x_q^2}{8}),
$$

and hence

$$
\sum_{t=[np_n/2]}^{[n/2]} \mathbb{P}(\frac{1}{\sqrt{n}}|\sum_{t=0}^{m} \bar{\epsilon}_t v_1(t)| > x_q) \quad \leq \quad n(1-p_n)\exp(-\frac{x_q^2}{4}) \leq \frac{C}{n^{\delta_1}(\ln n)^{1/4}} \longrightarrow 0.
$$

□

Remark 6. *Note that the above calculation can be imitated with ease to conclude that when $\mu = 0$,*

$$
\frac{\max_{1\leq k\leq [n/2]} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}}{\sqrt{\ln n}} \xrightarrow{\mathcal{P}} 1.
$$

The proof is same, with only the normalizing constants changed suitably.

Remark 7. *If we include λ_0 in the definition $M(n^{-1/2}SC_n, f)$ that is, if $\overline{M}(n^{-1/2}SC_n, f) = \max_{0\leq k\leq [n/2]} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$ then it is easy to see that if we assume the mean μ of $\{\epsilon_i\}$ to be non-zero then*

$$
\overline{M}(n^{-1/2}SC_n, f) - |\mu|\sqrt{n} \xrightarrow{\mathcal{D}} N(0,2).
$$

Remark 8. *In Theorem 9 we were unable to consider the convergence over $L_n^1 \cup L_n^2$. It is not clear if the maximum over the two subsets are asymptotically independent and hence it is not clear if we would continue to obtain the same limit. Observe that for example, if k is odd and k' is even then*

$$
Cov(\lambda_{k,x}, \lambda_{k',x}) = \frac{-D_{k,k'}}{n}\cot\frac{\pi(k+k')}{2n} - \frac{E_{k,kk'}}{n}\cot\frac{\pi(k'-k)}{2n}.
$$

So for this covariance terms going to zero we have to truncate the index set from below appropriately. For instance, in the Gaussian case we may consider the set $L' = \{(k,k') : 1 < k < [np_n/2],\ k + [nq_n/2] < k' < [np_n/2]\}$ with $q_n \to 0$, we can approximate it by the i.i.d. counterparts since $\sup_{k,k'\in L'} |Cov(\lambda_{k,x}, \lambda_{k',x})| \to 0$ as $n \to \infty$. The complicacy comes when dealing with the complement of L' since it has no longer small cardinality. We are looking into this problem actively.

3.3 k Circulant matrix

We assume $n = k^2 + 1$ and at first give a brief description of its eigenvalues.

$$
\nu = \nu_n := \cos(2\pi/n) + i\sin(2\pi/n),\ i^2 = -1 \ \text{ and } \ \lambda_k = \sum_{l=0}^{n-1} x_l \nu^{kl},\ \ 0 \leq j < n. \tag{3.21}
$$

For any positive integers k and n, let $p_1 < p_2 < \ldots < p_c$ be their common prime factors so that,

$$n = n' \prod_{q=1}^{c} p_q^{\beta_q} \quad \text{and} \quad k = k' \prod_{q=1}^{c} p_q^{\alpha_q}.$$

Here $\alpha_q,\ \beta_q \geq 1$ and $n',\ k',\ p_q$ are pairwise relatively prime. For any positive integer s, let $\mathbb{Z}_s = \{0, 1, 2, \ldots, s-1\}$. Define the following sets

$$S(x) = \{xk^b \bmod n' : b \geq 0\},\ 0 \leq x < n'.$$

Let $g_x = |S(x)|$. Define

$$\upsilon_{k,n'} := |\{x \in \mathbb{Z}_{n'} : g_x < g_1\}|.$$

We observe the following about the sets $S(x)$.

1. $S(x) = \{xk^b \bmod n' : 0 \leq b < |S(x)|\}$.
2. For $x \neq u$, either $S(x) = S(u)$ or, $S(x) \cap S(u) = \phi$. As a consequence, the distinct sets from the collection $\{S(x) : 0 \leq x < n'\}$ forms a partition of $\mathbb{Z}_{n'}$.

We shall call $\{S(x)\}$ the *eigenvalue partition* of $\{0, 1, 2, \ldots, n-1\}$ and we will denote the partitioning sets and their sizes by

$$\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{l-1}\}, \text{ and } \quad n_i = |\mathcal{P}_i|,\ 0 \leq i < l.$$

Define

$$y_j := \prod_{t \in \mathcal{P}_j} \lambda_{ty}, \quad j = 0, 1, \ldots, l-1 \quad \text{where} \quad y = n/n'.$$

Then the characteristic polynomial of $A_{k,n}$ is given by

$$\chi\left(A_{k,n}\right) = \lambda^{n-n'} \prod_{j=0}^{\ell-1} \left(\lambda^{n_j} - y_j\right), \tag{3.22}$$

and this provides a formula solution for the eigenvalues. By Lemma 7 of Bose, Mitra and Sen (2008), the eigenvalue partition of $\{0, 1, 2, \ldots, n-1\}$ contains exactly $\lfloor \frac{n}{4} \rfloor$ sets of size 4, say $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{\lfloor \frac{n}{4} \rfloor}\}$. Since each $\mathcal{P}_i$ is self-conjugate, we can find a set $\mathcal{A}_i \subset \mathcal{P}_i$ of size 2 such that

$$\mathcal{P}_j = \{x : x \in \mathcal{A}_j \text{ or } n - x \in \mathcal{A}_j\}.$$

Since we shall be using the bounds given in Walker (1965) we define a few relevant notation for convenience. Define,

$$I_{x,n}(\omega_j) = \frac{1}{n}\Big|\sum_{l=1}^{n} x_l e^{i\omega_j l}\Big|^2, \quad I_{\epsilon,n}(\omega_j) = \frac{1}{n}\Big|\sum_{l=1}^{n} \epsilon_l e^{i\omega_j l}\Big|^2,$$

$$J_{x,n}(\omega) = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} x_l e^{i\omega_j l}, \quad J_{\epsilon,n}(\omega) = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \epsilon_l e^{i\omega_j l},$$

$$\beta_{x,n}(t) = \prod_{j \in A_t} I_{x,n}(\omega_j), \quad \beta_{\epsilon,n}(t) = \prod_{j \in \mathcal{A}_t} I_{\epsilon,n}(\omega_j),$$

$$A(\omega_j) = \sum_{t=-\infty}^{\infty} a_t e^{i\omega_j t}, \quad T_n(\omega_j) = I_{x,n}(\omega_j) - |A(\omega_j)|^2 I_{\epsilon,n}(\omega_j),$$

$$\widetilde{\beta}_{x,n}(t) := \frac{\beta_{x,n}(t)}{\prod_{j\in\mathcal{A}_t} 2\pi f(\omega_j)} \text{ and } \mathrm{M}(n^{-1/2}A_{k,n}, f) = \max_{1\le t\le q} \big(\widetilde{\beta}_{x,n}(t)\big)^{1/4}.$$

Theorem 11. *Let $\{x_q\}$ be the two sided moving average process (3.1) where $E(\epsilon_i)=0$, $E(\epsilon_i^2)=1$, $E|\epsilon_i|^{2+\delta}<\infty$ for some $\delta>0$ and*

$$\sum_{j=-\infty}^{\infty} |a_j||j|^{1/2} < \infty \text{ and } f(\omega) > \alpha > 0 \text{ for all } \omega \in [0, 2\pi]. \tag{3.23}$$

Then

$$\frac{M(n^{-1/2}A_{k,n}, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

as $n\to\infty$ where $q=q(n)=\lfloor \frac{n}{4} \rfloor$ and c_q, d_q are same as defined in Theorem 6.

Proof. Observe that,

$$\widetilde{\beta}_{x,n}(t) := \frac{\beta_{x,n}(t)}{\prod_{j\in\mathcal{A}_t} 2\pi f(\omega_j)} = \beta_{\epsilon,n}(t) + R_n(t),$$

where

$$R_n(t) = I_{\epsilon,n}(\omega_{t_1})\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})} + I_{\epsilon,n}(\omega_{t_2})\frac{T_n(\omega_{t_1})}{2\pi f(\omega_{t_1})} + \frac{T_n(\omega_{t_1})}{2\pi f(\omega_{t_1})}\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}.$$

Let $q=\lfloor \frac{n}{4} \rfloor$. Recall that,

$$\|n^{-1/2}A_{k,n}\| = \max_{1\le t\le q} \big(\beta_{x,n}(t)\big)^{1/4} \text{ and } \mathrm{M}(n^{-1/2}A_{k,n}, f) = \max_{1\le t\le q} \big(\widetilde{\beta}_{x,n}(t)\big)^{1/4}.$$

We shall show $\max_{1\le t\le q} |\widetilde{\beta}_{x,n}(t) - \beta_{\epsilon,n}(t)| \to 0$ in probability.
Now

$$|\widetilde{\beta}_{x,n}(t) - \beta_{\epsilon,n}(t)| \le |I_{\epsilon,n}(\omega_{t_1})\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}| + |I_{\epsilon,n}(\omega_{t_2})\frac{T_n(\omega_{t_1})}{2\pi f(\omega_{t_1})}| + |\frac{T_n(\omega_{t_1})}{2\pi f(\omega_{t_1})}\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}|,$$

Note that

$$\max_{1\le t\le q} |I_{\epsilon,n}(\omega_{t_1})\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}| \le \frac{1}{2\pi\alpha} \max_{1\le t\le n} |I_{\epsilon,n}(\omega_t)| \max_{1\le t\le n} |T_n(\omega_t)|.$$

From Walker (1965) (page 112) we get

$$\max_{1\le t\le n} |T_n(\omega_t)| = O_p(n^{-\delta}(\ln n)^{1/2}).$$

Also it is known from Davis and Mikosch (1999) that

$$\max_{1\le t\le n} |I_{\epsilon,n}(\omega_t)| = O_p(\ln n).$$

Therefore

$$\max_{1\le t\le q} |I_{\epsilon,n}(\omega_{t_1})\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}| = O_p(n^{-\delta}(\ln n)^{3/2}) \text{ and } \max_{1\le t\le q} |\frac{T_n(\omega_{t_1})}{2\pi f(\omega_{t_1})}\frac{T_n(\omega_{t_2})}{2\pi f(\omega_{t_2})}| = O_p(n^{-2\delta}\ln n).$$

Combining all this we have

$$\max_{1\le t\le q} |R_n(t)| = \max_{1\le t\le q} |\widetilde{\beta}_{x,n}(t) - \beta_{\epsilon,n}(t)| = O_p(n^{-\delta}(\ln n)^{3/2}).$$

Note that

$$\big(\beta_{\epsilon,n}(t)\big)^{1/4} - |R_n(t)|^{1/4} \le \big(\widetilde{\beta}_{x,n}(t)\big)^{1/4} \le \big(\beta_{\epsilon,n}(t)\big)^{1/4} + |R_n(t)|^{1/4}$$

and hence

$$\big|\max_{1\le t\le q} \big(\widetilde{\beta}_{x,n}(t)\big)^{1/4} - \max_{1\le t\le q} \big(\beta_{\epsilon,n}(t)\big)^{1/4}\big| = O_p(n^{-\delta/4}(\ln n)^{3/8}).$$

From Theorem 6 we know

$$\frac{\max_{1\le t\le q} \big(\beta_{\epsilon,n}(t)\big)^{1/4} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

Hence

$$\frac{\mathrm{M}(n^{-1/2}A_{k,n}, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

□

4 Concluding Remarks

In Theorems 3 and 5 we saw that the nature of the limiting distribution depends on whether the input sequence has mean zero or not. Results from Adamczak (2008) and Bose and Sen (2007) suggest that the same should happen for the Toeplitz matrix. It would be interesting to find out the limiting distribution of the spectral norm of the Toeplitz matrix in general.

Theorem 4 shows that the joint distribution of the maximum and minimum of the eigenvalues of SC_n behave like the maximum and minimum of i.i.d. standard normal entries. It follows that the distribution of the range of the spectrum is the convolution of two Gumbel distributions. We are investigating what happens in general to the spectral gaps.

It will be interesting to see if results can be established for the spectral norm in the dependent case. The spectral density is expected to appear in some form in the limit. This seems to be a difficult problem.

For SC_n with inputs from linear process we have shown that the maximum over certain subsets converges in distribution to the Gumbel distribution. It is not clear what happens when maximum is taken over all the eigenvalues and this is an interesting problem.

Finally, for k circulant matrices, results are known only when $n = k^2 + 1$. It would be interesting to derive results for other cases where the structure of the eigenvalues are known.

We are currently working on the above issues.

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