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Spectral Properties of n-Normal Operators

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Abstract. For a bounded linear operator T on a complex Hilbert space and $n \in \mathbb{N}$, T is said to be n-normal if $T^*T^n = T^nT^*$. In this paper we show that if T is a 2-normal operator and satisfies $\sigma(T) \cap (-\sigma(T)) \subset \{0\}$, then T is isoloid and $\sigma(T) = \sigma_a(T)$. Under the same assumption, we show that if z and w are distinct eigenvalues of T, then $\ker(T - z) \perp \ker(T - w)$. And if non-zero number $z \in \mathbb{C}$ is an isolated point of $\sigma(T)$, then we show that $\ker(T - z)$ is a reducing subspace for T. We show that if T is a 2-normal operator satisfying $\sigma(T) \cap (-\sigma(T)) = \emptyset$, then Weyl's theorem holds for T. Similarly, we show spectral properties of n-normal operators under similar assumption. Finally, we introduce (n, m)-normal operators and show some properties of this kind of operators.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle\ ,\ \rangle$ and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, the spectrum, the approximate point spectrum and the point spectrum of T are denoted by $\sigma(T), \sigma_a(T)$ and $\sigma_p(T)$, respectively. The residual spectrum $\sigma_r(T)$ of T is $\sigma_r(T) = \{z \in \mathbb{C} : \exists x \in \mathcal{H}; x \neq 0, (T - z)^*x = 0\}$. It is well known that $\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$. For $T \in B(\mathcal{H})$, T^* denotes the adjoint operator of T. T is said to be *normal* and *n-normal* ($n \in \mathbb{N}$) if $T^*T = TT^*$ and $T^*T^n = T^nT^*$, respectively. Hence 1-normal is normal. For $T \in B(\mathcal{H})$, we denote $T \geq 0$ if $\langle Tx\,,\, x\rangle \geq 0$ for all $x \in \mathcal{H}$. An operator T is said to be *hyponormal* if $T^*T - TT^* \geq 0$, *quasi-nilpotent* if $\sigma(T) = \{0\}$, and *nilpotent* if there exists $k \in \mathbb{N}$ such that $T^k = 0$, respectively.

In [1], S.A. Alzraiqi and A.B. Patel introduced n-normal operators and showed interesting properties of this class. The class of n-normal operators is so wide. For example, n-nilpotent operators are clearly n-normal. Alzraiqi and Patel studied this condition

$$(*) \qquad \sigma(T) \bigcap (-\sigma(T)) = \emptyset.$$

Under the condition $(*)$, they proved some interesting results. But if an operator $T \in B(\mathcal{H})$ satisfies $(*)$, then the operator T is invertible automatically. We try to set a little bit weaker assumption than this condition $(*)$. The following result is well-known:

Theorem 1.1. (Stampfli [7], Theorem 2) *Let $T \in B(\mathcal{H})$ be hyponormal. If z is an isolated point of $\sigma(T)$, then $z \in \sigma_p(T)$.*

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An operator $T \in B(\mathcal{H})$ is said to be *isoloid* if every isolated point of $\sigma(T)$ belongs to the point spectrum of T. Hence, hyponormal operators are isoloid. Of course, there are many other classes of operators, weaker than hyponormal, which are isoloid. For example, let p be $0 < p \leq 1$. An operator $T \in B(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$. It is known that if T is p-hyponormal, then T is isoloid (see [4]).

Alzraiqi and Patel showed the following result which is of great significance for our work.

Theorem 1.2. (Alzraiqi and Patel [1], Proposition 2.2) *Let $T \in B(\mathcal{H})$ and $n \in \mathbb{N}$. Then T is n-normal if and only if T^n is normal.*

The following are the fundamental properties of n-normal operator T.

Theorem 1.3. (Alzraiqi and Patel [1], Proposition 2.6) *Let $T \in B(\mathcal{H})$ be an n-normal operator. Then the following statements hold.*
(1) *T^* is n-normal.*
(2) *If T^{-1} exists, then T^{-1} is n-normal.*
(3) *If $S \in B(\mathcal{H})$ is unitary equivalent to T, then S is n-normal.*
(4) *If M is a reducing subspace for T, then $T_{|M}$ is an n-normal operator on M.*

2. Spectral properties of 2-normal operators

For $T \in B(\mathcal{H})$, we set the following property:

$$(**) \qquad \sigma(T) \bigcap (-\sigma(T)) \subset \{0\}.$$

Then we begin with the following lemma.

Lemma 2.1. *Let $T \in B(\mathcal{H})$ satisfy $(**)$. If z is an isolated point of $\sigma(T)$, then z^2 is an isolated point of $\sigma(T^2)$.*

Proof. Assume that z is an isolated point of $\sigma(T)$ and z^2 is not isolated point of $\sigma(T^2)$. Then there exists a sequence $\{z_n\} \subset \sigma(T)$ such that $z_n^2 \rightarrow z^2$ $(n \rightarrow \infty)$ by the Spectral mapping theorem.
(1) If $z = 0$, then it is clear that $z_n \rightarrow 0$ $(n \rightarrow \infty)$. Hence, 0 is not an isolated point. It's a contradiction.
(2) Let $z \neq 0$. Since $(z_n + z)(z_n - z) \rightarrow 0$ $(n \rightarrow \infty)$, we may assume that (i) $z_n \rightarrow z$ $(n \rightarrow \infty)$ or (ii) $z_n \rightarrow -z$ $(n \rightarrow \infty)$.
Since z is an isolated point of $\sigma(T)$, (i) does not hold. In the case of (ii), since $z_n \in \sigma(T)$, $\lim_{n \rightarrow \infty} z_n = -z$ and $\sigma(T)$ is compact, we have $-z \in \sigma(T)$. Since $z \neq 0$ and $z \in \sigma(T) \cap (-\sigma(T))$, it's a contradiction to $(**)$ and proves the lemma. □

Our main result is the following.

Theorem 2.2. *Let $T \in B(\mathcal{H})$ be 2-normal and satisfy $(**)$. Then T is isoloid.*

Proof. We assume that z is an isolated point of $\sigma(T)$. Since T satisfies $(**)$, z^2 is an isolated point of $\sigma(T^2)$ by Lemma 2.1. Since T^2 is normal by Theorem 1.2, z^2 is in the point spectrum of T^2 by Theorem 1.2. Hence there exists a non-zero vector $x \in \mathcal{H}$ such that $T^2x = z^2x$. If $z = 0$, then it is clear that 0 is an eigenvalue. If $z \neq 0$, then $(T + z)(T - z)x = 0$ and $-z \notin \sigma(T)$. Since $T + z$ is invertible, we have $(T - z)x = 0$. Thus, z belongs to the point spectrum of T. □

Theorem 2.3. *Let $T \in B(\mathcal{H})$ be 2-normal and satisfy $(**)$. Then $\sigma(T) = \sigma_a(T)$.*

Proof. Since $\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$, it takes only to show $\sigma_r(T) \subset \sigma_a(T)$. Let $z \in \sigma_r(T)$. Then there exists a non-zero vector $x \in \mathcal{H}$ such that $T^*x = \bar{z}x$. Since $T^{*2}x = \bar{z}^2x$ and T^2 is normal, we have $T^2x = z^2x$.
(1) If $z \neq 0$, then $(T + z)(T - z)x = 0$. Since $-z \notin \sigma(T)$, it holds $(T - z)x = 0$ and hence $z \in \sigma_p(T)$.
(2) If $z = 0$, then $T^2x = 0$ and we have $0 \in \sigma_p(T)$.
Therefore, $\sigma(T) = \sigma_a(T)$. □

Theorem 2.4. *Let $T \in B(\mathcal{H})$ be 2-normal and satisfy (∗∗).*
(1) If z and w are distinct eigenvalues of T and $x, y \in \mathcal{H}$ are corresponding eigenvectors, respectively, then $\langle x, y\rangle = 0$.
(2) If z, w are distinct values of $\sigma_a(T)$ and $\{x_n\}$, $\{y_n\}$ are the sequences of unit vectors in $\mathcal{H}$ such that $(T - z)x_n \to 0$ and $(T - w)y_n \to 0$ $(n \to \infty)$, then $\lim_{n\to\infty} \langle x_n, y_n\rangle = 0$.

Proof. (1) follows from (2). So we will only show (2).
Since $(T^2 - z^2)x_n \to 0$ and $(T^2 - w^2)y_n \to 0$ and T^2 is normal, it holds $(T^{*2} - \overline{w}^2)y_n \to 0$. Hence, it holds

$$\lim_{n\to\infty} z^2\langle x_n, y_n\rangle = \lim_{n\to\infty}\langle z^2x_n, y_n\rangle = \lim_{n\to\infty}\langle T^2x_n, y_n\rangle = \lim_{n\to\infty}\langle x_n, T^{*2}y_n\rangle = \lim_{n\to\infty} w^2\langle x_n , y_n\rangle.$$

If $z^2 = w^2$, then $(z + w)(z - w) = 0$. Since $z \neq w$, we have $z = -w$. By (∗∗), this implies $z = w = 0$, which is imposible for distinct values. Hence, $\lim_{n\to\infty} \langle x_n, y_n\rangle = 0$. □

So, we have the following corollary.

Corollary 2.5. *Let $T \in B(\mathcal{H})$ be 2-normal and satisfy (∗∗). If z and w are distinct eigenvalues of T, then* $\ker(T - z) \perp \ker(T - w)$.

Let M be a subspace of $\mathcal{H}$. Then M is said to be a *reducing subspace* for T if $T(M) \subset M$ and $T^*(M) \subset M$, that is, M is an invariant subspace for T and T^*. Then we have the following result.

Theorem 2.6. *Let $T \in B(\mathcal{H})$ be 2-normal and satisfy (∗∗). If z is a non-zero eigenvalue of T, then* $\ker(T - z) = \ker(T^2 - z^2) = \ker(T^{*2} - \overline{z}^2) = \ker(T^* - \overline{z})$ *and hence* $\ker(T - z)$ *is a reducing subspace for T.*

Proof. First we show $\ker(T - z) = \ker(T^2 - z^2)$. Since it is obvious that $\ker(T - z) \subset \ker(T^2 - z^2)$, we show $\ker(T^2 - z^2) \subset \ker(T - z)$. Let $x \in \ker(T^2 - z^2)$, i.e., $(T^2 - z^2)x = 0$. Then $(T + z)(T - z)x = 0$. Since $z \neq 0$, by (∗∗) we have $-z \notin \sigma(T)$. Hence, it follows $(T - z)x = 0$ and $x \in \ker(T - z)$. Therefore, $\ker(T^2 - z^2) \subset \ker(T - z)$ and $\ker(T - z) = \ker(T^2 - z^2)$. Since T^2 is normal, it is clear $\ker(T^2 - z^2) = \ker(T^{*2} - \overline{z}^2)$. Since $-\overline{z} \notin \sigma(T^*)$, we have $\ker(T^{*2} - \overline{z}^2) = \ker(T^* - \overline{z})$. Finally by the above results, it is clear that $\ker(T - z)$ is a reducing subspace for T. □

Remark 2.7. *In general,* $\ker(T)$ *is not a reducing subspace for a 2-normal operator T.*
(1) Let T be as in Example 2.3 of [1], that is, let $\mathcal{H} = \ell^2$ and $\{e_j\}_{j=1}^{\infty}$ be the standard orthonormal basis of ℓ^2. Let T be defined by

$$Te_j = \begin{cases} e_1 & (j = 1) \\ e_{j+1} & (j = 2k) \\ 0 & (j = 2k + 1). \end{cases}$$

Then T is a 2-normal operator and satisfies (∗∗). Since $e_3 \in \ker(T)$ and $TT^*e_3 = e_3 \neq 0$, $\ker(T)$ does not reduce T. Let P be the orthogonal projection to the first coordinate. Since $T^2 = P$, it is clear $\ker(T) \subsetneq \ker(T^2) = \ker(P)$.
(2) We give this, even more simple, example. Let $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$. Then, since $S^2 = 0$ and $\sigma(S) = \{0\}$, T is 2-normal and satisfies (∗∗). Let $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $x \in \ker(S)$ and $SS^*x = x \neq 0$. Hence $\ker(S)$ does not reduce S and $\ker(S) \subsetneq \ker(S^2) = \mathbb{C}^2$. It is clear that $\sigma(S) = \{0\} \neq \{a + bi \in \mathbb{C} : |ab| \leq \frac{1}{2}\} = W(S)$, where $W(S) = \{\langle Sx, x\rangle : \|x\| = 1\}$ (the numerical range of S). Hence S is not convexoid, i.e., $\mathrm{co}\,\sigma(T) \subsetneq \overline{W(T)}$, where $\mathrm{co}\,\sigma(T)$ is the convex hull of $\sigma(T)$ and $\overline{W(T)}$ is the clouser of $W(T)$.

Remark 2.8. *Let p be $0 < p \leq 1$ and $k \in \mathbb{N}$. An operator $T \in B(\mathcal{H})$ is said to be (p, k)-quasihyponormal if $T^{*k}\Big((T^*T)^p - (TT^*)^p\Big)T^k \geq 0$. Let a non-zero z be an isolated point of the spectrum of a (p, k)-quasihyponormal operator T. Then $\ker(T - z)$ reduces T. But when 0 is an isolated point of the spectrum of a (p, k)-quasihyponormal operator T, in general, $\ker(T)$ does not reduce T. See [8] for details.*

Next we study Weyl's theorem. For $T \in B(\mathcal{H})$, the *Weyl spectrum* $\omega(T)$ is defined by

$$\omega(T) = \bigcap_{K \in C(\mathcal{H})} \sigma(T+K),$$

where $C(\mathcal{H})$ is the set of all compact operators on $\mathcal{H}$. Let $\pi_{00}(T)$ denote the set of all isolated eigenvalues of finite multiplicity of T. We say that *Weyl's theorem holds for* T if $\omega(T) = \sigma(T) - \pi_{00}(T)$. J.V. Baxley showed the following result.

Theorem 2.9. (Baxley [3], Lemma 3) *Let $T \in B(\mathcal{H})$ satisfy the following condition*
C-1: *"If $\{z_n\}$ is an infinite sequence of distinct points of the set of eigenvalues of finite multiplicity of T and $\{x_n\}$ is any sequence of corresponding normalized eigenvectors, then the sequence $\{x_n\}$ does not converge." Then*

$$\sigma(T) - \pi_{00}(T) \subset \omega(T).$$

If T is a 2-normal operator satisfying $(**)$, then T satisfies the condition C-1 by Corollary 2.5. Hence we have the following result by Theorem 2.9.

Theorem 2.10. *If $T \in B(\mathcal{H})$ is a 2-normal operator satisfying $(**)$, then*

$$\sigma(T) - \pi_{00}(T) \subset \omega(T).$$

For the converse inclusion, we show the following result.

Theorem 2.11. *If $T \in B(\mathcal{H})$ is a 2-normal operator satisfying $(**)$, then*

$$\omega(T) \subset \sigma(T) - \Big(\pi_{00}(T) - \{0\}\Big).$$

Proof. Let $z \in \pi_{00}(T) - \{0\}$. By Theorem 2.6, $\ker(T-z)$ is a reducing subspace of T. Hence we have the decomposition $T - z = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$ on $\ker(T-z) \oplus \ker(T-z)^{\perp}$. Since $T = \begin{pmatrix} z & 0 \\ 0 & S+z \end{pmatrix}$ and $T_{|\ker(T-z)^{\perp}}$ is 2-normal by Theorem 1.3 (4), $S+z$ is a 2-normal operator satisfying $(**)$ on $\ker(T-z)^{\perp}$. If $z \in \sigma(S+z)$, then $z \in \sigma_p(S+z)$ because z is an isolated point of $\sigma(S+z)$. It's a contradiction. Hence $z \notin \sigma(S+z)$ and hence S is invertible. Let $K = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Then $K \in C(\mathcal{H})$ and $T + K - z = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$ is an invertible operator. Therefore, $z \notin \omega(T)$. It completes the proof. $\square$

If T satisfies $(*)$, then T is invertible and $0 \notin \sigma(T)$. Hence we have the following result by Theorems 2.10 and 2.11.

Theorem 2.12. *If $T \in B(\mathcal{H})$ is a 2-normal operator satisfying $(*)$, then*

$$\omega(T) = \sigma(T) - \pi_{00}(T),$$

that is, Weyl's theorem holds for T.

3. *n*-normal operators

In this section, we show spectral properties of n-normal operators. Recall that, for $n \in \mathbb{N}$, T is said to be n-normal if $T^*T^n = TT^n$. First we extend Proposition 2.19 of [1] as follows:

Theorem 3.1. *The following statements are equivalent:*
(1) *$T - t$ is n-normal for all $t \ge 0$.*
(2) *T is normal.*
(3) *$T - z$ is n-normal for all $z \in \mathbb{C}$.*

Proof. It is sufficient to prove (1) $\Longrightarrow$ (2). Since T and $T-t$ are n-normal, it holds

$$\begin{aligned} & (T-t)^*(T-t)^n - (T-t)^n(T-t)^* \\ = & \sum_{j=1}^{n-1}(-1)^j\binom{n}{j}t^j(T^*T^{n-j}-T^{n-j}T^*) \\ = & (-1)^{n-1}nt^{n-1}(T^*T-TT^*) + \sum_{j=1}^{n-2}(-1)^j\binom{n}{j}t^j(T^*T^{n-j}-T^{n-j}T^*) = 0. \end{aligned}$$

Hence we have

$$(-1)^{n-1}n(T^*T-TT^*) + \sum_{j=1}^{n-2}(-1)^j\binom{n}{j}\frac{t^j}{t^{n-1}}(T^*T^{n-j}-T^{n-j}T^*) = 0.$$

Taking $t \to \infty$, it holds $T^*T - TT^* = 0$ and hence T is normal. □

For an n-normal operator $T \in B(\mathcal{H})$, we set the following property:

$$(***) \qquad \sigma(T) \bigcap \Big(\bigcup_{j=1}^{n-1} e^{\frac{2j\pi}{n}i}\sigma(T)\Big) \subset \{0\}.$$

Then we continue with the following lemma.

Lemma 3.2. *Let $T \in B(\mathcal{H})$ satisfy $(***)$. If z is an isolated point of $\sigma(T)$, then z^n is an isolated point of $\sigma(T^n)$.*

Proof. Assume that z is an isolated point of $\sigma(T)$ and z^n is not an isolated point of $\sigma(T^n)$. Then there exists a sequence $\{z_k\} \in \sigma(T)$ such that $z_k^n \to z^n$ $(k \to \infty)$ by the spectral mapping theorem.
(1) If $z = 0$, then it is clear that $z_k \to 0$ $(k \to \infty)$. Hence, 0 is not an isolated point. It's a contradiction.
(2) Let $z \neq 0$. Since

$$(z_k - e^{\frac{2\pi}{n}i}z)\cdot(z_k - e^{\frac{2\cdot 2\pi}{n}i}z)\cdots(z_k - e^{\frac{2(n-1)\pi}{n}i}z)\cdot(z_k - z) \to 0 \ (n \to \infty),$$

we may assume the following: (i) $z_k \to z$ $(k \to \infty)$ or (ii) there exists $j (j = 1, ..., n-1)$ such that $z_k \to e^{\frac{2j\pi}{n}i}z$ $(k \to \infty)$.
Since z is an isolated point of $\sigma(T)$, (i) does not hold. If $z_k \to e^{\frac{2j\pi}{n}i}z$ $(k \to \infty)$, then $z_k \to e^{\frac{2j\pi}{n}i}z$. Since $z_k \in \sigma(T)$ and $\sigma(T)$ is compact, $e^{\frac{2j\pi}{n}i}z \in \sigma(T)$. Since $z \neq 0$ and $z \in \sigma(T) \cap (e^{\frac{2j\pi}{n}i}\sigma(T))$, it's a contradiction and proves the lemma. □

If T is n-normal, then T^n is normal by Theorem 1.2. Hence, by Lemma 3.2, we have the following results. The proofs are similar to the proofs of Theorem 2.2,Theorem 2.3,Theorem 2.4, Theorem 2.6 and Corollary 2.5. So the proofs are omitted.

Theorem 3.3. *Let $T \in B(\mathcal{H})$ be n-normal and satisfy $(***)$. Then T is isoloid.*

Theorem 3.4. *Let $T \in B(\mathcal{H})$ be n-normal and satisfy $(***)$. Then $\sigma(T) = \sigma_a(T)$.*

Theorem 3.5. *Let $T \in B(\mathcal{H})$ be n-normal and satisfy $(***)$.*
(1) If z and w are distinct eigenvalues of T and $x, y \in \mathcal{H}$ are corresponding eigenvectors, respectively, then $\langle x, y\rangle = 0$.
(2) If z, w are distinct values of $\sigma_a(T)$ and $\{x_n\}$, $\{y_n\}$ are the sequences of unit vectors in $\mathcal{H}$ such that $(T-z)x_n \to 0$ and $(T-w)y_n \to 0$ $(n \to \infty)$, then $\lim_{n\to\infty}\langle x_n, y_n\rangle = 0$.

Corollary 3.6. *Let $T \in B(\mathcal{H})$ be n-normal and satisfy $(***)$. If z and w are distinct eigenvalues of T, then* $\ker(T-z) \perp \ker(T-w)$.

Theorem 3.7. *Let $T \in B(\mathcal{H})$ be n-normal and satisfy $(***)$. If z is a non-zero eigenvalue of T, then* $\ker(T-z) = \ker(T^n - z^n) = \ker(T^{*n} - \overline{z}^n) = \ker(T^* - \overline{z})$ *and hence* $\ker(T-z)$ *is a reducing subspace for T.*

Theorem 3.8. *If $T \in B(\mathcal{H})$ is an n-normal operator satisfying $(***)$, then*

$$\sigma(T) - \pi_{00}(T) \subset \omega(T) \subset \sigma(T) - \Big(\pi_{00}(T) - \{0\}\Big).$$

Moreover, T is invertible, and $\sigma(T) - \pi_{00}(T) = \omega(T)$, that is, Weyl's theorem holds for T.

4. (n, m)-normal operators

We begin with the definition of (n, m)-normal operators.

Definition 4.1. *For $T \in B(\mathcal{H})$ and $n, m \in \mathbb{N}$, T is said to be (n, m)-normal if*

$$T^{*m}T^n = T^nT^{*m}.$$

From the definition, it is clear that T is (n, m)-normal if and only if T is (m, n)-normal. Let $T \in B(\mathcal{H})$ be (n, m)-normal. Then the following hold clearly:

(1) T^* is (n, m)-normal.
(2) If T^{-1} exists, then T^{-1} is (n, m)-normal.
(3) If $S \in B(\mathcal{H})$ is unitary equivalent to T, then S is (n, m)-normal.
(4) If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ which reduces T, then $T_{|\mathcal{M}}$ is (n, m)-normal on $\mathcal{M}$.

Lemma 4.2. (1) *If $T \in B(\mathcal{H})$ is (n, m)-normal, then T^k is normal, where k is the least common multiple of n and m.*
(2) *If T^n is normal, then T is (n, m)-normal for every m.*

Proof. (1) Let $k = n \cdot j$ and $k = m \cdot \ell$. If T is (n, m)-normal, then

$$T^{*k}T^k = \overbrace{T^{*m} \cdots T^{*m}}^{\ell} \cdot \overbrace{T^n \cdots T^n}^{j} = T^n \cdots T^n \cdot T^{*m} \cdots T^{*m} = T^kT^{*k}.$$

Hence T^k is normal.
(2) Since T^n is normal and $T^m \cdot T^n = T^n \cdot T^m$, it follows from Fuglede's theorem that $T^{*m} \cdot T^n = T^n \cdot T^{*m}$. Hence, T is (n, m)-normal. □

Theorem 4.3. *If $T \in B(\mathcal{H})$ is quasi-nilpotent and (n, m)-normal, then T is nilpotent.*

Proof. Since $\sigma(T) = \{0\}$, we have $\sigma(T^k) = \{0\}$ for every $k \in \mathbb{N}$. Let k be the least common multiple of n and m. Then, by Lemma 4.2, T^k is normal. Hence $T^k = 0$. □

Theorem 4.4. *If $T, S \in B(\mathcal{H})$ are commuting (n, m)-normal operators, then TS is (k, j)-normal for every $j \in \mathbb{N}$, where k is the least common multiple of n and m.*

Proof. Since k is the least common multiple of n and m, by Lemma 4.2, $(TS)^k$ is normal. Since $(TS)^k$ commutes with $(TS)^j$ for every $j \in \mathbb{N}$. By Fuglede's theorem, it holds $(TS)^{*j}(TS)^k = (TS)^k(TS)^{*j}$. Hence TS is (k, j)-normal for every $j \in \mathbb{N}$. □

Theorem 4.5. *Let $T \in B(\mathcal{H})$ be (n, m)-normal and $(n+1, m)$-normal. If either T or T^* is injective, then T is m-normal.*

Proof. (1) Let T be injective. Since T is (n, m)-normal and $(n+1, m)$-normal, it holds

$$T^{n+1}T^{*m} = T^{*m}T^{n+1} = (T^{*m}T^n)T = T^nT^{*m}T.$$

Hence, we have $T^n(TT^{*m} - T^{*m}T) = 0$. Since T is injective, we have $TT^{*m} = T^{*m}T$ and $T^*T^m = T^mT^*$. Hence, T is m-normal.
(2) Let T^* be injective. Since it holds that T^* is (n, m)-normal and $(n+1, m)$-normal, we have T^* is m-normal by (1) and T is m-normal. □

Theorem 4.6. *For $T \in B(\mathcal{H})$, let $F = T^n + T^{*m}$ and $G = T^n - T^{*m}$. Then T is (n, m)-normal if and only if F commutes with G.*

Proof. Since

$$FG = T^{2n} - T^nT^{*m} + T^{*m}T^n - T^{*2m} \text{ and } GF = T^{2n} + T^nT^{*m} - T^{*m}T^n - T^{*2m}.$$

Hence, $FG = GF$ if and only if $T^{*m}T^n = T^nT^{*m}$. It completes the proof. □

Theorem 4.7. *For $T \in B(\mathcal{H})$, let $A = T^nT^{*m}$, $F = T^n + T^{*m}$ and $G = T^n - T^{*m}$. If T is (n, m)-normal, then A commutes with F and G.*

Proof. Since T is (n, m)-normal, we have

$$AF = T^nT^{*m}(T^n + T^{*m}) = T^nT^nT^{*m} + T^{*m}T^nT^{*m} = FA.$$

Similarly we have $AG = GA$. □

Theorem 4.8. *Let $T \in B(\mathcal{H})$ be an invertible (n, m)-normal operator. Then T and T^{-1} have a common nontrivial closed invariant subspace.*

Proof. Let k be the least common multiple of n and m. Then by Lemma 4.2, T^k is normal. Hence T^{-k} is also normal. Hence, T^k and T^{-k} have no hypercyclic vector by Corollary 4.5 of [6]. Hence, T and T^{-1} have no hypercyclic vector by [2]. Therefore, T and T^{-1} have a common nontrivial closed invariant subspace by [5]. □

Finally, we show results of the direct sum and the tensor product. The proof is easy. So we omit the proof.

Theorem 4.9. *If $T, S \in B(\mathcal{H})$ are (n, m)-normal, then $T \oplus S$ and $T \otimes S$ are (n, m)-normal on $\mathcal{H} \oplus \mathcal{H}$ and $\mathcal{H} \overline{\otimes} \mathcal{H}$, respectively.*

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