SPECTRAL PROPERTIES OF p-HYPONORMAL OPERATORS

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1. Introduction. Let $\mathscr{H}$ be a complex Hilbert space and $B(\mathscr{H})$ be the algebra of all bounded linear opeators on $\mathscr{H}$. An operator $T \in B(\mathscr{H})$ is said to be p-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$. If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is semi-hyponormal. It is well known that a p-hyponormal operator is q-hyponormal for $q \leq p$. Hyponormal operators have been studied by many authors. The semi-hyponormal operator was first introduced by D. Xia in [**7**]. The p-hyponormal operators have been studied by A. Aluthge in [**1**]. Let T be a p-hyponormal operator and $T = U|T|$ be a polar decomposition of T. If U is unitary, Aluthge in [**1**] proved the following properties.

(A1) The eigenspaces of U reduce T.

(A2) If $\sigma(U) \neq \mathbb{T}$, then the eigenspaces of $|T|$ reduce U, where $\sigma(T)$ is the spectrum of T and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

(A3) $r(T) = \|T\|$, where $r(T)$ and $\|T\|$ are the spectral radius and the operator norm of T, respectively.

Other related properties of a semi-hyponormal operator T are the following (see Xia's book [**8**]):

(X1) $\sigma(T) = \{z : \bar{z} \in \sigma_\pi(T^*)\}$;

(X2) $\sigma(|T|) \subset \pi_\rho(\sigma(T))$, where π_ρ is the mapping $\mathbb{C} \to \mathbb{R}^+$ such that $\pi_\rho(z) = |z|$ ($z \in \mathbb{C}$); i.e., Putnam's theorem holds.

The set of all p-hyponormal operators in $B(\mathscr{H})$ is denoted by p-H. Let p-HU denote the set of all operators in p-H with equal defect and nullity. Hence for $T \in p$-HU we may assume that the operator U in a polar decomposition $T = U|T|$ is unitary. We say that an operator T is a p-HU-operator if $T \in p$-HU.

In this paper we prove that (X1) and weakly Putnam's theorem hold for p-HU-operators. Let T be a p-HU-operator. If $r \in \sigma(T^*T) \cup \sigma(TT^*)$, then there exists $z \in \sigma(T)$ such that $|z|^2 \geq r$. Also we prove that doubly commuting n-tuples of p-HU-operators are jointly normaloid.

We need the following results.

THEOREM A (Th. 2 in [**1**]). *Let $T = U|T|$ be a p-Hu-operator. Then $\tilde{T} = |T|^{1/2} . U . |T|^{1/2}$ is $(p + \frac{1}{2})$-hyponormal. Hence $\tilde{T}$ is semi-hyponormal.*

THEOREM B (Th. 2.3 of p. 10 in [**8**]). *Let $T = U|T|$ be a semi-hyponormal operator on $\mathscr{H}$. If $Tx = r . e^{i\theta}x$ for a non-zero vector $x \in \mathscr{H}$, then $|T|x = rx$, $Ux = e^{i\theta}x$ and $T^*x = r . e^{-i\theta}x$.*

Also we need the following technique of Berberian.

THEOREM C. *Let $\mathscr{H}$ be a complex Hilbert space. Then there exist a Hilbert space $\mathscr{K} \supset \mathscr{H}$ and a map $\tau : B(\mathscr{H}) \to B(\mathscr{K})$ such that* (1) *τ is an isometric algebraic *-isomorphism preserving the order; i.e.,*

$$\tau(A^*) = \tau(A)^*, \qquad \tau(I) = I, \qquad \tau(\alpha A + \beta B) = \alpha\tau(A) + \beta\tau(B),$$

$\tau(AB) = \tau(A)\tau(B)$, $\|\tau(A)\| = \|A\|$ and $\tau(A) \leq \tau(B)$ whenever $A \leq B$, for all $A, B \in B(\mathscr{H})$ and $\alpha, \beta \in \mathbb{C}$;

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(2) *$\sigma(\tau(A))=\sigma(A)$ and $\sigma_\pi(A)=\sigma_\pi(\tau(A))=\sigma_p(\tau(A))$ for all $A\in B(\mathcal{H})$, where $\sigma(A)$, $\sigma_\pi(A)$ and $\sigma_p(A)$ are the spectrum, the approximate point spectrum and the point spectrum of A, respectively.*

See p. 15 in [**8**] for details. Hence, T is p-hyponormal if and only if $\tau(T)$ is. For an operator $T\in B(\mathcal{H})$, $z\in\mathbb{C}$ is in the normal approximate point spectrum $\sigma_{n\pi}(T)$ of T if there exists a sequence $\{x_k\}$ of unit vectors such that $(T-z)x_k\to 0$ and $(T-z)^*x_k\to 0$ as $k\to\infty$.

Though in Xia's book [**8**] this spectrum is called the joint approximate point spectrum, we use this word for n-tuples of operators.

2. p-Hyponormal operators. Throughout this paper, let p be $0<p<\frac{1}{2}$.

LEMMA 1. *Let T be a p-HU-operator. If $z\in\sigma_p(T)$, then $\bar{z}\in\sigma_p(T^*)$.*

Proof. Assume that $0\in\sigma_p(T)$. Then there exists a non-zero vector $x\in\mathcal{H}$ such that $Tx=0$. Since $|T|^2x=T^*Tx=0$ and $|T|\geq 0$, we have $(T^*T)^{1/2^k}x=0$ $(k=1,2,\ldots)$. For m in $\mathbb{N}$ such that $\dfrac{1}{m}<p$, we have $(T^*T)^{1/2^m}x=0$. It follows that $(T^*T)^px=0$. Since T is p-hyponormal, it follows that $(TT^*)^px=0$. Therefore $T^*x=0$.

Next assume that $z\in\sigma_p(T)$ for a non-zero $z\in\mathbb{C}$. Then there exists a non-zero vector $y\in\mathcal{H}$ such that $Ty=zy$. Let $T=U|T|$ be a polar decomposition of T with unitary operator U. Since $U|T|y=zy$, it follows that $|T|^{1/2}U|T|^{1/2}|T|^{1/2}y=z|T|^{1/2}y$. By Theorem A the operator $\tilde{T}=|T|^{1/2}U|T|^{1/2}$ is semi-hyponormal. Hence by Theorem B we have $\tilde{T}^*(|T|^{1/2}y)=|T|^{1/2}U^*|T|y=\bar{z}\,.\,|T|^{1/2}y$. Therefore $T^*(|T|y)=\bar{z}\,.\,|T|y$. Since $|T|y\neq 0$, we have $\bar{z}\in\sigma_p(T^*)$.

THEOREM 2. *Let T be a p-HU-operator. Then*

$$\sigma(T)=\{z:\bar{z}\in\sigma_\pi(T^*)\}.$$

Proof. Since we have $\sigma(T)=\sigma_\pi(T)\cup\{z:\bar{z}\in\sigma_p(T^*)\}$, we need only prove that $\sigma_\pi(T)\subset\{z:\bar{z}\in\sigma_\pi(T^*)\}$.

Assume that $z\in\sigma_\pi(T)$. Consider the mapping τ of Theorem C. Then we have $z\in\pi_p(\tau(T))$. Since $\tau(T)$ is a p-*HU*-operator, by Lemma 1 we have $\bar{z}\in\sigma_p(\tau(T)^*)$. Also since, by Theorem C, $\sigma_p(\tau(T)^*)=\sigma_\pi(T^*)$, it follows that $\bar{z}\in\sigma_\pi(T^*)$.

Next we prove that the weak form of Putnam's theorem holds for p-hyponormal operators. First we prove the following result.

LEMMA 3. *Let $T=UP\in B(\mathcal{H})$, U be unitary, $P\geq 0$ and $T^*T=P^2$. Let $r>0$, $|e^{i\theta}|=1$. Then $r\,.\,e^{i\theta}\sigma_{n\pi}(T)$ if and only if there exists a sequence $\{x_k\}$ of unit vectors in $\mathcal{H}$ such that*

$$\lim_{k\to\infty}\|(P-r)x_k\|=0 \quad \text{and} \quad \lim_{k\to\infty}\|(U-e^{i\theta})x_k\|=0. \qquad (*)$$

Proof. If $z\in\sigma_{n\pi}(T)$, then there exists a sequence $\{x_k\}$ of unit vectors in $\mathcal{H}$ such that $(T-r\,.\,e^{i\theta})x_k\to 0$ and $(T-r\,.\,e^{i\theta})^*x_k\to 0$ as $k\to\infty$. Since $T^*T=P^2$, $(P^2-r^2)x_k\to 0$ and $(P-r)x_k\to 0$ $(k\to\infty)$. Hence it follows that $(U-e^{i\theta})x_k\to 0$, because $r\neq 0$.

Conversely, suppose that $(*)$ holds. Since then U is unitary, we have $(U^*-e^{-i\theta})x_k\to 0$ $(k\to\infty)$. Hence $r\,.\,e^{i\theta}\in\sigma_{n\pi}(T)$.

THEOREM 4. *Let $T = U|T|$ be a p-HU-operator. If $r \in \sigma(T^*T)$, then there exist r' and θ such that $r \le r'$ and $\sqrt{r'} \,.\, e^{i\theta} \in \sigma(T)$.*

Proof. We need only prove that $p = \frac{1}{2^n}$. If $r = 0$, then it is clear that $0 \in \sigma(T)$. So let $r \neq 0$. Then $r \in \sigma(T^*T)$ and $r^p \in \sigma((T^*T)^p)$. Here put $S = U|T|^p$. Since then

$$S^*S = |T|^{2p} = (T^*T)^p \quad \text{and} \quad SS^* = U(T^*T)^pU^* = (TT^*)^p,$$

S is a hyponormal operator. Since $S = U|T|^p$ is a polar decomposition of S and $r^p \in \sigma(S^*S)$, by Putnam's theorem there exists θ such that $\sqrt{r^p} \,.\, e^{i\theta} \in \sigma(S)$. Hence there exists r_0 such that $\sqrt{r^p} \le r_0$ and $r_0 \,.\, e^{i\theta} \in \partial\sigma(S) \subset \sigma_\pi(S) \subset \sigma_{n\pi}(S)$, where $\partial\sigma(S)$ is the boundary of $\sigma(S)$. By Lemma 3 it follows that there exists a sequence $\{x_k\}$ of unit vectors in $\mathcal{H}$ such that

$$(|T|^p - r_0)x_k \to 0 \quad \text{and} \quad (U - e^{i\theta})x_k \to 0 \quad \text{as} \quad k \to \infty.$$

Since $p = \frac{1}{2^n}$, we have $(|T| - r_0^{2^n})x_k \to 0$ as $k \to \infty$. Let $r' = r_0^{2^{n+1}}$. Then $r' \,.\, e^{i\theta}$ is the desired number, and so the proof is complete.

REMARK. With the same assumption as in Theorem 4, we have: if $r \in \sigma(TT^*)$, then there exists $z \in \sigma(T)$ such that $|z|^2 = r$.

We have the following corollary.

COROLLARY 5. *Let T be a p-HU-operator. Then $r(T) = \|T\|$.*

Proof. Since $r(T^*T) = \|T\|^2$, the result follows from Theorem 4.

3. N-tuples of p-Hyponormal operators. In this section, we study doubly commuting n-tuples of p-hyponormal operators. First we will give some definitions. An n-tuple $\mathbb{T} = (T_1, \ldots, T_n)$ of operators is said to be a *doubly commuting n-tuple* if $T_i \cdot T_j = T_j \,.\, T_i$ and $T_i^* \,.\, T_j = T_j \,.\, T_i^*$, for every $i \neq j$. Let $\mathbb{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators on $\mathcal{H}$. We denote the Taylor spectrum of $\mathbb{T}$ by $\sigma(\mathbb{T})$ (see Taylor [**6**]). $z = (z_1, \ldots, z_n)$ is in the joint approximate point spectrum $\sigma_\pi(\mathbb{T})$ of $\mathbb{T}$ is there exists a sequence $\{x_k\}$ of unit vectors in $\mathcal{H}$ such that

$$\|(T_i - z_i)x_k\| \to 0 \quad \text{as} \quad k \to \infty,$$

for every $i = 1, 2, \ldots, n$. Further $z = (z_1, \ldots, z_n)$ is in the joint point spectrum $\sigma_p(\mathbb{T})$ of $\mathbb{T}$ if there exists a non-zero vector x such that $T_ix = z_ix$ for every $i = 1, 2, \ldots, n$. By Berberian's technique, we have the following result.

THEOREM D. *Let $\mathbb{T} = (T_1, \ldots, T_n)$ be an n-tuple of operators on $\mathcal{H}$. Let τ be the mapping of Theorem C. Then*

$$\sigma_\pi(\mathbb{T}) = \sigma_\pi(\tau(\mathbb{T})) = \sigma_p(\tau(\mathbb{T})),$$

where $\tau(\mathbb{T}) = (\tau(T_1), \ldots, \tau(T_n))$.

If an n-tuple $\mathbb{T} = (T_1, \ldots, T_n)$ is a doubly commuting n-tuple of p-HU-operators, then by Theorems 2 and 4 in Furuta [**4**] there exists unitary operators $U_1, \ldots, U_n$ with a polar decomposition $T_i = U_i|T_i|$ $(i = 1, \ldots, n)$ such that U_i and $|T_i|$ commute with U_j and $|T_j|$ for every $i \neq j$.

LEMMA 6. *Let $\mathbb{T} = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of p-HU-operators on $\mathcal{H}$. If $z = (z_1, \ldots, z_n) \in \sigma_p(\mathbb{T})$, then $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) \in \sigma_p(\mathbb{T}^*)$, where $\mathbb{T}^* = (T_1^*, \ldots, T_n^*)$.*

Proof. There exists a non-zero vector x in $\mathcal{H}$ such that $T_i x = z_i x$ $(i = 1, \ldots, n)$. We may assume that $z_1, \ldots, z_k$ are non-zero and $z_{k+1} = \ldots = z_n = 0$. From the proof of Lemma 1, we obtain

$$T_{k+1}^* x = \ldots = T_n^* x = 0.$$

Also from the proof of Lemma 1, we obtain $T_i^*(|T_i|\, x) = \bar{z}_i \,.\, |T_i|\, x$, where $|T_i|$ is the positive operator in a polar decomposition $T_i = U_i\, |T_i|$ $(i = 1, \ldots, k)$. Assume that $|T_1| \ldots |T_k|\, x = 0$. Since then $(T_1, \ldots, T_k)$ is a doubly commuting k-tuple of p-HU-operators, U_i and $|T_i|$ commute with U_j and $|T_j|$ for every $i \neq j$. Hence we have

$$T_1 \,.\, T_2 \ldots T_k x = 0.$$

It follows that $z_1 \ldots z_k = 0$. Since every $z_i \neq 0$ $(i = 1, \ldots, k)$, this is a contradiction. Therefore we have $|T_1| \ldots |T_k|\, x \neq 0$. For i $(i = 1, \ldots, k)$, we have

$$\begin{aligned} T_i^*(|T_1| \ldots |T_k|\, x) &= |T_1| \ldots |T_{i-1}| \,.\, |T_{i+1}| \ldots |T_k| \,.\, T_i^* \,.\, |T_i|\, x \\ &= \bar{z}_i(|T_1| \ldots |T_k|\, x). \end{aligned}$$

Since also T_i commutes with $|T_1| \ldots |T_k|$, we have

$$T_i^*(|T_1| \ldots |T_k|\, x) = 0 \qquad (i = k+1, \ldots, n).$$

Therefore it follows that $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) \in \sigma_p(\mathbb{T}^*)$.

THEOREM 7. *Let $\mathbb{T} = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of p-HU-operators on $\mathcal{H}$. Then*

$$\sigma(\mathbb{T}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : (\bar{z}_1, \ldots, \bar{z}_n) \in \sigma_\pi(\mathbb{T}^*)\}.$$

Proof. Since $\mathbb{T}$ is a doubly commuting n-tuple, by Corollary 3.3 in **[3]** it follows that if $(z_1, \ldots, z_n) \in \sigma(\mathbb{T})$, then there exist some partition $\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_s\} = \{1, \ldots, n\}$ and a sequence $\{x_k\}$ of unit vectors in $\mathcal{H}$ such that

$$(T_{i_\mu} - z_{i_\mu})x_k \to 0 \quad \text{and} \quad (T_{j_\nu} - z_{j_\nu})^* x_k \to 0 \quad \text{as} \quad k \to \infty,$$

for $\mu = 1, \ldots, m$ and $\nu = 1, \ldots s$. Consider the mapping τ of Theorem C. Then we have

$$(z_{i_1}, \ldots, z_{i_m}, \bar{z}_{j_1}, \ldots, \bar{z}_{j_s}) \in \sigma_p(\tau(S)),$$

where $\tau(S) = (\tau(T_{i_1}), \ldots, \tau(T_{i_m}), \tau(T_{j_1}^*), \ldots, \tau(T_{j_s}^*))$. Since $\tau(T_i)$ is a p-HU-operator for every i $(i = 1, \ldots, n)$, by Lemma 6 we have $(\bar{z}_1, \ldots, \bar{z}_n) \in \sigma_p(\tau(\mathbb{T}^*))$. Hence by Theorem D it follows that $(\bar{z}_1, \ldots, \bar{z}_n) \in \sigma_\pi(\mathbb{T}^*)$. It is clear that $\sigma_\pi(\mathbb{T}^*) \subset \sigma(T)$ and so the proof is complete.

THEOREM 8. *Let $\mathbb{T} = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of p-HU-operators on $\mathcal{H}$. If $(r_1, \ldots, r_n) \in \sigma(\mathbb{T}^*\mathbb{T}) \cup \sigma(\mathbb{T}\mathbb{T}^*)$, then there exists $(z_1, \ldots, z_n) \in \sigma(\mathbb{T})$ such that $|z_i|^2 \geq r_i$ $(i = 1, \ldots, n)$, where $\mathbb{T}^*\mathbb{T} = (T_1^*T_1, \ldots, T_n^*T_n)$ and $\mathbb{T}\mathbb{T}^* = (T_1T_1^*, \ldots, T_nT_n^*)$.*

Proof. We shall prove the theorem by induction. When $n = 1$, the theorem holds by Theorem 4. We assume that the theorem holds for every doubly commuting $(n-1)$-tuple of p-HU-operators. We assume that $(r_1, \ldots, r_n) \in \sigma(\mathbb{T}^*\mathbb{T})$. Since $\sigma(\mathbb{T}^*\mathbb{T}) = \sigma_\pi(\mathbb{T}^*\mathbb{T})$, we have $(\sqrt{r_1}, \ldots, \sqrt{r_n}) \in \sigma_\pi(|\mathbb{T}|)$, where $|\mathbb{T}| = ((|T_1|, \ldots, |T_n|)$. Consider the mapping τ of Theorem C. Let $\mathfrak{M} = \ker(|\tau(T_n)| - \sqrt{r_n})$ $(\neq \{0\})$. Then $\mathfrak{M}$ is a reducing subspace of $\tau(T_1), \ldots, \tau(T_{n-1})$ and $(\tau(T_1)_{|\mathfrak{M}}, \ldots, \tau(T_{n-1})_{|\mathfrak{M}})$ is a doubly commuting $(n-1)$-tuple of p-HU-operators on $\mathfrak{M}$. Since $\sum_{i=1}^{n} (|\tau(T_i)| - \sqrt{r_i})^2$ is not invertible, it follows that

$$\ker\left(\sum_{i=1}^{n} (|\tau(T_i)| - \sqrt{r_i})^2\right) = \left\{\bigcap_{i=1}^{n-1} \ker(|\tau(T_i)| - \sqrt{r_i})\right\} \cap \mathfrak{M} \neq \{0\}.$$

Hence it follows that $(\sqrt{r_1}, \ldots, \sqrt{r_{n-1}}) \in \sigma(R)$, where $R = (|\tau(T_1)_{|\mathfrak{M}}|, \ldots |\tau(T_{n-1})_{|\mathfrak{M}}|)$. So, by the induction hypothesis, there exists $(z_1, \ldots, z_{n-1}) \in \sigma(S)$ such that $|z_i| \geq \sqrt{r_i}$ $(i = 1, \ldots, n-1)$, where $S = (\tau(T_i)_{|\mathfrak{M}}, \ldots, \tau(T_{n-1})_{|\mathfrak{M}})$. Since by Theorem 7 it follows that $(\bar{z}_1, \ldots, \bar{z}_{n-1}) \in \sigma_p(S^*)$, there exists a non-zero vector x_0 in $\mathfrak{M}$ such that

$$\tau(T_i^*)x_0 = \bar{z}_i x_0 \quad (i = 1, \ldots, n-1).$$

Therefore $\sum_{i=1}^{n-1} (\tau(T_i) - z_i)(\tau(T_i) - z_i)^* + (|\tau(T_n)| - \sqrt{r_n})^2$ is not invertible. Hence

$$\ker\left(\sum_{i=1}^{n-1} (\tau(T_i) - z_i\right)((\tau(T_i) - z_i)^*) + (|\tau(T_n)| - \sqrt{r_n})^2) \neq \{0\}.$$

Let $\mathfrak{N} = \ker\left(\sum_{i=1}^{n-1} (\tau(T_i) - z_i\right)(\tau(T) - z_i)^*)$. Then $\mathfrak{N}$ reduces $\tau(T_n)$. Also since $\mathfrak{M} \cap \mathfrak{N} \neq \{0\}$, $\sqrt{r_n} \in \sigma(|\tau(T_n)_{|\mathcal{N}}|)$. Since $\tau(T_n)_{|\mathfrak{N}}$ is a p-HU-operator, by Theorem 2 it follows that there is a $z_n \in \mathbb{C}$ such that $(\tau(T_n)_{|\mathfrak{N}} - z_n)(\tau(T_n)_{|\mathfrak{N}} - z_n)^*$ is not invertible and $|z_n|^2 \geq r_n$. Since

$$\sum_{i=1}^{n} (\tau(T_i) - z_i)(\tau(T_i) - z_i)^*$$

is not invertible, this point $(z_1, \ldots, z_n)$ is in $\sigma(\mathbb{T})$ and satisfies

$$|z_i|^2 \geq r_i \quad (i = 1, \ldots, n).$$

In case of $(r_1, \ldots, r_n) \in \sigma(\mathbb{T}\mathbb{T}^*)$, the proof is similar. Thus the proof is complete.

For an n-tuple $\mathbb{T} = (T_1, \ldots, T_n)$ of operators on $\mathcal{H}$, the joint spectral radius $r(\mathbb{T})$ and the joint operator norm $\|\mathbb{T}\|$ of $\mathbb{T}$ are given by

$$r(\mathbb{T}) = \sup\left\{|z| = \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2} : z = (z_1, \ldots, z_n) \in \sigma(\mathbb{T})\right\}$$

and

$$\|\mathbb{T}\| = \sup\left\{\left(\sum_{i=1}^{n} \|T_i x\|^2\right)^{1/2} : x \in \mathcal{H}, \|x\| = 1\right\},$$

respectively. It always holds that $r(\mathbb{T}) \leq \|\mathbb{T}\|$ for every commuting n-tuple $\mathbb{T} = (T_1, \ldots, T_n)$ of operators.

THEOREM 9. *Let $\mathbb{T} = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of p-HU-operators on $\mathcal{H}$. Then $r(\mathbb{T}) = \|\mathbb{T}\|$; i.e., $\mathbb{T}$ is jointly normaloid.*

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Proof. Since $\mathbb{T}^*\mathbb{T} = (T_1^*T_1, \ldots, T_n^*T_n)$ is a commuting n-tuple of positive operators, $\mathbb{T}^*\mathbb{T}$ is jointly convexoid (see [2]). Also $\|\mathbb{T}\|^2 = \sup\left\{\sum_{i=1}^n (T_i^*T_ix, x) : x \in \mathscr{H}, \|x\| = 1\right\}$ and we can see that there exists $(r_1, \ldots, r_n) \in \sigma(\mathbb{T}^*\mathbb{T})$ such that $\sum_{i=1}^n r_i = \|\mathbb{T}\|^2$. By Theorem 8, it follows that there exists $(z_1, \ldots, z_n) \in \sigma(\mathbb{T})$ such that $\left(\sum_{i=1}^n |z_i|^2\right)^{1/2} \geq \|\mathbb{T}\|$. The converse inequality is clear and so the proof is complete.

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