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Spectral properties of Sturm-Liouville operators with discontinuities at finite points

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Abstract

Purpose: In this paper, we investigate a class of Sturm-Liouville operators with eigenparameter-dependent boundary conditions and transmission conditions at finite interior points.

Methods: By modifying the inner product in a suitable Krein space *K* associated with the problem, we generate a new self-adjoint operator *A* such that the eigenvalues of such a problem coincide with those of *A*.

Results: We construct its fundamental solutions, get the asymptotic formulae for its eigenvalues and fundamental solutions, discuss some properties of its spectrum, and obtain its Green function and the resolvent operator.

Conclusions: Three important conclusions can be drawn: (1) the new operator *A* is self-adjoint in the Krein space *K*; (2) if $\theta_i > 0$, $i = \overline{1, m}$, and $\rho_j > 0$, $j = 1, 2$, then, the eigenvalues of the problem (Equations 1 to 5) are analytically simple; (3) the residual spectrum of the operator *A* is empty, i.e., $\sigma_r(A) = \emptyset$.

Keywords: Sturm-Liouville operator; Transmission condition; Eigenparameter-dependent boundary condition

MSC (2000): 34L20; 47E05

Introduction

In recent years, more and more researchers are interested in the discontinuous Sturm-Liouville problem for its application in physics (see [1,2]). Such problems are connected with discontinuous material properties, such as heat and mass transfer, varied assortment of physical transfer problems, vibrating string problems when the string loaded additionally with point masses, and diffraction problems [3,4]. Moreover, there has been a growing interest in Sturm-Liouville problems with eigenparameter-dependent boundary conditions, i.e., the eigenparameter appears not only in the differential equations, but also in the boundary conditions of the problems (see [5-10]).

Here, we consider a class of Sturm-Liouville operators with eigenparameter-dependent boundary conditions and transmission conditions at finite points of discontinuity. We extend and generalize some approaches and results of the classic regular Sturm-Liouville problems to similar problems with discontinuities. By modifying the inner product in the direct sum of the Krein spaces and using the classical technics, we define a new self-adjoint operator A such that the eigenvalues of such a problem coincide with those of A. We construct its fundamental solutions, get the asymptotic formulae for its eigenvalues, discuss some properties of its spectrum, and obtain its Green function and the resolvent operator. Especially, we notice that the signs of $\theta_i (i = \overline{1, m})$ and ρ_j $(j = 1, 2)$ influence the spectrum properties of the operator A, promote and deepen the previous conclusions (see [6]).

In this study, we consider a discontinuous eigenvalue problem consisting of the Sturm-Liouville equation:

$$lu := -(a(x)u'(x))' + q(x)u(x) = \lambda u(x), \; x \in I, \quad (1)$$

where $I = [a, \xi_1) \cup (\xi_1, \xi_2) \cup \cdots \cup (\xi_m, b]$, $a(x) = a_1^2$ for $x \in [a, \xi_1)$, $a(x) = a_2^2$ for $x \in (\xi_1, \xi_2), \cdots, a(x) = a_{m+1}^2$ for $x \in (\xi_m, b]$, $a_1, a_2, \cdots, a_{m+1}$ are positive real constants; $\lambda \in \mathbb{C}$ is a complex eigenparameter; $q(x)$ is real-valued and continuous in I, and has finite limits $q(\xi_i \pm 0) := \lim_{x \to \xi_i \pm 0} q(x)$, $i = \overline{1, m}$; boundary conditions at the endpoints

$$l_1 u := \lambda(\alpha_1' u(a) - \alpha_2' u'(a)) - (\alpha_1 u(a) - \alpha_2 u'(a)) = 0, \quad (2)$$

$$l_2 u := \lambda(\beta_1' u(b) - \beta_2' u'(b)) + (\beta_1 u(b) - \beta_2 u'(b)) = 0 \quad (3)$$

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and transmission conditions at the points of discontinuity $x = \xi_i,\ i = \overline{1,m}$,

$$l_{2i+1}u := u(\xi_i+0) - \alpha_{i1}u(\xi_i-0) - \alpha_{i2}u'(\xi_i-0) = 0, \quad (4)$$

$$l_{2i+2}u := u'(\xi_i+0) - \beta_{i1}u(\xi_i-0) - \beta_{i2}u'(\xi_i-0) = 0, \quad (5)$$

where α_{ij}, β_{ij}, α_j, β_j, α_j', β_j' $(i = \overline{1,m},\ j = 1,2)$ are real numbers. Here, we assume that

$$\theta_i = \begin{vmatrix} \alpha_{i1} & \alpha_{i2} \\ \beta_{i1} & \beta_{i2} \end{vmatrix} \neq 0\ (i = \overline{1,m}),$$

$$\rho_1 = \begin{vmatrix} \alpha_1{}' & \alpha_1 \\ \alpha_2{}' & \alpha_2 \end{vmatrix} \neq 0,\ \rho_2 = \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} \neq 0.$$

In order to consider the problem (Equations 1 to 5), we define the inner product in $L^2(I)$ as

$$[f,g]_1 = \frac{\theta_1\theta_2\cdots\theta_m}{a_1^2}\int_a^{\xi_1} f_1\overline{g}_1 dx + \frac{\theta_2\theta_3\cdots\theta_m}{a_2^2} \times \int_{\xi_1}^{\xi_2} f_2\overline{g}_2 dx + \cdots + \frac{1}{a_{m+1}^2}\int_{\xi_m}^{b} f_{m+1}\overline{g}_{m+1} dx,$$

where $f_1(x) = f(x) \restriction_{[a,\xi_1)}$, $f_2(x) = f(x) \restriction_{(\xi_1,\xi_2)}$, $\cdots$, $f_{m+1}(x) = f(x) \restriction_{(\xi_m,b]}$. Obviously, the linear space $(L^2(I), [\cdot,\cdot]_1)$ is a modified Krein space.

Methods

The eigenparameter appears not only in the differential equations, but also in the boundary conditions of the problems. So by modifying the inner product in a suitable space K and using the classical technics, we define a new self-adjoint operator A such that the eigenvalues of such a problem coincide with those of A.

An operator formulation in the adequate Krein space

In the following, for simplicity, we set $\theta = \theta_1\theta_2\cdots\theta_m$. Define the special inner product in the direct sum of linear spaces $L^2(I) \oplus \mathbb{C}_{\theta\rho_1} \oplus \mathbb{C}_{\rho_2}$ by

$$[F,G] := [f,g]_1 + \frac{\theta}{\rho_1}\langle h,k\rangle + \frac{1}{\rho_2}\langle r,s\rangle$$

for $F := (f,h,r)$, $G := (g,k,s) \in L^2(I) \oplus \mathbb{C}_{\theta\rho_1} \oplus \mathbb{C}_{\rho_2}$. Then, $K := (L^2(I) \oplus \mathbb{C}_{\theta\rho_1} \oplus \mathbb{C}_{\rho_2}, [\cdot,\cdot])$ is the direct sum of modified Krein spaces.

A fundamental symmetry on the Krein space is given by

$$J := \begin{bmatrix} J_0 & 0 & 0 \\ 0 & \mathrm{sgn}\theta\cdot\mathrm{sgn}\rho_1 & 0 \\ 0 & 0 & \mathrm{sgn}\rho_2 \end{bmatrix},$$

where $\mathrm{sgn}\theta, \mathrm{sgn}\rho_j \in \{-1,1\} (j = 1,2)$ and $J_0 : L^2(I) \to L^2(I)$ is defined by

$$(J_0 f)(x) = \begin{cases} f(x)\mathrm{sgn}\theta,\ x \in [a,\xi_1), \\ f(x)\mathrm{sgn}(\theta_i\theta_{i+1}\cdots\theta_m),\ x \in (\xi_{i-1},\xi_i),\ i = \overline{2,m}, \\ f(x),\ x \in (\xi_m, b]. \end{cases}$$

Let $\langle\cdot,\cdot\rangle = [J\cdot,\cdot]$. Then, $\langle\cdot,\cdot\rangle$ is a positive definite inner product which turns K into a Hilbert space $H = (L^2(I) \oplus \mathbb{C}_{|\theta\rho_1|} \oplus \mathbb{C}_{|\rho_2|}, [J\cdot,\cdot])$.

We define the operator A in K as follows:

$$\begin{aligned} D(A) = \{&(f(x),h,r) \in K | f_1, f_1' \in AC_{loc}((a,\xi_1)), \\ &f_2, f_2' \in AC_{loc}((\xi_1,\xi_2)), \cdots, f_{m+1}, \\ &f_{m+1}' \in AC_{loc}((\xi_m,b)), lf \in L^2(I), l_{2i+j}f = 0, \\ &i = \overline{1,m}, j = 1,2,\ h = \alpha_1' f(a) - \alpha_2' f'(a), \\ &r = \beta_1' f(b) - \beta_2' f'(b)\}, \end{aligned}$$

$$AF = (lf, \alpha_1 f(a) - \alpha_2 f'(a), -(\beta_1 f(b) - \beta_2 f'(b)),$$

$$F = (f, \alpha_1' f(a) - \alpha_2' f'(a), \beta_1' f(b) - \beta_2' f'(b)) \in D(A).$$

Now, we can rewrite the considered problem (Equations 1 to 5) in the operator form as

$$AF = \lambda F.$$

From the above, we can easily obtain the following conclusion:

Theorem 1. The eigenvalues and eigenfunctions of the problem (Equations 1 to 5) are defined as the eigenvalues and the first components of the corresponding eigenelements of the operator A, respectively.

Theorem 2. (*cf.* Theorem 2.2 of [6]). The operator A is self-adjoint in the Krein space K.

Simplicity of eigenvalues

Lemma 1. Let the real valued function $q(x) \in C[a,b]$ be continuous on $[a,b]$ and $f(\lambda), g(\lambda)$ are given entire functions. Then, for $\forall\lambda \in \mathbb{C}$, Equation 1 has a unique solution $u = u(x,\lambda)$ satisfying the initial conditions

$$u(a) = f(\lambda),\ u'(a) = g(\lambda)\ (\text{or } u(b) = f(\lambda), u'(b) = g(\lambda)).$$

Let $\varphi_1(x,\lambda)$ be the solution of Equation 1 on the interval $[a,\xi_1)$, satisfying the initial conditions

$$\varphi_1(a,\lambda) = \lambda\alpha_2' - \alpha_2,\ \varphi_1'(a,\lambda) = \lambda\alpha_1' - \alpha_1.$$

By virtue of Lemma 1, after defining this solution, we can define the solutions $\varphi_{i+1}(x,\lambda)$ $(i = \overline{1,m-1})$ of Equation 1 on the interval $[\xi_i,\xi_{i+1})$ by the initial conditions

$$\varphi_{i+1}(\xi_i+0,\lambda) = \alpha_{i1}\varphi_i(\xi_i-0,\lambda) + \alpha_{i2}\varphi_i'(\xi_i-0,\lambda),$$
$$\varphi_{i+1}'(\xi_i+0,\lambda) = \beta_{i1}\varphi_i(\xi_i-0,\lambda) + \beta_{i2}\varphi_i'(\xi_i-0,\lambda).$$

After defining these solutions, we can define the final solution $\varphi_{m+1}(x,\lambda)$ of Equation 1 on the interval $[\xi_m, b]$ by the initial conditions

$$\varphi_{m+1}(\xi_m+0,\lambda)=\alpha_{m1}\varphi_m(\xi_m-0,\lambda)+\alpha_{m2}\varphi'_m(\xi_m-0,\lambda),$$
$$\varphi'_{m+1}(\xi_m+0,\lambda)=\beta_{m1}\varphi_m(\xi_m-0,\lambda)+\beta_{m2}\varphi'_m(\xi_m-0,\lambda).$$

Analogously, we shall define the solutions $\chi_{m+1}(x,\lambda)$ and $\chi_i(x,\lambda)$ $(i=\overline{1,m})$ by initial conditions

$$\chi_{m+1}(b,\lambda)=\lambda\beta'_2+\beta_2,\ \chi'_{m+1}(b,\lambda)=\lambda\beta'_1+\beta_1$$

and

$$\chi_i(\xi_i-0,\lambda)=\frac{\beta_{i2}\chi_{i+1}(\xi_i+0,\lambda)-\alpha_{i2}\chi'_{i+1}(\xi_i+0,\lambda)}{\theta_i},$$
$$\chi'_i(\xi_i-0,\lambda)=\frac{\beta_{i1}\chi_{i+1}(\xi_i+0,\lambda)-\alpha_{i1}\chi'_{i+1}(\xi_i+0,\lambda)}{-\theta_i}.$$

Let us consider the Wronskians

$$\omega_i(\lambda):=W_\lambda(\varphi_i,\chi_i;x):=\varphi_i\chi'_i-\varphi'_i\chi_i,\ x\in\Omega_i,\ i=\overline{1,m+1}$$

which are independent of x and are entire functions, where $\Omega_1=[a,\xi_1)$, $\Omega_2=(\xi_1,\xi_2)$, $\cdots$, $\Omega_{m+1}=(\xi_m,b]$. This sort of calculation gives $\omega_{i+1}(\lambda)=\theta_i\omega_i(\lambda)$ $(i=\overline{1,m})$. Now, we may introduce in consideration the characteristic function $\omega(\lambda)$ as $\omega(\lambda):=\omega_1(\lambda)$.

Theorem 3. The eigenvalues of the problem (Equations 1 to 5) consist of the zeros of function $\omega(\lambda)$.

Proof. Let $\omega(\lambda)=0$. Then, the functions $\varphi_1(x,\lambda)$ and $\chi_1(x,\lambda)$ linearly depended, i.e.,

$$\varphi_1(x,\lambda)=k\chi_1(x,\lambda)$$

for $k\neq 0$. Consequently, the function $k\chi_1(x,\lambda)$ also satisfied the boundary condition (Equation 2). So,

$$\begin{cases} k\chi_1(x,\lambda), x\in[a,\xi_1),\\ k\chi_i(x,\lambda),\ x\in(\xi_{i-1},\xi_i),\ i=\overline{2,m},\\ k\chi_{m+1}(x,\lambda),\ x\in(\xi_m,b] \end{cases}$$

is an eigenfunction of the problem (Equations 1 to 5) corresponding to eigenvalue λ.

Now, let $u(x)$ be any eigenfunction corresponding to eigenvalue λ, but $\omega(\lambda)\neq 0$. Then, the functions φ_1, χ_1 would be linearly independent on $[a,\xi_1)$. Similarly, φ_i, χ_i, $i=\overline{2,m}$, and φ_{m+1}, χ_{m+1} would also be linearly independent on (ξ_{i-1},ξ_i) and $(\xi_m,b]$, respectively. So, $u(x)$ may be represented in the following form:

$$\begin{cases} c_{11}\varphi_1(x,\lambda)+c_{12}\chi_1(x,\lambda),\ x\in[a,\xi_1),\\ c_{i1}\varphi_i(x,\lambda)+c_{i2}\chi_i(x,\lambda),\ x\in(\xi_{i-1},\xi_i),\ i=\overline{2,m},\\ c_{(m+1)1}\varphi_{m+1}(x,\lambda)+c_{(m+1)2}\chi_{m+1}(x,\lambda),\ x\in(\xi_m,b]. \end{cases}$$

According to transmission conditions at the points of discontinuities $x=\xi_i$, we have $c_{11}=c_{21}=\cdots=c_{(m+1)1}$, $c_{12}=c_{22}=\cdots=c_{(m+1)2}$. Thus,

$$u(x)=\begin{cases} c_{11}\varphi_1(x,\lambda)+c_{12}\chi_1(x,\lambda),\ x\in[a,\xi_1),\\ c_{11}\varphi_i(x,\lambda)+c_{12}\chi_i(x,\lambda),\ x\in(\xi_{i-1},\xi_i),\ i=\overline{2,m},\\ c_{11}\varphi_{m+1}(x,\lambda)+c_{12}\chi_{m+1}(x,\lambda),\ x\in(\xi_m,b], \end{cases}$$

where at least one of the constants c_{1k} $(k=1,2)$ is not zero.

Consider the true function

$$l_\nu(u(x))=0,\ \nu=1,2 \tag{6}$$

as the homogenous system of linear equations in the variables c_{1k} $(k=1,2)$, and taking into account the above initial conditions of the fundamental solutions, it follows that the determinant of this system is

$$\begin{vmatrix} 0 & \varphi_1(a)\chi'_1(a)-\varphi'_1(a)\chi_1(a)\\ \varphi_{m+1}(b)\chi'_{m+1}(b)-\varphi'_{m+1}(b)\chi_{m+1}(b) & 0 \end{vmatrix}$$

$$=-\theta\omega^2(\lambda)\neq 0.$$

Therefore, the system (Equation 6) has the only trivial solution $c_{1k}=0$ $(k=1,2)$. Thus, we get a contradiction, which completes the proof. □

Definition 1. The analytic multiplicity of an eigenvalue λ of the problem (Equations 1 to 5) is its order as a root of the characteristic equation $\omega(\lambda)=0$.

Theorem 4. Let $\theta_i>0$, $i=\overline{1,m}$, and $\rho_j>0$, $j=1,2$. Then, the eigenvalues of the problem (Equations 1 to 5) are analytically simple.

Proof. Let $\lambda=u+iv$. For convenience, set $\varphi=\varphi(x,\lambda)$, $\varphi_{1\lambda}=\frac{\partial\varphi_1}{\partial\lambda}$, $\varphi'_{1\lambda}=\frac{\partial\varphi'_1}{\partial\lambda}$, etc. We differentiate the equation $l\chi=\lambda\chi$ with respect to λ and have

$$l\chi_\lambda=\lambda\chi_\lambda+\chi. \tag{7}$$

By integration by parts, we get

$$\begin{aligned}[l\chi_\lambda,\varphi]_1-[\chi_\lambda,l\varphi]_1&=\theta(\chi_{1\lambda}\overline{\varphi}'_1-\chi'_{1\lambda}\overline{\varphi}_1)\Big|_a^{\xi_1}\\ &\quad+\theta_2\theta_3\cdots\theta_m(\chi_{2\lambda}\overline{\varphi}'_2-\chi'_{2\lambda}\overline{\varphi}_2)\Big|_{\xi_1}^{\xi_2}+\cdots\\ &\quad+(\chi_{(m+1)\lambda}\overline{\varphi}'_{m+1}-\chi'_{(m+1)\lambda}\overline{\varphi}_{m+1})\Big|_{\xi_m}^{b}.\end{aligned} \tag{8}$$

Substituting Equation 7 and $l\varphi=\lambda\varphi$ into the left side of Equation 8, we have

$$\lambda[\chi_\lambda,\varphi]_1+[\chi,\varphi]_1-[\chi_\lambda,\lambda\varphi]_1=[\chi,\varphi]_1+2iv[\chi_\lambda,\varphi]_1.$$

Moreover,

$$\theta(\chi_{1\lambda}\overline{\varphi}_1' - \chi_{1\lambda}'\overline{\varphi}_1)\Big|_a^{\xi_1} + \theta_2\theta_3\cdots\theta_m(\chi_{2\lambda}\overline{\varphi}_2' - \chi_{2\lambda}'\overline{\varphi}_2)\Big|_{\xi_1}^{\xi_2}$$
$$+\cdots+(\chi_{(m+1)\lambda}\overline{\varphi}_{m+1}' - \chi_{(m+1)\lambda}'\overline{\varphi}_{m+1})\Big|_{\xi_m}^{b}$$
$$=\theta((\lambda\alpha_2'-\alpha_2)\chi_{1\lambda}'(a,\lambda)-(\lambda\alpha_1'-\alpha_1)\chi_{1\lambda}(a,\lambda))$$
$$+(\beta_2'\overline{\varphi}_{m+1}'(b,\lambda)-\beta_1'\overline{\varphi}_{m+1}(b,\lambda)).$$

Note that

$$\omega'(\lambda)=\alpha_2'\chi_1'(a,\lambda)-\alpha_1'\chi_1(a,\lambda)+(\lambda\alpha_2'-\alpha_2)\chi_{1\lambda}'(a,\lambda)$$
$$-(\lambda\alpha_1'-\alpha_1)\chi_{1\lambda}(a,\lambda).$$

So, Equation 8 becomes

$$\theta\omega'(\lambda)=[\chi,\varphi]_1+2iv[\chi_\lambda,\varphi]_1+\theta(\alpha_2'\chi_1'(a,\lambda)-\alpha_1'\chi_1(a,\lambda))$$
$$-(\beta_2'\overline{\varphi}_3'(b,\lambda)-\beta_1'\overline{\varphi}_3(b,\lambda)). \tag{9}$$

Next, let μ be the arbitrary zero of $\omega(\lambda)$. Obviously, μ is real since

$$\omega(\mu)=\begin{vmatrix}\varphi_1(x,\mu) & \chi_1(x,\mu)\\ \varphi_1'(x,\mu) & \chi_1'(x,\mu)\end{vmatrix}=0.$$

We have $\varphi_i(x,\mu)=c_i\chi_i(x,\mu)$ $(c_i\neq 0)$, where $c_i\in\mathbb{C}$, $i=\overline{1,m+1}$. From

$$\varphi_2(\xi_1,\mu)=c_1(\alpha_{11}\chi_1(\xi_1,\mu)+\alpha_{12}\chi_1'(\xi_1,\mu))=c_1\chi_2(\xi_1,\mu),$$

we get $c_1=c_2\neq 0$. Similarly, we can obtain $c_1=c_2=\cdots=c_{m+1}\neq 0$. Thus, with a short calculation, Equation 9 becomes

$$\theta\omega'(\mu)=\overline{c}_1\left(\frac{\theta}{a_1^2}\int_a^{\xi_1}|\chi_1(x,\mu)|^2dx+\frac{\theta_2\cdots\theta_m}{a_2^2}\right.$$
$$\times\int_{\xi_1}^{\xi_2}|\chi_2(x,\mu)|^2dx+\cdots+\frac{1}{a_{m+1}^2}$$
$$\left.\times\int_{\xi_m}^{b}|\chi_{m+1}(x,\mu)|^2dx+\theta\rho_1+\rho_2\right).$$

Here, $\theta_i>0$, $\overline{i=1,m}$, $\rho_j>0$, $j=1,2$, and $\overline{c}_1\neq 0$. So, $\omega'(\mu)\neq 0$. Hence, the analytic multiplicity of μ is one. By Definition 1, the proof is completed. □

Results and discussion

Asymptotic formulae for fundamental solutions and eigenvalues

Lemma 2. Let $\lambda=s^2$, $s=\sigma+it$. Then, the following integral equations hold for $k=0,1$,

$$\frac{d^k}{dx^k}\varphi_1(x,\lambda)=(-\alpha_2+s^2\alpha_2')\frac{d^k}{dx^k}\cos\frac{s(x-a)}{a_1}$$
$$+\frac{a_1}{s}(-\alpha_1+s^2\alpha_1')\frac{d^k}{dx^k}\sin\frac{s(x-a)}{a_1}$$
$$+\frac{1}{a_1 s}\int_a^x\frac{d^k}{dx^k}\sin\frac{s(x-y)}{a_1}q(y)\varphi_1(y,\lambda)dy, \tag{10}$$

$$\frac{d^k}{dx^k}\varphi_{i+1}(x,\lambda)=(\alpha_{i1}\varphi_i(\xi_i)+\alpha_{i2}\varphi_i'(\xi_i))$$
$$\times\frac{d^k}{dx^k}\cos\frac{s(x-\xi_i)}{a_{i+1}}+\frac{a_{i+1}}{s}(\beta_{i1}\varphi_i(\xi_i)$$
$$+\beta_{i2}\varphi_i'(\xi_i))\frac{d^k}{dx^k}\sin\frac{s(x-\xi_i)}{a_{i+1}}+\frac{1}{a_{i+1}s}$$
$$\times\int_{\xi_i}^x\frac{d^k}{dx^k}\sin\frac{s(x-y)}{a_{i+1}}q(y)\varphi_{i+1}(y,\lambda)dy,$$
$$i=\overline{1,m}. \tag{11}$$

Proof. Regard $\varphi_1(x,\lambda)$ as the solution of the following non-homogeneous Cauchy problem:

$$\begin{cases}a_1^2u''(x)+s^2u(x)=q(x)\varphi_1(x,\lambda),\\ \varphi_1(a,\lambda)=-\alpha_2+s^2\alpha_2',\ \varphi_1'(a,\lambda)=-\alpha_1+s^2\alpha_1'.\end{cases}$$

Using the method of constant changing, $\varphi_1(x,\lambda)$ satisfies

$$\varphi_1(x,\lambda)=(-\alpha_2+s^2\alpha_2')\cos\frac{s(x-a)}{a_1}+\frac{a_1}{s}(-\alpha_1+s^2\alpha_1')$$
$$\times\sin\frac{s(x-a)}{a_1}+\frac{1}{a_1s}\int_a^x\sin\frac{s(x-y)}{a_1}(y)\varphi_1(x,\lambda)dy.$$

Then, differentiating it with respect to x, we have Equation 10. The proof for Equation 11 is similar. □

Lemma 3. Let $\lambda=s^2$, $\mathrm{Im}s=t$. Then, for $\alpha_2'\neq 0$,

$$\frac{d^k}{dx^k}\varphi_1(x,\lambda)=\alpha_2's^2\frac{d^k}{dx^k}\cos\frac{s(x-a)}{a_1}$$
$$+O(|s|^{k+1}e^{|t|\frac{x-a}{a_1}})\ (|\lambda|\to\infty), \tag{12}$$

$$\frac{d^k}{dx^k}\varphi_{i+1}(x,\lambda)=(-1)^i\frac{\alpha_{12}\cdots\alpha_{i2}\alpha_2's^{i+2}}{a_1\cdots a_i}\sin\frac{s(\xi_1-a)}{a_1}\cdots$$
$$\times\sin\frac{s(\xi_i-\xi_{i-1})}{a_i}\frac{d^k}{dx^k}\cos\frac{s(x-\xi_i)}{a_{i+1}}$$
$$+O(|s|^{k+i+1}e^{|t|(\frac{\xi_1-a}{a_1}+\cdots+\frac{x-\xi_i}{a_{i+1}})})\ (|\lambda|\to\infty),$$
$$i=\overline{1,m}, \tag{13}$$

while if $\alpha_2'=0$,

$$\frac{d^k}{dx^k}\varphi_1(x,\lambda)=\alpha_1's\frac{d^k}{dx^k}\sin\frac{s(x-a)}{a_1}$$
$$+O(|s|^ke^{|t|\frac{x-a}{a_1}})\ (|\lambda|\to\infty), \tag{14}$$

$$\frac{d^k}{dx^k}\varphi_{i+1}(x,\lambda) = (-1)^{i-1}\frac{\alpha_{12}\cdots\alpha_{i2}\alpha_1' s^{i+1}}{a_1\cdots a_i}\cos\frac{s(\xi_1-a)}{a_1} \times \sin\frac{s(\xi_2-\xi_1)}{a_2}\cdots\sin\frac{s(\xi_i-\xi_{i-1})}{a_i}\frac{d^k}{dx^k} \times \cos\frac{s(x-\xi_i)}{a_{i+1}} + O(|s|^{k+i}e^{|t|(\frac{\xi_1-a}{a_1}+\cdots+\frac{x-\xi_i}{a_{i+1}})}) \quad (|\lambda|\to\infty),\ i=\overline{1,m}, \tag{15}$$

$k = 0, 1$. Each of these asymptotic equalities holds uniformly for x.

Proof. The asymptotic formulas for $\varphi_1(x,\lambda)$ are found in the same way as those of [6]. Therefore, we shall formulate them without proof.

Let $\alpha_2' \neq 0$, substituting Equation 12 into Equation 11 (for $k = 0$), we have

$$\varphi_2(x,\lambda) = (\alpha_{11}\alpha_2' s^2\cos\frac{s(\xi_1-a)}{a_1} - \frac{\alpha_{12}\alpha_2' s^3}{a_1}\sin\frac{s(\xi_1-a)}{a_1}) \times \cos\frac{s(x-\xi_1)}{a_2} + \frac{a_2}{s}(\beta_{11}\alpha_2' s^2\cos\frac{s(\xi_1-a)}{a_1} - \frac{\beta_{12}\alpha_2' s^3}{a_1}\sin\frac{s(\xi_1-a)}{a_1})\sin\frac{s(x-\xi_1)}{a_2} + \frac{1}{a_2 s}\int_{\xi_1}^{x}\sin\frac{s(x-y)}{a_2}q(y)\varphi_2(y,\lambda)dy + O(|s|^2 e^{|t|(\frac{\xi_1-a}{a_1}+\frac{x-\xi_1}{a_2})}). \tag{16}$$

It is easy to show that $\varphi_2(x,\lambda) = O(|s|^3 e^{|t|(\frac{\xi_1-a}{a_1}+\frac{x-\xi_1}{a_2})})$. Substituting it into Equation 16 gives Equation 13 for $i = 1$ and $k = 0$. The other cases follow by applying the same procedure as in the case $i = 1$ and $k = 0$.

The proof of Equation 15 is similar to that of Equation 13, hence omitted. □

Theorem 5. Let $\lambda = s^2$, Im$s = t$. Then, the characteristic function $\omega(\lambda)$ has the following asymptotic representations:

Case 1 $\beta_2' \neq 0,\ \alpha_2' \neq 0$,

$$\omega(\lambda) = (-1)^m\frac{\alpha_{12}\alpha_{22}\cdots\alpha_{m2}\alpha_2'\beta_2' s^{m+5}}{a_1a_2\cdots a_{m+1}\theta}\sin\frac{s(\xi_1-a)}{a_1} \times\sin\frac{s(\xi_2-\xi_1)}{a_2}\cdots\sin\frac{s(\xi_m-\xi_{m-1})}{a_m}\sin\frac{s(b-\xi_m)}{a_{m+1}} + O(|s|^{m+4}e^{|t|(\frac{\xi_1-a}{a_1}+\frac{\xi_2-\xi_1}{a_2}+\cdots+\frac{b-\xi_m}{a_{m+1}})}).$$

Case 2 $\beta_2' \neq 0,\ \alpha_2' = 0$,

$$\omega(\lambda) = (-1)^m\frac{\alpha_{12}\alpha_{22}\cdots\alpha_{m2}\alpha_1'\beta_2' s^{m+4}}{a_1a_2\cdots a_m\theta}\cos\frac{s(\xi_1-a)}{a_1} \times\sin\frac{s(\xi_2-\xi_1)}{a_2}\cdots\sin\frac{s(\xi_m-\xi_{m-1})}{a_m}\sin\frac{s(b-\xi_m)}{a_{m+1}} + O(|s|^{m+3}e^{|t|(\frac{\xi_1-a}{a_1}+\frac{\xi_2-\xi_1}{a_2}+\cdots+\frac{b-\xi_m}{a_{m+1}})}).$$

Case 3 $\beta_2' = 0,\ \alpha_2' \neq 0$,

$$\omega(\lambda) = (-1)^{m+1}\frac{\alpha_{12}\alpha_{22}\cdots\alpha_{m2}\alpha_2'\beta_1' s^{m+4}}{a_1a_2\cdots a_{m+1}\theta}\sin\frac{s(\xi_1-a)}{a_1} \times\sin\frac{s(\xi_2-\xi_1)}{a_2}\cdots\sin\frac{s(\xi_m-\xi_{m-1})}{a_m}\cos\frac{s(b-\xi_m)}{a_{m+1}} + O(|s|^{m+3}e^{|t|(\frac{\xi_1-a}{a_1}+\frac{\xi_2-\xi_1}{a_2}+\cdots+\frac{b-\xi_m}{a_{m+1}})}).$$

Case 4 $\beta_2' = 0,\ \alpha_2' = 0$,

$$\omega(\lambda) = (-1)^{m+1}\frac{\alpha_{12}\alpha_{22}\cdots\alpha_{m2}\alpha_1'\beta_1' s^{m+3}}{a_1a_2\cdots a_m\theta}\cos\frac{s(\xi_1-a)}{a_1} \times\sin\frac{s(\xi_2-\xi_1)}{a_2}\cdots\sin\frac{s(\xi_m-\xi_{m-1})}{a_m}\cos\frac{s(b-\xi_m)}{a_{m+1}} + O(|s|^{m+2}e^{|t|(\frac{\xi_1-a}{a_1}+\frac{\xi_2-\xi_1}{a_2}+\cdots+\frac{b-\xi_m}{a_{m+1}})}).$$

Proof. The proof is obtained by substituting the asymptotic equalities $\frac{d^k}{dx^k}\varphi_{m+1}(x,\lambda)$ into the representation

$$\theta\omega(\lambda) = (\beta_1+\lambda\beta_1')\varphi_{m+1}(b,\lambda) - (\beta_2+\lambda\beta_2')\varphi_{m+1}'(b,\lambda).$$

□

Theorem 6. The following asymptotic formulas hold for the real eigenvalues of the boundary value transmission problem (Equations 1 to 5):

Case 1 $\beta_2' \neq 0,\ \alpha_2' \neq 0$,

$$\sqrt{\lambda_n^{(1)}} = \frac{a_1(n-1)\pi}{\xi_1-a} + O(\frac{1}{n}),\ \sqrt{\lambda_n^{(i)}} = \frac{a_i(n-1)\pi}{\xi_i-\xi_{i-1}} + O(\frac{1}{n}),\ \sqrt{\lambda_n^{(m+1)}} = \frac{a_{m+1}(n-1)\pi}{b-\xi_m} + O(\frac{1}{n}).$$

Case 2 $\beta_2' \neq 0,\ \alpha_2' = 0$,

$$\sqrt{\lambda_n^{(1)}} = \frac{a_1(n-\frac{1}{2})\pi}{\xi_1-a} + O(\frac{1}{n}),\ \sqrt{\lambda_n^{(i)}} = \frac{a_i(n-1)\pi}{\xi_i-\xi_{i-1}} + O(\frac{1}{n}),\ \sqrt{\lambda_n^{(m+1)}} = \frac{a_{m+1}(n-1)\pi}{b-\xi_m} + O(\frac{1}{n}).$$

Case 3 $\beta'_2 = 0,\ \alpha'_2 \neq 0,$

$$\sqrt{\lambda_n^{(1)}} = \frac{a_1(n-1)\pi}{\xi_1 - a} + O(\frac{1}{n}),\ \sqrt{\lambda_n^{(i)}} = \frac{a_i(n-1)\pi}{\xi_i - \xi_{i-1}} + O(\frac{1}{n}),\ \sqrt{\lambda_n^{(m+1)}} = \frac{a_{m+1}(n-\frac{1}{2})\pi}{b-\xi_m} + O(\frac{1}{n}).$$

Case 4 $\beta'_2 = 0,\ \alpha'_2 = 0,$

$$\sqrt{\lambda_n^{(1)}} = \frac{a_1(n-\frac{1}{2})\pi}{\xi_1 - a} + O(\frac{1}{n}),\ \sqrt{\lambda_n^{(i)}} = \frac{a_i(n-1)\pi}{\xi_i - \xi_{i-1}} + O(\frac{1}{n}),\ \sqrt{\lambda_n^{(m+1)}} = \frac{a_{m+1}(n-\frac{1}{2})\pi}{b-\xi_m} + O(\frac{1}{n}).$$

Here, $i = \overline{2, m}$.

Proof. By applying the known Rouche theorem, we can obtain these conclusions (*cf.* Theorem 2.3 of [11]). □

Corollary 1. The real eigenvalues of the problem (Equations 1 to 5) are bounded below.

Proof. Let $s = it$, i.e., $\lambda = -t^2$. In the above formulas, it follows that $\omega(-t^2) \to \infty$ as $t \to \infty$. Consequently, $\omega(-t^2) \neq 0$ for λ negative and sufficiently large in modulus. □

Green function and resolvent operator

Let us consider the following differential equation:

$$-(a(x)u'(x))' + q(x)u(x) - \lambda u(x) = -f(x),\ x \in I, \quad (17)$$

where $I = [a, \xi_1) \cup (\xi_1, \xi_2) \cup \cdots \cup (\xi_m, b]$, $a(x) = a_1^2$ for $x \in [a, \xi_1)$, $a(x) = a_2^2$ for $x \in (\xi_1, \xi_2), \cdots, a(x) = a_{m+1}^2$ for $x \in (\xi_m, b]$, $a_1, a_2, \cdots a_{m+1}$ are positive real constants, together with the eigenparameter-dependent boundary and transmission conditions (Equations 2 to 5).

We can represent the general solution of homogeneous differential equation (Equation 1), appropriate to Equation 17. By applying the standard method of variation of constants, we shall search the general solution of the non-homogeneous differential equation (Equation 17) in the form

$$U(x) = \begin{cases} C_{11}(x,\lambda)\varphi_1(x,\lambda) + C_{12}(x,\lambda)\chi_1(x,\lambda), x \in [a, \xi_1), \\ C_{i1}(x,\lambda)\varphi_i(x,\lambda) + C_{i2}(x,\lambda)\chi_i(x,\lambda),\ x \in (\xi_{i-1}, \xi_i),\ i = \overline{2, m}, \\ C_{(m+1)1}(x,\lambda)\varphi_{m+1}(x,\lambda) + C_{(m+1)2}(x,\lambda)\chi_{m+1}(x,\lambda],\ x \in (\xi_m, b), \end{cases} \quad (18)$$

where the functions $C_{kj}(x,\lambda)$ ($k = \overline{1, m+1}, j = 1, 2$) satisfy the linear system of equation

$$\begin{cases} C'_{11}(x,\lambda)\varphi_1(x,\lambda) + C'_{12}(x,\lambda)\chi_1(x,\lambda) = 0, \\ C'_{11}(x,\lambda)\varphi'_1(x,\lambda) + C'_{12}(x,\lambda)\chi'_1(x,\lambda) = f(x) \end{cases}$$

for $x \in [a, \xi_1)$,

$$\begin{cases} C'_{i1}(x,\lambda)\varphi_i(x,\lambda) + C'_{i2}(x,\lambda)\chi_i(x,\lambda) = 0, \\ C'_{i1}(x,\lambda)\varphi'_i(x,\lambda) + C'_{i2}(x,\lambda)\chi'_i(x,\lambda) = f(x) \end{cases}$$

for $x \in (\xi_{i-1}, \xi_i)$, and

$$\begin{cases} C'_{(m+1)1}(x,\lambda)\varphi_{m+1}(x,\lambda) + C'_{(m+1)2}(x,\lambda)\chi_{m+1}(x,\lambda) = 0, \\ C'_{(m+1)1}(x,\lambda)\varphi'_{m+1}(x,\lambda) + C'_{(m+1)2}(x,\lambda)\chi'_{m+1}(x,\lambda) = f(x) \end{cases}$$

for $x \in (\xi_{m+1}, b]$. Because the characteristic function $\omega(\lambda) \neq 0$, the following relations can be easily obtained:

$$C_{11}(x,\lambda) = \frac{1}{\omega_1(\lambda)} \int_x^{\xi_1} f\chi_1 dy + C_{11},$$
$$C_{12}(x,\lambda) = \frac{1}{\omega_1(\lambda)} \int_a^x f\varphi_1 dy + C_{12},\ x \in [a, \xi_1),$$

$$C_{i1}(x,\lambda) = \frac{1}{\omega_i(\lambda)} \int_x^{\xi_i} f\chi_i dy + C_{i1},$$
$$C_{i2}(x,\lambda) = \frac{1}{\omega_i(\lambda)} \int_{\xi_{i-1}}^x f\varphi_i dy + C_{i2},\ x \in (\xi_{i-1}, \xi_i),$$

$$C_{(m+1)1}(x,\lambda) = \frac{1}{\omega_{m+1}(\lambda)} \int_x^b f\chi_{m+1} dy + C_{(m+1)1},$$
$$C_{(m+1)2}(x,\lambda) = \frac{1}{\omega_{m+1}(\lambda)} \int_{\xi_m}^x f\varphi_{m+1} dy + C_{(m+1)2},\ x \in (\xi_m, b].$$

Here, C_{kj} ($k = \overline{1, m+1}, j = 1, 2$) are arbitrary constants.

In the following, for simplicity, set $\varphi(x) = \varphi(x,\lambda)$, etc. Substituting the above equations into Equation 18, the general solution $U(x,\lambda)$ of the non-homogeneous differential equation (Equation 17) is obtained as

$$\begin{cases} \dfrac{\varphi_1(x)}{\omega_1(\lambda)}\displaystyle\int_x^{\xi_1} f\chi_1 dy + \dfrac{\chi_1(x)}{\omega_1(\lambda)}\int_a^x f\varphi_1 dy \\ \qquad + C_{11}\varphi_1(x) + C_{12}\chi_1(x),\ x \in [a,\xi_1), \\ \dfrac{\varphi_i(x)}{\omega_i(\lambda)}\displaystyle\int_x^{\xi_i} f\chi_i dy + \dfrac{\chi_i(x)}{\omega_i(\lambda)}\int_{\xi_{i-1}}^x f\varphi_i dy + C_{i1}\varphi_i(x) \\ \qquad + C_{i2}\chi_i(x),\ x \in (\xi_{i-1},\xi_i), i = \overline{2,m}, \\ \dfrac{\varphi_{m+1}(x)}{\omega_{m+1}(\lambda)}\displaystyle\int_x^b f\chi_{m+1} dy + \dfrac{\chi_{m+1}(x)}{\omega_{m+1}(\lambda)}\int_{\xi_m}^x f\varphi_{m+1} dy \\ \qquad + C_{(m+1)1}\varphi_{m+1}(x) + C_{(m+1)2}\chi_{m+1}(x),\ x \in (\xi_m, b], \end{cases} \tag{19}$$

where C_{kj} ($k = \overline{1,m+1}, j = 1,2$) are arbitrary constants. By differentiating Equation 19, we have the representation of $U'(x,\lambda)$, i.e.,

$$\begin{cases} \dfrac{\varphi_1'(x)}{\omega_1(\lambda)}\displaystyle\int_x^{\xi_1} f\chi_1 dy + \dfrac{\chi_1'(x)}{\omega_1(\lambda)}\int_a^x f\varphi_1 dy \\ \qquad + C_{11}\varphi_1'(x) + C_{12}\chi_1'(x),\ x \in [a,\xi_1), \\ \dfrac{\varphi_i'(x)}{\omega_i(\lambda)}\displaystyle\int_x^{\xi_i} f\chi_i dy + \dfrac{\chi_i'(x)}{\omega_i(\lambda)}\int_{\xi_{i-1}}^x f\varphi_i dy \\ \qquad + C_{i1}\varphi_2'(x) + C_{i2}\chi_i'(x),\ x \in (\xi_{i-1},\xi_i), i = \overline{2,m}, \\ \dfrac{\varphi_{m+1}'(x)}{\omega_{m+1}(\lambda)}\displaystyle\int_x^b f\chi_{m+1} dy + \dfrac{\chi_{m+1}'(x)}{\omega_{m+1}(\lambda)}\int_{\xi_m}^x f\varphi_{m+1} dy \\ \qquad + C_{(m+1)1}\varphi_{m+1}'(x) + C_{(m+1)2}\chi_{m+1}(x),\ x \in (\xi_m, b]. \end{cases} \tag{20}$$

By using the system of Equation 19 and the proof process of Theorem 3, the following equalities are obtained for $l_v(U), v = \overline{1,2m+2}$:

$$l_1(U) = -C_{12}\omega_1(\lambda), + \tag{21}$$

$$l_2(U) = C_{(m+1)1}\omega_{m+1}(\lambda), \tag{22}$$

$$\begin{aligned} l_{2i+1}(U) &= \frac{\varphi_{i+1}(\xi_i)}{\omega_{i+1}(\lambda)}\int_{\xi_i}^{\xi_{i+1}} f\chi_{i+1}dy - \frac{\chi_{i+1}(\xi_i)}{\omega_i(\lambda)} \\ &\quad \times \int_{\xi_{i-1}}^{\xi_i} f\varphi_i dy - C_{i1}\varphi_{i+1}(\xi_i) \\ &-C_{i2}\chi_{i+1}(\xi_i)+C_{(i+1)1}\varphi_{i+1}(\xi_i)+C_{(i+1)2}\chi_{i+1}(\xi_{i+1}),\ i=\overline{1,m}, \end{aligned} \tag{23}$$

$$\begin{aligned} l_{2i+2}(U) &= \frac{\varphi_{i+1}'(\xi_i)}{\omega_{i+1}(\lambda)}\int_{\xi_i}^{\xi_{i+1}} f\chi_{i+1}dy - \frac{\chi_{i+1}'(\xi_i)}{\omega_i(\lambda)} \\ &\quad \times \int_{\xi_{i-1}}^{\xi_i} f\varphi_i dy - C_{i1}\varphi_{i+1}'(\xi_i) - C_{i2}\chi_{i+1}'(\xi_i) \\ &\quad + C_{(i+1)1}\varphi_{i+1}'(\xi_i) + C_{(i+1)2}\chi_{i+1}'(\xi_{i+1}), \\ &\quad i = \overline{1,m}. \end{aligned} \tag{24}$$

Because $U(x,\lambda)$ is a solution and $\omega(\lambda) \neq 0$, from the boundary condition (Equation 2) and equality (Equation 21), we have $C_{12} = 0$. Similarly, from the equality (Equation 22) and boundary condition (Equation 3), we have $C_{(m+1)1} = 0$.

On the other hand, by taking into account Equations 23 and 24 and transmission conditions, the following linear equation system according to the variables C_{ij} ($i = \overline{2,m}$, $j = 1,2$) is obtained:

$$\begin{cases} C_{i1}\varphi_{i+1}(\xi_i) + C_{i2}\chi_{i+1}(\xi_i) - C_{(i+1)1}\varphi_{i+1}(\xi_i) - C_{(i+1)2}\chi_{i+1}(\xi_i) \\ \quad = \dfrac{\varphi_{i+1}(\xi_i)}{\omega_{i+1}(\lambda)}\displaystyle\int_{\xi_i}^{\xi_{i+1}} f\chi_{i+1}dy - \dfrac{\chi_{i+1}(\xi_i)}{\omega_i(\lambda)}\int_{\xi_{i-1}}^{\xi_i} f\varphi_i dy, \\ C_{i1}\varphi_{i+1}(\xi_i) + C_{i2}\chi_{i+1}(\xi_i) - C_{(i+1)1}\varphi_{i+1}(\xi_i) - C_{(i+1)2}\chi_{i+1}(\xi_i) \\ \quad = \dfrac{\varphi_{i+1}(\xi_i)}{\omega_{i+1}(\lambda)}\displaystyle\int_{\xi_i}^{\xi_{i+1}} f\chi_{i+1}dy - \dfrac{\chi_{i+1}(\xi_i)}{\omega_i(\lambda)}\int_{\xi_{i-1}}^{\xi_i} f\varphi_i dy. \end{cases} \tag{25}$$

By using the definitions of solutions $\varphi_i(x,\lambda)$ and $\chi_i(x,\lambda)$ ($i = \overline{2,m+1}$), the following relation is obtained for the determinant of this linear equation system:

$$\begin{vmatrix} \varphi_2(\xi_1) & -\varphi_2(\xi_1) & -\chi_2(\xi_1) & 0 & \cdots & 0 & 0 & 0 & 0 \\ \varphi_2'(\xi_1) & -\varphi_2'(\xi_1) & -\chi_2'(\xi_1) & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \varphi_3(\xi_2) & -\chi_3(\xi_2) & \varphi_3(\xi_2) & \cdots & 0 & 0 & 0 & 0 \\ 0 & -\varphi_3'(\xi_2) & -\chi_3'(\xi_2) & \varphi_3'(\xi_2) & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\chi_m(\xi_{m-1}) & \varphi_m(\xi_{m-1}) & \chi_m(\xi_{m-1}) & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\chi_m'(\xi_{m-1}) & \varphi_m'(\xi_{m-1}) & \chi_m'(\xi_{m-1}) & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\varphi_{m+1}(\xi_m) & -\chi_{m+1}(\xi_m) & \chi_{m+1}(\xi_m) \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\varphi_{m+1}'(\xi_m) & -\chi_{m+1}'(\xi_m) & \chi_{m+1}'(\xi_m) \end{vmatrix}$$

$$= \prod_{i=2}^{m+1} \omega_i(\lambda).$$

Since this determinant is different from zero, the solution of Equation 25 is unique. If we solve the system (Equation 25), we get the following equalities:

$$C_{i1} = \frac{1}{\omega_{i+1}(\lambda)} \int_{\xi_i}^{\xi_{i+1}} f\chi_{i+1}dy + \cdots + \frac{1}{\omega_{m+1}(\lambda)} \times \int_{\xi_m}^{b} f\chi_{m+1}dy,\ i = \overline{1,m},$$

$$C_{(i+1)2} = \frac{1}{\omega_1(\lambda)} \int_a^{\xi_1} f\varphi_1 dy + \cdots + \frac{1}{\omega_i(\lambda)} \times \int_{\xi_{i-1}}^{\xi_i} f\varphi_i dy,\ i = \overline{1,m}.$$

Finally, by substituting the coefficients C_{kj} ($k = \overline{1,m+1}, j = 1,2$) in Equation 24, we can get the formulas of the resolvent $U(x,\lambda)$. Further, let

$$\varphi(x) = \begin{cases} \varphi_1(x),\ x \in [a,\xi_1), \\ \varphi_i(x),\ x \in (\xi_{i-1},\xi_i),\ i = \overline{2,m}, \\ \varphi_{m+1}(x),\ x \in (\xi_m, b], \end{cases}$$

$$\chi(x) = \begin{cases} \chi_1(x),\ x \in [a,\xi_1), \\ \chi_i(x),\ x \in (\xi_{i-1},\xi_i),\ i = \overline{2,m}, \\ \chi_{m+1}(x),\ x \in (\xi_m, b]. \end{cases}$$

Then,

$$U(x,\lambda) = \frac{\varphi(x)}{\omega_i(\lambda)} \int_x^b f\chi_i dy + \frac{\chi(x)}{\omega_i(\lambda)} \int_a^x f\varphi_i dy,\ i = \overline{1,m+1}. \tag{26}$$

Thus, the resolvent of the boundary-value transmission problem is obtained. We can find the Green function from the resolvent (Equation 31). Namely, denoting

$$G(x,y;\lambda) = \begin{cases} \dfrac{\varphi_i(y,\lambda)\chi(x,\lambda)}{\omega_i(\lambda)},\ a \le y \le x \le b,\ x \ne \xi_1,\xi_2, \\ \cdots,\xi_m,\ y \ne \xi_1,\xi_2,\cdots,\xi_m, \\ \qquad\qquad i = \overline{1,m+1}. \\ \dfrac{\varphi(x,\lambda)\chi_i(y,\lambda)}{\omega_i(\lambda)},\ a \le x \le y \le b,\ x \ne \xi_1,\xi_2, \\ \cdots,\xi_m,\ y \ne \xi_1,\xi_2,\cdots,\xi_m, \end{cases} \tag{27}$$

We can rewrite the resolvent (Equation 26) in the next form

$$U(x,\lambda) = \int_a^b G(x,y;\lambda) f(y) dy.$$

Let λ not be an eigenvalue of A. It is obvious that the operator equation

$$(\lambda I - A)U = F,\ F = (f(x), f_1, f_2) \in K$$

is equal to the following problem:

$$lu := -(a(x)u'(x))' + q(x)u(x) = \lambda u(x) - f(x) \tag{28}$$

to hold in $I = [a,\xi_1) \cup (\xi_1,\xi_2) \cup \cdots \cup (\xi_m, b]$, subject to the eigenparameter-dependent boundary conditions

$$\lambda(\alpha_1' u(a) - \alpha_2' u'(a)) - (\alpha_1 u(a) - \alpha_2 u'(a)) = f_1, \tag{29}$$

$$\lambda(\beta_1' u(b) - \beta_2' u'(b)) + (\beta_1 u(b) - \beta_2 u'(b)) = f_2 \tag{30}$$

and transmission conditions (Equations 4 and 5). The general solution $V(x,\lambda)$ of Equation 28 can be represented as

$$\begin{cases} \dfrac{\varphi_1(x)}{\omega_1(\lambda)} \displaystyle\int_x^{\xi_1} f\chi_1 dy + \dfrac{\chi_1(x)}{\omega_1(\lambda)} \displaystyle\int_a^x f\varphi_1 dy \\ \quad + D_{11}\varphi_1(x) + D_{12}\chi_1(x),\ x \in [a,\xi_1), \\ \dfrac{\varphi_i(x)}{\omega_i(\lambda)} \displaystyle\int_x^{\xi_i} f\chi_i dy + \dfrac{\chi_i(x)}{\omega_i(\lambda)} \displaystyle\int_{\xi_{i-1}}^x f\varphi_i dy \\ \quad + D_{i1}\varphi_i(x) + D_{i2}\chi_i(x),\ x \in (\xi_{i-1},\xi_i), i = \overline{2,m}, \\ \dfrac{\varphi_{m+1}(x)}{\omega_{m+1}(\lambda)} \displaystyle\int_x^b f\chi_{m+1} dy + \dfrac{\chi_{m+1}(x)}{\omega_{m+1}(\lambda)} \displaystyle\int_{\xi_m}^x f\varphi_{m+1} dy \\ \quad + D_{(m+1)1}\varphi_{m+1}(x) + D_{(m+1)2}\chi_{m+1}(x),\ x \in (\xi_m, b], \end{cases} \tag{31}$$

where, D_{kj} ($k = \overline{1,m+1}, j = 1,2$) are arbitrary constants. Substituting Equation 31 into Equations 29 and 30 as well as Equations 4 and 5, we obtain

$$D_{12} = -\frac{f_1}{\omega_1(\lambda)}, D_{(m+1)1} = \frac{f_2}{\omega_{m+1}(\lambda)},$$

$$D_{i1} = \frac{1}{\omega_{i+1}(\lambda)} \int_{\xi_i}^{\xi_{i+1}} f\chi_{i+1} dy + \cdots + \frac{1}{\omega_{m+1}(\lambda)} \times \int_{\xi_m}^b f\chi_{m+1} dy + \frac{f_2}{\omega_{m+1}(\lambda)},\ i = \overline{1,m},$$

$$D_{(i+1)2} = \frac{1}{\omega_1(\lambda)} \int_a^{\xi_1} f\varphi_1 dy + \cdots + \frac{1}{\omega_i(\lambda)} \times \int_{\xi_{i-1}}^{\xi_i} f\varphi_i dy - \frac{f_1}{\omega_1(\lambda)},\ i = \overline{1,m}.$$

Substitution of these equivalents into Equation 31, we get

$$V(x,\lambda) = U(x,\lambda) - \frac{\chi(x)}{\omega_1(\lambda)} f_1 + \frac{\varphi(x)}{\omega_{m+1}(\lambda)} f_2. \tag{32}$$

Let

$$N_1(f) = \alpha_1 f(a) - \alpha_2 f'(a),\ N_1'(f) = \alpha_1' f(a) - \alpha_2' f'(a),$$

$$N_2(f) = \beta_1 f(b) - \beta_2 f'(b),\ N_2'(f) = \beta_1' f(b) - \beta_2' f'(b).$$

Then, the formula (Equation 32) can also be written as

$$V(x,\lambda) = \int_a^b f(y)G(x,y,\lambda)dy + \frac{1}{\rho_1}N_1'(G(x,\cdot,\lambda))f_1 + \frac{1}{\rho_2}N_2'(G(x,\cdot,\lambda))f_2, \quad (33)$$

where $G(x,y,\lambda)$ is the same with Equation 27.

Now, denoting

$$\widetilde{G}_{x,\lambda} = \begin{pmatrix} G(x,\cdot,\lambda) \\ N_1'(G(x,\cdot,\lambda)) \\ N_2'(G(x,\cdot,\lambda)) \end{pmatrix},$$

$$F_p = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix}, \ \overline{F}_p = \begin{pmatrix} \overline{f(x)} \\ \overline{f_1} \\ \overline{f_2} \end{pmatrix}.$$

The formula (Equation 33) takes the form

$$V(x,\lambda) = [\widetilde{G}_{x,\lambda}, \overline{F}_p].$$

So, the resolvent of the operator $R(\lambda,A) = (\lambda I - A)^{-1}$ can be represented in the form

$$R(\lambda,A)F = \begin{pmatrix} [\widetilde{G}_{x,\lambda}, \overline{F}_p] \\ N_1'[\widetilde{G}_{x,\lambda}, \overline{F}_p] \\ N_2'[\widetilde{G}_{x,\lambda}, \overline{F}_p] \end{pmatrix}.$$

Conclusions

Three important conclusions can be really drawn: (1) the residual spectrum of the operator A is empty, i.e., $\sigma_r(A) = \emptyset$; (2) if $\theta_i > 0$ $(i = \overline{1,m})$ and $\rho_j > 0$ $(j = 1,2)$, a.e., then the operator A has only real point spectrum, i.e., $\sigma(A) = \sigma_p(A) \subset \mathbb{R}$; (3) if $B = JA > 0$, then the point spectrum of the operator A is all real, i.e., $\sigma_p(A) \subset \mathbb{R}$.

Competing interests

Both authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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