Geophys. J. R. astr. Soc. (1976) 47, 225–249

Spectral representation of the Love wave operator

M. H. Kazi* *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW*

Received 1975 November 6

Summary. This paper is concerned with the spectral representation of the two-dimensional Love wave operator, associated with the propagation of monochromatic *SH* waves in a laterally uniform layered strip or half-space. A method is presented for obtaining a complete set of proper or improper eigenfunctions belonging to the operator, in terms of which the displacement field may be represented generally. The method is illustrated by means of two-layer models of an infinite strip, overlying another infinite strip or a half-space, with constant rigidity and density within each layer.

1 Introduction

In many problems of diffraction, reflection and transmission of Love waves at a horizontally discontinuous change in elevation (vertical step) or in material properties of layered structures, we need to express the displacements on either side of the discontinuity, in terms of a complete set of eigenfunctions, proper or improper, associated with the Love wave operator. We may wish to do this, for instance, in order to be able to use such powerful techniques as the integral equation formulation together with the application of the Schwinger–Levine variational principle, which has proved effective in the treatment of acoustical (Miles 1946), electromagnetic (Marcuvitz 1951) and water wave (Miles 1967) problems. Ordinarily it is straightforward to determine the eigenfunctions belonging to the discrete spectrum, but the calculation of the improper eigenfunctions for continuous spectrum presents greater problems. We, therefore, present in this paper the spectral representation of the two-dimensional Love wave operator, associated with the propagation of monochromatic *SH* waves in a laterally uniform layered strip or half-space. In the former case, we obtain a regular Sturm–Liouville system, and in the latter case a singular Sturm–Liouville system, both with discontinuous coefficients and corresponding interface conditions. Such systems have been discussed in detail by Sangren (1953) and Stallard (1955). Hudson (1962) has applied Sturm–Liouville oscillation theorems to the Love wave operator for such a medium.

* On leave from Department of Mathematics, Punjab University, Lahore, Pakistan.

The spectrum of eigenvalues, when the layered structure is of finite depth, is discrete (i.e. a point-spectrum), whereas the spectrum for a layered half-space is the disjoint union of the point-spectrum, giving rise to proper eigenfunctions, and the continuous spectrum, which yields improper eigenfunctions. These eigenfunctions (proper as well as improper) may be generated from a Green function. The essential step is to construct a Green function G as a function of a certain parameter λ, and to integrate it around a large circle $|\lambda| = R$ in the complex λ-plane. For the case of a strip of finite depth G is meromorphic, and the sum of the residues at simple poles, lying on the real axis, gives the representation of the delta function as a series in the eigenfunctions. For the half-space problem, we have in addition to poles, a branch-point singularity. The sum of residues at the poles and the contribution from the branch-cut yield the representation of the delta function in terms of proper and improper eigenfunctions. The method which we use here is a straightforward modification of the method described by Friedman (1956) and Stakgold (1967).

2 Equations of motion

We wish to represent the two-dimensional motion of a laterally homogeneous structure in a general way; the motion will consist of waves propagating along the x-axis (see Fig. 1). The medium to be considered is made up of $N + 1$ parallel, isotropic and solid layers. It may be finite in depth, with the $N + 1$ layers contained between two parallel surfaces or may be semi-infinite in depth with N layers overlying a half-space. The upper plane surface is supposed to be stress-free. The lower surface in the case of finite depth may be stress-free or may be rigidly fixed. In the case of a layered half-space, we assume that the displacements are square-integrable as a function of depth, relative to the rigidity function, over the semi-infinite interval.

The density ρ and the rigidity μ of the material in each layer may vary continuously with depth, but are uniform in any direction parallel to the boundaries. The density ρ and the rigidity μ may, however, be discontinuous across the N plane interfaces.

We choose the axes in such a way that the xy-plane coincides with the upper free surface, the positive z-axis is directed into the medium and all displacements are uniform in the y-direction. The various layers and the interfaces are numbered from the free surface as shown in Fig. 1.

The N interfaces have the equations $z = z_i$, $i = 1, 2, \ldots, N$, the upper free plane-surface has the equation $z = z_0 = 0$ and the lower bounding plane-surface, in the case of finite depth, has the equation $z = z_{N+1} = H$.

We shall consider horizontally polarized shear waves only, which means that there are no displacements in the x and z directions and the motion is in the y-direction only. Let

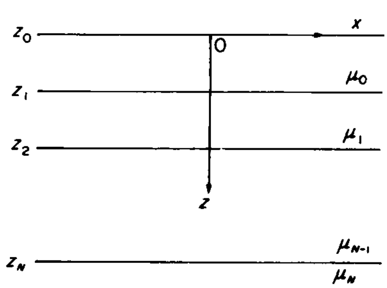

Figure 1. Geometry of the problem.

$v(x, z, t)$ be the y-component of displacement. It must satisfy the differential equation

$$\rho(z)\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu(z)\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu(z)\frac{\partial v}{\partial z}\right), \tag{2.1}$$

where

$$\left.\begin{aligned}\mu(z) &= \mu_i(z),\\ &\text{and}\\ \rho(z) &= \rho_i(z),\end{aligned}\right\} \tag{2.2}$$

when $z \in I_i = \{z\colon z_i < z < z_{i+1}\}$, $i = 0, 1, \ldots, N$. (In the case of a layered half-space, we take $I_N = \{z\colon z_N < z\}$.) We assume that the functions $\mu_i(z)$, $\partial\mu_i/\partial z$ and $\rho_i(z)$ are continuous and μ_i and ρ_i are positive in the subintervals I_i, $i = 0, 1, \ldots, N$ and that their limits, when $z \to z_i$ or z_{i+1} are finite.

In order to obtain a general representation, we first of all examine harmonic waves, travelling in the positive x-direction with positive real frequency ω and wave number k:

$$v(x, z, t) = V(z)\exp[i(\omega t - kx)]. \tag{2.3}$$

(We shall assume ω to be fixed and choose k to satisfy the propagation conditions.) Equation (2.1) becomes

$$\begin{aligned}L(V) &\equiv \frac{d}{dz}\left(\mu(z)\frac{dV}{dz}\right) + (\omega^2\rho(z) - k^2\mu(z))\,V = 0,\\ V(z) &\equiv V_i(z),\ z \in I_i,\end{aligned} \tag{2.4}$$

L being the Love wave operator.

The *interface conditions* of welded contact between the layers are given by the equations:

$$V_k(z_k) = V_{k-1}(z_k), \qquad k = 1, 2, \ldots, N, \tag{2.5}$$

and

$$\mu_k(z_k)\,V_k'(z_k) = \mu_{k-1}(z_k)\,V_{k-1}'(z_k), \tag{2.6}$$

where ($'$) denotes differentiation with respect to z.

The *end-conditions* for the *finite* depth problem will be chosen as

$$\left.\begin{aligned}V_0'(0) &= 0\\ V_N(H) &= 0,\end{aligned}\right\} \tag{2.7}$$

or

$$\left.\begin{aligned}V_0'(0) &= 0\\ V_N'(H) &= 0.\end{aligned}\right\} \tag{2.8}$$

For the layered *half-space* problem, we shall assume, in addition to the end-condition $V_0'(0) = 0$, the following condition:

$$\int_0^\infty \mu(z)\,|V(z)|^2 dz < \infty \tag{2.9}$$

The system (2.4)–(2.8) is a *regular* Sturm–Liouville (S–L) system, and the system (2.4)–(2.6), (2.9) is a *singular* S–L system, with N points of discontinuity and corresponding interface conditions. Such systems have been discussed in detail by Sangren (1953) and Stallard (1955). Hudson (1962) has applied standard Sturm–Liouville theory to investigate the existence of Love waves in a heterogeneous medium and to find the form of variation of the wavelength and the group velocity of the waves with period. Here, we shall describe a technique, based upon the use of Green's function, to obtain the representation of any displacement in terms of the proper and improper eigenfunctions belonging to the system. We will seek wider application by taking a more general form of the S–L problem.

3 Regular S–L systems with discontinuous coefficients. The finite depth problem

Consider the differential equations

$$L(y) - \lambda py = -(r(z)y'(z))' + q(z)y - \lambda p(z)y = 0 \tag{3.1}$$

defined over an interval (a, b), where

$$L \equiv -\frac{d}{dz}\left[r(z)\frac{d}{dz}\right] + q(z),$$

and $(')$ denotes differentiation with respect to z; λ is a parameter, real or complex. (Equation (3.1) reduces to (2.4) when $p(z) = -r(z) = \mu(z)$, $q(z) = \rho(z)\,\omega^2$ and $\lambda' = k^2$.)

Let

$$a = z_0 < z_1 < z_2 < \ldots < z_{N+1} = b,$$

and $p(z) \equiv p_j(z)$, $q(z) \equiv q_j(z)$, $r(z) \equiv r_j(z)$ and $y(z) \equiv y_j(z)$, when $z \in I_j = \{z: z_j < z < z_{j+1}\}$, $j = 0, 1, \ldots, N$. We assume that in each subinterval I_j, the functions $p_j(z)$, $q_j(z)$ and $r_j(z)$ are continuous and single-valued and that their limits when z tends to z_j or z_{j+1} exist and are finite. Moreover, $r(z)$ does not vanish at any point of a subinterval and preserves the same sign in all the subintervals.

We suppose that at each interface or point of discontinuity z_k, $k = 1, 2, \ldots, N$, the y_j satisfy the conditions

$$y_k(z_k) = d_k y_{k-1}(z_k) + e_k y'_{k-1}(z_k) \tag{3.2a}$$

$$y'_k(z_k) = b_k y_{k-1}(z_k) + c_k y'_{k-1}(z_k), \qquad d_k c_k - b_k e_k \neq 0, \tag{3.2b}$$

and at end-points $y(z)$ satisfies the Sturmian conditions:

$$B_a(y) \equiv \alpha_1 y(a) + \alpha_2 y'(a) = 0, \tag{3.3a}$$

$$B_b(y) \equiv \beta_1 y(b) + \beta_2 y'(b) = 0, \tag{3.3b}$$

where

$$|\alpha_1| + |\alpha_2| \neq 0 \quad \text{and} \quad |\beta_1| + |\beta_2| \neq 0.$$

The system (3.1)–(3.3) is the *regular*, self-adjoint S–L system with N points of discontinuity. The usual results of ordinary S–L theory for regular systems hold for discontinuous, regular, S–L systems with interface conditions after necessary modification. We quote the following results, the proofs of most of which can be found in Sangren (1953):

(1) The eigenvalues λ_n of the system are real, if the following conditions are satisfied: (a) $r(z)$ preserves the same sign and does not vanish throughout its domain of definition,

(b) $p_i(z)$ is positive or identically zero in the subinterval I_i, $i = 0, 1, \ldots, N$ and for at least one subinterval $p_i(z) > 0$, (c) $c_j d_j - b_j e_j > 0, j = 1, 2, \ldots, N$.

(2) The eigenfunctions form a complete orthonormal set. The modified orthonormality condition is given by:

$$\langle \phi_m(z), \phi_n(z) \rangle \equiv \int_a^b A(z)\, p(z)\, \phi_m(z)\, \bar{\phi}_n(z)\, dz = \delta_{mn}, \tag{3.4}$$

where $\phi_m(z)$ and $\phi_n(z)$ are the normalized eigenfunctions belonging to the distinct eigenvalues λ_m and λ_n, respectively, $\bar{\phi}_n(z)$ denotes the complex conjugate of $\phi_n(z)$, and $A(z)$ is a piecewise constant function defined by

$$A(z) = A_i, \; z \in I_i, \quad i = 0, 1, 2, \ldots, N, \tag{3.5}$$

where

$$A_i = \frac{r_{i-1}(z_i)}{r_i(z_i)} (c_i d_i - b_i e_i)^{-1} \cdot A_{i-1}, \quad i = 1, 2, \ldots, N, \tag{3.6}$$

and A_0 is a non-zero constant which may be chosen arbitrarily. Condition (3.6) ensures the self-adjointness of the system.

(3) The spectrum of eigenvalues is discrete.

We now return to the system (2.4)–(2.8). It is a self-adjoint, regular S–L system with N points of discontinuity and N interface conditions with

$$p_i = -r_i = \mu_i(z), \; i = 0, 1, 2, \ldots, N, \; q_i = \rho_i \omega^2, \; \lambda = k^2$$

$$d_k = 1, e_k = 0, b_k = 0 \quad \text{and} \quad c_k = \mu_{k-1}(z_k)/\mu_k(z_k), \; k = 1, 2, \ldots, N.$$

Hence

$$A_i = \frac{r_{i-1}(z_i)}{r_i(z_i)} (c_i d_i - b_i e_i)^{-1} \cdot A_{i-1} = A_{i-1}, \quad i = 1, 2, \ldots, N.$$

We choose $A_0 = 1$, thus giving $A_i = 1, i = 1, 2, \ldots, N$. The conditions (a), (b), (c), given above are all satisfied. Hence the results (1), (2) and (3) are valid and the orthonormality condition of eigenfunctions becomes:

$$\langle \phi_m(z), \phi_n(z) \rangle = \int_0^H \mu(z)\, \phi_m(z)\, \overline{\phi_n(z)}\, dz = \delta_{mn}. \tag{3.7}$$

Hudson (1962) has also shown that the Love waves exist for any variation in density, such that it is integrable as a function of depth, and any variation in rigidity such that it is a piecewise continuous function of depth.

4 Green's function for regular S–L systems. The formula for spectral representation

Green's function $G(z, \zeta; \lambda)$ for the S–L system (3.1)–(3.3) satisfies the following conditions:

($G1$) $G_j(z, \zeta; \lambda)$, the restriction of the function $G(z, \zeta; \lambda)$ to the interval $z_j \leqslant z \leqslant z_{j+1}$, $j = 0, 1, \ldots, N$, is continuous at each point of the interval.

($G2$) Each $G_j(z, \zeta; \lambda)$ possesses a continuous first-order derivative for $j \neq k$, where k is such that $\zeta \in I_k$; G_k possesses a continuous first-order derivative at each point of the interval I_k except at the point $z = \zeta$, where it has a jump discontinuity, given by:

$$G_k'(\zeta^+, \zeta; \lambda) - G_k'(\zeta^-, \zeta; \lambda) = -\frac{1}{r(\zeta)} \tag{4.1}$$

($G3$) $L(G_j) = 0, z \in I_j, j \neq k; L(G_k) = 0, z \neq \zeta$.

($G4$) $G(z, \zeta; \lambda)$ satisfies the interface conditions (3.2) and the end-point conditions (3.3).

The *inhomogeneous* problem, where y satisfies

$$L(y) - \lambda p(z) y = f(z) \tag{4.2}$$

together with equations (3.2) and (3.3) can be solved with the help of Green's function of the associated homogeneous problem. $y(z)$ is given by

$$y(z) = \int_a^b G(z, \zeta; \lambda) f(\zeta)\, d\zeta. \tag{4.3}$$

We can also set up the series expansion of $y(z)$ in terms of the complete set of normalized eigenfunctions $\{\phi_n(z)\}$ of the homogeneous system. Let

$$y(z) = \sum_n \alpha_n \phi_n(z)\cdot \tag{4.4}$$

Forming the generalized inner product (relative to the weight function $A(z)p(z)$) of equation (4.4) with $\phi_k(z)$, we obtain:

$$\langle y(z), \phi_k(z) \rangle = \sum_n \alpha_n \langle \phi_n, \phi_k \rangle = \alpha_k,$$

whence

$$y(z) = \sum_n \langle y, \phi_n \rangle \phi_n(z). \tag{4.5}$$

In particular, if $y = \delta(z - \zeta)$, then

$$\begin{aligned} \langle \delta(z-\zeta), \phi_n \rangle &= \int_a^b A(z) p(z) \bar{\phi}_n(z) \delta(z - \zeta)\, dz \\ &= A(\zeta) p(\zeta) \bar{\phi}_n(\zeta)\cdot \end{aligned} \tag{4.6}$$

Equations (4.5) and (4.6) yield the following representation of the delta function:

$$\frac{\delta(z-\zeta)}{A(\zeta)p(\zeta)} = \sum_n \phi_n(z) \overline{\phi_n(\zeta)}. \tag{4.7}$$

Next, we find the bilinear series for Green's function. On multiplying the differential equation in (4.2) by $A(z)\overline{\phi_n}(z)$ and integrating with respect to z between the limits a to b, we get:

$$\int_a^b A(z) \overline{\phi_n(z)} \cdot L(y)\, dz - \lambda \langle y, \phi_n(z) \rangle = \int_a^b A(z) \overline{\phi_n(z)} f(z)\, dz. \tag{4.8}$$

But

$$\int_a^b L(y) A(z) \overline{\phi_n(z)}\, dz = \int_a^b y(z) \cdot A(z)\, L(\overline{\phi_n(z)})\, dz$$

$$= \lambda_n \int_a^b y(z) A(z) p(z) \cdot \overline{\phi_n(z)}\, dz$$

$$= \lambda_n \langle y(z), \phi_n(z) \rangle, \tag{4.9}$$

whence equations (4.8) and (4.9) yield

$$(\lambda_n - \lambda) \langle y(z), \phi_n(z) \rangle = \alpha_n (\lambda_n - \lambda) = \int_a^b A(z) f(z)\, \overline{\phi_n(z)}\, dz. \tag{4.10}$$

If $\lambda \neq \lambda_n$, then (4.10) implies

$$\alpha_n = \frac{1}{(\lambda_n - \lambda)} \int_a^b A(z) f(z)\, \overline{\phi_n(z)}\, dz. \tag{4.11}$$

From (4.4) and (4.11), we obtain

$$y(z) = \sum_n \frac{\phi_n(z)}{\lambda_n - \lambda} \int_a^b A(z) f(z)\, \overline{\phi_n(z)}\, dz \tag{4.12}$$

which is the unique solution of (4.2).

In the case $f(z) = \delta(z - \zeta)$, $y(z) \equiv G(z, \zeta; \lambda)$ and so

$$G(z, \zeta; \lambda) = \sum_n \frac{A(\zeta)\, \phi_n(z)\, \overline{\phi_n(z)}}{\lambda_n - \lambda} \tag{4.13}$$

which is the bilinear series for $G(z, \zeta; \lambda)$. It can be shown (Stakgold 1967) that Green's function $G(z, \zeta; \lambda)$ is a meromorphic function of λ with simple poles located at the points $\lambda_1, \lambda_2, \ldots, \lambda_n \ldots$ on the real axis. On integrating (4.13) around a large circle $|\lambda| = R$ in the complex λ-plane and using (4.7), we find

$$\lim_{R \to \infty} \cdot \frac{1}{2\pi i} \oint_{|\lambda| = R} G(z, \zeta; \lambda)\, d\lambda = - \sum A(\zeta)\, \phi_n(z)\, \overline{\phi_n(\zeta)} = - \frac{\delta(z - \zeta)}{p(\zeta)}. \tag{4.14}$$

(The limiting process is such that R does not take the values $|\lambda_n|$.)

For the system (2.4)–(2.8), the formula (4.14) becomes:

$$\lim_{R \to \infty} \frac{1}{2\pi i} \oint_{|\lambda| = R} G(z, \zeta; \lambda)\, d\lambda = - \sum \phi_n(z)\, \overline{\phi_n(\zeta)} = - \frac{\delta(z - \zeta)}{\mu(\zeta)} \tag{4.15}$$

The equation (4.15) is useful, because it enables us to find the spectrum and the corresponding eigenfunctions from the knowledge of Green's function. In order to determine the eigenfunctions of the problem, we integrate the Green function $G(z, \zeta; \lambda)$ over a large circle of radius R in the complex λ-plane, and find the residues at the poles. The set of poies, all lying on the real axis in the complex λ-plane, constitutes the spectrum of eigenvalues. The corresponding eigenfunctions normalized according to the condition:

$$\langle \phi_m, \phi_n \rangle = \delta_{mn} \tag{4.16}$$

may then be identified from the sum of the residues. The representation of the delta function as a sum over eigenfunctions can be utilized to obtain similar representation for any function $u(z)$. We shall illustrate the method by means of a two-layer model, with piecewise constant distribution of the rigidity and the density functions, in Section 7.

5 The condition at infinity for the layered half-space problem

If in the system (3.1)–(3.3), (3.6), a or b is infinite, or if on the finite interval $r(x)$ vanishes or $q(x)$ or $p(x)$ become infinite at the end-points, then the S–L system is called a self-adjoint *singular* S–L system with discontinuous coefficients and interface conditions.

Since the interval is semi-infinite in the layered half-space problem, the system (2.4)–(2.7), (2.9) is a singular S–L problem with N points of discontinuity and corresponding interface conditions. The method for finding the spectral representation of the Love wave operator associated with the layered half-space problem is based on the integration of Green's function in complex wave-number domain, as in the case of the finite depth problem.

First of all, let us consider the Green function $G(z, \zeta; \lambda)$ which is the solution of the system:

$$\left.\begin{aligned} &L(G) - \lambda p(z) G = \delta(z - \zeta), \\ &B_a(G) = 0, \\ &\text{interface conditions (3.2)}, \end{aligned}\right\} \tag{5.1}$$

together with a condition at infinity. It is the condition at infinity which presents certain difficulties for the construction of a unique solution to the singular problem. The difficulty may be investigated by the following argument.

Let $\phi(z, \lambda)$ and $\psi(z, \lambda)$ be two linearly independent solutions of the associated, homogeneous differential equation:

$$L(u) - \lambda p(z) u = 0, \tag{5.2}$$

satisfying the interface conditions (3.2) and the following initial conditions at the point a;

$$\phi(a, \lambda) = -\alpha_2, \qquad \phi'(a, \lambda) = \frac{\alpha_1}{r(a)}, \tag{5.3a}$$

$$\psi(a, \lambda) = \alpha_1, \qquad \psi'(a, \lambda) = \frac{\alpha_2}{r(a)}, \tag{5.3b}$$

where α_1 and α_2 are real and $|\alpha_1| + |\alpha_2| \neq 0$.

Each solution of 5.2 (except the multiples of ψ) can be expressed as a multiple of

$$u = \phi + m\psi, \tag{5.4}$$

where m is a complex number.

Further, we impose the following boundary condition at a point $b_0 < \infty$:

$$\beta_1 u(b_0, \lambda) + \beta_2 r(b_0)\, u'(b_0, \lambda) = 0, \tag{5.5}$$

where β_1 and β_2 are arbitrary real numbers and $|\beta_1| + |\beta_2| \neq 0$, and investigate the limit as $b_0 \to \infty$. Following the argument given by Stakgold 1967, Section 4.4), it can be shown that $u = \phi + m\psi$ satisfies (5.5), if and only if m satisfies the equation:

$$r(b_0)\, [W(\phi, \bar{\phi}; b_0) + m\bar{m} W(\psi, \bar{\psi}; b_0) + m W(\psi, \bar{\phi}; b_0) + \bar{m} \cdot W(\phi, \bar{\psi}; b_0)] = 0, \tag{5.6}$$

where bar denotes the complex conjugate and $W(\phi, \bar{\psi}; b_0)$ is the Wronskian evaluated at the point b_0. As the ratio $h = \beta_1/\beta_2$ takes up all real values (and provided $\mathscr{I}(\lambda) \neq 0$, m describes a circle in the complex plane with

$$\text{centre } S = -\frac{W(\phi, \bar{\psi}; b_0)}{W(\psi, \bar{\psi}; b_0)} \tag{5.7}$$

and

$$\text{radius } R = \frac{(\alpha_1^2 + \alpha_2^2)^{1/2}}{2\,|\mathscr{I}(\lambda)|\left[\int_a^{b_0} A(z)\,p(z)\,|\psi|^2 dz\right]}. \tag{5.8}$$

As b_0 increases with λ fixed, the radius R decreases and each circle is contained in those for smaller values of b_0. Hence in the limit as $b_0 \to \infty$, the circles approach a *limit-circle* $C_\infty(\lambda)$ or a *limit-point* $m_\infty(\lambda)$. It can be further shown that if $m = m_\infty(\lambda)$ or m lies on $C_\infty(\lambda)$, then the solution $u = m\psi + \phi$ is square-integrable over the interval (a, ∞), relative to the weight function $A(z)\,p(z)$. It follows from (5.8) that in the limit-point case there is just one solution $u = \phi(z, \lambda) + m_\infty(\lambda)\,\psi(z, \lambda)$, which satisfies the condition:

$$\|u\| = \int_a^\infty A(z)\,p(z)\,|u(z)|^2 dz < \infty, \tag{5.9}$$

and in the limit-circle case the independent solutions ψ and $\phi + m\psi$ and therefore every solution of (5.2) satisfies the condition (5.9). We recall that we assumed $\mathscr{I}(\lambda) \neq 0$. It can be shown (Stakgold 1967, Section 4.4), that if for some value of λ every solution of (5.2) satisfies the condition (5.9), then so does every solution for any other value of λ. We conclude therefore, that in the limit-circle case all solutions are square-integrable, relative to the weight function $A(z)\,p(z)$, over the interval (a, ∞). This is the extension to singular S–L systems with discontinuous coefficients and interface conditions of the following fundamental theorem due to Weyl. Defining the s-norm to be

$$\|u\|_s = \left[\int_a^b s\,|u|^2 dz\right]^{1/2},$$

we have:

WEYL'S THEOREM

Let a be a regular point and b a singular point and consider the equation

$$(-pu')' + qu - \lambda su = 0, \qquad a \leqslant z < b, \tag{5.10}$$

where p, q, s are continuous for $a \leqslant z < b$.

1. If for some particular value of λ, every solution is of finite s-norm over (a, b), then for any other value of λ, every solution is again of finite s-norm over (a, b).
2. For every λ with $\mathscr{I}(\lambda) \neq 0$, there exists at least one solution of finite s-norm over (a, b).

The ordinary Love wave operator, when the substratum is a homogeneous half-space, for example, is in the limit-point case at infinity, and so the boundary condition at infinity may be taken to the requirement that the solution must be of finite μ-norm. This confines the solution in exactly the same way as a homogeneous boundary condition at a finite point and therefore with equations (5.1) define the solution precisely.

6 Singular S–L systems with discontinuous coefficients. Spectral representation for the layered half-space problem

The Green function $G(z, \zeta; \lambda)$ is characterized as the only solution of the system (5.1) which satisfies:

$$\int_a^\infty A(z)\, p(z)\, |G|^2 dz < \infty, \tag{6.1}$$

when $\mathscr{I}(\lambda) \neq 0$ and the singularity at infinity is of the limit-point type. For real values of λ, we apply the principle of analytic continuation and define the Green function as the limit of G, regarded as a function of complex λ, as λ approaches the real axis. As λ may approach the real axis from either side, we may ensure uniqueness by supposing, for example, that $\mathscr{I}(\sqrt{\lambda}) > 0$ – see Friedman (1956, pp. 230–231).

We follow Stakgold (Section 4.4 loc. cit.) in assuming the validity of the formula

$$-\frac{\delta(z-\zeta)}{p(\zeta)} = \lim_{R \to \infty} \frac{1}{2\pi i} \oint_{|\lambda| = R} G(z, \zeta; \lambda)\, d\lambda \tag{4.14}$$

in the singular as well as the regular case.

Unlike the Green function in regular case, this Green function $G(z, \zeta; \lambda)$, regarded as a function of λ in the complex λ-plane has, in addition to poles, branch-point singularities described by Friedman (pp. 214 loc. cit.) and Stakgold (Section 4.4 loc. cit.) the integral reduces to a sum of residues at the poles plus the integral along the branch-cut. The spectrum (of eigenvalues) is the disjoint union of the discrete spectrum and the continuous spectrum. The points belonging to the discrete spectrum are real and arise from the poles of the Green function. The continuous spectrum is the set of points on the branch-cut along a portion of the real axis. We obtain the following representation of the delta function:

$$\delta(z-\zeta) = \sum A(\zeta)\, p(\zeta)\, \phi_n(z)\, \overline{\phi_n(\zeta)} + \int A(\zeta)\, p(\zeta)\, \psi(z, \lambda)\, \overline{\psi(\zeta, \lambda)}\, d\lambda, \tag{6.2}$$

where $\phi_n(z)$ are the eigenfunctions and $\psi(z, \lambda)$ are the improper eigenfunctions.

The normalization of eigenfunctions $\{\phi_n(z)\}$ may be achieved by the condition

$$\langle \phi_m(z), \phi_n(z) \rangle = \int_a^\infty A(z)\, p(z)\, \phi_m(z)\, \overline{\phi_n(z)}\, dz = \delta_{mn}. \tag{6.3}$$

In order to obtain the normalization conditions for the improper eigenfunctions $\{\psi(z, \lambda)\}$, we multiply (6.2) by $\psi(\zeta, \lambda')$ and integrate from a to ∞ to get

$$\psi(z, \lambda') = \sum \phi_n(z) \langle \psi(\zeta, \lambda'), \phi_n(\zeta) \rangle + \int \psi(z, \lambda) \langle \psi(\zeta, \lambda'), \psi(\zeta, \lambda) \rangle\, d\lambda. \tag{6.4}$$

For (6.4) to be valid, we must have

$$\langle \psi(z, \lambda'), \phi_n(z) \rangle = \int_a^\infty A(z)\, \psi(z, \lambda')\, \overline{\phi_n(z)}\, dz = 0 \tag{6.5}$$

and

$$\langle \psi(z, \lambda'), \psi(z, \lambda)\rangle = \int_a^{\infty} A(z)\, p(z)\, \psi(z, \lambda')\, \overline{\psi(z, \lambda)}\, dz = \delta(\lambda - \lambda'). \tag{6.6}$$

Hence the improper eigenfunctions must be normalized according to (6.5) and (6.6). For other methods of obtaining improper eigenfunctions along with their proper normalizations, reference may be made to Kato (1966) and Friedman (1956).

The spectral representation of the Love wave operator for a layered half-space can be found by the procedure given above. For the system (2.4)–(2.6), (2.9), the formulae (6.2), (6.3), (6.5), (6.6) become:

$$\delta(z - \zeta) = \sum \mu(\zeta)\, \phi_n(z)\, \overline{\phi_n(\zeta)} + \int \mu(\zeta)\, \psi(z, \lambda)\, \overline{\psi(\zeta, \lambda)}\, d\lambda, \tag{6.7}$$

$$\langle \phi_m, \phi_n\rangle = \int_0^{\infty} \mu(z)\, \phi_m(z)\, \overline{\phi_n(z)}\, dz = \delta_{mn}, \tag{6.8}$$

$$\langle \psi(z, \lambda'), \phi_m(z)\rangle = \int_0^{\infty} \mu(z)\, \psi(z, \lambda')\, \overline{\phi_m(z)}\, dz = 0, \tag{6.9}$$

and

$$\langle \psi(z, \lambda'), \psi(z, \lambda)\rangle = \int_0^{\infty} \mu(z)\, \psi(z, \lambda')\, \overline{\psi(z, \lambda)}\, dz = \delta(\lambda - \lambda'). \tag{6.10}$$

Any function $V(z)$, which is square-integrable, relative to the weight function $u(z)$, can be expressed as

$$V(z) = \sum_n \langle V, \phi_n\rangle\, \phi_n(z) + \int \langle V, \psi\rangle\, \psi(z, \lambda)\, d\lambda. \tag{6.11}$$

We shall find the spectral representation of the Love wave operator for a homogeneous surface layer, overlying a homogeneous half-space, in Section 8.

7 Example I: Spectral representation of the Love wave operator for an infinite layered strip

As an illustration of the method, outlined in Section 4, for finding the spectral representation of the Love wave operator associated with layered strips, we consider an infinite strip consisting of a layer of depth $H - h$, rigidity μ_2, shear velocity β_2 and density ρ_2, overlaid by another infinite strip, consisting of a layer of depth $h(<H)$, density ρ_1, rigidity $\mu_1\,(<\mu_2)$ and shear velocity $\beta_1\,(<\beta_2)$ (see Fig. 2). We suppose the density and the rigidity of each layer to be constant and the top and the bottom plane-surfaces to be stress-free.

Let

$$v(x, z, t) = V(z) \exp\,[i(\omega t - kx)]$$

where

$$\begin{aligned} V(z) &\equiv V_1(z), \quad 0 < z < h \\ &\equiv V_2(z), \quad h < z < H. \end{aligned}$$

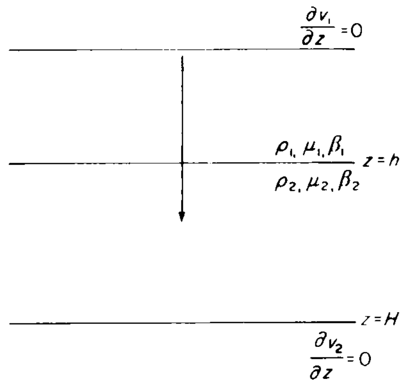

Figure 2.

Then $V_1(z)$ and $V_2(z)$ satisfy the following equations:

$$\frac{d^2V_1}{dz^2} + \sigma_1^2 V_1 = 0, \quad \sigma_1^2 = \left(\frac{\omega^2}{\beta_1^2} - \lambda\right), \quad \lambda = k^2, \quad \beta_1^2 = \frac{\mu_1}{\rho_1}, \quad 0 < z < h, \tag{7.1}$$

$$\frac{d^2V_2}{dz^2} - \sigma_2^2 V_2 = 0, \quad \sigma_2^2 = \left(\lambda - \frac{\omega^2}{\beta_2^2}\right), \quad \beta_2^2 = \frac{\mu_2}{\rho_2}, \quad h < z < H, \tag{7.2}$$

$$V_1(h) = V_2(h), \tag{7.3a}$$

$$\mu_1 V_1'(h) = \mu_2 V_2'(h), \tag{7.3b}$$

$$V_1'(0) = 0, \tag{7.4a}$$

$$V_2'(H) = 0. \tag{7.4b}$$

First we find the Green function $G(z, \zeta; \lambda)$ belonging to the system (7.1)–(7.4).

(a) GREEN'S FUNCTION

Let

$$G(z, \zeta; \lambda) = G_{ij} \tag{7.5}$$

where $i, j = 1, 2$; the subscript i refers to the z-interval and the subscript j refers to the ζ-interval, and '1', '2' refer to the intervals $(0, h)$ and (h, H), respectively (see Fig. 3). Then G_{ij} determine the Green function completely.

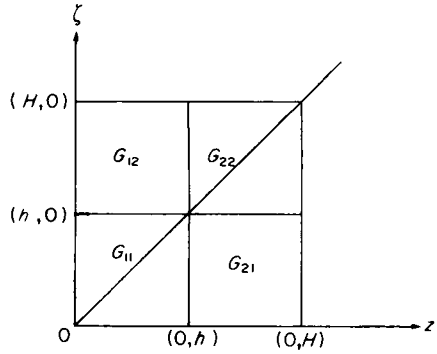

Figure 3. The Character of Green's Function.

(i) If $0 \leqslant \zeta \leqslant h$ then G_{11} and G_{21} satisfy the differential equations:

$$\frac{\partial^2 G_{11}}{\partial z^2} + \sigma_1^2 G_{11} = \delta(z-\zeta) \tag{7.6}$$

and

$$\frac{\partial^2 G_{21}}{\partial z^2} - \sigma_2^2 G_{21} = 0, \tag{7.7}$$

together with the following conditions:

$$\frac{\partial G_{11}}{\partial z} = 0 \quad \text{at} \quad z = 0, \tag{7.8a}$$

$$G_{11} = G_{21} \quad \text{at} \quad z = h, \tag{7.8b}$$

$$\mu_1 \frac{\partial G_{11}}{\partial z} = \mu_2 \frac{\partial G_{21}}{\partial z} \quad \text{at} \quad z = h, \tag{7.8c}$$

$$\frac{\partial G_{21}}{\partial z} = 0 \quad \text{at} \quad z = H, \tag{7.8d}$$

$$G_{11}(z, \zeta+0; \lambda) = G_{11}(z, \zeta-0; \lambda) \tag{7.8e}$$

and

$$\lim_{z \to \zeta+0} \frac{\partial G_{11}}{\partial z} - \lim_{z \to \zeta-0} \frac{\partial G_{11}}{\partial z} = \frac{1}{\mu_1}. \tag{7.8f}$$

We find

$$G_{11}(z, \zeta; \lambda) = \frac{\gamma_1}{\mu_1 \sigma_1} \cos \sigma_1 \zeta \cos \sigma_1 z + \frac{1}{\mu_1 \sigma_1} [\sin \sigma_1 \zeta \cdot \cos \sigma_1 z \cdot \theta(\zeta - z) + \cos \sigma_1 \zeta \sin \sigma_1 z \cdot \theta(z - \zeta)], \tag{7.9}$$

and

$$G_{21}(z, \zeta; \lambda) = \frac{\cos \sigma_1 \zeta}{\mu_1 \sigma_1} \{\gamma_1 \cos \sigma_1 h + \sin \sigma_1 h\} \cdot \frac{\cosh \sigma_2 (z - H)}{\cosh \sigma_2 (H - h)}, \tag{7.10}$$

where

$$\gamma_1 = \frac{\mu_1 \sigma_1 + \mu_2 \sigma_2 \tan \sigma_1 h \tanh \sigma_2 (H - h)}{\mu_1 \sigma_1 \tan \sigma_1 h - \mu_2 \sigma_2 \tanh \sigma_2 (H - h)}, \tag{7.11}$$

and

$$\left.\begin{aligned} \theta(\zeta - z) &= 1, \quad \zeta > z \\ &= 0, \quad \zeta < z \end{aligned}\right\}$$

is the Heaviside unit function.

(ii) If $h \leqslant \zeta \leqslant H$, then G_{22} and G_{12} satisfy the differential equations:

$$\frac{\partial^2 G_{12}}{\partial z^2} + \sigma_1^2 G_{12} = 0 \tag{7.12}$$

and

$$\frac{\partial^2 G_{22}}{\partial z^2} - \sigma_2^2 G_{22} = 0 \tag{7.13}$$

together with conditions (7.8a)–(7.8d), given above, and the conditions:

$$G_{22}(z, \zeta + 0; \lambda) = G_{22}(z, \zeta - 0; \lambda), \tag{7.8e'}$$

and

$$\lim_{z \to \zeta + 0} \frac{\partial G_{22}}{\partial z} - \lim_{z \to \zeta - 0} \frac{\partial G_{22}}{\partial z} = \frac{1}{\mu_2}. \tag{7.8f'}$$

We find

$$G_{12}(z, \zeta; \lambda) = -\frac{1}{\mu_2 \sigma_2} \cosh \sigma_2 (H - \zeta) \cdot \{\gamma_2 \cosh \sigma_2 (H - h) - \sinh \sigma_2 (H - h)\} \frac{\cos \sigma_1 z}{\cos \sigma_1 h}, \tag{7.14}$$

and

$$G_{22}(z, \zeta; \lambda) = -\frac{\gamma_2}{\mu_2 \sigma_2} \cosh \sigma_2 (z - H) \cdot \cosh \sigma_2 (\zeta - H) - \frac{1}{\mu_2 \sigma_2}$$
$$\times [\{\sinh (z - H) \sigma_2 \cdot \cosh (H - \zeta) \sigma_2 \cdot \theta (\zeta - z) + \sinh (\zeta - H) \sigma_2$$
$$\times \cosh (z - H) \sigma_2 \cdot \theta (z - \zeta)\}], \tag{7.15}$$

where

$$\gamma_2 = \frac{\mu_1 \sigma_1 \tan \sigma_1 h \tanh (H - h) \sigma_2 - \mu_2 \sigma_2}{\mu_1 \sigma_1 \tan \sigma_1 h - \mu_2 \sigma_2 \tanh \sigma_2 (H - h)}. \tag{7.16}$$

It may be noted that

$$G_{12}(\zeta, z; \lambda) = G_{21}(z, \zeta; \lambda) = \frac{1}{\mu_1 \sigma_1 \tan \sigma_1 h - \mu_2 \sigma_2 \tanh \sigma_2 (H - h)} \frac{\cos \sigma_1 \zeta}{\cos \sigma_1 h} \frac{\cosh \sigma_2 (z - H)}{\cosh \sigma_2 (H - h)}, \tag{7.17}$$

and that the Green function is symmetric, as shown in Fig. 3.

(b) SPECTRAL REPRESENTATION

We use formula (4.15):

$$\lim_{R \to \infty} \frac{1}{2\pi i} \oint_{|\lambda| = R} G(z, \zeta; \lambda) \, d\lambda = -\sum \phi^{(n)}(z) \overline{\phi^{(n)}(\zeta)} = -\frac{\delta (z - \zeta)}{\mu(\zeta)} \tag{4.15}$$

to find the spectrum and the corresponding orthonormal and complete set of eigenfunctions $\{\phi_n(z)\}$.

(i) Consider

$$I_{11} = \lim_{R \to \infty} \frac{-1}{2\pi i} \oint_{|\lambda| = R} G_{11}(z, \zeta; \lambda) d\lambda$$
$$= \lim_{R \to \infty} \frac{-1}{2\pi i} \oint_{|\lambda| = R} \left[\frac{\gamma_1}{\mu_1 \sigma_1} \cos \sigma_1 \zeta \cos \sigma_1 z + \frac{1}{\mu_1 \sigma_1} \{\sin \sigma_1 \zeta \cdot \cos \sigma_1 z \cdot \theta (\zeta - z)\right.$$
$$\left. + \cos \sigma_1 \zeta \cdot \sin \sigma_1 z \cdot \theta (z - \zeta)\}\right] d\lambda \tag{7.18}$$

where γ_1 is given by (7.11).

We note that the integrand is an even function of σ_1 and σ_2 and therefore the points $\lambda = \omega^2/\beta_1^2$ and $\lambda = \omega^2/\beta_2^2$ are *not* branch-points of the integrand. The only singularities of G_{11} are poles, which are the roots of the equation

$$\mu_1\sigma_1 \tan\sigma_1 h - \mu_2\sigma_2 \tanh\sigma_2(H-h) = 0 \tag{7.19}$$

the LHS being the denominator of γ_1. All the poles are simple and are located on the real axis.

On evaluating (7.18) as the sum of the residues at the poles:

$$I_{11} = -\sum_{n=1}^{\infty} \frac{\mu_1\sigma_1^{(n)} + \mu_2\sigma_2^{(n)} \tan\sigma_1^{(n)}h \cdot \tanh\sigma_2^{(n)}(H-h)}{\mu_1\sigma_1^{(n)}(\partial/\partial\lambda)\,[\mu_1\sigma_1\tan\sigma_1 h - \mu_2\sigma_2\tanh\sigma_2(H-h)]_{\lambda=\lambda_n}} \cdot \cos\sigma_1^{(n)}\zeta \cdot \cos\sigma_1^{(n)}z \tag{7.20}$$

where

$$\sigma_1^{(n)} = \left(\frac{\omega^2}{\beta_1^2} - \lambda_n\right)^{1/2}, \quad \sigma_2^{(n)} = \left(\lambda_n - \frac{\omega^2}{\beta_2^2}\right)^{1/2},$$

$\{\lambda_n\}$ $(n = 1, 2 \ldots)$ being the infinite set of roots of equation (7.19).

Since

$$\left.\frac{\partial\sigma_1}{\partial\lambda}\right|_{\lambda_n} = -\frac{1}{2\sigma_1^{(n)}} \quad \text{and} \quad \left.\frac{\partial\sigma_2}{\partial\lambda}\right|_{\lambda_n} = \frac{1}{2\sigma_2^{(n)}},$$

we have

$$\frac{\partial}{\partial\lambda}[\mu_1\sigma_1\tan\sigma_1 h - \mu_2\sigma_2\tanh\sigma_2(H-h)]_{\lambda_n}$$

$$= -\frac{1}{2}\left[\frac{\mu_1}{\sigma_1^{(n)}}\tan\sigma_1^{(n)}h + \mu_1 h\sec^2\sigma_1^{(n)}h + \frac{\mu_2}{\sigma_2^{(n)}}\tanh\sigma_2^{(n)}(H-h) + \mu_2(H-h)\operatorname{sech}^2\sigma_2^{(n)}(H-h)\right]. \tag{7.21}$$

It is convenient to express (7.21) in terms of the surface phase and group velocities. If we regard equation (7.19), which is also the dispersion equation between λ and ω, we have:

$$\text{phase velocity } C_n = \frac{\omega}{\lambda_n^{1/2}} \tag{7.22}$$

and

$$\text{group velocity } U_n = 2\lambda_n^{1/2}\frac{d\omega}{d\lambda_n}. \tag{7.23}$$

We obtain

$$\frac{\partial}{\partial\lambda}[\mu_1\sigma_1\tan\sigma_1 h - \mu_2\sigma_2\tanh\sigma_2(H-h)]_{\lambda_n} = -\frac{1}{2}\left[\frac{\mu_2}{\sigma_2^{(n)}}\tanh\sigma_2^{(n)}(H-h) + \mu_2(H-h) \times \operatorname{sech}^2\sigma_2^{(n)}(H-h)\right](\beta_1^{-2}-\beta_2^{-2})(\beta_1^{-2}-U_n^{-1}C_n^{-1})^{-1}. \tag{7.24}$$

Substituting (7.24) in (7.20) and simplifying, we get

$$I_{11} = \sum_{n=1}^{\infty}\phi_1^{(n)}(\zeta)\,\phi_1^{(n)}(z), \tag{7.25}$$

where

$$\phi_1^{(n)}(z) = D_n \frac{\cos \sigma_1^{(n)} z}{\cos \sigma_1^{(n)} h}, \tag{7.26}$$

and

$$D_n = 2\left(\frac{\sigma_2^{(n)}}{\mu_2}\right)^{1/2} \left(\frac{\beta_1^{-2} - U_n^{-1} C_n^{-1}}{\beta_1^{-2} - \beta_2^{-2}}\right)^{1/2} \cdot \frac{\cosh \sigma_2^{(n)} \cdot (H-h)}{\{\sinh 2\sigma_2^{(n)}(H-h) + 2\sigma_2^{(n)} \cdot (H-h)\}^{1/2}}. \tag{7.27}$$

(ii) Next, we consider

$$I_{22} = \lim_{R \to \infty} \frac{-1}{2\pi i} \oint_{|\lambda| = R} G_{22}(z, \zeta; \lambda)\, d\lambda$$

$$= \lim_{R \to \infty} \frac{1}{2\pi i} \oint_{|\lambda| = R} \frac{1}{\mu_2 \sigma_2} [\gamma_2 \cosh \sigma_2(z-H) \cdot \cosh \sigma_2(\zeta - H) + \sinh(z-H)\sigma_2$$

$$\times \cosh \sigma_2(\zeta - H) \cdot \theta(\zeta - z) + \sinh(\zeta - H)\sigma_2$$

$$\times \cosh(z-H)\sigma_2 \cdot \theta(z-\zeta)]\, d\lambda, \tag{7.28}$$

where γ_2 is given by (7.16).

Again the integrand is an even function of σ_1 and σ_2, and so the points $\lambda = \omega^2/\beta_1^2$ and $\lambda = \omega^2/\beta_2^2$ are not branch-points of the integrand. The poles of the integrand of I_{22} are the same as those of I_{11}. Calculating the sum of the residues at the poles, we get

$$I_{22} = \sum_{n=1}^{\infty} \frac{\mu_1^{(n)} \sigma_1^{(n)} \tan \sigma_1^{(n)} h \cdot \tanh \sigma_2^{(n)} (H-h) - \mu_2 \sigma_2^{(n)}}{\mu_2 \sigma_2^{(n)} \, (\partial/\partial\lambda)\, [\mu_1 \sigma_1 \tan \sigma_1 h - \mu_2 \sigma_2 \tanh \sigma_2 (H-h)]_{\lambda = \lambda_n}} \cosh \sigma_2^{(n)} (z-H)$$

$$\times \cosh \sigma_2^{(n)} (\zeta - H),$$

$$= \sum_{n=1}^{\infty} \frac{2\sigma_2^{(n)}}{\mu_2} \frac{\beta_1^{-2} - U_n^{-1} C_n^{-1}}{\beta_1^{-2} - \beta_2^{-2}} \cdot \frac{2 \cosh \sigma_2^{(n)} (H-z) \cdot \cosh \sigma_2^{(n)} (H - \zeta)}{[\sinh 2\sigma_2^{(n)} (H-h) + 2\sigma_2^{(n)} (H-h)]},$$

(using (7.24) and simplifying)

$$= \sum_{n=1}^{\infty} \phi_2^{(n)}(z) \cdot \phi_2^{(n)}(\zeta), \tag{7.29}$$

where

$$\phi_2^{(n)}(z) = D_n \frac{\cosh \sigma_2^{(n)} (z-H)}{\cosh \sigma_2^{(n)} (H-h)}, \tag{7.30}$$

and D_n is given by (7.27).

(iii) We now consider:

$$I_{21} = \lim_{R \to \infty} \frac{-1}{2\pi i} \oint_{|\lambda| = R} G_{21}(z, \zeta; \lambda)\, d\lambda$$

$$= \lim_{R \to \infty} \frac{-1}{2\pi i} \oint_{|\lambda| = R} \frac{\cos \sigma_1 \zeta \cdot \cosh \sigma_2 (z-H)\, d\lambda}{\cos \sigma_1 h \cosh \sigma_2 (H-h) \{\mu_1 \sigma_1 \tan \sigma_1 h - \mu_2 \sigma_2 \tanh \sigma_2 (H-h)\}}. \tag{7.31}$$

Again, the poles of the integrand are the same as those of the integrands in I_{11}, I_{22}, and $\lambda = \omega^2/\beta_1^2$, ω^2/β_2^2 are not branch-points of the integrand.

On evaluating (7.31) as the sum of the residues at the poles – using (8.24) – we get

$$I_{21} = \sum_{n=1}^{\infty} \phi_1^{(n)}(\zeta) \cdot \phi_2^{(n)}(z), \tag{7.32}$$

where $\phi_1^{(n)}$ and $\phi_2^{(n)}$ are given by (7.26) and (7.30).

Since $G_{21}(z, \zeta; \lambda) = G_{12}(\zeta, z; \lambda)$, we also have:

$$I_{12} = \sum_{n=1}^{\infty} \phi_1^{(n)}(z) \cdot \phi_2^{(n)}(\zeta) = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{12}(z, \zeta; \lambda)\, d\lambda. \tag{7.33}$$

From (4.15), (7.25), (7.29), (7.32)–(7.33), we obtain the following representation of the delta function:

$$\delta(z-\zeta) = \sum_{n=1}^{\infty} \mu(\zeta)\, \phi^{(n)}(z)\, \phi^{(n)}(\zeta), \tag{7.34}$$

where

$$\left.\begin{aligned} \phi^{(n)}(z) &= \phi_1^{(n)}(z), \quad 0 \leqslant z \leqslant h, \\ &= \phi_2^{(n)}(z). \quad h \leqslant z \leqslant H, \end{aligned}\right\} \tag{7.35}$$

$$\left.\begin{aligned} \mu(z) &= \mu_1, \quad 0 \leqslant z < h \\ &= \mu_2, \quad h < z \leqslant H \end{aligned}\right\} \tag{7.36}$$

and $\phi_1^{(n)}(z)$ and $\phi_2^{(n)}(z)$ are given by (7.26) and (7.30).

If $f(z)$ is any function of finite μ-norm, then on multiplying (7.34) by $f(\zeta)$ and integrating with respect to ζ over the interval $(0, H)$, we obtain the following representation for $f(z)$:

$$f(z) = \sum_{n=1}^{\infty} f_n \phi^{(n)}(z), \tag{7.37}$$

where

$$f_n = \langle f, \phi^{(n)} \rangle = \int_0^H \mu(\zeta)\, f(\zeta)\, \phi^{(n)}(\zeta)\, d\zeta. \tag{7.38}$$

In particular, when $f(z) = \phi^{(m)}(z)$ we get the following orthonormality relations:

$$\langle \phi^{(m)}, \phi^{(n)} \rangle = \int_0^H \mu(z)\, \phi^{(m)}(z)\, \phi^{(n)}(z)\, dz = \delta_{mn}, \tag{7.39}$$

which may be verified by direct integration.

8 Example II: Spectral representation of the Love wave operator for a half-space

We shall now find the spectral representation of the Love wave operator, associated with a homogeneous layer of thickness h, density ρ_1, rigidity μ_1, shear velocity β_1, overlying a homogeneous half-space of density ρ_2, rigidity $\mu_2 > \mu_1$ and shear velocity $\beta_2 > \beta_1$ (see Fig. 4). The method we shall follow is described in Section 6. The system of equations is given by (7.1)–(7.4a) with (7.4b) replaced by the following condition:

$$\int_0^\infty \mu(z) \mid V(z) \mid^2 dz < \infty. \tag{8.1}$$

(a) GREEN'S FUNCTION

Let

$$G(z, \zeta; \lambda) = G_{ij} \tag{8.2}$$

where G_{ij} have the same meaning as before – see explanation after (7.5) – with the interval (h, H) replaced by (h, ∞).

(i) If $0 \leqslant \zeta \leqslant h$ then G_{11} and G_{21} satisfy the differential equations (7.6) and (7.7) together with conditions (7.8a)–(7.8c), (7.8e)–(7.8f) and the condition

$$\int_0^\infty \mu(z) \mid G(z) \mid^2 dz < \infty. \tag{7.8d$'$}$$

First of all, we observe that, since $V_2 = \exp(-\sigma_2 z)$ and $V_2 = \exp(\sigma_2 z)$ are two linearly independent solutions of the differential equation in $z > h$, it follows that one of the solutions $v(z)$ is not of finite μ-norm. Hence the Love wave operator is in the limit-point case at infinity, as noted earlier in Section 6.

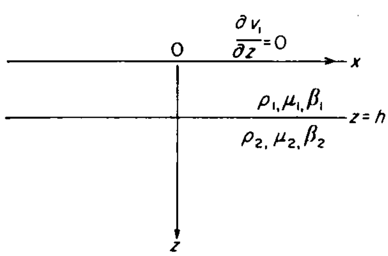

Figure 4.

We find

$$G_{11}(z, \zeta; \lambda) = \frac{\gamma_1'}{\mu_1 \sigma_1} \cos \sigma_1 \zeta \cdot \cos \sigma_1 z + \frac{1}{\mu_1 \sigma_1}$$
$$\times [\sin \sigma_1 \zeta \cos \sigma_1 z \cdot \theta(\zeta - z) + \cos \sigma_1 \zeta \sin \sigma_1 z \cdot \theta(z - \zeta)] \tag{8.3}$$

and

$$G_{21}(z, \zeta; \lambda) = \frac{1}{\mu_1 \sigma_1} \cos \sigma_1 \zeta \cdot \{\gamma_1' \cos \sigma_1 h + \sin \sigma_1 h\} \cdot \exp[-\sigma_2 (z - h)], \tag{8.4}$$

where

$$\gamma_1' = \frac{\mu_1 \sigma_1 + \mu_2 \sigma_2 \tan \sigma_1 h}{\mu_1 \sigma_1 \tan \sigma_1 h - \mu_2 \sigma_2}, \tag{8.5}$$

and choosing

$\operatorname{Re}\sigma_2 > 0$ for $\mathscr{I}(\lambda) \neq 0$.

(ii) If $h \leqslant \zeta$ then G_{12} and G_{22} satisfy the differential equations (7.12), (7.13) together with the conditions (7.8a)–(7.8c), (7.8d′)–(7.8f′).

We obtain

$$G_{12}(z,\zeta;\lambda) = \frac{1}{\cos\sigma_1 h}\,\frac{1}{2\mu_2\sigma_2}\cdot\exp\left[-\sigma_2(\zeta-h)\right]\cdot\left[\gamma_2'-1\right]\cos\sigma_1 z, \tag{8.6}$$

and

$$G_{22}(z,\zeta;\lambda) = \frac{1}{2\mu_2\sigma_2}\left[\gamma_2'\cdot\exp\left[-\sigma_2(z+\zeta-2h)\right] - \exp\left(-\sigma_2|\zeta-z|\right)\right], \tag{8.7}$$

where

$$\gamma_2' = \frac{\mu_1\sigma_1\tan\sigma_1 h + \mu_2\sigma_2}{\mu_1\sigma_1\tan\sigma_1 h - \mu_2\sigma_2}.$$

We note that

$$G_{12}(\zeta,z;\lambda) = G_{21}(z,\zeta;\lambda) = \frac{\cos\sigma_1\zeta\cdot\exp\left[-\sigma_2(z-h)\right]}{\cos\sigma_1 h\cdot\left[\mu_1\sigma_1\tan\sigma_1 h-\mu_2\sigma_2\right]} \tag{8.8}$$

and that the Green function G is symmetric.

(b) SPECTRAL REPRESENTATION

In order to find the discrete and the continuous spectrum along with the corresponding eigenfunctions $\{\phi_n(z)\}$ and improper eigenfunctions $\{\psi(z,\lambda)\}$ we use formulae (4.14) and (6.2):

$$\lim_{R\to\infty}\frac{-1}{2\pi i}\oint_{|\lambda|=R} G(z,\zeta;\lambda)\,d\lambda = \sum\phi^{(n)}(z)\,\overline{\phi^n(\zeta)} + \int\psi(z,\lambda)\,\overline{\psi(\zeta,\lambda)}\,d\lambda = \frac{\delta(z-\zeta)}{\mu(\zeta)}. \tag{8.9}$$

(i) First, we consider

$$\begin{aligned} I_{11}' &= \lim_{R\to\infty}\frac{-1}{2\pi i}\oint_{|\lambda|=R} G_{11}(z,\zeta;\lambda)\,d\lambda \\ &= \lim_{R\to\infty}\frac{-1}{2\pi i}\oint_{|\lambda|=R}\left[\frac{\gamma_1'}{\mu_1\sigma_1}\cos\sigma_1\zeta\cos\sigma_1 z + \frac{1}{\mu_1\sigma_1}\right. \\ &\qquad \left.\times\left\{\sin\sigma_1\zeta\cdot\cos\sigma_1 z\cdot\theta(\zeta-z) + \cos\sigma_1\zeta\cdot\sin\sigma_1 z\cdot\theta(z-\zeta)\right\}\right] d\lambda. \end{aligned} \tag{8.10}$$

We note that $\lambda = \omega^2/\beta_2^2$ is the only branch-point singularity of the integrand and the poles of the integrand are the roots of the equation

$$\mu_1\sigma_1\tan\sigma_1 h = \mu_2\sigma_2, \tag{8.11}$$

which is the period equation for the Love waves and appears in the denominator of γ_1'.

The poles are all simple, finite in number, and are located in the open interval $[\omega^2/\beta_2^2, \omega^2/\beta_1^2]$. The set of these poles constitutes the discrete spectrum. The continuous spectrum is related to the integral over the branch-lines ℓ^+, ℓ^{-1} and the path of integration is shown in Fig. 5.

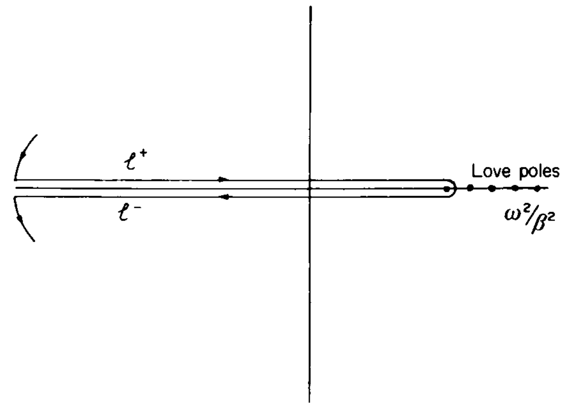

Figure 5. The Contour of Integration.

The condition $\mathrm{Re}(\sigma_2) > 0$ means that on the branch-line ℓ^+, $\sigma_2 = is_2$ and on ℓ^-, $\sigma_2 = -is_2$, where $s_2 = (\omega^2/\beta_2^2 - \lambda)^{1/2}$ is real and positive for $\lambda < \omega^2/\beta_2^2$.

The contribution to I'_{11} from ℓ^+ and ℓ^- is

$$\int (G_{11}^+ - G_{11}^-)\, d\lambda = \int \frac{1}{\mu_1 \sigma_1} (\gamma_1'^+ - \gamma_1'^-) \cos \sigma_1 \zeta \cos \sigma_1 z\, d\lambda, \tag{8.12}$$

where the superscripts + and − refer to the values at the branches ℓ^+ and ℓ^- respectively.

But

$$\gamma_1'^+ - \gamma_1'^- = \frac{2i\mu_1\mu_2\sigma_1 s_2}{\mu_1^2\sigma_1^2 \sin^2 \sigma_1 h + \mu_2^2 s_2^2 \cos^2 \sigma_1 h}, \tag{8.13}$$

and so the branch-line contribution to the integral (8.10) is given by

$$-\frac{\mu_2}{\pi} \int_{-\infty}^{\omega^2/\beta_2^2} \frac{s_2}{\{\mu_1^2\sigma_1^2 \sin^2 \sigma_1 h + \mu_2^2 s_2^2 \cos^2 \sigma_1 h\}} \cos \sigma_1 \zeta \cos \sigma_1 z\, d\lambda$$

$$= -\int_{-\infty}^{\omega^2/\beta_2^2} \psi_1(z, \lambda)\, \psi_1(\zeta, \lambda)\, d\lambda \tag{8.14}$$

where

$$\psi_1(z, \lambda) = G_\lambda \frac{\cos \sigma_1 z}{\cos \sigma_1 h}, \tag{8.15}$$

and

$$G_\lambda = \frac{\sin \theta}{\sqrt{\pi}\mu_2 s_2}, \tag{8.16}$$

$$\theta = \tan^{-1}\left(\frac{\mu_2 s_2 \cot \sigma_1 h}{\mu_1 \sigma_1}\right). \tag{8.17}$$

The sum of the residues at the finite number of the poles, which lie in the interval $[\omega^2/\beta_2^2, \omega^2/\beta_1^2]$ is given by:

$$\sum_{n=1}^{N} \frac{\mu_1 \sigma_1^{(n)} + \mu_2 \sigma_2^{(n)} \tan \sigma_1^{(n)} h}{\mu_1 \sigma_1^{(n)} (\partial/\partial\lambda)[\mu_1 \sigma_1 \tan \sigma_1 h - \mu_2 \sigma_2]_{\lambda=\lambda_n}} \cos \sigma_1^{(n)} \zeta \cdot \cos \sigma_1^{(n)} z, \tag{8.18}$$

where

$$\sigma_1^{(n)} = \left(\frac{\omega^2}{\beta_1^2} - \lambda_n\right)^{1/2},$$

and

$$\sigma_2^{(n)} = \left(\lambda_n - \frac{\omega^2}{\beta_2^2}\right)^{1/2}.$$

Now

$$\frac{\partial}{\partial\lambda}[\mu_1 \sigma_1 \tan \sigma_1 h - \mu_2 \sigma_2]_{\lambda_n} = -\frac{1}{2}\frac{\mu_2}{\sigma_2^{(n)}} (\beta_1^{-2} - \beta_2^{-2})(\beta_1^{-2} - U_n^{-1} C_n^{-1})^{-1}, \tag{8.19}$$

where C_n and U_n denote the surface phase and group velocities respectively.

From (8.18) and (8.19), we obtain the contributions from the poles as

$$\sum_{n=1}^{N} \phi_1^{(n)}(z)\, \phi_1^{(n)}(\zeta), \tag{8.20}$$

where

$$\phi_1^{(n)}(z) = F_n \frac{\cos \sigma_1^{(n)} z}{\cos \sigma_1^{(n)} h}, \tag{8.21}$$

and

$$F_n = \left\{\frac{2\sigma_2^{(n)}}{\mu_2}\left(\frac{\beta_1^{-2} - U_n^{-1} C_n^{-1}}{\beta_1^{-2} - \beta_2^{-2}}\right)\right\}^{1/2}. \tag{8.22}$$

From (8.10), (8.14) and (8.20) we obtain

$$I_{11}' = \sum_{n=1}^{N} \phi_1^{(n)}(z)\, \phi_1^{(n)}(\zeta) - \int_{-\infty}^{\omega^2/\beta_2^2} \psi_1(z,\lambda)\, \psi_1(\zeta,\lambda)\, d\lambda, \tag{8.23}$$

where ψ_1 is given by (8.15) and $\phi_1^{(n)}$ is given by (8.21).

(ii) Next, we consider

$$\begin{aligned} I_{22}' &= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{22}(z,\zeta;\lambda)\, d\lambda \\ &= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{1}{2\sigma_2 \mu_2} [\gamma_2' \exp\{-\sigma_2(z+\zeta-2h)\} - \exp(-\sigma_2|\zeta - z|)]\, d\lambda. \end{aligned} \tag{8.24}$$

Again the point $\lambda = \omega^2/\beta_2^2$ is the only branch-point of the integrand and the poles, being the zeros of the denominators of the factor γ_2' are the same as for I_{11}'.

On $\mathcal{L}^+$ and $\mathcal{L}^-$, we have

$$G_{22}^+ = G_{22}^{-*}, \tag{8.25}$$

where * denotes the complex conjugate, and so

$$G_{22}^+ - G_{22}^- = 2i\mathscr{I}(G_{22}^+).$$

Moreover,

$$\gamma_2'^+ = \frac{\mu_1\sigma_1 \tan\sigma_1 h + i\mu_2 s_2}{\mu_1\sigma_1 \tan\sigma_1 h - i\mu_2 s_2} = \exp(2i\theta), \tag{8.26}$$

where θ is given by (8.17).

Thus

$$G_{22}^+ - G_{22}^- = \frac{i}{\mu_2 s_2}\left[-\cos\{s_2(z+\zeta-2h)-2\theta\} + \cos\{s_2(\zeta - z)\}\right],$$

$$= \frac{2i}{\mu_2 s_2}\sin\{\theta - s_2(\zeta - h)\}\cdot\sin\{\theta - s_2(z-h)\}. \tag{8.27}$$

Hence the branch-line contribution is

$$-\frac{1}{\pi}\int_{-\infty}^{\omega^2/\beta_2^2}\frac{1}{\mu_2 s_2}\sin\{\theta - s_2(\zeta-h)\}\cdot\sin\{\theta - s_2(z-h)\}\,d\lambda$$

$$= -\int_{-\infty}^{\omega^2/\beta_2^2}\psi_2(z,\lambda)\,\psi_2(\zeta,\lambda)\,d\lambda, \tag{8.28}$$

where

$$\psi_2(z,\lambda) = \frac{G_\lambda\cdot\sin\{\theta - s_2(z-h)\}}{\sin\theta}, \tag{8.29}$$

and G_λ is given by (8.16).

Contribution from the poles is given by

$$-\sum_{n=1}^{N}\frac{\{\mu_1\sigma_1^{(n)}\tan\sigma_1^{(n)}h + \mu_2\sigma_2^{(n)}\}\exp[-\sigma_2^{(n)}(z+\zeta-2h)]}{2\mu_2\sigma_2^{(n)}\cdot(\partial/\partial\lambda)\,[\mu_1\sigma_1\tan\sigma_1 h - \mu_2\sigma_2]_{\lambda_n}} = \sum_{n=1}^{N}\phi_2^{(n)}(z)\,\phi_2^{(n)}(\zeta), \tag{8.30}$$

where

$$\phi_2^{(n)}(z) = F_n\exp[\sigma_2^{(n)}(h-z)], \tag{8.31}$$

and F_n is given by (8.22).

From (8.24), (8.28) and (8.30), we obtain

$$I_{22}' = \sum_{n=1}^{N}\phi_2^{(n)}(z)\,\phi_2^{(n)}(\zeta) - \int_{-\infty}^{\omega^2/\beta_2^2}\psi_2(z,\lambda)\,\psi_2(\zeta,\lambda)\,d\lambda, \tag{8.32}$$

where $\phi_2^{(n)}(z)$ is given by (8.31) and $\psi_2(z,\lambda)$ is given by (8.29).

(iii) Finally, we consider

$$I'_{21} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{21}(z,\zeta;\lambda)\,d\lambda = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{\cos\sigma_1\zeta}{\cos\sigma_1 h} \times \frac{\exp[-\sigma_2(z-h)]}{\mu_1\sigma_1\tan\sigma_1 h - \mu_2\sigma_2}\,d\lambda, \tag{8.33}$$

and note that $\lambda = \omega^2/\beta_2^2$ is the only branch-point of the integrand. The poles of the integrand are the same as those for I'_{11} and I'_{22}.

Now

$$G^+_{21} - G^-_{21} = 2i\frac{\cos\sigma_1\zeta}{\cos\sigma_1 h}\cdot\frac{\sin\theta}{\mu_2 s_2}\cdot\sin\{\theta - s_2(z-h)\},$$

and so the branch-line contribution

$$= -\frac{1}{\pi}\int_{-\infty}^{\omega^2/\beta_2^2} \psi_1(\zeta,\lambda)\,\psi_2(z,\lambda)\,d\lambda, \tag{8.34}$$

where ψ_1 and ψ_2 are given by (8.15) and (8.29), respectively.

Contribution from the poles

$$= -\sum_{n=1}^{N} \frac{\cos\sigma_1^{(n)}\zeta}{\cos\sigma_1^{(n)}h}\cdot\frac{\exp[-\sigma_2^{(n)}(z-h)]}{(\partial/\partial\lambda)[\mu_1\sigma_1\tan\sigma_1 h - \mu_2\sigma_2]_{\lambda_n}} \tag{8.35}$$

$$= \sum_{n=1}^{N} \phi_1^{(n)}(\zeta)\,\phi_2^{(n)}(z),$$

where ϕ_1 and ϕ_2 are given by (8.21) and (8.31), respectively. Hence,

$$I'_{21} = \sum_{n=1}^{N} \phi_1^{(n)}(\zeta)\,\phi_2^{(n)}(z) - \frac{1}{\pi}\int_{-\infty}^{\omega^2/\beta_2^2} \psi_1(\zeta,\lambda)\,\psi_2(z,\lambda)\,d\lambda. \tag{8.36}$$

Moreover, since $G_{21}(z,\zeta;\lambda) = G_{12}(\zeta,z;\lambda)$, we have

$$I'_{12} = \lim_{R\to\infty}\frac{-1}{2\pi i}\oint_{|\lambda|=R} G_{12}\,d\lambda = \sum_{n=1}^{N}\phi_1^{(n)}(z)\,\phi_2^{(n)}(\zeta) - \frac{1}{\pi}\int_{-\infty}^{\omega^2/\beta_2^2}\psi_1(z,\lambda)\,\psi_2(\zeta,\lambda)\,d\lambda. \tag{8.37}$$

From (8.9), (8.23), (8.32), (8.36) and (8.37), we obtain the following representation of the delta function:

$$\delta(z-\zeta) = \sum_{n=1}^{N}\mu(\zeta)\,\phi^{(n)}(z)\,\phi^{(n)}(\zeta) - \int_{-\infty}^{\omega^2/\beta_2^2}\mu(\zeta)\,\psi(z,\lambda)\,\psi(\zeta,\lambda)\,d\lambda, \tag{8.38}$$

where

$$\left.\begin{aligned}\phi^{(n)}(z) &= \phi_1^{(n)}(z), \quad 0 \leqslant z \leqslant h,\\ &= \phi_2^{(n)}(z), \quad h \leqslant z,\end{aligned}\right\} \tag{8.39}$$

(with $\mu(z)$, $\phi_1^{(n)}(z)$, $\phi_2^{(n)}(z)$ given by (7.36), (8.21) and (8.31), respectively), are the normalized proper eigenfunctions, and

$$\left.\begin{aligned}\psi(z,\lambda) &= \psi_1(z,\lambda),\ 0 \leqslant z \leqslant h,\\ &= \psi_2(z,\lambda),\ z \geqslant h,\end{aligned}\right\} \tag{8.40}$$

(with $\psi_1(z,\lambda)$ and $\psi_2(z,\lambda)$ given by (8.15) and (8.29), respectively), are normalized improper eigenfunctions. It can be verified after some effort, that the normalizing factors, which have arisen for eigenfunctions and improper eigenfunctions, are indeed correct.

If $f(z)$ is of finite μ-norm over the interval $(0,\infty)$, then the representation of $f(z)$ in terms of eigenfunctions $\{\phi_n(z)\}$ and improper eigenfunctions $\{\psi(z,\lambda)\}$ can be obtained on multiplying (8.38) by $f(\zeta)$ and integrating with respect to ζ from 0 to ∞.

We get

$$f(z) = \sum_{n=1}^{N} f_n\,\phi^{(n)}(z) - \int_{-\infty}^{\omega^2/\beta_2^2} f_\lambda\,\psi(\lambda,z)\,dz, \tag{8.41}$$

where

$$f_n = \langle f, \phi^{(n)}\rangle = \int_0^\infty \mu(\zeta)\,f(\zeta)\,\phi^{(n)}(\zeta)\,d\zeta, \tag{8.42}$$

and

$$f_\lambda = \langle f, \psi(\zeta,\lambda)\rangle = \int_0^\infty \mu(\zeta)\,f(\zeta)\,\psi(\zeta,\lambda)\,d\zeta. \tag{8.43}$$

In particular, if $f(z) = \phi^{(m)}(z)$ or $\psi(z,\lambda')$, then (8.41)–(8.43) yield the following orthonormality relations:

$$\int_0^\infty \mu(z)\,\phi^{(m)}(z)\,\phi^{(n)}(z)\,dz = \delta_{mn} = \langle\phi^{(m)},\phi^{(n)}\rangle,\quad 1 \leqslant m,n \leqslant N, \tag{8.44a}$$

$$\int_0^\infty \mu(z)\,\psi(z,\lambda)\,\psi(z,\lambda')\,dz = \delta(\lambda-\lambda') = \langle\psi(z,\lambda),\psi(z,\lambda')\rangle,\quad -\infty \leqslant \lambda,\lambda' < \omega^2/\beta_2^2, \tag{8.44b}$$

$$\int_0^\infty \mu(z)\,\phi^{(m)}(z)\,\psi(z,\lambda)\,dz = 0 = \langle\phi^{(m)},\psi\rangle,\ 1 \leqslant m \leqslant N,\quad -\infty \leqslant \lambda < \omega^2/\beta_2^2. \tag{8.44c}$$

9 Conclusions

The spectral representation enables us to tackle a class of problems associated with the transmission and reflection of Love waves at a horizontally discontinuous change in elevation (vertical step) or in material properties of layered structures. Alsop (1966) has given an approximate method for calculating reflection and transmission coefficients for Love waves incident on a vertical discontinuity, which is based upon the representation of motion in terms of the discrete Love modes only. The continuous part of the spectrum neglected by Alsop (1966) is related to body waves and is of considerable significance.

The method of integral representation and Schwinger–Levine variational principle used by Miles (1946, 1967) in the treatment of acoustical and water wave problems, and Marcuvitz (1951) in Electromagnetic problems, presupposes the existence of a complete set of proper and improper eigenfunctions, in terms of which the displacements on either side of the discontinuity may be expressed. The present paper fulfils the necessary prerequisite for the application of the method which will be described in a subsequent paper.

Acknowledgments

The author wishes to thank Dr J. A. Hudson and Dr P. W. Buchen for many valuable and helpful discussions and suggestions, and also to the British Council for the award of a T.A.T.D. fellowship.

References

Alsop, L. E., 1966. *J. geophys. Res.*, **71**, 3969.
Friedman, B., 1956. *Principles and techniques of applied mathematics*, Wiley, New York.
Hudson, J. A., 1962. *Geophys. J. R. astr. Soc.*, **6**, 131.
Kato, T., 1966. *Perturbation theory for linear operators*, Springer.
Marcuvitz, N., 1951. *Waveguide handbook*, McGraw-Hill Book Company, New York.
Miles, J. W., 1946. *J. acoust. Soc. Am.*, **17**, 259–84.
Miles, J. W., 1967. *J. fluid Mech.*, **28**, Part 4, p. 755.
Sangren, W. C., 1953. *ORNL 1566*, Oakridge National Laboratory, USA.
Stakgold, I., 1967. *Boundary value problems of mathematical physics*, Volume I, Macmillan.
Stallard, F. W., 1955. *ORNL 1876*, Oakridge National Laboratory, USA.