

# Spectral theory of a Neumann-Poincaré-type operator and analysis of anomalous localized resonance II\*

Habib Ammari<sup>†</sup>    Giulio Ciruolo<sup>‡</sup>    Hyeonbae Kang<sup>§</sup>    Hyundae Lee<sup>§</sup>  
Graeme W. Milton<sup>¶</sup>

December 21, 2012

## Abstract

If a body of dielectric material is coated by a plasmonic structure of negative dielectric constant with nonzero loss parameter, then cloaking by anomalous localized resonance (CALR) may occur as the loss parameter tends to zero. The aim of this paper is to investigate this phenomenon in two and three dimensions when the coated structure is radial, and the core, shell and matrix are isotropic materials. In two dimensions, we show that if the real part of the permittivity of the shell is  $-1$  (under the assumption that the permittivity of the background is 1), then CALR takes place. If it is different from  $-1$ , then CALR does not occur. In three dimensions, we show that CALR does not occur. The analysis of this paper reveals that occurrence of CALR is determined by the eigenvalue distribution of the Neumann-Poincaré-type operator associated with the structure.

## 1 Introduction

If a body of dielectric material is coated by a plasmonic structure of negative dielectric constant (with nonzero loss parameter), then anomalous localized resonance may occur as the loss parameter tends to zero. This phenomena, first discovered by Nicorovici, McPhedran and Milton [36] (see also [33]), is responsible for the subwavelength focussing properties of superlenses [38], and also occurs in magnetoelectric and thermoelectric systems [33]. The fields blow-up in a localized region, which moves as the position of the source is moved, which is why it is termed anomalous localized resonance. Remarkably, as found by Milton and Nicorovici [30] the localized resonant fields created by a source can act back on the source and cloak it. This invisibility cloaking has attracted much attention [30, 37, 8, 31, 35, 32, 27, 7, 11, 34, 2, 20, 40].

To state the problem and results in a precise way, let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$ ,  $d = 2, 3$ , and  $D$  be a domain whose closure is contained in  $\Omega$ . For a given loss parameter  $\delta > 0$ , the permittivity distribution in  $\mathbb{R}^d$  is given by

$$\epsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ \epsilon_s + i\delta & \text{in } \Omega \setminus \overline{D}, \\ \epsilon_c & \text{in } D, \end{cases} \quad (1.1)$$

---

\*This work was supported by the ERC Advanced Grant Project MULTIMOD-267184 and NRF grants No. 2010-0004091, and 2010-0017532, and by the NSF through grants DMS-0707978 and DMS-1211359.

<sup>†</sup>Department of Mathematics and Applications, Ecole Normale Supérieure, 45 Rue d'Ulm, 75005 Paris, France (habib.ammari@ens.fr).

<sup>‡</sup>Dipartimento di Matematica e Informatica, Università di Palermo Via Archirafi 34, 90123, Palermo, Italy (g.ciraolo@math.unipa.it).

<sup>§</sup>Department of Mathematics, Inha University, Incheon 402-751, Korea (hbkang@inha.ac.kr, hdlee@inha.ac.kr).

<sup>¶</sup>Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA (milton@math.utah.edu).

where  $-\epsilon_s$  and  $\epsilon_c$  are positive. We may consider the configuration as a core with permittivity  $\epsilon_c$  coated by the shell  $\Omega \setminus \overline{D}$  with permittivity  $\epsilon_s + i\delta$ . For a given function  $f$  compactly supported in  $\mathbb{R}^d \setminus \overline{\Omega}$  satisfying

$$\int_{\mathbb{R}^2} f dx = 0 \quad (1.2)$$

(which is required by conservation of charge), we consider the following dielectric problem:

$$\nabla \cdot \epsilon_\delta \nabla V_\delta = f \quad \text{in } \mathbb{R}^d, \quad (1.3)$$

with the decay condition  $V_\delta(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . The problem of cloaking by anomalous localized resonance (CALR) can be formulated as the problem of identifying the sources  $f$  such that first

$$E_\delta := \int_{\Omega \setminus D} \delta |\nabla V_\delta|^2 dx \rightarrow \infty \quad \text{as } \delta \rightarrow 0, \quad (1.4)$$

and second,  $V_\delta/\sqrt{E_\delta}$  goes to zero outside some radius  $a$ , as  $\delta \rightarrow 0$ :

$$|V_\delta(x)/\sqrt{E_\delta}| \rightarrow 0 \quad \text{as } \delta \rightarrow 0 \quad \text{when } |x| > a. \quad (1.5)$$

Physically the quantity  $E_\delta$  is proportional to the electromagnetic power dissipated into heat by the time harmonic electrical field averaged over time. Using integration by parts we have the identity

$$E_\delta = \Im \int_{\mathbb{R}^d} (\epsilon_\delta \nabla V_\delta) \cdot \nabla \overline{V_\delta} dx = -\Im \int_{\mathbb{R}^d} f \overline{V_\delta} dx \quad (1.6)$$

( $\Im$  denotes the imaginary part) which equates the power dissipated into heat with the electromagnetic power produced by the source, where  $\overline{V_\delta}$  is the complex conjugate of  $V_\delta$ . Hence (1.4) implies an infinite amount of energy dissipated per unit time in the limit  $\delta \rightarrow 0$  which is unphysical. If we rescale the source  $f$  by a factor of  $1/\sqrt{E_\delta}$  then the source will produce the same power independent of  $\delta$  and the new associated potential  $V_\delta/\sqrt{E_\delta}$  will, by (1.5), approach zero outside the radius  $a$ : cloaking due to anomalous localized resonance (CALR) occurs.

In the recent paper [2] the authors develop a spectral approach to analyze the CALR phenomenon. In particular, they show that if  $D$  and  $\Omega$  are concentric disks in  $\mathbb{R}^2$  and  $\epsilon_c = -\epsilon_s = 1$ , then there is a critical radius  $r_*$  such that for any source  $f$  supported outside  $r_*$  CALR does not occur, and for sources  $f$  satisfying a mild condition CALR takes place. The critical radius  $r_*$  is given by

$$r_* = \sqrt{r_e^3/r_i}, \quad (1.7)$$

where  $r_e$  and  $r_i$  are the radii of  $\Omega$  and  $D$ , respectively. It is worth mentioning that these results were extended in [20] to the case when the core  $D$  is not radial by a different method based on a variational approach.

The purpose of this paper is to extend some of the results in [2] in two directions. We consider the case when  $\epsilon_c$  and  $-\epsilon_s$  are not both 1 and we consider CALR in three dimensions. The results of this paper are threefold: Let  $\Omega$  and  $D$  be concentric disks or balls in  $\mathbb{R}^d$  of radii  $r_e$  and  $r_i$ , respectively. Then, the following results hold:

- If  $d = 2$  and  $\epsilon_s = -1$ , then CALR occurs. When  $\epsilon_c = 1$  the critical radius  $r_*$  is given by (1.7) and when  $\epsilon_c \neq 1$  the critical radius is

$$r_* = \frac{r_e^2}{r_i}. \quad (1.8)$$

That is, for almost any source  $f$  supported inside  $r_*$  CALR occurs and for any source  $f$  supported outside  $r_*$  CALR does not occur. When  $\epsilon_c \neq 1$  the cloaking radius  $r_e^2/r_i$  matches that found in [30] for a single dipolar source (see figure 5 there and accompanying text).

- If  $\epsilon_s \neq -1$ , then CALR does not occur.
- If  $d = 3$ , then CALR does not occur whatever  $\epsilon_s$  and  $\epsilon_c$  are.

We emphasize that the result on non-occurrence of CALR in three dimensions holds only when the dielectric constant  $\epsilon_s$  is constant. In the recent work [3] we show that CALR does occur in three dimensions if we use a shell with non-constant (anisotropic) dielectric constant.

It turns out that the occurrence of CALR depends on the distribution of eigenvalues of the Neumann-Poincaré (NP) operator associated with the structure (see the next section for the definition of the NP operator). The NP operator is compact with its eigenvalues accumulating towards 0. It is proved in [2] that in two dimensions the NP operator associated with the circular structure has the eigenvalues  $\pm\rho^n$  for  $n = 1, 2, \dots$ , where  $\rho = r_i/r_e$ . We show that in three dimensions the NP operator associated with the spherical structure has the eigenvalues

$$\pm \frac{1}{2(2n+1)} \sqrt{1 + 4n(n+1)\rho^{2n+1}}, \quad n = 0, 1, \dots \quad (1.9)$$

The exponential convergence of the eigenvalues in two dimensions is responsible for the occurrence of CALR and the slow convergence (at the rate  $1/n$ ) in three dimensions is responsible for the non-occurrence.

## 2 Layer potential formulation

Let  $G$  be the fundamental solution to the Laplacian in  $\mathbb{R}^d$  which is given by

$$G(x) = \begin{cases} \frac{1}{2\pi} \ln|x|, & d = 2, \\ -\frac{1}{4\pi} \frac{1}{|x|}, & d = 3. \end{cases}$$

Let  $\Gamma_i := \partial D$  and  $\Gamma_e := \partial\Omega$ . For  $\Gamma = \Gamma_i$  or  $\Gamma_e$ , we denote the single layer potential of a function  $\varphi \in L^2(\Gamma)$  as  $\mathcal{S}_\Gamma[\varphi]$ , where

$$\mathcal{S}_\Gamma[\varphi](x) := \int_\Gamma G(x-y)\varphi(y) d\sigma(y), \quad x \in \mathbb{R}^d.$$

We also define the boundary integral operator  $\mathcal{K}_\Gamma$  on  $L^2(\Gamma)$  by

$$\mathcal{K}_\Gamma[\varphi](x) := \int_\Gamma \frac{\partial G(x-y)}{\partial\nu(y)} \varphi(y) d\sigma(y), \quad x \in \Gamma,$$

and let  $\mathcal{K}_\Gamma^*$  be the  $L^2$ -adjoint of  $\mathcal{K}_\Gamma$ . Hence, the operator  $\mathcal{K}_\Gamma^*$  is given by

$$\mathcal{K}_\Gamma^*[\varphi](x) = \int_\Gamma \frac{\partial G(x-y)}{\partial\nu(x)} \varphi(y) d\sigma(y), \quad \varphi \in L^2(\Gamma).$$

The operators  $\mathcal{K}_\Gamma$  and  $\mathcal{K}_\Gamma^*$  are sometimes called Neumann-Poincaré operators. These operators are compact in  $L^2(\Gamma)$  if  $\Gamma$  is  $\mathcal{C}^{1,\alpha}$  for some  $\alpha > 0$ .

The following notation will be used throughout this paper. For a function  $u$  defined on  $\mathbb{R}^d \setminus \Gamma$ , we denote

$$u|_\pm(x) := \lim_{t \rightarrow 0^+} u(x \pm t\nu(x)), \quad x \in \Gamma,$$

and

$$\frac{\partial u}{\partial\nu}\Big|_\pm(x) := \lim_{t \rightarrow 0^+} \langle \nabla u(x \pm t\nu(x)), \nu(x) \rangle, \quad x \in \Gamma,$$

if the limits exist. Here and throughout this paper,  $\langle \cdot, \cdot \rangle$  denotes the scalar product on  $\mathbb{R}^d$ .

The following jump formula relates the traces of the normal derivative of the single layer potential to the operator  $\mathcal{K}_\Gamma^*$ . We have

$$\frac{\partial}{\partial \nu} \mathcal{S}_\Gamma[\varphi] \Big|_{\pm}(x) = \left( \pm \frac{1}{2} I + \mathcal{K}_\Gamma^* \right) [\varphi](x), \quad x \in \Gamma. \quad (2.1)$$

Here,  $\nu$  is the outward unit normal vector field to  $\Gamma$ . See, for example, [5, 12].

Let  $F$  be the Newtonian potential of  $f$ , *i.e.*,

$$F(x) = \int_{\mathbb{R}^d} G(x-y)f(y)dy, \quad x \in \mathbb{R}^d. \quad (2.2)$$

Then  $F$  satisfies  $\Delta F = f$  in  $\mathbb{R}^d$ , and the solution  $V_\delta$  to (1.3) may be represented as

$$V_\delta(x) = F(x) + \mathcal{S}_{\Gamma_i}[\varphi_i](x) + \mathcal{S}_{\Gamma_e}[\varphi_e](x) \quad (2.3)$$

for some functions  $\varphi_i \in L_0^2(\Gamma_i)$  and  $\varphi_e \in L_0^2(\Gamma_e)$  ( $L_0^2$  is the collection of all square integrable functions with zero mean-value). The transmission conditions along the interfaces  $\Gamma_e$  and  $\Gamma_i$  satisfied by  $V_\delta$  read

$$\begin{aligned} (\epsilon_s + i\delta) \frac{\partial V_\delta}{\partial \nu} \Big|_+ &= \epsilon_c \frac{\partial V_\delta}{\partial \nu} \Big|_- \quad \text{on } \Gamma_i, \\ \frac{\partial V_\delta}{\partial \nu} \Big|_+ &= (\epsilon_s + i\delta) \frac{\partial V_\delta}{\partial \nu} \Big|_- \quad \text{on } \Gamma_e. \end{aligned}$$

Hence the pair of potentials  $(\varphi_i, \varphi_e)$  is the solution to the following system of integral equations:

$$\begin{cases} (\epsilon_s + i\delta) \frac{\partial \mathcal{S}_{\Gamma_i}[\varphi_i]}{\partial \nu_i} \Big|_+ - \epsilon_c \frac{\partial \mathcal{S}_{\Gamma_i}[\varphi_i]}{\partial \nu_i} \Big|_- + (\epsilon_s - \epsilon_c + i\delta) \frac{\partial \mathcal{S}_{\Gamma_e}[\varphi_e]}{\partial \nu_e} = (-\epsilon_s + \epsilon_c - i\delta) \frac{\partial F}{\partial \nu_i} & \text{on } \Gamma_i, \\ (-1 + \epsilon_s + i\delta) \frac{\partial \mathcal{S}_{\Gamma_i}[\varphi_i]}{\partial \nu_e} - \frac{\partial \mathcal{S}_{\Gamma_e}[\varphi_e]}{\partial \nu_e} \Big|_+ + (\epsilon_s + i\delta) \frac{\partial \mathcal{S}_{\Gamma_e}[\varphi_e]}{\partial \nu_e} \Big|_- = (1 - \epsilon_s - i\delta) \frac{\partial F}{\partial \nu_e} & \text{on } \Gamma_e. \end{cases}$$

Note that we have used the notation  $\nu_i$  and  $\nu_e$  to indicate the outward normal on  $\Gamma_i$  and  $\Gamma_e$ , respectively. Using the jump formula (2.1) for the normal derivative of the single layer potentials, the above equations can be rewritten as

$$\begin{bmatrix} z_i^\delta I - \mathcal{K}_{\Gamma_i}^* & -\frac{\partial}{\partial \nu_i} \mathcal{S}_{\Gamma_e} \\ \frac{\partial}{\partial \nu_e} \mathcal{S}_{\Gamma_i} & z_e^\delta I + \mathcal{K}_{\Gamma_e}^* \end{bmatrix} \begin{bmatrix} \varphi_i \\ \varphi_e \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial \nu_i} \\ -\frac{\partial F}{\partial \nu_e} \end{bmatrix} \quad (2.4)$$

on  $\mathcal{H}_0 = L_0^2(\Gamma_i) \times L_0^2(\Gamma_e)$ , where we set

$$z_i^\delta = \frac{\epsilon_c + \epsilon_s + i\delta}{2(\epsilon_c - \epsilon_s - i\delta)}, \quad z_e^\delta = \frac{1 + \epsilon_s + i\delta}{2(1 - \epsilon_s - i\delta)}. \quad (2.5)$$

Let  $\mathcal{H} = L^2(\Gamma_i) \times L^2(\Gamma_e)$  and let the Neumann-Poincaré-type operator  $\mathbb{K}^* : \mathcal{H} \rightarrow \mathcal{H}$  be defined by

$$\mathbb{K}^* := \begin{bmatrix} -\mathcal{K}_{\Gamma_i}^* & -\frac{\partial}{\partial \nu_i} \mathcal{S}_{\Gamma_e} \\ \frac{\partial}{\partial \nu_e} \mathcal{S}_{\Gamma_i} & \mathcal{K}_{\Gamma_e}^* \end{bmatrix}, \quad (2.6)$$

and let

$$\Phi := \begin{bmatrix} \varphi_i \\ \varphi_e \end{bmatrix}, \quad g := \begin{bmatrix} \frac{\partial F}{\partial \nu_i} \\ -\frac{\partial F}{\partial \nu_e} \end{bmatrix}. \quad (2.7)$$

Then, (2.4) can be rewritten in the form

$$(\mathbb{I}^\delta + \mathbb{K}^*)\Phi = g, \quad (2.8)$$

where  $\mathbb{I}^\delta$  is given by

$$\mathbb{I}^\delta = \begin{bmatrix} z_i^\delta I & 0 \\ 0 & z_e^\delta I \end{bmatrix}. \quad (2.9)$$

### 3 Eigenvalues of the NP operator

It is proved in [2] that for arbitrary-shaped domains  $\Omega$  and  $D$  the spectrum of the NP operator  $\mathbb{K}^*$  lies in  $[-1/2, 1/2]$ , and if  $\Omega$  and  $D$  are concentric disks, the eigenvalues of  $\mathbb{K}^*$  on  $\mathcal{H}_0$  are  $\pm\rho^n/2$ ,  $n = 1, 2, \dots$ . In this section we compute the eigenvalues of  $\mathbb{K}^*$  on  $\mathcal{H}$  when  $\Omega$  and  $D$  are concentric disks or balls.

#### 3.1 Two dimensions

Let  $\Gamma = \{|x| = r_0\}$  in two dimensions. It is known that for each integer  $n$

$$\mathcal{S}_\Gamma[e^{in\theta}](x) = \begin{cases} -\frac{r_0}{2|n|} \left(\frac{r}{r_0}\right)^{|n|} e^{in\theta} & \text{if } |x| = r < r_0, \\ -\frac{r_0}{2|n|} \left(\frac{r_0}{r}\right)^{|n|} e^{in\theta} & \text{if } |x| = r > r_0. \end{cases} \quad (3.1)$$

Moreover,

$$\mathcal{K}_\Gamma^*[e^{in\theta}] = 0 \quad \forall n \neq 0, \quad (3.2)$$

and

$$\mathcal{K}_\Gamma[1] = \frac{1}{2}. \quad (3.3)$$

In other words,  $\mathcal{K}_\Gamma$  is a rank 1 operator whose only non-zero eigenvalue is  $1/2$ .

Using (3.2), it is proved that eigenvalues of  $\mathbb{K}^*$  on  $\mathcal{H}_0$  are  $\pm\rho^2/2$  (see [2]). We now show that  $\pm 1/2$  are also eigenvalues of  $\mathbb{K}^*$  on  $\mathcal{H}_0$ . These eigenvalues are of interest in relation to estimation of stress concentration [4]. Using (3.3) we have

$$\mathcal{S}_\Gamma[1](x) = \begin{cases} \log r_0 & \text{if } |x| = r < r_0, \\ \log |x| & \text{if } |x| = r > r_0, \end{cases} \quad (3.4)$$

and hence

$$\frac{\partial}{\partial r} \mathcal{S}_\Gamma[1](x) = \begin{cases} 0 & \text{if } |x| = r < r_0, \\ \frac{1}{r} & \text{if } |x| = r > r_0. \end{cases} \quad (3.5)$$

It then follows that

$$\mathbb{K}^* \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{r_e} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \quad (3.6)$$

where  $a$  and  $b$  are constants. So  $\pm 1/2$  are eigenvalues of  $\mathbb{K}^*$ .

We summarize our findings in the following proposition.

**Proposition 3.1** *The eigenvalues of  $\mathbb{K}^*$  defined on concentric circles in two dimensions are*

$$-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\rho^n, \frac{1}{2}\rho^n, \quad n = 1, 2, \dots, \quad (3.7)$$

and corresponding eigenfunctions are

$$\begin{bmatrix} 1 \\ -\frac{1}{r_e} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} e^{\pm in\theta} \\ \rho e^{\pm in\theta} \end{bmatrix}, \begin{bmatrix} e^{\pm in\theta} \\ -\rho e^{\pm in\theta} \end{bmatrix}, \quad n = 1, 2, \dots \quad (3.8)$$

### 3.2 Three dimensions

Let  $Y_n^m(\hat{x})$  ( $m = -n, -n+1, \dots, 0, 1, \dots, n$ ) be the orthonormal spherical harmonics of degree  $n$ . Here  $\hat{x} = \frac{x}{|x|}$ . Then  $|x|^n Y_n^m(\hat{x})$  is harmonic in  $\mathbb{R}^3$ .

**Lemma 3.2** *Let  $\Gamma = \{|x| = r_0\}$  in three dimensions. We have for  $n = 0, 1, \dots$*

$$\mathcal{K}_\Gamma^*[Y_n^m](x) = \frac{1}{2(2n+1)} Y_n^m(\hat{x}), \quad |x| = r_0, \quad m = -n, \dots, n. \quad (3.9)$$

*Proof.* It is proved in [18, Lemma 2.3] that

$$\mathcal{K}_\Gamma^*[\varphi](x) = -\frac{1}{2r_0} \mathcal{S}_\Gamma[\varphi](x), \quad |x| = r_0 \quad (3.10)$$

for any function  $\varphi \in L^2(\Gamma)$ . So it follows from (2.1) that

$$\frac{\partial}{\partial r} \mathcal{S}_\Gamma[\varphi]|_-(x) + \frac{1}{2r_0} \mathcal{S}_\Gamma[\varphi](x) = -\frac{1}{2} \varphi(x), \quad |x| = r_0. \quad (3.11)$$

Let  $\varphi(x) = Y_n^m(\hat{x})$ . Then since  $\mathcal{S}_\Gamma[Y_n^m](x)$  and  $|x|^n Y_n^m(\hat{x})$  are harmonic functions in  $\{|x| < r_0\}$ , we have

$$\mathcal{S}_\Gamma[Y_n^m](x) = -\frac{1}{2n+1} \frac{r^n}{r_0^{n-1}} Y_n^m(\hat{x}), \quad \text{for } |x| = r \leq r_0, \quad (3.12)$$

and (3.9) follows from (3.10).  $\square$

Lemma 3.2 says that the eigenvalues of  $\mathcal{K}_\Gamma^*$  on  $L^2(\Gamma)$  when  $\Gamma$  is a sphere are  $\frac{1}{2(2n+1)}$ ,  $n = 0, 1, \dots$ , and their multiplicities are  $2n+1$ .

By (3.12), we have

$$\frac{\partial}{\partial \nu_i} \mathcal{S}_{\Gamma_e}[Y_n^m](x) = -\frac{n}{2n+1} \left(\frac{r_i}{r_e}\right)^{n-1} Y_n^m(\hat{x}), \quad |x| = r_i. \quad (3.13)$$

Similarly, we have

$$\mathcal{S}_{\Gamma_i}[Y_n^m](x) = -\frac{1}{2n+1} \frac{r_i^{n+2}}{r^{n+1}} Y_n^m(\hat{x}), \quad \text{for } |x| = r \geq r_i,$$

and hence

$$\frac{\partial}{\partial \nu_e} \mathcal{S}_{\Gamma_i}[Y_n^m](x) = \frac{n+1}{2n+1} \left(\frac{r_i}{r_e}\right)^{n+2} Y_n^m(\hat{x}), \quad |x| = r_e. \quad (3.14)$$

We now have for constants  $a$  and  $b$

$$\mathbb{K}^* \begin{bmatrix} aY_n^m \\ bY_n^m \end{bmatrix} = \begin{bmatrix} \left(-\frac{a}{2(2n+1)} + b\frac{n}{2n+1}\rho^{n-1}\right) Y_n^m \\ \left(a\frac{n+1}{2n+1}\rho^{n+2} + \frac{b}{2(2n+1)}\right) Y_n^m \end{bmatrix} = \begin{bmatrix} -\frac{1}{2(2n+1)} & \frac{n}{2n+1}\rho^{n-1} \\ \frac{n+1}{2n+1}\rho^{n+2} & \frac{1}{2(2n+1)} \end{bmatrix} \begin{bmatrix} aY_n^m \\ bY_n^m \end{bmatrix}. \quad (3.15)$$

Thus we have the following result.

**Proposition 3.3** *The eigenvalues of  $\mathbb{K}^*$  defined on two concentric spheres are*

$$\pm \frac{1}{2(2n+1)} \sqrt{1 + 4n(n+1)\rho^{2n+1}}, \quad n = 0, 1, \dots, \quad (3.16)$$

and corresponding eigenfunctions are

$$\left[ \frac{(\sqrt{1 + 4n(n+1)\rho^{2n+1}} - 1)Y_n^m}{2(n+1)\rho^{n+2}Y_n^m} \right], \quad \left[ \frac{(-\sqrt{1 + 4n(n+1)\rho^{2n+1}} - 1)Y_n^m}{2(n+1)\rho^{n+2}Y_n^m} \right], \quad m = -n, \dots, n, \quad (3.17)$$

respectively.

It is quite interesting to observe that if we let  $\frac{1}{2} = \lambda_0 \geq \lambda_1 \geq \dots$  be the eigenvalues of  $\mathcal{K}_\Gamma$  for a disk or a sphere enumerated according to their multiplicities, then the eigenvalues  $\mu_n$  of  $\mathbb{K}^*$  satisfy

$$\mu_n = \pm \lambda_n + O(\rho^n). \quad (3.18)$$

## 4 Anomalous localized resonance in two dimensions

In this section we consider the CALR when the domains  $\Omega$  and  $D$  are concentric disks. We first observe that  $z_i^\delta$  and  $z_e^\delta$  converges to non-zero numbers as  $\delta$  tends to 0 if  $\epsilon_c \neq -\epsilon_s \neq 1$ . So, in this case CALR does not occur regardless of the location of the source. Furthermore, if  $\epsilon_c = \epsilon_s = 1$ , a thorough study was done in [2]. It is proved in [2] that if the source  $f$  is supported inside the critical radius  $r_* = \sqrt{r_e^3/r_i}$ , then the weak CALR occurs, namely,

$$\limsup_{\delta \rightarrow 0} E_\delta = \infty. \quad (4.1)$$

Moreover, if  $F$  is the Newtonian potential of  $f$  and the Fourier coefficients  $g_e^n$  of  $-\frac{\partial F}{\partial \nu_e}$  satisfies the following gap property:

[GP] There exists a sequence  $\{n_k\}$  with  $|n_1| < |n_2| < \dots$  such that

$$\lim_{k \rightarrow \infty} \rho^{|n_{k+1}| - |n_k|} \frac{|g_e^{n_k}|^2}{|n_k| \rho^{|n_k|}} = \infty,$$

then CALR occurs, namely

$$\lim_{\delta \rightarrow 0} E_\delta = \infty, \quad (4.2)$$

and  $V_\delta/\sqrt{E_\delta}$  goes to zero outside the radius  $\sqrt{r_e^3/r_i}$ .

The remaining two cases are when  $\epsilon_c \neq -\epsilon_s = 1$  and  $\epsilon_c = -\epsilon_s \neq 1$ . In these cases, we have the following theorem.

**Theorem 4.1** (i) *If  $\epsilon_c = -\epsilon_s \neq 1$ , then CALR does not occur, i.e.,*

$$E_\delta \leq C \quad (4.3)$$

for some  $C > 0$ . (We note, however, that there will be CALR for appropriately placed sources inside the core, as can be seen from the fact that the equations are invariant under conformal transformations, and in particular under the inverse transformation  $1/z$  where  $z = x_1 + ix_2$ , which in effect interchanges the roles of the matrix and core.)

(ii) *If  $\epsilon_c \neq -\epsilon_s = 1$ , then weak CALR occurs and the critical radius is  $r_* = r_e^2 r_i^{-1}$ , i.e., if the source function is supported inside  $r_*$  (and its Newtonian potential does not extend harmonically to  $\mathbb{R}^2$ ), then*

$$\limsup_{\delta \rightarrow 0} E_\delta = \infty, \quad (4.4)$$

and there exists a constant  $C$  such that

$$|V_\delta(x)| < C \quad (4.5)$$

for all  $x$  with  $|x| > r_e^3/r_i^2$ .

(iii) In addition to the assumptions of (ii), the Fourier coefficients  $g_e^n$  of  $-\frac{\partial F}{\partial \nu_e}$  satisfies the following gap property:

[GP2] There exists a sequence  $\{n_k\}$  with  $|n_1| < |n_2| < \dots$  such that

$$\lim_{k \rightarrow \infty} \rho^{2(|n_{k+1}| - |n_k|)} \frac{|g_e^{n_k}|^2}{|n_k| \rho^{|n_k|}} = \infty,$$

then the CALR occurs, i.e.,

$$\lim_{\delta \rightarrow 0} E_\delta = \infty, \quad (4.6)$$

and  $V_\delta/\sqrt{E_\delta}$  goes to zero outside the radius  $r_e^3/r_i^2$ , as implied by (4.5).

Before proving Theorem 4.1 we make a remark on the Gap Properties [GP] and [GP2]. One can easily see that [GP] is weaker than [GP2], namely, if [GP] holds, so does [GP2].

The rest of this section is devoted to the proof of Theorem 4.1. As was proved in [2], we have

$$\begin{aligned} \frac{\partial}{\partial \nu_e} \mathcal{S}_{\Gamma_i}[e^{in\theta}](x) &= \frac{1}{2} \rho^{|n|+1} e^{in\theta}, \\ \frac{\partial}{\partial \nu_i} \mathcal{S}_{\Gamma_e}[e^{in\theta}](x) &= -\frac{1}{2} \rho^{|n|-1} e^{in\theta}. \end{aligned}$$

Using these identities, one can see that if  $g$  defined by (2.7) has the Fourier series expansion

$$g = \sum_{n \neq 0} \begin{bmatrix} g_i^n \\ g_e^n \end{bmatrix} e^{in\theta},$$

then the integral equations (2.8) are equivalent to

$$\begin{cases} z_i^\delta \varphi_i^n + \frac{\rho^{|n|-1}}{2} \varphi_e^n = g_i^n, \\ z_e^\delta \varphi_e^n + \frac{\rho^{|n|+1}}{2} \varphi_i^n = g_e^n \end{cases} \quad (4.7)$$

for every  $|n| \geq 1$ . It is readily seen that the solution  $\Phi = (\varphi_i, \varphi_e)$  to (4.7) is given by

$$\begin{aligned} \varphi_i &= 2 \sum_{n \neq 0} \frac{2z_e^\delta g_i^n - \rho^{|n|-1} g_e^n}{4z_i^\delta z_e^\delta - \rho^{2|n|}} e^{in\theta}, \\ \varphi_e &= 2 \sum_{n \neq 0} \frac{2z_i^\delta g_e^n - \rho^{|n|+1} g_i^n}{4z_i^\delta z_e^\delta - \rho^{2|n|}} e^{in\theta}. \end{aligned}$$

If the source is located outside the structure, i.e.,  $f$  is supported in  $|x| > r_e$ , then the Newtonian potential of  $f$ ,  $F$ , is harmonic in  $|x| \leq r_e$  and

$$F(x) = c - \sum_{n \neq 0} \frac{g_e^n}{|n| r_e^{|n|-1}} r^{|n|} e^{in\theta}, \quad |x| \leq r_e. \quad (4.8)$$

Thus we have

$$g_i^n = -g_e^n \rho^{|n|-1}. \quad (4.9)$$



So we have

$$\begin{cases} \varphi_i = -2 \sum_{n \neq 0} \frac{(2z_e^\delta + 1)\rho^{|n|-1}g_e^n}{4z_i^\delta z_e^\delta - \rho^{2|n|}} e^{in\theta}, \\ \varphi_e = 2 \sum_{n \neq 0} \frac{(2z_i^\delta + \rho^{2|n|})g_e^n}{4z_i^\delta z_e^\delta - \rho^{2|n|}} e^{in\theta}. \end{cases} \quad (4.10)$$

Therefore, from (3.1) we find that

$$\mathcal{S}_{\Gamma_i}[\varphi_i](x) + \mathcal{S}_{\Gamma_e}[\varphi_e](x) = \sum_{n \neq 0} \frac{2(r_i^{2|n|}z_e^\delta - r_e^{2|n|}z_i^\delta)}{|n|r_e^{|n|-1}(4z_i^\delta z_e^\delta - \rho^{2|n|})} \frac{g_e^n}{r^{|n|}} e^{in\theta}, \quad r_e < r = |x|, \quad (4.11)$$

and

$$\mathcal{S}_{\Gamma_i}[\varphi_i](x) = - \sum_{n \neq 0} \frac{r_i^{2|n|}(2z_e^\delta + 1)}{|n|r_e^{|n|-1}(\rho^{2|n|} - 4z_i^\delta z_e^\delta)} \frac{g_e^n}{r^{|n|}} e^{in\theta}, \quad r_i < r = |x| < r_e, \quad (4.12)$$

$$\mathcal{S}_{\Gamma_e}[\varphi_e](x) = \sum_{n \neq 0} \frac{(2z_i^\delta + \rho^{2|n|})}{|n|r_e^{|n|-1}(\rho^{2|n|} - 4z_i^\delta z_e^\delta)} g_e^n r^{|n|} e^{in\theta}, \quad r_i < r = |x| < r_e. \quad (4.13)$$

We obtain the following lemma.

**Lemma 4.2** *There exists  $\delta_0$  such that*

$$E_\delta \approx \begin{cases} \sum_{n \neq 0} \frac{\delta |g_e^n|^2}{|n|(\delta^2 + \rho^{4|n|})}, & \text{if } \epsilon_c \neq \epsilon_s = 1, \\ \sum_{n \neq 0} \frac{\delta \rho^{2|n|} |g_e^n|^2}{|n|(\delta^2 + \rho^{4|n|})}, & \text{if } \epsilon_c = \epsilon_s \neq 1, \end{cases} \quad (4.14)$$

uniformly in  $\delta \leq \delta_0$ .

*Proof.* Using (4.8), (4.12), and (4.13), one can see that

$$V_\delta(x) = c + r_e \sum_{n \neq 0} \left[ \frac{r_i^{2|n|}}{r^{|n|}} - 2z_i^\delta r^{|n|} \right] \frac{(2z_e^\delta + 1)g_e^n e^{in\theta}}{|n|r_e^{|n|}(4z_i^\delta z_e^\delta - \rho^{2|n|})}.$$

We check that

$$\left| \nabla \left( \left( \frac{r_i^{2|n|}}{r^{|n|}} - 2z_i^\delta r^{|n|} \right) e^{in\theta} \right) \right|^2 = \frac{2|n|^2}{r^2} \left| \frac{r_i^{2|n|}}{r^{|n|}} - 2z_i^\delta r^{|n|} \right|^2.$$

Then straightforward computations yield that

$$\int_{B_e \setminus B_i} \delta |\nabla V_\delta|^2 \approx \sum_{n \neq 0} \delta \left| \frac{2z_e^\delta + 1}{4z_i^\delta z_e^\delta - \rho^{2|n|}} \right|^2 (4|z_i^\delta|^2 + \rho^{2|n|}) \frac{|g_e^n|^2}{|n|}.$$

If  $\delta$  is sufficiently small, then one can also easily show that

$$|4z_i^\delta z_e^\delta - \rho^{2|n|}| \approx \delta + \rho^{2|n|}.$$

Therefore we get (4.14) and the proof is complete.  $\square$

First, if  $\epsilon_c = -\epsilon_s \neq 1$ , then

$$E_\delta \approx \sum_{n \neq 0} \frac{\delta \rho^{2|n|} |g_e^n|^2}{|n|(\delta^2 + \rho^{4|n|})} \leq \sum_{n \neq 0} \frac{|g_e^n|^2}{2|n|} \leq \frac{1}{2} \left\| \frac{\partial F}{\partial v_e} \right\|_{L^2(\Gamma_e)} \leq C \|f\|_{L^2(\mathbb{R}^2)}.$$

Suppose that  $\epsilon_c \neq -\epsilon_s = 1$ , and let

$$N_\delta = \frac{\log \delta}{2 \log \rho}. \quad (4.15)$$

If  $|n| \leq N_\delta$ , then  $\delta \leq \rho^{2|n|}$ , and hence

$$\sum_{n \neq 0} \frac{\delta |g_e^n|^2}{|n|(\delta^2 + \rho^{4|n|})} \geq \sum_{0 \neq |n| \leq N_\delta} \frac{\delta |g_e^n|^2}{|n|(\delta^2 + \rho^{4|n|})} \geq \sum_{0 \neq |n| \leq N_\delta} \frac{\delta |g_e^n|^2}{|n| \rho^{4|n|}}. \quad (4.16)$$

If the following holds

$$\limsup_{n \rightarrow \infty} \frac{|g_e^n|^2}{|n| \rho^{2|n|}} = \infty, \quad (4.17)$$

then one can show as in [2] that there is a sequence  $\{|n_k|\}$  such that

$$\lim_{k \rightarrow \infty} E_{\rho^{|n_k|}} = \infty. \quad (4.18)$$

Suppose that the source function  $f$  is supported inside the critical radius  $r_* = r_e^2 r_i^{-1}$  (and outside  $r_e$ ). Then its Newtonian potential  $F$  cannot be extended harmonically in  $|x| < r_*$  in general. So, if  $F$  is given by

$$F = c - \sum_{n \neq 0} a_n r^{|n|} e^{in\theta}, \quad r < r_e + \epsilon \quad (4.19)$$

for some  $\epsilon > 0$ , then the radius of convergence of the series is less than  $r_*$ . Thus we have

$$\limsup_{|n| \rightarrow \infty} |a_n|^2 r_*^{2|n|} = \infty. \quad (4.20)$$

Since  $g_e^n = |n| a_n r_e^{|n|-1}$ , (4.17) holds.

By (4.11), we know

$$|V_\delta| \leq |F| + C \sum_{n \neq 0} \frac{r_e^{|n|}}{\delta + \rho^{2|n|}} \frac{|g_e^n|}{r^{|n|}} \leq |F| + C \sum_{n \neq 0} \frac{r_e^{3|n|}}{r_i^{2|n|}} \frac{|g_e^n|}{r^{|n|}} \leq C' \quad (4.21)$$

if  $r > r_e^3 / r_i^2$ . Thus (ii) is proved.

We now prove (iii). We emphasize that [GP2] implies (4.17), but the converse may not be true. On the other hand [GP2] holds if

$$\lim_{n \rightarrow \infty} \frac{|g_e^n|^2}{|n| \rho^{2|n|}} = \infty. \quad (4.22)$$

So we may regard the condition [GP2] something between (4.17) and (4.22).

Suppose that [GP2] holds. If we take  $\delta = \rho^{2\alpha}$  and let  $k(\alpha)$  be the number such that

$$|n_{k(\alpha)}| \leq \alpha < |n_{k(\alpha)+1}|,$$

then

$$\sum_{0 \neq |n| \leq N_\delta} \frac{\delta |g_e^n|^2}{|n| \rho^{4|n|}} = \rho^{2\alpha} \sum_{0 \neq |n| \leq \alpha} \frac{|g_e^n|^2}{|n| \rho^{4|n|}} \geq \rho^{2(|n_{k(\alpha)+1}| - |n_{k(\alpha)}|)} \frac{|g_e^{n_{k(\alpha)}}|^2}{|n_{k(\alpha)}| \rho^{2|n_{k(\alpha)}|}} \rightarrow \infty, \quad (4.23)$$

as  $\alpha \rightarrow \infty$ . Combined with Lemma 4.2 and (4.16), it gives us (iii).

## 5 Non-occurrence of CALR in 3D

In this section we show that CALR does not occur in a radially symmetric three dimensional coated sphere structure when the core, matrix and shell are isotropic. We have the following theorem.

**Theorem 5.1** *Suppose that  $\Gamma_e$  and  $\Gamma_i$  are concentric spheres. For any  $\epsilon_c$  and  $\epsilon_s$ , there is a constant  $C$  independent of  $\delta$  such that if  $V_\delta$  is the solution to (1.3), then*

$$\int_{\Omega \setminus D} \delta |\nabla V_\delta|^2 \leq C \|f\|_{L^2(\mathbb{R}^3)}^2. \quad (5.1)$$

*Proof.* Suppose that  $\frac{\partial}{\partial \nu_e} F$  has the Fourier series expansion

$$\frac{\partial}{\partial \nu_e} F = - \sum_{n=0}^{\infty} \sum_{m=-n}^n g_{mn}^e Y_m^n. \quad (5.2)$$

Then one can show as in (4.9) that

$$\frac{\partial}{\partial \nu_i} F = - \sum_{n=0}^{\infty} \sum_{m=-n}^n g_{mn}^e \rho^{n-1} Y_m^n. \quad (5.3)$$

By solving the integral equation (2.4) using (3.15), we obtain

$$\varphi_i = - \sum_{n=0}^{\infty} \sum_{m=-n}^n \Delta_n^{-1} \rho^{n-1} \left( z_e^\delta + \frac{1}{2} \right) g_{mn}^e Y_m^n, \quad (5.4)$$

$$\varphi_e = \sum_{n=0}^{\infty} \sum_{m=-n}^n \Delta_n^{-1} \rho^{n-1} \left( z_i^\delta - \frac{1}{2(2n+1)} + \frac{n+1}{2(2n+1)} \rho^{2n+1} \right) g_{mn}^e Y_m^n, \quad (5.5)$$

where

$$\Delta_n := \left( z_i^\delta - \frac{1}{2(2n+1)} \right) \left( z_e^\delta + \frac{1}{2(2n+1)} \right) + \frac{n(n+1)}{(2n+1)^2} \rho^{2n+1}.$$

Suppose for simplicity that  $\epsilon_c = -\epsilon_s = 1$ , so that  $z_i^\delta$  and  $z_e^\delta$  given by (2.5) simplify to

$$z_i^\delta = z_e^\delta = \frac{i\delta}{2(2-i\delta)}.$$

Then one can see that if  $\delta$  is sufficiently small, then

$$|\Delta_n| \approx \delta^2 + n^{-2}.$$

So we have

$$\begin{aligned} \delta \|\varphi_i\|_{L^2(\Gamma_i)}^2 &\leq C \sum_{n=0}^{\infty} \sum_{m=-n}^n \frac{\delta \rho^{2n}}{(\delta^2 + n^{-2})^2} |g_{mn}^e|^2 \\ &\leq C \sum_{n=0}^{\infty} \sum_{m=-n}^n n^3 \rho^{2n} |g_{mn}^e|^2 \\ &\leq C \sum_{n=0}^{\infty} |g_{mn}^e|^2 \leq C \|f\|_{L^2(\mathbb{R}^3)}^2. \end{aligned}$$

and

$$\delta \|\varphi_e\|_{L^2(\Gamma_e)}^2 \leq C \sum_{n=0}^{\infty} \sum_{m=-n}^n \frac{\delta \rho^{2n}}{\delta^2 + n^{-2}} |g_{mn}^e|^2 \leq C \|f\|_{L^2(\mathbb{R}^3)}^2.$$

Therefore we have

$$\begin{aligned}
\int_{\Omega \setminus D} \delta |\nabla V_\delta|^2 &= \int_{\Omega \setminus D} \delta |\nabla F|^2 + \int_{\Omega \setminus D} \delta |\nabla (\mathcal{S}_{\Gamma_i}[\varphi_i] + \mathcal{S}_{\Gamma_e}[\varphi_e])|^2 \\
&\leq \int_{\Omega \setminus D} \delta |\nabla F|^2 + \int_{\Omega \setminus D} \delta |\nabla (\mathcal{S}_{\Gamma_i}[\varphi_i] + \mathcal{S}_{\Gamma_e}[\varphi_e])|^2 \\
&\leq \int_{\Omega \setminus D} \delta |\nabla F|^2 + \delta (\|\varphi_i\|_{L^2(\Gamma_i)}^2 + \|\varphi_e\|_{L^2(\Gamma_e)}^2) \leq C \|f\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}$$

If  $\epsilon_s \neq -1$  or/and  $\epsilon_c \neq 1$ , then the same argument can be applied to arrive at (5.1). This completes the proof.  $\square$

## References

- [1] A. Alú and N. Engheta, Achieving transparency with plasmonic and metamaterial coatings, *Phys. Rev. E* 72 (2005), 0166623.
- [2] H. Ammari, G. Ciraolo, H. Kang, H. Lee and G.W. Milton, Spectral theory of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance, *Arch. Rat. Mech. Anal.*, to appear.
- [3] H. Ammari, G. Ciraolo, H. Kang, H. Lee and G.W. Milton, Anomalous localized resonance using a folded geometry in three dimensions, in preparation.
- [4] H. Ammari, G. Ciraolo, H. Kang, H. Lee and K. Yun, Spectral analysis of the Neumann-Poincaré operator and characterization of the stress concentration in anti-plane elasticity, *Arch. Rat. Mech. Anal.*, to appear.
- [5] H. Ammari and H. Kang, *Polarization and Moment Tensors with Applications to Inverse Problems and Effective Medium Theory*, Applied Mathematical Sciences, Vol. 162, Springer-Verlag, New York, 2007.
- [6] H. Ammari, H. Kang, H. Lee, and M. Lim, Enhancement of near cloaking using generalized polarization tensors vanishing structures. Part I: The conductivity problem, *Comm. Math. Phys.*, to appear.
- [7] G. Bouchitté and B. Schweizer, Cloaking of small objects by anomalous localized resonance, *Quart. J. Mech. Appl. Math.* 63 (2010), 437–463.
- [8] O.P. Bruno and S. Lintner, Superlens-cloaking of small dielectric bodies in the quasi-static regime, *J. Appl. Phys.* 102 (2007), 124502.
- [9] K. Bryan and T. Leise, Impedance Imaging, inverse problems, and Harry Potter’s Cloak, *SIAM Rev.* 52 (2010), 359–377.
- [10] W. Cai, U.K. Chettiar, A.V. Kildishev, V.M. Shalaev and G.W. Milton, Non-magnetic cloak with minimized scattering, *Appl. Phys. Lett.* 91 (2007), 111105.
- [11] J.-W. Dong, H.H. Zheng, Y. Lai, H.-Z. Wang, and C. T. Chan, Metamaterial slab as a lens, a cloak, or an intermediate, *Phys. Rev. B* 83 (2011), 115124.
- [12] G.B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, NJ, 1976.
- [13] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, Cloaking devices, electromagnetic wormholes, and transformation optics, *SIAM Rev.* 51 (2009), 3–33.

- [14] A. Greenleaf, M. Lassas, and G. Uhlmann, On nonuniqueness for Calderón’s inverse problem, *Math. Res. Lett.* 10 (2003), 685–693.
- [15] F. Guevara Vasquez and G. W. Milton and D. Onofrei, Broadband exterior cloaking, *Opt. Express* 17 (2009), 14800–14805.
- [16] V. Isakov, *Inverse Source Problems*, Math. Surveys and Monograph Series Vol. 34, AMS, Providence, RI, 1990.
- [17] H. Kang and J-K. Seo, Layer potential techniques for the inverse conductivity problems, *Inverse Problems*, 12 (1996), 267–278.
- [18] H. Kang and J.-K. Seo, The inverse conductivity problem with one measurement: uniqueness for balls”, *SIAM J. of Applied Math.* Vol 59 (1999), pp. 1533-1539
- [19] O.D. Kellogg, *Foundations of Potential Theory*, Dover, New York, 1953.
- [20] R.V. Kohn, J. Lu, B. Schweizer, and M.I. Weinstein, A variational perspective on cloaking by anomalous localized resonance, preprint, arXiv:1210.4823.
- [21] R. V. Kohn, D. Onofrei, M. S. Vogelius, and M. I. Weinstein, Cloaking via change of variables for the Helmholtz equation, *Comm. Pure Appl. Math.* 63 (2010), 973–1016.
- [22] R. V. Kohn, H. Shen, M. S. Vogelius, and M. I. Weinstein, Cloaking via change of variables in electric impedance tomography, *Inverse Problems* 24 (2008), 015016.
- [23] Y. Lai, H. Chen, Z.-Q. Zhang, and C. T. Chan, Complementary media invisibility cloak that cloaks objects at a distance outside the cloaking shell, *Phys. Rev. Lett.* 102 (2009), 093901.
- [24] U. Leonhardt, Optical conforming mapping, *Science*, 312 (2006), 5781, 1777–1780.
- [25] U. Leonhardt and T. Tyc, Broadband invisibility by non-euclidean cloaking, *Science* 323 (2009), 110–111.
- [26] H. Liu, Virtual reshaping and invisibility in obstacle scattering, *Inverse Problems* 25 (2009), 044006.
- [27] R.C. McPhedran, N.-A.P. Nicorovici, L.C. Botten, and G.W. Milton, Cloaking by plasmonic resonance among systems of particles: cooperation or combat? *C.R. Phys.* 10 (2009), 391–399.
- [28] D.A.B. Miller, On perfect cloaking, *Opt. Express* 14 (2006) 12457–12466.
- [29] G.W. Milton, N.-A.P. Nicorovici, R.C. McPhedran, and V.A.Podolskiy, A proof of superlensing in the quasistatic regime, and limitations of superlenses in this regime due to anomalous localized resonance, *Proc. R. Soc. A* 461 (2005), 3999–4034.
- [30] G.W. Milton and N.-A.P. Nicorovici, On the cloaking effects associated with anomalous localized resonance, *Proc. R. Soc. A* 462 (2006), 3027–3059.
- [31] G.W. Milton, N.-A.P. Nicorovici, and R.C. McPhedran, Opaque perfect lenses, *Physica B* 394 (2007), 171–175.
- [32] G.W. Milton, N.-A.P. Nicorovici, R.C. McPhedran, K. Cherednichenko, and Z. Jacob, Solutions in folded geometries, and associated cloaking due to anomalous resonance, *New. J. Phys.* 10 (2008), 115021.
- [33] G.W. Milton, N.-A.P. Nicorovici, R.C. McPhedran, and V.A. Podolskiy, A proof of superlensing in the quasistatic regime, and limitations of superlenses in this regime due to anomalous localized resonance, *Proc. R. Soc. A* 461 (2005), 3999–4034.

- [34] N.-A.P. Nicorovici, R.C. McPhedran, and L.C. Botten, Relative local density of states and cloaking in finite clusters of coated cylinders, *Wave. Random Complex* 21 (2011) 248–277.
- [35] N.-A.P. Nicorovici, R.C. McPhedran, S. Enoch, and G. Tayeb, Finite wavelength cloaking by plasmonic resonance, *New. J. Phys.* 10 (2008), 115020.
- [36] N.-A.P. Nicorovici, R.C. McPhedran, and G.W. Milton, Optical and dielectric properties of partially resonant composites, *Phys. Rev. B* 49 (1994), 8479–8482.
- [37] N.-A.P. Nicorovici, G.W. Milton, R.C. McPhedran, and L.C. Botten, Quasistatic cloaking of two-dimensional polarizable discrete systems by anomalous resonance, *Optics Express* 15 (2007), 6314–6323.
- [38] J.B. Pendry, Negative refraction makes a perfect lens, *Phys. Rev. Lett.* 85 (2000), 3966–3969.
- [39] J.B. Pendry, D. Schurig, and D.R. Smith, Controlling electromagnetic fields, *Science*, 312 (2006), 1780–1782.
- [40] M. Xiao, X. Huang, J.-W. Dong, and C. T. Chan, On the time evolution of the cloaking effect of a metamaterial slab, *Opt. Lett.* 37 (2012), 4594–4596.