SPECTRAL THEORY OF REDUCIBLE NONNEGATIVE MATRICES: A GRAPH THEORETIC APPROACH

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After reviewing the classical Perron-Frobenius theory of irreducible matrices we turn to the reducible case and discuss it in terms of underlying graphs.

Graph of A

$$A \in \mathbb{R}^{nn}_{+},\ A \geq 0$$

$\mathcal{G}(A)$: *Graph* of A

Vertex set $\{1, \ldots, n\}$

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Vertex set $\{1, \ldots, n\}$

$$i \to j \quad : \quad a_{ij} > 0$$

$$i \xrightarrow{*} j \quad : \quad \exists (i_1, \ldots, i_k) \quad i \to i_1 \to \cdots \to i_k \to j$$
$$\text{or} \quad i = j$$

Irreducibility

A irreducible:

$\mathcal{G}(A)$ strongly connected ($\forall i, j, \quad i \xrightarrow{*} j$):

$\Longleftrightarrow$

NOT, after permutation similarity,

$$\begin{pmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{pmatrix}$$

with A_{11}, A_{22} square, really there

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$\iff$
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(0) irreducible

$$\rho(A) = \max\{|\lambda| \ : \ \lambda \in spec(A)\}$$

spectral radius of $A \in \mathbb{R}^{nn}$

Irreducible Perron-Frobenius

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Perron (1907, 1907) Frobenius (1908, 1909, 1912)

Theorem

$A \geq 0$, irreducible,
THEN

- $0 < \rho(A) \in \mathrm{spec}(A), \ (A \neq (0))$

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- $\exists$ unique $x, \; Ax = \rho x, \; \& \; x > 0$

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- $\rho(A)$ *simple eigenvalue*
- $\exists$ unique $x, \ Ax = \rho x, \ \& \ x > 0$
- *x is the only nonnegative evector*

Theorem

$A \geq 0$

THEN

- $\rho(A) \in \text{ spec}(A)$,

By continuity

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Much, much more may be said about reducible nonneg A

Frobenius Normal Form (FNF)

collect strong conn cpts of $\mathcal{G}(A)$

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After permutation similarity

$$A = \begin{bmatrix} A_{11} & 0 & \dots & \dots & 0 \\ A_{21} & A_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ A_{k1} & A_{k2} & \dots & \dots & A_{kk} \end{bmatrix}$$

each diagonal block irreducible

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$\mathcal{R}(A)$: *Reduced Graph* of A

Vertex set $\{1, \dots, k\}$ (classes) $\quad i \to j \iff A_{ij} \gneq 0$

i has access to j in $\mathbb{R}(A)$: $\quad i \stackrel{*}{\to} j$ in $\mathcal{R}(A)$

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i has access to j in $\mathbb{R}(A)$: $\quad i \xrightarrow{*} j$ in $\mathcal{R}(A)$

partial order of classes

Each vertex marked with its Perron root (spec rad)

Example

$$\begin{bmatrix} A_{11} & \cdot & \cdot & \cdot \\ 0 & A_{22} & \cdot & \cdot \\ A_{31} & A_{32} & A_{33} & \cdot \\ ? & ? & A_{43} & A_{44} \end{bmatrix}$$

Marked reduced graph

Each vertex marked with its Perron root (spec rad)

Example

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(ρ_1) $\quad$ (ρ_2)

\ $\quad$ /

(ρ_3)

|

(ρ_4)

$$\rho_i = \rho(A_{ii})$$

QUESTIONS

- Nonnegativity of eigenvectors
- Nonnegativity of generalized eigenvectors: $(A - \lambda I)^k x = 0$
- Nonnegativity of basis for generalized eigenspace for $\rho(A)$
- Nonnegativity of Jordan basis for ρ
- Relation of Jordan form to graph structure for ρ

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- Nonnegativity of Jordan basis for ρ
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We explore how the nonnegativity, combinatorial, spectral properties inter-relate, see e.g. LAA 84 (1986), 161 - 189.

Definition

Vertex i of is a $\mathcal{R}(A)$ is a *distinguished vertex* if

$$i \stackrel{*}{\leftarrow} j \quad \Longrightarrow \quad \rho_i > \rho_j$$

Theorem

Let A be a nonnegative matrix in FNF. Then the nonnegative eigenvectors of A correspond to the distinguish vertices of A:

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These are linearly independent, and for any part evalue, extremals of the cone of nonneg evectors. (Carlson 1963)

Example

$$\begin{bmatrix} A_{11} & \cdot & \cdot & \cdot \\ 0 & A_{22} & \cdot & \cdot \\ A_{31} & A_{32} & A_{33} & \cdot \\ ? & ? & A_{43} & A_{44} \end{bmatrix}$$

$$\rho_1 > \rho_3 = \rho_4 > \rho_2$$

$$(\rho_1)^{**} \qquad\qquad (\rho_2)$$

$$(\rho_3)^{*}$$

$$(\rho_4)^{**}$$

continuation

$$\rho_1 > \rho_3 = \rho_4 > \rho_2$$

$$(\rho_1)^{**} \qquad\qquad (\rho_2)$$

$$\diagdown \qquad\qquad \diagup$$

$$(\rho_3)^{*}$$

$$|$$

$$(\rho_4)^{**}$$

ρ_1	ρ_4
$+$	0
0	0
$+$	0
$+$	$+$

Warning! Nonnegative eigenvectors!

$$\begin{pmatrix} 0 & \cdot & \cdot \\ 0 & 0 & \cdot \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{matrix} (0) & & (0) \\ \backslash & & / \\ & (0) & \end{matrix}$$

Eigenvectors

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Reminder: Jordan Form

Jordan block (of size 4):

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

Theorem

Over the complex numbers, every matrix is similar to a direct sum of Jordan blocks.

$\mathrm{ind}_\lambda(A)$: = max size of J–block for λ
$= \min\{k \;:\; \mathcal{N} = \mathcal{N}(\lambda I - A)^{k+1} = \mathcal{N}(\lambda I - A)^k\}$

$\mathcal{N}$ – generalized nullspace of A

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$\mathcal{N}$ – generalized nullspace of A

Q: Does the red graph determine the J-form for ρ?

Special case 1: S 1952/56

Theorem

$A \in \mathbb{R}_+^{nn}$. TFAE:

(a) $dim(\mathcal{N}((A - \rho I)) = 1$

(a') *All Jordan block for ρ are size 1*

(b) *The ρ classes are trivially ordered.*

(a) & (a') are complex algebra
(b) is combinatorial

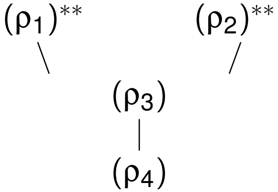
Example

$$\begin{bmatrix} A_{11} & \cdot & \cdot & \cdot \\ 0 & A_{22} & \cdot & \cdot \\ A_{31} & A_{32} & A_{33} & \cdot \\ ? & ? & A_{43} & A_{44} \end{bmatrix}$$

$$\begin{array}{ccc} (\rho_1)^{**} & & (\rho_2)^{**} \\ \backslash & & / \\ & (\rho_3) & \\ & | & \\ & (\rho_4) & \end{array}$$

$$(\rho =)\ \rho_1 \ = \rho_2 \ > \rho_3 \ = \rho_4$$

J-form for ρ is $(1, 1)$



$$\begin{matrix} + & 0 \\ 0 & + \\ + & + \\ + & + \end{matrix}$$

These are the *only* evecs for ρ

Theorem

$A \in \mathbb{R}^{nn}_{+}$. TFAE:

(a) *dim null$(A - \rho I) = \mathrm{mult}_{\rho}(A)$*

(a') *There is only one Jordan block for ρ*

(b) *The ρ classes are linearly ordered.*

Example

$$(\rho_1)^{**} \qquad\qquad (\rho_2)$$

$$\diagdown \qquad\qquad \diagup$$

$$(\rho_3)$$

$$|$$

$$(\rho_4)^{**}$$

$$(\rho =)\ \rho_1 \ = \rho_4 > \rho_2 \ = \rho_3$$

x	z
$+$	0
0	0
$+$	0
$+$	$+$

$$(\rho I - A)x = z, \quad (\rho I - A)z = 0$$

$$(\rho_1)^{**} \qquad\qquad (\rho_2)$$

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x	z
$+$	0
0	0
$+$	0
$+$	$+$

$$(\rho I - A)x = z, \quad (\rho I - A)z = 0$$

J-form form for ρ is (2)

General case??

Example that stopped me in 1952

$$\begin{bmatrix} 0 & \cdot & \cdot & \cdot \\ . & 0 & \cdot & \cdot \\ 1 & 1 & 0 & \cdot \\ a & 1 & \cdot & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & \cdot & \cdot & \cdot \\ . & 0 & \cdot & \cdot \\ 1 & 1 & 0 & \cdot \\ a & 1 & \cdot & 0 \end{bmatrix}$$

$$\begin{matrix} 0 & & 0 \\ & \times & \\ 0 & & 0 \end{matrix}$$

Jordan form

$a \neq 1 \quad (2,2)$

$a = 1 \quad (2,1,1)$

General case??

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Jordan form

$a \neq 1 \quad (2,2)$

$a = 1 \quad (2,1,1)$

Hershkowitz-S (1991)

"Solved" the problem using majorization

$\text{ind}_\rho(A)$: = max size of J–block for A
= $\min\{k : \mathcal{N} = \mathcal{N}(\rho I - A)^{k+1} = \mathcal{N}(\rho I - A)^k\}$

Theorem

ind_ρ = max length of chain of ρ classes

Example

$$\begin{bmatrix} 0 & \cdot & \cdot & \cdot \\ . & 0 & \cdot & \cdot \\ 1 & 1 & 0 & \cdot \\ a & 1 & \cdot & 0 \end{bmatrix}$$

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$$\begin{matrix} 0 & & 0 \\ & \times & \\ 0 & & 0 \end{matrix}$$

max chain of 0 classes = 2

Example

$$\begin{bmatrix} 0 & \cdot & \cdot & \cdot \\ . & 0 & \cdot & \cdot \\ 1 & 1 & 0 & \cdot \\ a & 1 & \cdot & 0 \end{bmatrix}$$

$$\begin{matrix} 0 & & 0 \\ & \times & \\ 0 & & 0 \end{matrix}$$

max chain of 0 classes = 2

Jordan form:
either $(2,2)$ or $(2,1,1)$
either case $\text{ind}_0 = 2$

generalized eigenvectors

x a gen evector of A for λ

$$(A - \lambda I)^r x = 0, \; r > 0$$

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$$\mathcal{N}_\lambda(A) := \{x : (A - \lambda I)^r x = 0, \; r \geq n\}$$

x a gen evector of A for λ

$$(A-\lambda I)^r x = 0, \; r > 0$$

$$\mathcal{N}_\lambda(A) := \{x : (A-\lambda I)^r x = 0, \; r \geq n\}$$

i is a *semi-distinguished vertex*:

$$i \stackrel{*}{\leftarrow} j \quad \Longrightarrow \quad \rho_i \geq \rho_j$$

Rothblum(1975), Richman-S(1978), Hershkowitz-S(1988)

Theorem

Let $\lambda \geq 0$. Suppose the semi-dist vertices of A with $\rho_i = \lambda$ are $i_1 < \ldots < i_s$. Then there exist x^p, $p = 1, \ldots, s$ in $\mathcal{N}(A)$ such that

$$\begin{array}{ll} x_j^p > 0 & \text{if} \quad i_p \stackrel{*}{\leftarrow} j \\ x_j^p = 0 & \text{otherwise} \end{array}$$

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$$\begin{array}{ll} x_j^p > 0 & \text{if} \quad i_p \stackrel{*}{\leftarrow} j \\ x_j^p = 0 & \quad \text{otherwise} \end{array}$$

and such that

$$(A - \lambda I)x^p = \sum_q c_{pq} x^q$$

where

$$\begin{array}{ll} c_{pq} > 0 & \text{if} \quad i_p \stackrel{*}{\leftarrow} i_q, \ q \neq p \\ c_{pq} = 0 & \quad \text{otherwise} \end{array}$$

Example

$$\begin{bmatrix} 0 & \cdot & \cdot & \cdot \\ . & 0 & \cdot & \cdot \\ 1 & 1 & 0 & \cdot \\ 1 & 1 & \cdot & 0 \end{bmatrix}$$

Example

$$\begin{bmatrix} 0 & . & . & . \\ . & 0 & . & . \\ 1 & 1 & 0 & . \\ 1 & 1 & . & 0 \end{bmatrix}$$

$$0^* \qquad 0^*$$

$$\times$$

$$0^{**} \qquad 0^{**}$$

Example

$$\begin{bmatrix} 0 & \cdot & \cdot & \cdot \\ . & 0 & \cdot & \cdot \\ 1 & 1 & 0 & \cdot \\ 1 & 1 & \cdot & 0 \end{bmatrix}$$

$$0^* \qquad 0^*$$

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x^1	x^2	x^3	x^4
1	0	0	0
0	1	0	0
1	1	1	0
1	1	0	1

Example

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$$Ax^1 = Ax^2 = x^3 + x^4$$

$$Ax^3 = Ax^4 = 0$$

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$$Ax^1 = Ax^2 = x^3 + x^4$$

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$$Ax^3 = Ax^4 = 0$$

These vectors span $\mathcal{N}_0$ but are not lin indep

Rothblum(1975)

Theorem

The gen null space for $\rho(A)$ has a nonneg basis

proofs?

By Frobenius tracedown method:
Solve successively equations for $x_i \geq 0$ of the form

$$(A_{ii} - \rho_i I_{ii}) x_i = b_i$$

where $b_i \geq 0$.

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where $b_i \geq 0$.

Carlson1963
$Ax + b = \rho x$ given reducible $A \geq 0$ and $b \geq 0$.

H.S The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and related properties: A survey, Lin. Alg. Appl. 84 (1986), 161-189.

D. Hershkowitz and H.S, On the existence of matrices with prescribed height and level characteristics, Israel Math J. 75 (1991), 105-117.

two references

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THANK YOU