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SPECTRAL THEORY OF THE LINEARIZED VLASOV-POISSON EQUATION

BY

PIERRE DEGOND

ABSTRACT. We study the spectral theory of the linearized Vlasov-Poisson equation, in order to prove that its solution behaves, for large times, like a sum of plane waves. To obtain such an expansion involving damped waves, we must find an analytical extension of the resolvent of the equation. Then, the poles of this extension are no longer eigenvalues and must be interpreted as eigenmodes, associated to "generalized eigenfunctions" which actually are linear functionals on a Banach space of analytic functions.

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Introduction. This paper is concerned with the linearized Vlasov-Poisson equation

$$(1) \qquad \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial x} f_0'(v) = 0; \quad \frac{\partial^2 \phi}{\partial x^2} = \int f(v, v, t)\, dv; \quad f\,|_{t=0} = g.$$

This equation describes the behaviour of the electron distribution function in a plasma: given a uniform steady distribution $f_0(v)$ and an initial perturbation $g(x, v)$, the equation describes the evolution of this perturbation $f(x, v, t)$ under the action of the electrostatic potential $\phi(x, t)$, generated by the charges of the electrons.

This work (cf. [**3**]) is an attempt at a mathematical explanation of Landau damping in terms of eigenmodes and scattering theory. Landau damping arose when physicists solved equation (1) by means of a Fourier-Laplace transform (cf. [**1**, Chapter 8]) and wondered whether the potential ϕ behaved like plane waves when t is large. They found that there is no way to exhibit damped plane waves except by providing an analytical extension of the Laplace transform of f, which is only possible by assuming that f_0 and g are analytic with respect to v. Although very strong, these hypotheses are verified in numerous physical situations for instance in the case of a Maxwellian distribution: $f_0(v) = \exp(-v^2)$.

In this paper we show that the potential $\phi(x, t)$ actually admits the expansion

$$(2) \qquad \phi(x, t) = \sum_{s=1}^{S} c_s e^{\lambda_s t + i n_s x} + \mathcal{O}(e^{rt}), \qquad r < \operatorname*{Min}_{1 \leqslant s \leqslant S} (\operatorname{Re} \lambda_s),$$

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in which r can be negative, and we proceed as follows:

Equation (1), which, for convenience, we write as

$$\dot{f} = T \cdot f; \quad f(0) = g, \tag{3}$$

is solved by semigroup theory: $f(t) = \exp(tT) \cdot g$. The Dunford formula [**4**] then relates it to the resolvent $R_\lambda = (\lambda - T)^{-1}$. A deformation of the path of integration in the Dunford formula allows us to give an asymptotic expansion for f:

$$f(x, v, t) = \sum_{s=1}^{S} \alpha_s(v) e^{\lambda_s t + i n_s x} + \mathcal{O}(e^{rt}), \qquad r < \min_{1 \leqslant s \leqslant S} (\operatorname{Re} \lambda_s), \tag{4}$$

where r is positive, λ_s are eigenvalues of T, and $\alpha_s(v)$ are well-defined functions of v.

However, the continuous spectrum of T forbids us to use eigenvalues λ_s such that $\operatorname{Re} \lambda_s < 0$, and formula (4) only involves unstable modes. In order to get rid of this restriction, we construct an analytical extension $\tilde{R}_\lambda$ of the resolvent across the continuous spectrum; but we must then choose the initial data g in a Banach space Γ of analytic functions and consider $\tilde{R}_\lambda g$ as a linear functional on Γ. Therefore, we can obtain an expansion (4) involving damped waves (i.e. r may be negative) having the remarkable properties that the λ_s for $\operatorname{Re} \lambda_s < 0$ are no longer eigenvalues of T, and that the corresponding $\alpha_s(v)$ are linear functionals on Γ. These λ_s may be interpreted as eigenmodes, thanks to a suitable generalization of T to Γ', the dual of Γ.

We try then to explain the appearance of such elements of Γ' by means of a partial Fourier transform with respect to v. Indeed, the transform of some elements φ of Γ' can be defined as a genuine function $\check{\varphi}(\xi)$ growing exponentially fast at infinity.[1] Thus, from formula (4), we obtain

$$\check{f}(x, \xi, t) = \sum_{s=1}^{S} \check{\alpha}_s(\xi) e^{\lambda_s t + i n_s x} + \mathcal{O}(e^{rt}). \tag{5}$$

Because of the exponential growth of $\check{\alpha}_s(\xi)$ for $\operatorname{Re} \lambda_s < 0$, expansion (5) is valid uniformly for ξ in an arbitrary compact subset of $\mathbf{R}$.

This behaviour may be compared to that of the wave equation outside an obstacle with no trapped rays. In [**2**], Lax and Phillips obtain for the solution of such an equation, the expansion, uniform for $|y| \leqslant R$, with arbitrary R,

$$u(y, t) = \sum_{s=1}^{S} e^{\lambda_s t} W_s(y) + \mathcal{O}(e^{rt}), \qquad r < \min_{1 \leqslant s \leqslant S} (\operatorname{Re} \lambda_s), \tag{6}$$

where $W_s(y)$ are exponentially growing functions. This analogy will be developed in the conclusion.

Landau damping is examined in [**1**, Chapter 8]. But beside this approach, another successful theory has been developed by Van Kampen [**8**] and Case [**9**], using a "normal mode expansion". The link between the two theories has been established by Trocheris [**10**] in an interesting paper, where he uses linear functionals on spaces of analytic functions. In [**3**] we try to relate the normal mode expansion to the Dunford formula.

[1] The notation $\hat{\varphi}$ will be used for the partial Fourier transform, with respect to the space variable.

1. Solution and spectral theory of Vlasov-Poisson equations in a 1-dimensional periodic domain.

1.1. *The equations and the associated semigroup*. Let $x \in X = [0, 2\pi]$ be the spatial coordinate, $v \in \mathbf{R}$ the velocity, and $t > 0$, the time. Let $f_0(v)$ be the steady state of the plasma, and let $F_0(v) = f_0'(v)$ be its derivative which is a continuous, real-valued function, verifying the following hypotheses:

H1: $F_0 \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$;

H2: F_0 is Hölder continuous in the neighbourhood of every point of $\mathbf{R}$.

We seek a pair of functions $f(x, v, t)$ and $\phi(x, t)$ solutions of the Cauchy boundary value problem:

(7)

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial x}F_0(v) = 0, \quad \frac{\partial^2 \phi}{\partial x^2} = \int_{-\infty}^{+\infty} f(x, v, t)\, dv, \quad f(x, v, 0) = g(x, v).$$

(8) $$f(0, v, t) = f(2\pi, v, t), \quad \phi(0, t) = \phi(2\pi, t) = 0, \quad \frac{\partial \phi}{\partial x}(0, t) = \frac{\partial \phi}{\partial x}(2\pi, t).$$

The periodic boundary conditions (8) imply the conservation of charge $\iint_{X\times\mathbf{R}} f(x, v, t)\, dx\, dv = 0, \forall t \geqslant 0$.

Now, denote by $\mathscr{A}$ the Banach space $\mathscr{A} = \{\varphi \in L^1(x \times \mathbf{R}) \mid \iint \varphi(x, v)\, dx\, dv = 0\}$ equipped with the L^1 norm.

A is the unbounded operator such that $A\varphi = -v(\partial\varphi/\partial x)$ with domain

$$D(A) = \{\varphi \in \mathscr{A} \mid v(\partial\varphi/\partial x) \in \mathscr{A},\ \varphi(0, v) = \varphi(2\pi, v)\text{a.e.}\}.$$

K is the compact bounded operator of $\mathscr{A}$, defined by

$$K\varphi = -d\phi/dx \cdot F_0,$$

where ϕ is given by

$$\frac{d^2\phi}{dx^2} = \int_{-\infty}^{+\infty} \varphi(x, v)\, dv, \quad \phi(0) = \phi(2\pi) = 0, \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi).$$

Finally $T = A + K$ is the unbounded operator associated with equation (7). The following property results from a classical perturbation theorem [**5**, p. 497] and allows us to solve equations (7) and (8).

PROPOSITION 1.1. (a) *A generates a strongly continuous group of isometries of $\mathscr{A}$ given by*

$$\exp(tA) \cdot \varphi(x, v) = \varphi_*(x - vt, v), \quad \forall\varphi \in \mathscr{A},$$

where φ_* denotes the periodic extension of φ to $\mathbf{R}$:

$$\varphi_*(x, v) = \varphi(x - 2k\pi, v), \quad \textit{if } x \in]2k\pi, 2(k+1)\pi[.$$

(b) *T generates a strongly continuous group on $\mathscr{A}$.*

REMARK 1.1. There are two kinds of physically reasonable boundary conditions for the Vlasov-Poisson equations. The first one consists of choosing $x \in \mathbf{R}$ and imposing some decay at infinity. The second one is the periodic condition (8). Both allow the use of Fourier analysis in order to simplify the computations, but the latter gives a better framework to study the existence of solutions, and the spectral theory, essentially because T is a compact perturbation of the transport operator A. □

1.2. *Spectrum and resolvent of T.* The plasma dispersion function $D(n, \lambda)$ appears naturally in the Fourier analysis of our equation. It will be defined as follows:

DEFINITION 1.1. *For $\lambda \in C \setminus i\mathbf{R}$ and $n \in \mathbf{Z}^*$, then*

$$D(n, \lambda) = 1 - \frac{i}{n} \int_{-\infty}^{+\infty} \frac{F_0(v)\, dv}{\lambda + inv}. \tag{9}$$

Formula (9) *still has a meaning for pairs $(i\eta, n)$ in $i\mathbf{R} \times \mathbf{Z}^*$ such that $F_0(-\eta/n) = 0$, and defines $D(n, i\eta)$.*

The following lemma is straightforward.

LEMMA 1.1. (a) $\lim_{|n| \to +\infty} D(n, \lambda) = 1$ *uniformly for λ in $S_\alpha = \{\lambda \in C \mid |\mathrm{Re}\, \lambda| \geqslant \alpha\}$, where α is arbitrary positive.*

(b) *For any fixed $n \in \mathbf{Z}^*$, $\lim_{\lambda \to \infty; \lambda \in S_\alpha} D(n, \lambda) = 1$.*

(c) *Let λ_0 in $C \setminus i\mathbf{R}$ be given such that there is no integer n verifying $D(n, \lambda_0) = 0$. Then, for λ in a small enough neighbourhood of λ_0, one has*

$$|D(n, \lambda)| \geqslant D_0 > 0, \quad \forall n \in \mathbf{Z}^*. \tag{10}$$

Formula (10) *is also valid for λ in a neighbourhood of ∞ in S_α.*

As T generates a group, its spectrum $\sigma(T)$ is contained in a strip $\{\lambda \in C \mid |\mathrm{Re}\, \lambda| \leqslant \omega_0\}$. We denote, respectively, by $\sigma_p(T)$, $\sigma_R(T)$, $\sigma_C(T)$, the point, residual, and continous spectra of T.

THEOREM 1.1. *For φ in $\mathscr{A}$, let*

$$\hat{\varphi}_n(v) = (2\pi)^{-1} \int_X e^{-inx} \varphi(x, v)\, dx$$

be its nth Fourier coefficient.

(a) *For λ not in the spectrum of T, the resolvent $R_\lambda = (\lambda - T)^{-1}$ satisfies $\widehat{(R_\lambda g)}_0 = \hat{g}_0/\lambda$ and*

$$\widehat{(R_\lambda g)}_n = \frac{\hat{g}_n(v)}{\lambda + inv} + \frac{i}{nD(n, \lambda)} \frac{F_0(v)}{\lambda + inv} \int_{-\infty}^{+\infty} \frac{\hat{g}_n(w)\, dw}{\lambda + inw}, \quad \text{if } n \neq 0. \tag{11}$$

(b) *The spectrum of T is given by*

$$\sigma(T) = i\mathbf{R} \cup \{\lambda \in C \mid \exists n \in \mathbf{Z}^*;\ D(n, \lambda) = 0\}.$$

It is symmetric with respect to the real and imaginary axes, and its intersection with any S_α is a finite set.

(c) *λ belongs to $\sigma_p(T)$ if and only if $\lambda = 0$ or there exists n in $\mathbf{Z}^*$ such that $D(n, \lambda) = 0$. Moreover, if $\mathrm{Re}\, \lambda \neq 0$, the set of solutions n of the equation $D(n, \lambda) = 0$ is finite and denoted by $\{n_1, \dots, n_p\}$. Then a basis of the eigenspace $E_\lambda(T)$ is given by*

$$\{\exp(in_j x)\varphi_j(v),\ j = 1, \dots, p\}, \quad \textit{where } \varphi_j(v) = F_0(v)/(\lambda + in_j v),$$

(d) $\sigma_R(T) = \sigma_p(T) \setminus i\mathbf{R}^*$; $\sigma_C(T) = i\mathbf{R}^*$.

REMARK 1.2. Figure 1 shows a possible configuration of $\sigma(T)$. Of course, it depends on f_0 and for instance in [**1**, p. 445], it is proved that for a Maxwellian $f_0(v) = \exp(-v^2)$, one has $\sigma_P(T) = \sigma_R(T) = \{0\}$, $\sigma_C(T) = i\mathbf{R}^*$. $\square$

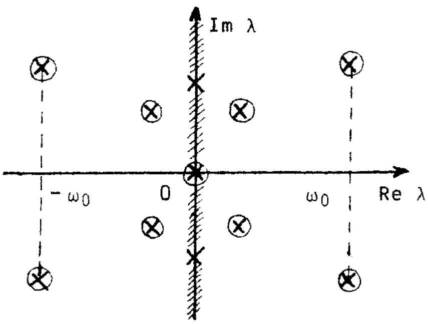


FIGURE 1. Spectrum of T. ($\times$ = Point spectrum; $\bigcirc$ = Residual spectrum; ▨ = Continuous spectrum; ω_0 = type of $\exp(tT)$.)

PROOF OF THEOREM 1.1. (a) Let λ be such that $\operatorname{Re}\lambda \neq 0$ and that $D(n, \lambda) \neq 0$ for every integer n in $\mathbf{Z}^*$, and let g be in $\mathscr{A}$. We look for a unique solution φ of the spectral equations

$$\lambda\varphi + v \cdot \frac{\partial \varphi}{\partial x} + \frac{d\phi}{dx} \cdot F_0 = g, \tag{12}$$

$$\frac{d^2\phi}{dx^2} = \int \varphi(x, v)\, dv, \quad \phi(0) = \phi(2\pi) = 0, \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi).$$

Taking Fourier coefficients of (12) we get

$$(\lambda + inv)\hat{\varphi}_n + in\hat{\phi}_n F_0 = \hat{g}_n \quad \text{and} \quad -n^2\hat{\phi}_n = \int \hat{\varphi}_n(v)\, dv. \tag{13}$$

So, we obtain $\hat{\varphi}_0 = \hat{g}_0/\lambda$ and, for $n \neq 0$, we divide by $(\lambda + inv)$ and integrate with respect to v. Therefore

$$D(n, \lambda) \int \hat{\varphi}_n(v)\, dv = \int \frac{\hat{g}_n(v)\, dv}{\lambda + inv}.$$

As $D(n, \lambda) \neq 0$, we can then obtain $\hat{\phi}_n$, and replacing it in (13), we get equation (11) for $\hat{\varphi}_n$.

Conversely we must now show that formula (11) defines a function of $D(A)$ verifying equations (12). But $\hat{g}_n(v)/(\lambda + inv)$ are the Fourier coefficients of the resolvent $\rho_\lambda g = (\lambda - A)^{-1} g$, which is well defined in $\mathscr{A}$ for $\operatorname{Re}\lambda \neq 0$, because A generates a group of isometries. The other term,

$$\left(\widehat{\alpha_\lambda g}\right)_n(v) = \frac{i}{nD(n, \lambda)} \frac{F_0(v)}{\lambda + inv} \int \frac{\hat{g}_n(w)\, dw}{\lambda + inw}, \qquad n \neq 0,$$

defines an absolutely convergent Fourier series in $\mathscr{A}$, as the expression

$$\iint_{X \times \mathbf{R}} \left| \left(\widehat{\alpha_\lambda g}\right)_n e^{inx} \right| dx\, dv \leqslant \frac{1}{nD_0(\lambda)} \|F_0\|_{L^2} \left(\frac{\pi}{n \operatorname{Re}\lambda} \right)^{1/2} \|g\|_{L^1} \cdot \frac{1}{2\pi \operatorname{Re}\lambda}$$

is of order $n^{-3/2}$ (use Lemma 1.1). This completes the proof of point (a).

(b) Formula (11) shows that $(\widehat{R_\lambda g})_n$ is no longer defined when $D(n, \lambda) = 0$, and this provides point spectrum (cf. (c)). On the other hand, when $i\lambda$ is real, $(\widehat{R_\lambda g})_n$ is not an integrable function of v, and this gives continuous spectrum (cf. (d)). So, one has

$$\sigma(T) = i\mathbf{R} \cup \{\lambda \in C \mid \exists n \in \mathbf{Z}^*;\ D(n, \lambda) = 0\}.$$

Separating real and imaginary parts, the equation $D(n, \lambda) = 0$ becomes, for $\lambda = \xi + in$,

$$\int \frac{F_0(v)\,dv}{\xi^2 + (\eta + nv)^2} = 0 \quad \text{and} \quad \int \frac{vF_0(v)\,dv}{\xi^2 + (\eta + nv)^2} = 1.$$

These equations are invariant under the transformations $\xi \to -\xi$ and $(\eta, n) \to (-\eta, -n)$, so that $\sigma(T)$ is symmetric with respect to the real and the imaginary axes.

On the other hand, $\sigma(T) \cap S_\alpha$ is finite, thanks to Lemma 1.1 and to the analyticity of D with respect to λ.

(c) Take an eigenvalue λ, such that $\operatorname{Re}\lambda \neq 0$, with corresponding eigenvector φ. The same computations as above lead to $\hat{\varphi}_0 = 0$ and $(\lambda + inv)\hat{\varphi}_n + in\hat{\phi}_n F_0 = 0$; $D(n, \lambda)\hat{\phi}_n = 0$, if $n \neq 0$.

As φ is not zero, not every $\hat{\varphi}_n$ can be zero either, which implies that $D(n, \lambda) = 0$ for some n. By virtue of Lemma 1.1(b), this equation only admits a finite number of solutions, $n_1, \dots, n_p$, so that φ reads, with arbitrary C_j's

$$\varphi(x, v) = \sum_{j=1}^{p} \frac{C_j \exp(in_j x) F_0(v)}{\lambda + in_j v}.$$

0 is an eigenvalue, because every odd integrable function of v is in the null space of T.

If $\lambda = i\eta$, $\eta \in \mathbf{R}^*$, the integrability of φ with respect to v implies that $F_0(-\eta/n) = 0$ for any n such that $\hat{\varphi}_n \neq 0$. This implies that $D(n, i\eta)$ is well defined and allows the same computations as in the case $\operatorname{Re}\lambda \neq 0$.

(d) The elements of $\sigma_R(T)$ are the eigenvalues of the adjoint T^* of T, which is defined on $\mathscr{A}^* = L^\infty(X \times \mathbf{R})/\mathbf{R}$ (space of bounded functions defined up to an additive constant) by

$$T^*\theta = v\frac{\partial \theta}{\partial x} - \chi_\theta(x),$$

where

$$\chi_\theta(x) = -\int_0^x \int_{-\infty}^{+\infty} F_0(w)\theta(y, w)\,dy\,dw + \frac{x}{2\pi}\iint_{X \times \mathbf{R}} F_0(w)\theta(y, w)\,dy\,dw.$$

Then computations similar to those of part (c) yield the result.

From what precedes, in order to get $\sigma_C(T) = i\mathbf{R}^*$, it suffices to prove that $i\mathbf{R}^* \subset \sigma(T)$. Now it is easily shown that the equation

$$(i\eta - T)\varphi = \exp(ix - v^2)/(\eta + v)^{1/2}$$

has no solution $\varphi \in \mathscr{A}$, proving that $i\eta \in \sigma(T)$.

1.3. *Asymptotic behaviour of* $\exp(tT)$.

Simplifying hypotheses 1.1. *We assume that for each* $n \in \mathbf{Z}^*$, *the zeroes of the analytic function* $\lambda \to D(n, \lambda)$ $(\operatorname{Re}\lambda \neq 0)$ *are simple. This implies that the Laurent development of* R_λ *around an eigenvalue* λ_0 *is* (*with the notations of Theorem* 1.1(a))

$$R_\lambda g = \frac{1}{\lambda - \lambda_0}\left\{\sum_{j=1}^{p} \langle C_j, g\rangle \exp(in_j x)\varphi_j(v)\right\} + O(1) \tag{14}$$

where

$$\langle C_j, g\rangle = \frac{i}{n_j}\left(\frac{\partial D}{\partial \lambda}\right)^{-1}(n_j, \lambda_0)\cdot\int \frac{\hat{g}_{n_j}(w)\,dw}{\lambda_0 + in_j w}. \tag{15}$$

In the general case (detailed in [3]), one can link the multiplicity of a zero of $D(n, \cdot)$ to the algebraic multiplicity of the corresponding eigenvalue.

PROPOSITION 1.2. *Let r be a real number such that $0 < r < \omega_0$, and that no eigenvalue of T lies on the line $\operatorname{Re}\lambda = r$. Let λ_s $(s = 1, \dots, S)$ be the eigenvalues of T such that $r < \operatorname{Re}\lambda_s \leqslant \omega_0$, which are in finite number by virtue of Theorem* 1.1(d). *Denote by the superscript s the quantities p, n_j, C_j associated with λ_s in formulae* (14) *and* (15). *Then, the following expansion is valid, in the L^1 norm, for $g \in D(A)$, when t goes to ∞ (cf. Figure* 2):

$$e^{tT}g = \sum_{s=1}^{S}\sum_{j=1}^{P_s} e^{\lambda_s t + in_j^s x}\langle C_j^s, g\rangle \frac{F_0(v)}{\lambda_s + in_j^s v} + O(e^{rt}). \tag{16}$$

PROOF. By the theorem of separation of the spectrum (cf. [5, p. 178]), one can split T into its bounded projection T_1 associated with the compact subset of $\sigma(T)$: $\Sigma = \{\lambda_s, s = 1, \dots, S\}$, and an unbounded remainder T_2. Then, the Dunford formula applied to T_1 gives

$$e^{tT}g = \sum_{s=1}^{S} \operatorname{Res}\left(e^{\lambda t}R_\lambda g, \lambda_s\right) + e^{tT_2}g.$$

By a classical estimate, we get $\exp(tT_2)g = O(e^{rt})$ in $\mathscr{A}$, and then expansion (14) leads to the result. □

REMARK 1.3. The expansion (16) only involves unstable waves, because $\operatorname{Re}\lambda_s > 0$. Moreover, in the case of the Maxwellian f_0, Corollary 1.1 does not give any indication by virtue of Remark 1.2.

In fact, the continuous spectrum appears as a barrier which prevents us from picking up the contributions of the eigenvalues λ such that $\operatorname{Re}\lambda < 0$. In order to overcome this difficulty, one must extend the resolvent R_λ across the imaginary axis.

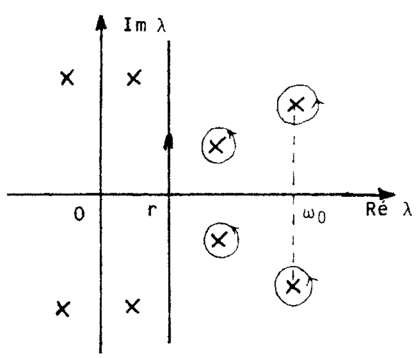


FIGURE 2. Eigenvalues used in expansion (15) are encircled.

To this purpose, we refer to formula (11), with $n = 1$, for simplicity, and we try to extend the term $\hat{g}_1(v)/(\lambda + iv)$ or for instance $G_\lambda(v) = 1/(\lambda + iv)$ $(\operatorname{Re}\lambda > 0)$. Then, if we define

$$G_{i\eta}(v) = \frac{1}{i}\,\mathrm{P.V.}\left(\frac{1}{v+\eta}\right) + \pi\delta(v+\eta).$$

We have the limit in the sense of distributions

$$G_\lambda \overset{\mathcal{D}}{\to} G_{i\eta} \quad \text{as } \lambda \to i\eta,\ \operatorname{Re}\lambda > 0$$

(P.V. denoting the Cauchy principal value). Moreover, if h is a regular function, then

$$\langle G_\lambda, h\rangle = \int \frac{h(v)\,dv}{\lambda + iv} \qquad (\operatorname{Re}\lambda > 0)$$

is the Hilbert transform of h. It is well known [**6**] that this function may be extended into an analytic function on the half-plane $\operatorname{Re}\lambda > -\alpha$ $(\alpha > 0)$, if one assumes that $h(v)$ is analytic in a strip $-\alpha < \operatorname{Im} v < 0$.

So G_λ and consequently $\tilde{R}_\lambda g$, must now be understood as linear functionals on a space of analytic functions. This is shown in detail in the next chapter. □

2. Analytical extension of the resolvent.

2.1. *The spaces G and* Γ. Let $Q_\alpha = \{z \in C \mid |\operatorname{Im} z| \leqslant \alpha\}$, $\Pi_\alpha = \{z \in C \mid \operatorname{Re} z > -\alpha\}$ where $\alpha > 0$ is fixed. The restriction of a function defined on Q_α, to the line $\operatorname{Im} z = \eta$ $(\eta \in [-\alpha, \alpha])$, will be denoted by f_η.

DEFINITION 2.1. *G (resp. Γ) is the Banach space of analytic functions f: $Q_\alpha \to C$ (resp. $L^2(X)$) such that*

$$\|f\|_{G\ (resp.\,\Gamma)} = \sup_{\eta \in [-\alpha, +\alpha]} \|f_\eta\|_{L^2(\mathbf{R})\ (resp.\ L^2(X\times\mathbf{R}))} < +\infty.$$

These well-known Hardy type spaces are studied in [**7**, Chapter 5]. In particular G is a closed subspace of the Banach space $C^0([-\alpha, +\alpha], L^2(\mathbf{R}))$, which answers the question of the definition of the boundary values f_α and $f_{-\alpha}$. From [**7**], we also

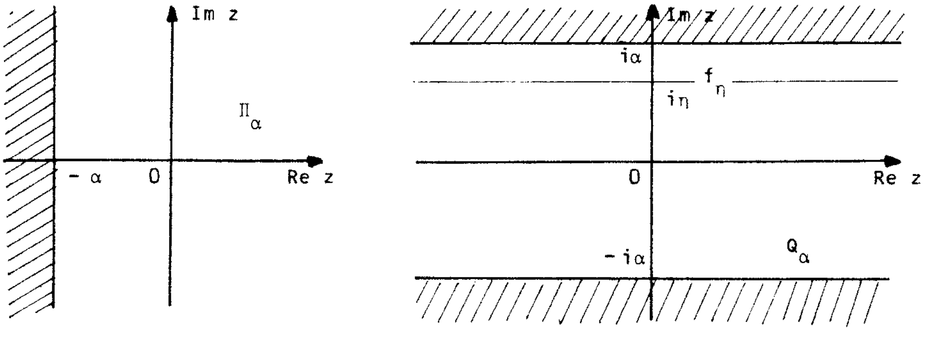


FIGURE 3. Π_α and Q_α

deduce the

PROPOSITION 2.1. *Let $f \in G$. Then the Hilbert transforms $H^+f(z)$ and $H^-f(z)$ defined respectively for* $\operatorname{Im} z > 0$ *and* $\operatorname{Im} z < 0$ *by the formula* $\int f(v)/(v-z)\,dv$ *may be analytically extended respectively to* $\operatorname{Im} z > -\alpha$ *and* $\operatorname{Im} z < \alpha$, *thanks to formulas*

$$H^+f(z) = \int_{-i\alpha-\infty}^{-i\alpha+\infty} \frac{f(\zeta)\,d\zeta}{\zeta - z} \quad \text{and} \quad H^-f(z) = \int_{i\alpha-\infty}^{i\alpha+\infty} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

and we have

$$(17) \qquad \forall z \in Q_\alpha, \quad f(z) = \frac{1}{2i\pi}\left(H^+f(z) - H^-f(z)\right) \quad (\textit{Plemelj formula}).$$

The dual spaces will play a fundamental part. In order to avoid conjugates in the Parseval formula, we define the duality in $L^2(X \times \mathbf{R})$ by

$$\langle g, h\rangle = \iint_{X\times\mathbf{R}} g(x,v)h(-x,v)\,dx\,dv.$$

Then

PROPOSITION 2.2. (a) *The restriction, to the real axis, of a function f of G or Γ defines continuous injections $J\colon \Gamma \to L^2(X\times\mathbf{R})$ or $J\colon G \to L^2(\mathbf{R})$.*

(b) *The mapping $J\colon L^2(X\times\mathbf{R}) \to \Gamma'$ defined by*

$$\langle Jg, h\rangle = \iint_{X\times\mathbf{R}} g(x,v)h(-x,v)\,dx\,dv, \quad \forall g \in L^2(X\times\mathbf{R}), \forall h \in \Gamma,$$

is a continuous injection. (A similar property is true for $J\colon L^2(\mathbf{R}) \to G'$.)

PROOF. (a) Follows from the analytical extension principle.

(b) Let $g \in L^2(X\times\mathbf{R})$ be such that $Jg = 0$. Take $h(x,z) = \varphi(-x)/(z - z_0)$, with $\varphi \in L^2(X)$ and $|\operatorname{Im} z_0| > \alpha$.

Then, $\langle Jg, h\rangle = 0$ means that

$$\int_{\mathbf{R}} (v-z)^{-1}\left[\int_X \varphi(x)g(x,v)\,dx\right]dv = 0, \quad \forall z \text{ such that } |\operatorname{Im} z| > \alpha$$

and by the analytical extension principle, the same result holds for all z. This means that the Hilbert transform of the L^2-function $\int_X \varphi(x)g(x,v)\,dx$ vanishes identically so that [**6**]

$$\int_X \varphi(x)g(x,v)\,dx = 0, \quad \forall \varphi \in L^2(X).$$

Hence $g = 0$. $\square$

We do not try to characterize the elements of Γ', because we shall only use the most regular ones, namely those g^* which can be written

$$(18) \qquad \langle g^*, h\rangle = \sum_{j=1}^{m} \iint_{X\times\mathbf{R}} g_j(x,v)h_{\eta_j}(-x,v)\,dx\,dv, \quad \forall h \in \Gamma$$

where $m \in \mathbf{N}$, $g_j \in L^2(X\times\mathbf{R})$, and $\eta_j \in [-\alpha, +\alpha]$ for $j = 1,\ldots,m$.

We call Γ_1' the subset of all functionals of type (18) and G_1', the analogous with respect to G'. For instance, formula (17) shows that the Dirac mass δ_z defined for $z \in Q_\alpha$ by $\langle \delta_z, f \rangle = f(z)$ belongs to G_1'.

Finally, we notice the following obvious facts, relative to the Fourier analysis of Γ and Γ'.

LEMMA 2.1. *If $f \in \Gamma$ then $\hat{f}_n \in G$, and $\|\hat{f}_n\|_G \leqslant \|f\|_\Gamma/\sqrt{2\pi}$. Moreover $(\widehat{f_\eta})_n = (\hat{f}_n)_\eta$, which we denote by $\hat{f}_{\eta,n}$.*

However, the Fourier series $\sum \hat{f}_n \exp(inx)$ do not generally converge in Γ.

DEFINITION 2.2. *For $g^* \in G'$, we denote by $\exp(inx)g^* \in \Gamma'$ the functional $\langle \exp(inx)g^*, h\rangle = 2\pi\langle g^* \hat{h}_n\rangle, \forall h \in \Gamma$.*

This definition coincides with the usual one when g^* is a genuine function (namely when g^* is in the range of J).

2.2. *The analytical extension of the resolvent $\tilde{R}_\lambda$.* In order to bring out the main difficulties, we begin with the extension of the resolvent $\rho_\lambda = (\lambda - A)^{-1}$, whose coefficients are

$$\left(\widehat{\rho_\lambda g}\right)_n(v) = \hat{g}_n(v)/(\lambda + inv) \quad \text{for } g \in \mathscr{A}.$$

Let $\varepsilon_n = \operatorname{sgn}(n)$, for $n \neq 0$, and $\varepsilon_0 = 0$, and denote by $B(X, Y)$ the space of linear bounded operators from X to Y.

THEOREM 2.1. *For $\lambda \neq 0$, $\operatorname{Re}\lambda > -\alpha$, one defines $\tilde{\rho}_\lambda \in B(\Gamma, \Gamma')$ by the formula*

$$\langle \tilde{\rho}_\lambda g, h\rangle = 2\pi \sum_{n\in\mathbf{Z}} \int_{-i\varepsilon_n\alpha-\infty}^{-i\varepsilon_n\alpha+\infty} \frac{\hat{g}_n(\zeta)\hat{h}_n(\zeta)}{\lambda + in\zeta}\, d\zeta; \quad \forall g, h \in \Gamma. \tag{19}$$

(a) *The mapping $\lambda \to \tilde{\rho}_\lambda$ is uniformly meromorphic*[2] *with unique pole* 0, *of order* 1, *and residue σ such that $\sigma g = \hat{g}_0$.*

(b) *If $g \in \mathscr{A} \cap \Gamma$, and $\operatorname{Re}\lambda > 0$, then $\rho_\lambda g$ and $\tilde{\rho}_\lambda g$ coincide: $J\rho_\lambda g = \tilde{\rho}_\lambda g$; on the contrary, this is false for $\operatorname{Re}\lambda < 0$.*

PROOF. (a) The mapping $\tilde{\rho}_\lambda$ belongs to $B(\Gamma, \Gamma')$ because, thanks to the Cauchy-Schwarz inequality, and Parseval's formula, one gets

$$\begin{aligned}|\langle \tilde{\rho}_\lambda g, h\rangle| &\leqslant \frac{2\pi}{\operatorname{Re}\lambda + \alpha}\left(\|g_{-\alpha}\|_{L^2}\|h_{-\alpha}\|_{L^2} + \|g_\alpha\|_{L^2}\|h_\alpha\|_{L^2}\right) + \frac{1}{|\lambda|}\|g\|_{L^2}\|h\|_{L^2}\\ &\leqslant C(\lambda)\|g\|_\Gamma\|h\|_\Gamma.\end{aligned}$$

Then expanding $1/(\lambda + in\zeta)$ in powers of $(\lambda - \lambda_0)$ (for any $\lambda_0 \in \Pi_\alpha$) gives

$$\tilde{\rho}_\lambda = \frac{\sigma}{\lambda} + \sum_{p\geqslant 0} \tilde{\rho}_{\lambda_0}^{(p)}(\lambda - \lambda_0)^p$$

where $\tilde{\rho}_{\lambda_0}^{(p)} \in B(\Gamma, \Gamma')$ is such that

$$\langle \tilde{\rho}_{\lambda_0}^{(p)} g, h\rangle = 2\pi(-1)^p \sum_{n\in\mathbf{Z}^*} \int_{-i\varepsilon_n\alpha-\infty}^{-i\varepsilon_n\alpha+\infty} \frac{\hat{g}_n(\zeta)\hat{h}_n(\zeta)}{(\lambda_0 + in\zeta)^{p+1}}\, d\zeta, \quad \forall g, h \in \Gamma.$$

This proves the analyticity.

[2] This means "meromorphic in the uniform topology of the operators of $B(\Gamma, \Gamma')$".

(b) Proposition 2.1 applies, but it is worthwhile going into details. For $\operatorname{Re}\lambda > 0$ the function $\hat{g}_n(\zeta)\hat{h}_n(\zeta)/(\lambda + in\zeta)$ has a unique pole at $i\lambda/n$. Then the Cauchy formula allows us to deform the integration path according to Figure 4. Thus

$$\langle \tilde{\rho}_\lambda g, h\rangle = 2\pi \sum_{n\in\mathbf{Z}} \int_{-\infty}^{+\infty} \frac{\hat{g}_n(v)\hat{h}_n(v)}{\lambda + inv}\, dv \tag{20}$$

$$= \iint_{X\times\mathbf{R}} \rho_\lambda g(x,v) h(-x,v)\, dx\, dv,$$

proving that $\tilde{\rho}_\lambda g = J\rho_\lambda g$.

REMARK 2.1. For $\operatorname{Re}\lambda < 0$, the pole $i\lambda/n$ lies on the other side of the real axis, so that a residue appears in formula (20). Actually $(\widehat{\tilde{\rho}_\lambda g})_n$ is the sum of an L^2 function and a complex Dirac mass:

$$\left(\widehat{\tilde{\rho}_\lambda g}\right)_n = \left(\widehat{\rho_\lambda g}\right)_n + \frac{2\Pi}{|n|}\hat{g}_n\left(\frac{i\lambda}{n}\right)\delta_{i\lambda/n}. \quad \square$$

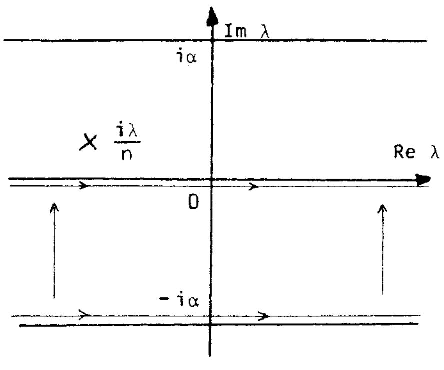
Since the operator $\tilde{\rho}_\lambda$ is an extension of $\rho_\lambda = (\lambda + v\partial/\partial x)^{-1}$, its inverse will provide us an extension of $(\lambda + v\partial/\partial x)$ to Γ'.

PROPOSITION 2.3. (a) $\tilde{\rho}_\lambda$ *is injective. Its inverse is an unbounded operator from* Γ' *to* Γ, *denoted by* $(\lambda + \zeta\cdot\partial/\partial x)$, *whose domain* Δ_λ *is the range of* $\tilde{\rho}_\lambda$.

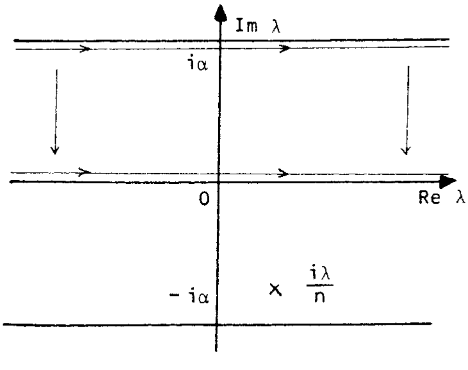
(b) *Let* $\Delta = \{g \in \Gamma \mid \zeta\cdot\partial g/\partial x \in \Gamma,\ g(0,\zeta) = g(2\pi,\zeta)\ \forall\zeta \in Q_\alpha\}$. *Then* $J(\Delta) \subset \Delta_\lambda$, *and for such functions, the operator* $(\lambda + \zeta\partial/\partial x)$ *defined above, coincides with the usual* $\lambda g + \zeta\partial g/\partial x$.

PROOF. The proof of (a) is similar to that of Proposition 2.2, and the proof of (b), to that of Theorem 2.1(b). $\square$

We notice that the domain Δ_λ of the operator $(\lambda + \zeta\partial/\partial x)$ depends on λ, and that this operator is not defined for $\lambda = 0$, because zero is a pole of $\tilde{\rho}_\lambda$. So there is no natural way to give a sense to the operator $\zeta\partial/\partial x$.



Case $n > 0 \Rightarrow \operatorname{Im}(i\lambda/n) > 0$
Deformation of the integration path from $\operatorname{Im}\lambda = -\alpha$ to $\operatorname{Im}\lambda = 0$



Case $n < 0 \Rightarrow \operatorname{Im}(I\lambda/n) < 0$
Deformation of the integration path from $\operatorname{Im}\lambda = \alpha$ to $\operatorname{Im}\lambda = 0$

FIGURE 4

Now we perform the extension of $D(n,\lambda)$ under
HYPOTHESIS H3. *F_0 extends to Q_α and is an element of G.*

PROPOSITION 2.4. *There exists an analytical extension of $D(n,\cdot)$ defined on Π_α, which reads*

$$\tilde{D}(n,\lambda) = 1 - \frac{i}{n}\int_{-i\varepsilon_n\alpha-\infty}^{-i\varepsilon_n\alpha+\infty} \frac{F_0(\zeta)\,d\zeta}{\lambda + in\zeta}. \tag{21}$$

Then $\tilde{D}$ has the same properties as D, namely

(a) *If $\mathscr{P} = \{\lambda \in \Pi_\alpha \mid \exists n \in \mathbf{Z}^*,\ \tilde{D}(n,\lambda) = 0\}$, then $\mathscr{P} \cap \overline{\Pi_\beta}$ is a finite set, $\forall \beta < \alpha$.*

(b) *The inequality $|\tilde{D}(n,\lambda)| \geqslant D_0 > 0$, $\forall n \in \mathbf{Z}^*$ is valid in the neighbourhood of any $\lambda_0 \in \Pi_\alpha \setminus \mathscr{P}$, and also in the neighbourhood of ∞ in $\overline{\Pi_\beta}$, $\forall \beta < \alpha$.*

We can now go to the extension of the resolvent. The proof of Theorem 2.2 is similar to that of Theorem 2.1 and is omitted.

THEOREM 2.2. *For $\lambda \in \Pi_\alpha \setminus (\mathscr{P} \cup \{0\})$, define $\tilde{R}_\lambda \in B(\Gamma, \Gamma')$ by*

$$\langle \tilde{R}_\lambda g, h\rangle = \langle \tilde{\rho}_\lambda g, h\rangle + 2\pi \sum_{n\in\mathbf{Z}^*} \frac{i}{n\tilde{D}(n,\lambda)} \int_{-i\varepsilon_n\alpha-\infty}^{-i\varepsilon_n\alpha+\infty} \frac{F_0(\zeta)\hat{h}_n(\zeta)}{\lambda + in\zeta}\,d\zeta \cdot \int_{-i\varepsilon_n\alpha-\infty}^{-i\varepsilon_n\alpha+\infty} \frac{\hat{g}_n(\zeta')\,d\zeta'}{\lambda + in\zeta'}. \tag{22}$$

Then $\tilde{R}_\lambda$ is uniformly meromorphic on Π_α with poles in $\mathscr{P} \cup \{0\}$ and is an extension of R_λ in the same manner as $\tilde{\rho}_\lambda$.

$$\tilde{R}_\lambda g = J(R_\lambda g) \quad \forall \lambda, \operatorname{Re}\lambda > 0, \forall g \in \mathscr{A} \cap \Gamma.$$

More indications about the poles are provided by a
SYMPLIFYING HYPOTHESIS 2.1. *We assume that for any $n \in \mathbf{Z}^*$, the zeroes of $\tilde{D}(n,\lambda)$ are simple.*

So, let $\lambda_0 \in \mathscr{P}$, and denote by $\{n_j,\ j = 1,\ldots,p\}$, the finite set of solutions of the equation $\tilde{D}(n,\lambda_0) = 0$. Then, the Laurent development of $\tilde{R}_\lambda$ near λ_0 is written

$$\tilde{R}_\lambda g = \frac{1}{\lambda - \lambda_0}\left\{\sum_{j=1}^{p} \langle \tilde{C}_j, g\rangle \exp(in_j x)\tilde{\varphi}_j\right\} + O(1) \tag{23}$$

where $\tilde{\varphi}_j \in G'$ and $\tilde{C}_j \in \Gamma'$ are defined according to

$$\langle \tilde{\varphi}_j, h\rangle = \int_{-i\varepsilon_{n_j}\alpha-\infty}^{-i\varepsilon_{n_j}\alpha+\infty} \frac{F_0(\zeta)h(\zeta)\,d\zeta}{\lambda_0 + in_j\zeta}, \quad \forall h \in G \tag{24}$$

and

$$\langle \tilde{C}_j, g\rangle = \frac{i}{n_j}\left(\frac{\partial \tilde{D}}{\partial \lambda}\right)^{-1}(n_j,\lambda_0) \cdot \int_{-i\varepsilon_{n_j}\alpha-\infty}^{-i\varepsilon_{n_j}\alpha+\infty} \frac{\hat{g}_{n_j}(\zeta)\,d\zeta}{\lambda_0 + in_j\zeta}. \tag{25}$$

REMARK 2.2. The poles λ_0, such that $\operatorname{Re}\lambda_0 > 0$ are eigenvalues of T, and the corresponding $\tilde{\varphi}_j$'s coincide with the eigenvalue φ_j, up to the injection J, because we can perform a deformation of the integration paths in formula (24), as shown on Figure 4.

On the other hand, when $\operatorname{Re}\lambda_0 < 0$, λ_0 is no more an eigenvalue of T, neither is $\tilde{\varphi}_j$ a genuine function of v. As pointed out in Remark 2.1, $\tilde{\varphi}_j$ is the sum of an L^2-function and of a complex Dirac mass. Thus, the concepts of eigenvalue and eigenfunction may no longer be used.

However, we shall generalize $(\lambda - T)$; so that it operates in Γ', in order to formalize the idea that $\tilde{\varphi}_j$ is "almost" an eigenfunction. This is the aim of the next paragraph. □

2.3. *The generalized spectral equation*. We begin to give a sense to $\int g^*\,dv$ for $g^* \in \Gamma'$.

DEFINITION 2.3. *If $g^* \in \Gamma'$ is defined according to* (18) *and furthermore, if $g_j \in L^1(X \times \mathbf{R})$, $(j = 1, \dots, p)$, then*

(26)
$$\int g^*\,dv = \sum_{j=1}^{m} \int_{-\infty}^{+\infty} g_j(x, v)\,dv \in L^1(X).$$

LEMMA 2.3. *Let $g \in \mathscr{A} \cap \Gamma$. Then $\int \tilde{\rho}_\lambda g\,d\lambda$ is well defined by* (26) *and is an element of $L^2(X)$.*

PROOF. Formula (19) gives, for $g \in \mathscr{A} \cap \Gamma$ and $h \in \Gamma$,

$$\langle \tilde{\rho}_\lambda g, h\rangle = \int \hat{g}_0 \hat{h}_0\,dv + \iint_{X\times\mathbf{R}} \left(\psi^-(x,v)h_{-\alpha}(-x,v) + \psi^+(x,v)h_\alpha(-x,v)\right)dx\,dv$$

where ψ^- and ψ^+ are defined by their Fourier coefficients

$$\hat{\psi}_n^-(v) = \frac{\hat{g}_{-\alpha,n}(v)}{\lambda + |n|\alpha + inv} \quad \text{if } n > 0,$$
$$\hat{\psi}_n^+(v) = \frac{\hat{g}_{\alpha,n}(v)}{\lambda + |n|\alpha + inv} \quad \text{if } n < 0,$$

the other coefficients of ψ^+ and ψ^- being zero. It is easy to see that there exists $\chi(v)$ in $L^2(\mathbf{R})$ such that

$$\frac{1}{|\lambda + |n|\alpha + inv|} \leqslant \chi(v) \quad \forall n \in \mathbf{Z}^*.$$

Thus

$$\int_{-\infty}^{+\infty} \left(\sum_{n>0} |\hat{\psi}_n^-(v)|^2\right)^{1/2} dv \leqslant \int_{-\infty}^{+\infty} \chi(v)\|g_{-\alpha}(v)\|_{L^2(X)}\,dv \leqslant \|g\|_\Gamma \|\chi\|_{L^2}.$$

Therefore ψ^- and ψ^+ belong to $L^1(\mathbf{R}, L^2(X))$. Then, as $\int \hat{g}_0(v)\,dv = 0$, we obtain the existence of $\int \tilde{\rho}_\lambda g\,dv$ in $L^2(X)$. □

THEOREM 2.3. *Let $g \in \mathscr{A} \cap \Gamma$, and $\lambda \in \Pi_\alpha \setminus \mathscr{P}$. Then $\tilde{\varphi} = \tilde{R}_\lambda g$ is a solution of the generalized spectral equation*

(27)
$$\left(\lambda + \zeta\frac{\partial}{\partial x}\right)\tilde{\varphi} + \frac{d\phi}{dx}\cdot F_0 = g; \quad \frac{d^2\phi}{dx^2} = \int \tilde{\varphi}\,dv,$$
$$\phi(0) = \phi(2\pi) = 0, \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi),$$

where $(\lambda + \zeta\partial/\partial x)$ *is defined in Proposition* 2.3. *Furthermore*, $\exp(in_j x)\tilde{\varphi}_j$ *defined at formula* (24) *is solution of equation* (27) *where* $\lambda = \lambda_0$ *and* $g = 0$.

PROOF. Formula (22) shows that $\tilde{\varphi} = \tilde{\rho}_\lambda(g + \psi F_0)$, where $\psi(x)$ is an $L^2(X)$ function defined by

$$\hat{\psi}_0 = 0 \quad \text{and} \quad \hat{\psi}_n = \frac{i}{n\tilde{D}(n,\lambda)} \int_{-i\varepsilon_n\alpha - \infty}^{-i\varepsilon_n\alpha + \infty} \frac{\hat{g}_n(\zeta')\, d\zeta'}{\lambda + in\zeta'} \quad \text{for } n \neq 0.$$

Then $g + \psi F_0 \in \mathscr{A}$, so that Lemma 2.3 applies, and one checks that $\psi = -d\phi/dx$. Then, thanks to the definition of $(\lambda + \zeta\partial/\partial x)$, one gets (27). □

Theorem 2.3 shows that the extension $\tilde{R}_\lambda$ is still, in a generalized sense, a resolvent of T, and that its poles λ_0 with $\operatorname{Re}\lambda_0 < 0$ are generalized eigenvalues that we may call eigenmodes.

In the next section we use this extension $\tilde{R}_\lambda$ to give an expansion of type (16) involving both unstable and stable waves. However, semigroup techniques fail in this context, and we must prove some estimates about $\tilde{R}_\lambda$.

2.4 *Generalized asymptotic behaviour of* $\exp(tT)$.

LEMMA 2.4 (*estimate of* $\tilde{R}_\lambda$). *One has for every* $g \in \mathscr{A}$ *and* $r > -\alpha$

$$\lim_{\substack{\lambda \to \infty \\ \operatorname{Re}\lambda \geq r}} \|\tilde{R}_\lambda g\|_{\Gamma'} = 0.$$

PROOF. Using the convergence of the series and of the integrals, we may reduce formulae (19) and (22) to finite sums with respect to n, and integrals upon bounded domains with respect to ζ. Then the Lebesgue theorem allows us to take the limit when λ goes to infinity, which is obviously zero. Details are left to the reader. □

LEMMA 2.5 (*estimate of the remainder*). *Let* $g \in \Gamma$ *and* r $(-\alpha < r < 0)$ *be such that no element of* $\mathscr{P}$ *lies on the line* $\operatorname{Re}\lambda = r$. *Then*

$$\left\| \frac{1}{2i\pi} \int_{r-i\infty}^{r+i\infty} e^{\lambda t}\tilde{R}_\lambda g\, d\lambda \right\|_{\Gamma'} \leq C(t_0) e^{rt} \|g\|_\Gamma \quad \forall t \geq t_0 > 0. \tag{28}$$

PROOF. Let

$$S_r(t) = \frac{1}{2i\pi} \int_{r-i\infty}^{r+i\infty} e^{\lambda t}\tilde{R}_\lambda g\, d\lambda = \frac{1}{2\pi} e^{rt}\varphi(t). \tag{29}$$

Then (28) is equivalent to $\|\varphi(t)\|_{\Gamma'} \leq C(t_0)\|g\|_\Gamma$.

We integrate by parts and split $\varphi(t)$ into two operators $\varphi_1(t)$ and $\varphi_2(t)$, which read as follows for $h \in \Gamma$, $\lambda = r + i\eta$ (the integrals with respect to ζ are performed on the line $\operatorname{Im}\zeta = -\varepsilon_n\alpha$).

$$\langle \varphi_1(t), h\rangle = \frac{2\pi}{t} \int_{-\infty}^{+\infty} e^{i\eta t} \left(\sum_{\mathbf{Z}^*} \int \frac{\hat{g}_n(\zeta)\hat{h}_n(\zeta)}{(\lambda + in\zeta)^2}\, d\zeta \right) d\eta, \quad \forall h \in \Gamma, \tag{30}$$

(31) $\langle \varphi_2(t), h\rangle$

$$= \frac{2\pi}{t} \int_{-\infty}^{+\infty} e^{i\eta t} \frac{d}{d\lambda} \left(\frac{i}{n\tilde{D}(n,\lambda)} \int \frac{F_0(\zeta)\hat{h}_n(\zeta)}{\lambda + in\zeta}\, d\zeta \int \frac{\hat{g}_n(\zeta')}{\lambda + in\zeta'}\, d\zeta' \right) d\eta.$$

Thanks to an inversion of summations, and to the estimate

$$(32)\qquad \int d\eta/|r+i(\eta+n\zeta)|^2 \leqslant \pi/(r+\alpha), \quad \text{one obtains}$$
$$|\langle \varphi_1(t),h\rangle| \leqslant \pi\|g\|_\Gamma\|h\|_\Gamma/t(r+\alpha) \leqslant C(t_0)\|g\|_\Gamma\|h\|_\Gamma.$$

Differentiating with respect to λ in (31) we get a sum of three terms, the first one of which is

$$\frac{2\pi}{t}\int_{-\infty}^{+\infty} e^{i\eta t}\left\{\sum_{\mathbf{Z}^*}\frac{-i}{n\tilde{D}(n,\lambda)^2}\int\frac{F_0(\zeta_3)\,d\zeta_3}{(\lambda+in\zeta_3)^2}\cdot\int\frac{\hat{g}_n(\zeta_1)\,d\zeta_1}{\lambda+in\zeta_1}\int\frac{F_0(\zeta_2)\hat{h}_n(\zeta_2)}{\lambda+in\lambda_2}\,d\zeta_2\right\}d\eta.$$

With the help of Proposition 2.4 its modulus is estimate by

$$(33)\qquad \frac{C\|F_0\|_G}{tD_0(r+\alpha)}\sum_{\mathbf{Z}^*}\left(\int|\hat{g}_n(\zeta_1)|^2\,d\zeta_1\right)^{1/2}\cdot\int_{-\infty}^{+\infty}d\eta\left(\int\left|\frac{F_0(\zeta_3)}{\lambda+in\zeta_3}\right|^2d\zeta_3\right)^{1/2}\left(\int\left|\frac{\hat{h}_n(\zeta_2)}{\lambda+in\zeta_2}\right|^2d\zeta_2\right)^{1/2}.$$

Applying now the Cauchy-Schwarz inequality to the integral with respect to η, and using estimate (32) and Parseval's formula, one gets that (33) is less than or equal to

$$C(r,\alpha,D_0,F_0)\|g\|_\Gamma\|h\|_\Gamma/t \leqslant C(t_0)\|g\|_\Gamma\|h\|_\Gamma \quad \forall t\geqslant t_0.$$

A similar method can be applied so as to estimate the other terms, and to prove the final result. □

We are now able to prove the following

THEOREM. 2.4. *As in Proposition* 1.2, *the superscipts s indicate quantities associated to the pole* λ_s *of* $\tilde{R}_\lambda$. *We define*

$$D_1(A)=\left\{g\in\mathscr{A}\cap L^2(X\times\mathbf{R})\,\middle|\,v\frac{\partial g}{\partial x}\in\mathscr{A}\cap L^2(X\times\mathbf{R}),\ g(0,v)=g(2\pi,v)\,a.e.\right\}.$$

Now, let $g\in D_1(A)\cap\Gamma$ *and* r $(-\alpha<r<0)$ *be such that no element of* $\mathscr{P}$ *lies on the line* $\operatorname{Re}\lambda=r$. *Denote by* $\{\lambda_1,\dots,\lambda_s\}$ *the finite set of the elements* λ *of* $\mathscr{P}$, *such that* $r<\operatorname{Re}\lambda\leqslant\omega_0$.

Then, the following asymptotic expansion is valid in the strong topology of Γ', *when t goes to infinity*:

$$(34)\qquad J(e^{tT}g)=\hat{g}_0+\sum_{s=1}^{S}\sum_{j=1}^{P_s}e^{\lambda_s t+in_j^s x}\langle\tilde{C}_j^s,g\rangle\tilde{\varphi}_j^s(v)+\mathcal{O}(e^{rt}).$$

PROOF. $D_1(A)$ is the domain of A and T, as operators in $\mathscr{A}\cap L^2(X\times\mathbf{R})$. Thus, when g belongs to $D_1(A)\cap\Gamma$, the following Dunford integral converges in the strong topology of Γ':

$$J(e^{tT}g)=\frac{1}{2i\pi}\int_{p-i\infty}^{p+i\infty}e^{\lambda t}\tilde{R}_\lambda g\,d\lambda \quad \forall t>0,\forall p>\omega_0.$$

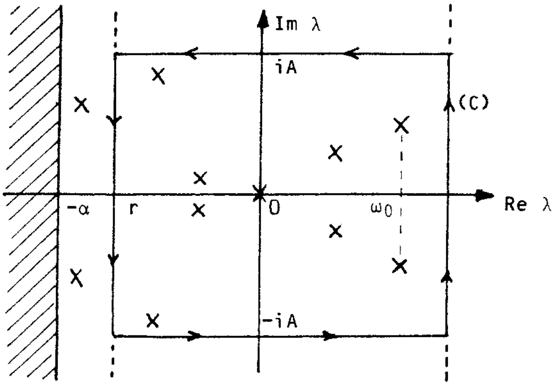


FIGURE 5. The path (C)

Now, apply the theorem of residues with the path (C) depicted in Figure 5. We obtain, according to (23),

$$\frac{1}{2i\pi}\int_{(C)} e^{\lambda t}\tilde{R}_\lambda g\,d\lambda = \hat{g}_0 + \sum_{s=1}^{S}\sum_{j=1}^{p_s} e^{\lambda_s t + i n_j^s x}\langle \tilde{C}_j^s, g\rangle \tilde{\varphi}_j^s. \tag{35}$$

Then, thanks to Lemma 2.4, the integrals along the segments of the horizontal lines $\operatorname{Im}\lambda = \pm A$ go to zero when A goes to infinity. Furthermore, Lemma 2.5 asserts that the integral along the infinite line $\operatorname{Re}\lambda = r$ is of order e^{rt}. So, putting $A = \infty$ in (35) gives (34). □

REMARK 2.3. Theorem 2.4 answers the questions that arose at Remark 1.4. Indeed, expansion (34) involves both stable and unstable waves. But we must point out the fact that the magnitude of the stable waves depends on the velocity v through the functionals $\tilde{\varphi}_j^s$.

Before looking for similar expansions of the potential, we shall interpret the appearance of such functionals by means of a Fourier transform with respect to the velocity. □

3. Fourier transform with respect to the velocity.

3.1. *Fourier transform in* Γ *and* Γ'. For $g \in L^2(X \times \mathbf{R})$, we denote by $\check{g}(x,\xi)$ its Fourier transform with respect to v:

$$\check{g}(x,\xi) = \int g(x,v)\exp(-iv\cdot\xi)\,dv.$$

If $g \in \Gamma$, then one gets a kind of Paley-Wiener property:

$$\check{g}(x,\xi) = \exp(\eta\xi)(\check{g}_\eta)(x,\xi). \tag{36}$$

DEFINITION 3.1. *If the element* g^* *of* Γ_1' *is given by* (18) *then*

$$\check{g}^*(x,\xi) = \sum_{j=1}^{m}\int_{-\infty}^{+\infty} g_j(x,v)e^{-i(v+i\eta_j)\xi}\,dv = \sum_{j=1}^{m}\check{g}_j(x,\xi)e^{\eta_j\xi}. \tag{37}$$

REMARK 3.1. This definition seems very natural, but the exponential growth of $\check{g}^*$, when ξ goes to infinity, explains why g^* cannot be considered as an ordinary distribution. Analogous definitions can be stated for G, and for instance, one has $\check{\delta}_z(\xi) = \exp(-iz\xi)$.

But a difficulty arises from the nonuniqueness of expression (18) for g^*. However, one has

PROPOSITION 3.1. *Let g^* in Γ_1' be given by formula* (18). *then g^* is identically zero if and only if its Fourier transform $\check{g}^*$ given by* (37) *is identically zero.*

PROOF. It uses the same techniques as Proposition 2.2, and is omitted (cf. [**3**]). □

3.2. *Fourier transform of the resolvent and of its extension.* Theorem 3.1 clarifies the meaning of the analytical extension of the resolvent. We denote by $H_0(\xi)$ the Fourier transform of $F_0(v)$.

THEOREM 3.1. (a) *Let* $\operatorname{Re}\lambda$ *be positive and g belong to $\mathscr{A}$. Then one has for $n \neq 0$:*

$$(38)\quad \widetilde{(R_\lambda g)}_n(\xi) = \int_\xi^{\varepsilon_n\infty} e^{-\lambda(s-\xi)/n}\check{g}_n(s)\,\frac{ds}{n} + \frac{i}{nD(n,\lambda)}\int_\xi^{\varepsilon_n\infty} e^{-\lambda(s-\xi)/n}H_0(s)\,\frac{ds}{n}\cdot\int_0^{\varepsilon_n\infty} e^{-\lambda s/n}\check{g}_n(s)\,\frac{ds}{n}.$$

(b) *Let* $\operatorname{Re}\lambda > -\alpha$ *and $g \in \mathscr{A}\cap\Gamma$. Then the Fourier transform of $\tilde{R}_\lambda g$, which is an element of Γ_1', is still given by formula* (38).

REMARK 3.2. In statement (a), $\widetilde{(R_\lambda g)}_n$ is a continuous function of ξ going to zero at infinity, whereas $(\tilde{R}_\lambda g)_n$ (for $-\alpha < \operatorname{Re}\lambda < 0$) grows exponentially, when ξ tends to infinity. Furthermore, in the latter case, the convergence of the integrals in (38) is due to formula (36).

However, the same formula (38) gives the resolvent and its extension and they both satisfy the spectral equation

$$\lambda\check{\varphi}_n - n\frac{\partial\check{\varphi}_n}{\partial\xi} - \frac{i}{n}\check{\varphi}_n(0)\cdot H_0 = \check{g}_n \quad \text{if } n \neq 0.$$

Similar properties are true for the eigenfunctions $\varphi_j^s(v)$, and their generalization $\tilde{\varphi}_j^s(v)$. These considerations strengthen our feeling that $\tilde{R}_\lambda$ is the right extension of R_λ and that its poles are genuine "generalized eigenvalues" of operator T.

PROOF OF THEOREM 3.1. (a) The problem reduces to looking for the Fourier transform of $\psi_n(v) = \varphi(v)/(\lambda + inv)$ where $\varphi \in L^1(\mathbf{R})$. Now $\check{\psi}_n$ is a solution of the equation

$$\left(\lambda - n\frac{\partial}{\partial\xi}\right)\check{\psi}_n = \check{\varphi};\quad \check{\psi}_n(-\infty) = \check{\psi}_n(+\infty) = 0.$$

Thus

$$(39)\qquad \check{\psi}_n(\xi) = \int_\xi^{\varepsilon_n\infty} e^{-\lambda(s-\xi)/n}\check{\varphi}_n(s)\,\frac{ds}{n}.$$

(b) Now, we prove that the Fourier transform of the functional $\tilde{\psi}_n \in G_1'$ such that

$$\langle\tilde{\psi}_n, h\rangle = \int_{-i\varepsilon_n\alpha-\infty}^{-i\varepsilon_n\alpha+\infty} \frac{\varphi(\zeta)h(\zeta)}{\lambda + in\zeta}\,d\zeta \quad \forall h \in G$$

(where $\varphi \in G \cap L^1(\mathbf{R})$) is still given by (39). According to Definition 3.1, one gets

$$\check{\tilde{\psi}}_n(\xi) = e^{-\varepsilon_n \alpha \xi} \int_{-\infty}^{+\infty} \frac{\varphi_{-\varepsilon_n \alpha}(v) \exp(-iv\xi)\, dv}{(\lambda + |n|\alpha) + inv}.$$

But, since $\operatorname{Re}(\lambda + |n|\alpha) > 0$, the proof of (a) applies and gives

$$\check{\tilde{\psi}}_n(\xi) = e^{-\varepsilon_n \alpha \xi} \int_{\xi}^{\varepsilon_n \infty} e^{-(\lambda + |n|\alpha)(s-\xi)/n} \check{\varphi}_{-\varepsilon_n \alpha}(s) \frac{ds}{n}.$$

Formula (36) then leads to (39).

3.3. *Asymptotic behaviour of the fourier transform of the distribution function and of the potential.*

THEOREM 3.2. *Assume that the hypotheses of Theorem 2.4 are satisfied, and furthermore, that $g_{-\alpha}$ and $g_\alpha \in L^1(X \times \mathbf{R})$. Then the velocity Fourier transform of* $\exp(tT) \cdot g$ *expands according to*

$$J(\widetilde{\exp(tT)} g)(x, \xi) = \sum_{s,j} e^{\lambda_s t + i n_j^s x} \langle \tilde{C}_j^s, g \rangle \check{\varphi}_j^s(\xi) + \check{g}_0(\xi) + \mathcal{O}(e^{rt}) \tag{40}$$

in $C^0([-R, +R]_\xi, L^1(X))$ for every arbitrary $R > 0$. Then the following expansion holds for the potential ϕ in $W^{2,1}(X)$:

$$\phi(x, t) = \sum_{s,j} e^{\lambda_s t + i n_j^s x} \cdot \frac{i \langle \tilde{C}_j^s, g \rangle}{n_j^s} + \mathcal{O}(e^{rt}). \tag{41}$$

PROOF. The velocity Fourier transform of equation (34) gives (40). However, the main problem is to estimate the Fourier transform of the integral remainder $S_r(t)$ (formula (29)) in the $C^0([-R, R], L^1(X))$-norm. This is not a consequence of Lemma 2.5, because the Fourier transform is not continuous on Γ', and estimate (40) is actually sharper. Keeping the notations of Lemma 2.5, we shall prove that

$$\|\check{\varphi}_1(t)\|_{C^0([-R,R], L^1(X))} \leqslant C, \quad \forall t \geqslant t_0 > 0 \tag{42}$$

where $C = C(t_0, g, r, R)$. A similar estimate can be proved for φ_2, by means of the same techniques, and is omitted (cf. [**3**]).

Indeed, one can invert the summations in (30) and compute the integral with respect to η, by the residue theorem. Then one has

$$\langle \varphi_1(t), h \rangle = (2\pi)^2 e^{-rt} \sum_{n \in \mathbf{Z}^*} \int_{-i\varepsilon_n \alpha - \infty}^{-i\varepsilon_n \alpha + \infty} \hat{g}_n(\zeta) \hat{h}_n(\zeta) e^{-in\zeta t}\, d\zeta.$$

Thus

$$\langle \varphi_1(t), h \rangle = 2\pi e^{-rt} \iint_{X \times \mathbf{R}} (\gamma^-(x, v) h_{-\alpha}(-x, v) + \gamma^+(x, v) h_\alpha(-x, v))\, dx\, dv$$

where, one can easily check by the aid of Fourier coefficients that

$$\gamma^{\mp}(x, v) = \int_X g_{\mp\alpha}(y - vt, v) \frac{e^{\pm i(x-y) - \alpha t}}{1 - e^{\pm i(x-y) - \alpha t}}\, dy.$$

This proves that $\varphi_1(t)$ is an element of Γ_1', whose Fourier transform is

$$\check{\varphi}_1(t, x, \xi) = 2\pi \exp(-rt)(\check{\gamma}^-(x, \xi) \exp(-\alpha\xi) + \check{\gamma}^+(x, \xi) \exp(\alpha\xi)).$$

Then

$$\|\check{\varphi}_1(t)\|_{C^0([-R,R],L^1(X))} \leqslant C(t_0)e^{-(r+\alpha)t}e^{\alpha R}\left(\|g_\alpha\|_{L^1} + \|g_{-\alpha}\|_{L^1}\right)$$

and since $r + \alpha > 0$, one gets (42).

As far as ϕ is concerned, put $\xi = 0$ in (40) and integrate twice with respect to x. Since $D(n_j^s, \lambda_s) = 0$, then one has $\check{\varphi}_j^s(0) = -in_j^s$, which leads to (41). □

Expansion (41) was the aim of this paper. However, expansion (40) presents interesting analogies with Lax and Phillips' scattering theory, which we detail in the conclusion.

3.4. *Comparison between Vlasov's equation and scattering theory.* One of the aims of the scattering theory [**2**] is to obtain an asymptotic behaviour for the wave equation outside a bounded obstacle. Although the total energy of the solution remains constant, the dispersion towards infinity leads to a local decay of the L^∞ norm of the solution. This decay is well expressed by expansion (6).

The same phenomena actually arises for Vlasov's equation. Indeed, for any fixed x, expansion (40) is only valid locally in ξ, because of the exponential growth of the $\check{\varphi}_j^s(\xi)$. Moreover, the velocity Fourier transform of Vlasov's equation is a perturbed transport equation

$$\frac{\partial \check{f}_n}{\partial t} - n\frac{\partial \check{f}_n}{\partial \xi} - \frac{i}{n}\check{f}_n(0)H_0 = 0$$

whose solution propagates towards infinity in the ξ-space. Thus, in a stable periodic plasma, the energy disperses towards the high oscillatory modes in the velocity space, which explains the appearance of "analytic distributions". This phenomena is known in physics as "phase mixing" and is sometimes considered as unphysical (cf. [**1**, §8.7]).

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Centre de Mathématiques, Ecole Polytéchnique, 91128 Palaiseau Cedex, France