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Spectral Theory, Zeta Functions and the Distribution of Periodic Points for Collet–Eckmann Maps

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Abstract. We study unimodal interval maps T with negative Schwarzian derivative satisfying the Collet–Eckmann condition $|DT^n(Tc)| \geqq K\lambda_c^n$ for some constants $K > 0$ and $\lambda_c > 1$ (c is the critical point of T). We prove exponential mixing properties of the unique invariant probability density of T, describe the long term behaviour of typical (in the sense of Lebesgue measure) trajectories by Central Limit and Large Deviations Theorems for partial sum processes of the form $S_n = \sum_{i=0}^{n-1} f(T^i x)$, and study the distribution of "typical" periodic orbits, also in the sense of a Central Limit Theorem and a Large Deviations Theorem.

This is achieved by proving quasicompactness of the Perron Frobenius operator and of similar transfer operators for the Markov extension of T and relating the isolated eigenvalues of these operators to the poles of the corresponding Ruelle zeta functions.

1. Introduction

During the last years considerable progress was made towards the understanding of the metric structure of general unimodal maps with negative Schwarzian derivative (henceforth called S-unimodal maps). The likely limit set in the sense of Milnor [Mi] was described and related to the conservativeness/transitiveness of the map with respect to Lebesgue measure [BL1, BL3, GJ, Ma, K4]. The ergodicity of S-unimodal maps without stable periodic orbit was proved in [BL2, BL3, Ma]. (For a discussion of these results see [HK3].) Also new sufficient or equivalent conditions for the existence of invariant probability densities were found. (The uniqueness of such invariant densities follows from the ergodicity of T.) Before we describe some of these results, we introduce the class of maps we are going to investigate:

> T: $[0, 1] \to [0, 1]$ is of class C^3 and has a unique nondegenerate critical point c of order l, see (1.2).

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We also assume that T has negative Schwarzian derivative, i.e.

$$ST := \frac{T'''}{T'} - \frac{3}{2}\left(\frac{T''}{T'}\right)^2 \leqq 0 \quad \text{except at } c \text{ where } T' = 0\,.$$

For such maps Collet–Eckmann [CE] proved:

If $\liminf_{n\to\infty} \sqrt[n]{|DT^n(Tc)|} > 1$, then T has an invariant probability density.

Indeed, they used some additional assumption, which was removed in [N3]. Recently, Nowicki–van Strien [NvS2] strengthened this to:

If $\sum_{n=1}^{\infty} |DT^n(Tc)|^{-1/l} < \infty$, then T has an invariant probability density.

For $x \in [0, 1]$, let $\bar{\lambda}(x) := \limsup_{n\to\infty} \sqrt[n]{|DT^n(x)|}$. With this notation Keller proved in [K4]:

If $\bar{\lambda}(x) > 1$ for a set of points x of positive Lebesgue measure, then T has an invariant probability density.

Finally, combining this with observations from Nowicki [N1], the following result is shown in [K4]:

If T is uniformly hyperbolic on periodic points, i.e. if

$$\inf\{\bar{\lambda}(z) \colon T^n z = z \text{ for some } n \in \mathbf{N}\} > 1\,,$$

then T has an invariant probability density.

As a consequence of the general metric theory of S-unimodal maps [BL3, K4, Le] it is known that an invariant probability density, if it exists at all, gives rise to a measure preserving dynamical system which is mixing (and even weakly Bernoulli) up to a finite rotation. This means that there is a finite disjoint collection of p intervals $I_0, \ldots, I_{p-1}$ which are cyclically permuted by T, and T^p, restricted to any of these intervals, is unimodal and mixing. If $p = 1$, T is called nonrenormalizable, otherwise we say T is finitely renormalizable.

None of these results, however, answers the following two questions in case T has an invariant probability density:

1. Suppose T is mixing with respect to its invariant density. What is its mixing rate in terms of correlation decay or coefficients of weak Bernoulliness (= coefficients of absolute regularity)?
2. How are "typical" periodic orbits distributed?

In this paper we attempt to answer these two questions for S-unimodal maps satisfying the Collet–Eckmann condition

$$\lambda_c > 1, \quad \text{where } \lambda_c := \liminf_{n\to\infty} \sqrt[n]{|DT^n(Tc)|}\,, \tag{1.1}$$

henceforth called *Collet–Eckmann maps* (C–E maps). Benedicks–Carleson [BC] proved that the set of parameters a for which the map $x \mapsto ax(1-x)$ satisfies the condition (1.1), has positive Lebesgue measure. It is widely believed that these maps are very close to uniformly hyperbolic ones, i.e. to maps T for which there is some $k>0$ such that $|(T^k)'(x)|>1$ uniformly for all $x\in[0,1]$. Very general transfer operators and zeta functions for uniformly hyperbolic maps were studied in [BK], and the results proved there open the road to answering the above two questions for such maps (cf. also [K5]). In this paper we apply the same strategy of proof to Collet–Eckmann maps T satisfying the following additional regularity assumptions:

$T(0)=T(1)=0$, and for each $\beta_0>0$ there is a constant $M>0$ such that for all $\beta\in[0,\beta_0]$,

$$1.\ M^{-1}<\left(\frac{|x-c|^{l-1}}{|DT(x)|}\right)^{\beta}<M \quad \text{for all } x,$$

$$2.\ \mathrm{var}_{[0,1]}\left(\frac{|x-c|^{l-1}}{|DT(x)|}\right)^{\beta}<M \quad \text{and} \tag{1.2}$$

$$3.\ \mathrm{var}_{[0,u]}\left(\frac{|Tx-Tu|}{|x-u||DT(x)|}\right)^{\beta},\ \mathrm{var}_{[v,1]}\left(\frac{|Tx-Tv|}{|x-v||DT(x)|}\right)^{\beta}<M \quad \text{if } u<c<v\,.$$

These conditions are satisfied e.g. if T is a polynomial map with vanishing derivatives at c of all orders up to $l-1$, but also for $T(x)=a(1-|2x-1|^l)$ with real $l>1$. In both cases conditions 1 and 2 are easily checked. For condition 3 one should observe that the expressions of interest are bounded by 1 if both, x and u (respectively v), are close to c, and that the derivatives of these expressions have a bounded number of sign changes.

Our main results determine the spectrum of transfer operators associated with T (Theorem 2.1) and relate poles of dynamical zeta-functions of T to isolated eigenvalues of these operators (Theorem 2.2). As it involves the Markov extension of T, we can give a precise formulation of it only after some preparations in Sect. 2. Here we formulate some consequences of these theorems, which can be stated more directly:

Let T be a nonrenormalizable Collet–Eckmann map, and denote Lebesgue measure on $[0,1]$ by m. Then T has an invariant probability density h, and for $\mu=h\cdot m$, the dynamical system (T,μ) is mixing.

Theorem 1.1. *(T,μ) has exponentially decaying correlations, namely: There are constants $C>0$ and $\rho<1$ such that for any measurable $F,G:[0,1]\to\mathbf{C}$ with F of bounded variation and $\int|G(x)|^{l+\delta}dx<\infty$ for some $\delta>0$ and any $n\in\mathbf{N}$ holds*

$$\left|\int F\cdot(G\circ T^n)\,d\mu-\int F\,d\mu\cdot\int G\,d\mu\right|\leqq C\cdot\rho^n\cdot\mathrm{var}(F)\cdot\|G(x)\|_{l+\delta}\,.$$

Here $\mathrm{var}(F)$ *denotes the variation of F over* $[0,1]$.

Remark 1.1. For Misiurewicz maps (maps for which c is not an accumulation point of $(T^nc)_{n>0}$) an estimate of this type is contained in [Zi1]. For maps of Benedicks–Carleson type (maps with $\lambda_c>1$ and $|T^nc-c|>r^n$ for some $r>0$ and all $n\geqq 1$, see [BC]) L.S. Young announced a result like Theorem 1.1 during a conference on Lyapunov Exponents in May 1990 at Oberwolfach. It is published in her preprint [Yo], which we received, after this paper was submitted. In both situations also a central limit theorem like our Theorem 1.2.1 is proved.

Remark 1.2. The invariant density $h(x)$ can be estimated from above as follows:

$$h(x) \leqq \text{const} \cdot \sum_{i=1}^{\infty} |DT^{i-1}(Tc)|^{-1/l} |x - T^i c|^{-(1-1/l)} .$$

This is proved, even for more general maps, in [N4]. From our construction of the density h we can derive only a weaker estimate. L.S. Young [Yo], however, obtained from her construction an estimate similar to the one above. For Misiurewicz maps an estimate of this type was already obtained by Szewc [Sz].

Now consider $F:[0,1] \to \mathbf{R}$ of bounded variation or $F(x) = \log|T'(x)|$ and define random variables $S_n(x) = \sum_{i=0}^{n-1} F(T^i x)$ on the probability space $([0,1], m)$. It follows from Theorem 1.1 that

$$0 \leqq \sigma_F^2 := \text{Var}(F) + 2 \cdot \sum_{n=1}^{\infty} \text{Cov}(F, F \circ T^n) < \infty ,$$

where Var and Cov denote variance and covariance respectively (compare e.g. [Rou]). (If $F = \log|T'|$, which is not of bounded variation, one has to approximate F by $F_n = \max\{F, -n\}$.)

Theorem 1.2. 1. *The process $(S_n)_{n>0}$ satisfies the following central limit theorem:*

$$\text{Law}(n^{-1/2} S_n - \textstyle\int F\, d\mu) \Rightarrow \mathcal{N}(0, \sigma_F^2) .$$

($\mathcal{N}(0,0)$ *denotes the point mass in* 0.)
2. *Suppose $\sigma_F^2 > 0$. Then $(S_n)_{n>0}$ satisfies the following large deviations estimate: For each sufficiently small $\varepsilon > 0$ there is $-\infty \leqq \alpha(\varepsilon) < 0$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log m \left\{ \left| \frac{1}{n} S_n - \int F\, d\mu \right| > \varepsilon \right\} = \alpha(\varepsilon) .$$

Remark 1.3. The possible occurrence of $\sigma_F^2 = 0$ is discussed in [Rou, PUZ]. In particular, $\sigma_F^2 > 0$ if F is an indicator function. For Misiurewicz maps (see Remark 1.2) and for $F = \log|T'|$ see also [Zi2].

Remark 1.4. Stronger limit theorems such as invariance principles can be proved along the lines of [HK1] and [Ry]: First use the spectral properties of $\hat{\mathscr{L}}_{\hat{\psi}}$ (see Corollary 2.1) to show that the itinerary process $(I_n)_{n\in\mathbf{N}}$ (defined by $I_n(x) = L$ if $T^n x < c$ and $I_n(x) = R$ if $T^n x > c$) is absolutely regular (= weakly Bernoulli) with exponential mixing rate, and then apply general results from the theory of stationary stochastic processes. For Misiurewicz maps this was done in [Zi1]. The proof there relies on a spectral representation theorem for Perron–Frobenius operators of Misiurewicz maps given in [Sz] which is similar to our Theorem 2.1. However, we must say that we were not able to follow all the arguments used in [Sz], namely his assertions (5.30) and (6.18).

Remark 1.5. Using our Proposition 4.1, one can proceed as in [Rou] to prove convergence rates in the central limit theorem and a local limit theorem.

Remark 1.6. If T is a finitely renormalizable Collet–Eckmann map, then, as we remarked above, it has a periodic interval I of some period p, and the dynamical system $(I, T^p, \mu|_I)$ is mixing. Theorems 1.1 and 1.2 hold also for this system.

For the next theorem let $\mathrm{Per}_n = \{x \in [0, 1] : T^n x = x\}$. The sets Per_n are finite, and we consider discrete probability distributions ν_n on Per_n with probabilities $\nu_n(x)$ proportional to $|DT^n(x)|^{-1}$. These distributions reflect the fact that a periodic orbit is detected all the easier the more stable it is.

Consider again $F: [0, 1] \to \mathbf{C}$ of bounded variation or $F = \log|T'|$, and define random variables S'_n on the probability space (Per_n, ν_n) by

$$S'_n(x) = \sum_{i=0}^{n-1} F(T^i x) .$$

Theorem 1.3. 1. *The process* $(S'_n)_{n>0}$ *satisfies the following central limit theorem:*

$$\mathrm{Law}(n^{-1/2} S'_n - \int F\, d\mu) \Rightarrow \mathcal{N}(0, \sigma_F^2) .$$

2. *Suppose* $\sigma_F^2 > 0$. *Then* $(S'_n)_{n>0}$ *satisfies the following large deviations estimate: For each sufficiently small* $\varepsilon > 0$ *there is* $-\infty \leqq \alpha(\varepsilon) < 0$ *such that*

$$\lim_{n \to \infty} \frac{1}{n} \log \nu_n \left\{ \left| \frac{1}{n} S'_n - \int F\, d\mu \right| > \varepsilon \right\} = \alpha(\varepsilon) .$$

Remark 1.7. If $F = \log|T'|$, the theorem says that ν_n-typical periodic orbits have Lyapunov exponents very close to the exponent of m-a.e. trajectory.

Analogues of Remarks 1.5 and 1.6 apply also to $(S'_n)_{n \in \mathbf{N}}$.

Remark 1.8. Dynamical zeta functions and their relations to transfer operators have been first studied by Ruelle, see e.g. [Rue]. A very complete account of transfer operators, zeta functions and the distribution of periodic orbits for subshifts of finite type and Axiom A systems is the recent book by Parry and Pollicott [PP]. Results like Theorem 1.3 for Axiom A flows were obtained without using zeta functions by Lalley [La].

2. Outline of the Main Results and Scheme of the Proofs

The strategy of proofs is the following: We extend the system $([0, 1], T)$ to a system $(\hat{X}, \hat{T})$ (Sects. 2.1, 3), which inherits local properties (such as metric and derivative) of $([0, 1], T)$. On $\hat{X}$ we introduce a new metric (Sect. 6.2) in which some iterate of $\hat{T}$ is uniformly expanding (Proposition 6.3). We find isolated eigenvalues of the Perron–Frobenius operator (a particular transfer operator) in the new metric (2.1) and deduce mixing properties of the invariant density on $\hat{X}$ and on $[0, 1]$ (Corollary 2.1 and Appendix B).

In order to prove Theorem 1.3 we use zeta functions (Sect. 2.3) which are related to the characteristic functions of the measures ν_n on periodic points. We prove a 1-1 correspondence between poles of zeta functions and isolated eigenvalues of corresponding transfer operators (Theorem 2.2), and using analytic perturbation theory for linear operators (Proposition 4.2 and Sect. 5)) we deduce Theorem 1.3.

2.1. Markov Extensions. An essential tool in this paper is the Markov exstension of T, which was used in [K4] to study Lyapunov exponents of maps with negative Schwarzian derivative and in [BK] to investigate the relation between the poles of zeta-functions and eigenvalues of transfer operators for T in (abstract) hyperbolic situations.

The Markov extension of T (more exactly: of the dynamical system $([0, 1], T)$) is a dynamical system $(\hat{X}, \hat{T})$ together with a factor map $\pi: \hat{X} \to [0, 1]$. The state space $\hat{X}$ is a countable union of intervals $\hat{D}_i$, $(i \in \mathbf{N})$, which are disjoint copies of subintervals D_i of $[0, 1]$. One should think of it as an infinite tower of intervals over the basis $[0, 1]$. π maps all points $\hat{x} \in \hat{X}$ from the same vertical fiber to the same base point $x = \pi(\hat{x})$. The requirement $\pi \circ \hat{T} = T \circ \pi$ means that $\hat{T}$ acts horizontally just like T but may (and will) push a point $\hat{x}$ from one level to another. Hence it makes sense to talk about the derivative $\hat{T}'$ of $\hat{T}$, namely $\hat{T}' = T' \circ \pi$. Also, as $\hat{X}$ is a countable union of intervals, it is natural to specify a Lebesgue measure $\hat{m}$ on $\hat{X}$ by $\hat{m}_{|\hat{D}_i} \circ \pi^{-1} = m_{|D_i} (i \in \mathbf{N})$. Further details of this construction are given in Sect. 3.

2.2. Transfer Operators. Many aspects of the dynamics of T and $\hat{T}$ can be described by *transfer operators* $\mathscr{L}_\varphi$ and $\hat{\mathscr{L}}_{\hat{\varphi}}$ associated with these transformations. For $\varphi \in \mathbf{C}^{[0,1]}$ we define

$$\mathscr{L}_\varphi: \mathbf{C}^{[0,1]} \to \mathbf{C}^{[0,1]}, \quad \mathscr{L}_\varphi f(x) = \sum_{y \in T^{-1}x} \varphi(y) f(y) ,$$

and analogously $\hat{\mathscr{L}}_{\hat{\varphi}}: \mathbf{C}^{\hat{X}} \to \mathbf{C}^{\hat{X}}$ for $\hat{\varphi} \in \mathbf{C}^{\hat{X}}$. $\mathscr{L}_\varphi$ is obviously a well defined linear operator, whereas one has to be careful in defining $\hat{\mathscr{L}}_{\hat{\varphi}}$ since its definition involves a possibly infinite summation ($\hat{T}$ is an infinite-to-one transformation). Occasionally we write $\hat{\mathscr{L}}[\hat{\varphi}]$ instead of $\hat{\mathscr{L}}_{\hat{\varphi}}$.

Of immediate interest are the transfer functions $\psi = 1/|T'|$ and $\hat{\psi} = 1/|\hat{T}'|$. They not only give rise to positive operators, but $\mathscr{L}_\psi$ also has the property that for any two bounded measurable $f, g: [0, 1] \to \mathbf{C}$,

$$\int f \cdot (g \circ T)\, dm = \int (\mathscr{L}_\psi f) \cdot g\, dm .$$

In particular, if h is a probability density on $[0, 1]$ satisfying $\mathscr{L}_\psi h = h$, then $\mu = hm$ is a T-invariant measure and mixing properties of the system (T, μ) are reflected by spectral properties of $\mathscr{L}_\psi$, cf. [HK1]. This operator is traditionally called Perron–Frobenius operator.

The same holds for $\hat{\mathscr{L}}_{\hat{\psi}}$ with respect to the measure $\hat{m}$, the Lebesgue measure on $\hat{X}$. Also (cf. [K2, Lemma 4.6]),

$$\text{if } \hat{\mathscr{L}}_{\hat{\psi}} \hat{h} = \hat{h} \quad \text{and} \quad h(x) = \sum_{\hat{x} \in \pi^{-1}x} \hat{h}(\hat{x}), \quad \text{then } \mathscr{L}_\psi h = h . \tag{2.1}$$

Unfortunately, in the case of maps with critical points, the special transfer functions ψ and $\hat{\psi}$ are unbounded and may give rise to transfer operators with very unpleasant spectral properties. One way to overcome this difficulty is to consider a transfer function $\hat{\Psi}$ which is multiplicatively cohomologous to $\hat{\psi}$,

$$\hat{\Psi} = \hat{\psi} \cdot \frac{\hat{w}}{\hat{w} \circ \hat{T}} \tag{2.2}$$

with a weight function $\hat{w}$, $\hat{w}(\hat{x}) \neq 0$ for all $\hat{x}$. It follows from the definition of $\hat{\mathscr{L}}_{\hat{\Psi}}$ that

$$\hat{\mathscr{L}}_{\hat{\psi}}(\hat{w} \cdot \hat{f}) = \hat{w} \cdot \hat{\mathscr{L}}_{\hat{\Psi}}(\hat{f}) . \tag{2.3}$$

In Sect. 4 we investigate $\hat{\mathscr{L}}_{\hat{\Psi}}$ for particular weight function $\hat{w}$, for which this operator leaves the space $\widehat{BV}$ of functions of bounded variation $\hat{f}: \hat{X} \to \mathbf{C}$ invariant. More precisely, as in [BK] we define

$$\widehat{BV} := \{\hat{f} \in \mathbf{C}^{\hat{X}} : \|\hat{f}\|_{\widehat{BV}} < \infty\} ,$$

where

$$\|\hat{f}\|_{\widehat{BV}} := \sum_{i \in \mathbf{N}} \left(\operatorname{var}_{\hat{D}_i}(\hat{f}) + \sup_{\hat{D}_i} |\hat{f}| \right) .$$

$\widehat{BV}$ equipped with the norm $\|\cdot\|_{\widehat{BV}}$ is Banach space.

Recall that

$$\lambda_c = \liminf_{n \to \infty} \sqrt[n]{|DT^n(Tc)|} \text{, and let} \tag{2.4}$$

$$\lambda_{\text{per}} := \inf\{\bar{\lambda}(z) : T^n z = z \text{ for some } n \in \mathbf{N}\} > 1 , \tag{2.5}$$

$$\lambda_\eta := \liminf_{n \to \infty} \{|\eta|^{-1/n} : \eta \text{ is the biggest monotonicity interval of } T^n\} . \tag{2.6}$$

Denote

$$\lambda_H := \min\{\lambda_c, \lambda_{\text{per}}\} \text{ and } \lambda_E := \min\{\lambda_H^{1/l}, \lambda_\eta\} . \tag{2.7}$$

In Sect. 6 we discuss the relations between the various λ's, and we see that $\lambda_E > 1$ follows from our basic assumption $\lambda_c > 1$.

Along the lines of [BK] we prove the following spectral theorem in Sect. 4.1:

Theorem 2.1. *Suppose T is a Collet–Eckmann map satisfying* (1.2). *For each $\Theta > \lambda_E^{-1}$ there is a weight function $\hat{w}$ defining $\hat{\Psi}$ such that the transfer operator $\hat{\mathscr{L}}_{\hat{\Psi}} : \widehat{BV} \to \widehat{BV}$ is quasicompact with spectral radius* 1 *and essential spectral radius $r_{\text{ess}} < \Theta$.*

This means that, for each $\Theta > \lambda_E^{-1}$, the weight function can be chosen such that the operator $\hat{\mathscr{L}}_{\hat{\varphi}}$ can be decomposed as

$$\hat{\mathscr{L}}_{\hat{\Psi}} = \sum_{i=1}^{N(\Theta)} \rho_i(\hat{\mathscr{P}}_i + \hat{\mathscr{N}}_i) + \hat{\mathscr{P}}\hat{\mathscr{L}} , \tag{2.8}$$

where $\hat{\mathscr{P}}_i$, for $i = 1, \ldots, N(\Theta)$, and $\hat{\mathscr{P}}$ are projections commuting with $\hat{\mathscr{L}}_{\hat{\Psi}}$ and such that $\hat{\mathscr{P}}_i\hat{\mathscr{P}}_j = \hat{\mathscr{P}}_i\hat{\mathscr{P}} = 0$ for $i \neq j$ and $\hat{\mathscr{P}} + \sum_i \hat{\mathscr{P}}_i = \text{Id}$. For each $i = 1, \ldots, N(\Theta)$ we have, $|\rho_i| > \Theta$, $\operatorname{rank}(\hat{\mathscr{P}}_i) < \infty$, and $\hat{\mathscr{N}}_i$ is nilpotent with $\hat{\mathscr{P}}_i\hat{\mathscr{N}}_i = \hat{\mathscr{N}}_i\hat{\mathscr{P}}_i = \hat{\mathscr{N}}_i$. Finally $\|\hat{\mathscr{P}}\hat{\mathscr{L}}^n\|_{\widehat{BV}} \leq \text{const} \cdot \Theta^n$.

Remark 2.1. Further special features of the particular transfer function $\hat{\Psi}$ are: $\rho_1 = 1$, and the set $\{\rho_i : |\rho_i| = 1\}$ is a cyclic group of simple eigenvalues. In particular, the corresponding $\hat{\mathscr{N}}_i$ are identically 0.

Remark 2.2. One cannot expect a better estimate of the essential spectral radius than $r_{\text{ess}} \leq \lambda_{\text{per}}^{-1}$. For the full parabola $Tx = 4x(1 - x)$, which is conjugated to the tent map with slope 2, the essential spectral radius equals 1/2, and one has $\lambda_{\text{per}} = 2 = \sqrt{4} = \lambda_c^{1/l}$.

Although we are quite far from proving it, we conjecture that the essential spectral radius is equal to $\lambda_{\mathrm{per}}^{-1}$ if T is uniformly hyperbolic on periodic points. Even more, there might be a functional analytic setting such that this statement makes sense for any S-unimodal map including those with stable periodic orbits and those with a solenoidal attractor.

Let $\widehat{BV}_{\hat{w}} = \{\hat{f}\hat{w} : \hat{f} \in \widehat{BV}\}$, and for $\hat{g} \in \widehat{BV}_{\hat{w}}$ let $\|\hat{g}\|_{\hat{w}} := \|\hat{g}\hat{w}^{-1}\|_{\widehat{BV}}$. It is obvious that $(\widehat{BV}_{\hat{w}}, \|\cdot\|_{\hat{w}})$ is a Banach space, isometrically isomorphic to $(\widehat{BV}, \|\cdot\|_{\widehat{BV}})$. Because of relation (2.3) we obtain immediately

Corollary 2.1. *$\hat{\mathscr{L}}_{\hat{\psi}} : \widehat{BV}_{\hat{w}} \to \widehat{BV}_{\hat{w}}$ is quasicompact with spectral radius* 1 *and essential spectral radius $r_{\mathrm{ess}} < 1$. The spectral decomposition from* (2.8) *carries over to $\hat{\mathscr{L}}_{\hat{\psi}}$.*

Theorem 1.1 follows from Corollary 2.1 by a standard calculation (see Appendix B). The central limit part of Theorem 1.2 can be deduced from Theorem 2.1 as in [K1], where a central limit theorem of Gordin is applied. There is, however, a more classical way to do this, which was introduced in [Rou], and which can be modified to yield a proof of the large deviations part of Theorem 1.2. It uses the relation between more general transfer operators and Fourier or Laplace transforms of the random variables S_n introduced in Sect. 1.

For $F : [0, 1] \to \mathbf{C}$ of bounded variation and $\beta, t \in \mathbf{C}$ let

$$\varphi(x) = \psi^{\beta}(x) \cdot e^{t \cdot F(x)}, \quad \hat{\varphi}(\hat{x}) = \hat{\psi}^{\beta}(\hat{x}) \cdot e^{t \cdot F(\pi\hat{x})}, \quad \hat{\Phi}(\hat{x}) = \hat{\Psi}^{\beta}(\hat{x}) \cdot e^{t \cdot F(\pi\hat{x})} . \tag{2.9}$$

In Sect. 4 we actually prove that, for β close to 1 and sufficiently small $|t|$, Theorem 2.1 and Corollary 2.1 remain valid for $\hat{\mathscr{L}}[\hat{\Phi}]$ and $\hat{\mathscr{L}}[\hat{\varphi}]$ instead of $\hat{\mathscr{L}}[\hat{\Psi}]$ and $\hat{\mathscr{L}}[\hat{\psi}]$ except that the spectral radius needs no longer be 1. (For arbitrary $\beta, t \in \mathbf{C}$ it may happen that the essential spectral radius and the spectral radius coincide, i.e. that the operator has no isolated leading eigenvalues.) This fact, combined with general analytic perturbation theory, is used in Sect. 5 to prove Theorem 1.2 from the Introduction. Let $\beta = 1$. The transfer operators are linked to these probabilistic results by the fact that $\int \hat{\mathscr{L}}^n[\hat{\varphi}]\hat{h}\,d\hat{m}$ (which is a function of t) is just the Fourier or Laplace transform of S_n when t is purely imaginary or purely real respectively. For the function $F = \log|T'|$, which is not of bounded variation, we set $t = 0$, and $\int \hat{\mathscr{L}}^n[\hat{\varphi}]\hat{h}\,d\hat{m}$, as a function of $\beta - 1$, is again the transform of S_n.

For a more comprehensive discussion of the various approaches to probabilistic limit theorems for mixing transformations see [K2, Sect. 9].

2.3. Zeta Functions. In order to relate the distribution of typical periodic orbits to the invariant measure μ, we study dynamic zeta functions

$$\zeta[\varphi](z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \zeta_n[\varphi] \right),$$

where

$$\zeta_n[\varphi] = \sum_{x \in \mathrm{Per}_n} (\varphi(x) \cdot \varphi(Tx) \cdot \ldots \cdot \varphi(T^{n-1}x)) .$$

$\hat{\zeta}[\hat{\varphi}](z)$, and $\hat{\zeta}[\hat{\Phi}](z)$ are defined analogously. In fact, using $\hat{\varphi} = \varphi \circ \pi$,

$$\zeta[\varphi] = \hat{\zeta}[\hat{\varphi}] = \hat{\zeta}[\hat{\Phi}] . \tag{2.10}$$

The first equality is a consequence of elementary facts about Markov extensions of unimodal maps, see (3.5), whereas the second one follows immediately from the relation

$$\hat{\Phi}(\hat{x}) = \varphi(\pi\hat{x}) \cdot \left(\frac{\hat{w}(\hat{x})}{\hat{w}(\hat{T}\hat{x})} \right)^{\beta},$$

see (2.9) and (2.2). Our main result is

Theorem 2.2. *Suppose T is a Collet–Eckmann map satisfying* (1.2). *Let φ and $\hat{\Phi}$ be transfer functions as in* (2.9) *depending on parameters β and t, and assume $\Re\beta > 0$. Define*

$$\vartheta = \vartheta[\hat{\Phi}] = \lim_{n\to\infty} \sqrt[n]{\sup_{\hat{x}} |\hat{\Phi}(\hat{x}) \cdot \hat{\Phi}(\hat{T}\hat{x}) \cdot \ldots \cdot \hat{\Phi}(\hat{T}^{n-1}\hat{x})|}\,.$$

Then

1. *$\zeta[\varphi](z) = \hat{\zeta}[\hat{\Phi}](z)$ is meromorphic and nonzero in $\{|z| < \vartheta^{-1}\}$.*
2 *$\mathscr{L}_{\hat{\Phi}}: \widehat{BV} \to \widehat{BV}$ has essential spectral radius $r_{\mathrm{ess}} \leqq \vartheta$, and if $|z| < \vartheta^{-1}$, then z is a pole of ζ with multiplicity k if and only if z^{-1} is an eigenvalue of $\mathscr{L}_{\hat{\Phi}}$ with multiplicity k.*
3. *For each $\Theta > \lambda_E^{-1}$ there is a weight function $\hat{w}$ such that for small t and β close to 1 holds $\vartheta[\hat{\Phi}] > \Theta$.*

Assertions 1 and 2 follow directly from Proposition 4.3. For $t = 0$ and $\beta = 1$, the third assertion is a direct consequence of Proposition 6.3, and the argument extends to t and $\beta - 1$ close to 0, because $\vartheta[\hat{\Phi}]$ depends continuously on t and β.

The probabilistic results on the distribution of typical periodic orbits given in the introduction follow from this theorem similarly as Theorem 1.2 follows from the spectral representation: Defining φ_t as in (2.9), it is easily checked that $\zeta_n[\varphi_t]/\zeta_n[\psi]$ (as a function of t) is just the Fourier respectively Laplace transform of the random variable S_n' defined in the introduction. Theorem 2.2 allows to expand this transform in powers of the isolated eigenvalues of $\hat{\mathscr{L}}[\hat{\varphi}_t]$ and $\hat{\mathscr{L}}[\hat{\psi}]$, and analytic perturbation theory for isolated eigenvalues links these facts to the probabilistic statements made in Theorem 1.3. More details are given in Sect. 5, and a general account of these ideas (presumably a kind of folklore knowledge) is [K5].

3. Markov Extensions

In this section we define the Markov extension of the dynamical system ([0, 1], T). As a purely topological construction it was introduced in a series of papers by Hofbauer, see e.g. [Ho1, Ho2]. A piecewise smooth version of it was used in [K2] and [BK] for investigating transfer operators and zeta functions of piecewise expanding maps, and that proved also useful for studying ergodic properties of S-unimodal maps.

Compared to ([0, 1], T) the Markov extension has two advantages: The critical trajectory has no accumulation points, and the extension has a countable Markov partition, where each member of the partition is an interval and is mapped onto a finite union of intervals from the partition. The price one must pay for these convenient properties is the non-compactness of the state space.

Before we start the construction of the Markov exstension, we must, for technical reasons, modify our original system ([0, 1], T) on the countable set of points $\bigcup_{n\geq 0} T^{-n}\{c\}$. We can either double all these points and extend T to the enlarged space by taking one-sided limits (see e.g. [BK, Sect. 1]), or we delete this set from the state space (cf. [K4, Sect. 3]). In neither case the zeta functions are affected, because the critical point cannot be periodic under the C–E assumption. Also the spectral properties of transfer operators are unchanged by this procedure, see [BK, Prop. 1.1]. Therefore, abusing slightly the notations, we shall forget this doubling. We write (a, b) for any intervals with endpoints a and b.

3.1. Cylinders and Their Images. Let $\mathscr{Z}_n$ be the partition of [0, 1] into maximal intervals of monotonicity of T^n. We call elements of $\mathscr{Z}_n$ cylinders of order n and denote them by η. $\eta_n[x]$ is the cylinder of order n containing x. Note that cylinders are open-closed after doubling c and its preimages. In particular, $\eta_n[x]$ is unambigously defined in this modified state space.

If $\eta\in\mathscr{Z}_n$, then (with two exceptions) $\eta=(\alpha,\beta)$, with $DT^n(\alpha)=DT^n(\beta)=0$, and there exist $r\neq s$, $0\leq r,s<n$ such that $T^r(\alpha)=c=T^s(\beta)$. Therefore $T^n(\eta)=(c_{n-r},c_{n-s})$, where c_m denotes $T^m(c)$. The two exceptions are the first and the last cylinder (in the sense of the order on the interval [0, 1]) which are of the form $(0,z)$ and $(z',1)$ with $T^{n-1}(z)=T^{n-1}(z')=c$ and $T^n(0,z)=T^n(z',1)=(0,c_1)$.

Let us call the intervals $\eta_k[c^+]$ and $\eta_k[c^-]$ the central cylinders (of order $k>0$), where c^+ and c^- are the points obtained from c by doubling. The images of these two cylinders

$$D_k := T^k(\eta_k[c^+]) = T^k(\eta_k[c^-]) = (c_k, c_{\bar{k}})$$

coincide, $\bar{k}$ being well defined for any $k>1$. Namely we have $\eta_k[c^+]=(c^+,\alpha)$ with $T^s\alpha=c^\pm$ for some $s<k$, and $\bar{k}=k-s<k$. As T^k is monotone on $\eta_k[c^+]$, it follows that $T^{\bar{k}}$ is monotone on $T^{k-\bar{k}}\eta_k[c^+]$, which contains $c^\pm$. But this means that $T^{k-\bar{k}}\eta_k[c^\pm]$ is contained in $\eta_{\bar{k}}[c^+]$ or in $\eta_{\bar{k}}[c^-]$. Hence

$$D_k = T^k(\eta_k[c^+]) \subseteq T^{\bar{k}}(\eta_{\bar{k}}[c^\pm]) = D_{\bar{k}} \quad \text{and} \quad c\in D_{k-\bar{k}}\,. \tag{3.1}$$

Additionally we denote $D_0=[0,1]$, $D_1=(0,c_1)$ and $c_{\bar{1}}=0$. It follows that for the two exceptional cylinders of order n, $T^n(\eta)=D_1$. For any other $\eta\in\mathscr{Z}_n$ there exist $0\leq r<s<n$ with $T^n(\eta)=(c_{n-r},c_{n-s})$ and such that $T^r(\eta)$ is a central cylinder of order $n-r$, $T^n(\eta)=D_{n-r}$ and $\overline{n-r}=n-s$.

Consider $\eta\in\mathscr{Z}_n$ with $T^n(\eta)=D_k=(c_k,c_{\bar{k}})$, $0<k\leq n$. If $c\in D_k$, then $\eta\notin\mathscr{Z}_{n+1}$, but $\eta=\eta^+\cup\eta^-$ (up to doubling of the endpoints). Both η^+,η^- are in $\mathscr{Z}_{n+1}$, and $T^{n+1}(\eta^+)=D_{k+1}=(c_{k+1},c_1)$, $T^{n+1}(\eta^-)=D_{\bar{k}+1}=(c_{\bar{k}+1},c_1)$. In this case $\overline{k+1}=1=\overline{\bar{k}+1}$. On the other hand, if $c\notin D_k$, then $\eta\in\mathscr{Z}_{n+1}$, $T^{n+1}(\eta)=D_{k+1}=(c_{k+1},c_{\bar{k}+1})$ and $\overline{k+1}=\bar{k}+1$.

3.2. The Extension. Now we define the extension of T acting on the tower $\hat{X}\subset[0,1]\times\mathbb{N}$. Let $\hat{X}:=\bigcup_{k=0}^{\infty}\hat{D}_k$ be the union of disjoint copies of the intervals

D_k, where $\hat{D}_k = \{\langle x, k\rangle | x \in D_k\}$. Denote $\hat{\mathscr{D}} = \{\hat{D}_k\}_{k \geq 0}$. We define $\hat{T}: \hat{X} \to \hat{X}$ in the following way: Let $\hat{x} = \langle x, k\rangle \in \hat{D}_k$,

$$\hat{T}\hat{x} = \begin{cases} \langle Tx, 1\rangle, & \text{if } k = 0\,, \\ \langle Tx, k+1\rangle, & \text{if } k > 0 \text{ and } c \notin D_k \text{ or } x \in (c_k, c) \subset D_k\,, \\ \langle Tx, \bar{k}+1\rangle, & \text{if } k > 0 \text{ and } c \in D_k \text{ and } x \in (c, c_{\bar{k}})\,. \end{cases}$$

Recall that $\hat{T}^n$ is well defined, because we doubled c and all its preimages. Let us call the two natural projections π and κ, so that $\hat{x} = \langle \pi(\hat{x}), \kappa(\hat{x})\rangle$. We say that $\hat{x}$ is climbing if $\kappa(\hat{T}\hat{x}) = \kappa(\hat{x}) + 1$ and jumping otherwise (then $\kappa(\hat{T}\hat{x}) = \bar{k}(\hat{x}) + 1 \leq \kappa(\hat{x})$, equality possible only for $\kappa = 1$ or 2). If $c \in D_k$, $k > 0$, then we say that $\hat{D}_k$ is a splitting level – both climbing and jumping are here possible. $\hat{D}_{k-\bar{k}}$ is always a splitting level.

As $\hat{T}$ acts locally just like T (modulo climbing and jumping), $\hat{T}'(\hat{x}) = T'(\pi(\hat{x}))$, and the critical points of $\hat{T}^n$ are the preimages of the critical point of T^n under π^{-1}. The critical values of $\hat{T}^n$ are the endpoints of $\hat{D}_k$, $1 < k \leq n$ and the point $\langle c_1, 1\rangle$. One of the endpoints of $\hat{D}_k$, namely $\langle c_k, k\rangle$, is climbing for ever, the other ($\langle c_{\bar{k}}, k\rangle$) is climbing to the next splitting level, say $\hat{D}_s$, and then jumps back to $\langle c_{\bar{s}+1}, \bar{s}+1\rangle$.

So all critical points of $\hat{T}^n$, $n > 0$, are eventually climbing forever and they are the only points with this property, because T and hence $\hat{T}$ has no homtervals (= intervals on which all iterates of T are monotone).

The notion of cylinders extends as follows: $\hat{\eta} \in \hat{\mathscr{Z}}_n$ iff $\hat{\eta} = \hat{D}_k \cap \pi^{-1}\eta$ for some k and $\eta \in \mathscr{Z}_n$. Such $\hat{\eta}$'s are maximal intervals of monotonicity of $\hat{T}^n$.

The system $(\hat{X}, \hat{T})$ has Markov property in the following sense: By definition $\hat{X} := \bigcup_{\mathscr{D}} \hat{D}_k$ and $\hat{T}\hat{D}_k = \hat{D}_{k+1}$ if there is no splitting or $\hat{T}\hat{D}_k = \hat{D}_{k+1} \cup \hat{D}_{\bar{k}+1}$ if $\hat{D}_k$ is a splitting level. For a cylinder $\hat{\eta} = \hat{D}_k \cap \pi^{-1}\eta \in \hat{\mathscr{Z}}_1$ this means that $\hat{T}\hat{\eta} = \hat{D}_j$, where $D_j = T(D_k \cap \eta)$, and by induction one infers:

$$\text{If } \hat{\eta} = \hat{D}_k \cap \pi^{-1}\eta \text{ with } \eta \in \mathscr{Z}_n, \text{ then } \hat{T}^n\hat{\eta} = \hat{D}_j \in \hat{\mathscr{D}} \text{ and } D_j = T^n(D_k \cap \eta)\,. \tag{3.2}$$

We note the following consequences of this assertion for later reference:

$$\hat{T}^n\hat{\eta} \in \hat{\mathscr{D}} \text{ for } \eta \in \mathscr{Z}_n. \tag{3.3}$$

If $\pi\hat{\eta} = \pi\hat{\eta}'$ for $\hat{\eta}, \hat{\eta}' \in \hat{\mathscr{Z}}_n$, then $\hat{T}^n\hat{\eta} = \hat{T}^n\hat{\eta}'$. (If c is eventually periodic with period p, then it might be necessary to restrict to $n > p$. Otherwise $D_i \neq D_j$ whenever $i \neq j$.) Any $\hat{x} \in \hat{\eta}$ has a *brother* $\hat{x}' \in \hat{\eta}'$ such that $\pi\hat{x} = \pi\hat{x}'$ and $\hat{T}^n\hat{x} = \hat{T}^n\hat{x}'$ and hence $D\hat{T}^n(\hat{x}) = D\hat{T}^n(\hat{x}')$. (3.4)

Hence the trajectories of two points of the same fiber (i.e. in $\pi^{-1}x$ for some $x \in [0, 1]$) which are in the interior of the tower (i.e. are not endpoints of an interval $\hat{D}_k$) will collapse after some iterations. As a consequence we have:

If $\hat{T}^n\hat{x} = \hat{x}$, then $T^n(\pi\hat{x}) = \pi\hat{x}$, and conversely, if $T^n x = x$, then there is a unique $\hat{x} \in \pi^{-1}x$ such that $\hat{T}^n\hat{x} = \hat{x}$. (3.5)

The first of these assertions is trivial. The second one is proved as follows: If $x = 0$, then $\hat{T}\langle x, 0\rangle = \hat{T}\langle x, 1\rangle = \langle x, 1\rangle$, and for $k > 1$ either $\langle x, k\rangle \notin \hat{D}_k$, or (in the case of a full unimodal map) $x = c_k$ and hence $\hat{T}\langle x, k\rangle = \langle x, k+1\rangle$. So suppose $x \neq 0$ and consider $\langle x, 0\rangle \in \hat{D}_0$. As c is not periodic, $x \notin \bigcup_{k \geq 0} T^{-k}\{c\}$, and as the only points in $\hat{D}_0$ which are mapped under some iteration of $\hat{T}$ to an endpoint of some $\hat{D}_k$ are of the form $\langle z, 0\rangle$ with $z \in \bigcup_{k \geqq 0} T^{-k}\{c\}$, the orbit of $\langle x, 0\rangle$ is disjoint from the set of endpoints of the intervals $\hat{D}_k$. Now (3.4) implies that there is some $p > 0$ such that $\hat{T}^{pn}\langle x, 0\rangle = \hat{T}^{pn}(\hat{T}^n\langle x, 0\rangle)$, which means that $\hat{x} := \hat{T}^{pn}\langle x, 0\rangle$ is periodic under $\hat{T}^n$ and $\pi(\hat{x}) = T^{pn}x = x$. Suppose there is a further point $\hat{y} \in \pi^{-1}x$ with $\hat{T}^n\hat{y} = \hat{y}$. By the same arguments as above, there is some $p' > 0$ such that $\hat{x} = \hat{T}^{p'n}\hat{x} = \hat{T}^{p'n}\hat{y} = \hat{y}$.

The following is obtained from [HK2, Lemma 4.i] as in [BK, Lemma 3.2]. The proof is easy, and we suggest the reader to try it on its own in order to see, whether he understood the basic features of Markov extensions:

$$\text{If } \hat{\eta} \in \hat{\mathscr{Z}}_n \text{ and } \hat{T}^n\hat{\eta} \supseteq \hat{\eta}, \text{ then } \kappa(\hat{\eta}) \leqq 2n\,. \tag{3.6}$$

Consider now the splitting levels. For any $\hat{D}_k \in \mathscr{D}$ with $k > 0$ there exist $r < k \leqq s$ such that $\hat{D}_r$ and $\hat{D}_s$ are splitting levels (i.e. $c \in D_r$, D_s) and there is no splitting level in between. In particular, if $k > 1$, then $r = k - \bar{k} = s - \bar{s}$.

We call D_k^+ the subinterval of D_k which climbs onto (c, c_s) and D_k^- its complement which climbs onto $(c_{\bar{s}}, c)$, where $\hat{D}_s$, as just mentioned, is the first splitting level above (or equal to) $\hat{D}_k$. Clearly $D_k^+ = (z, c_k)$ and $D_k^- = (c_{\bar{k}}, z)$ with $T^{s-k}(z) = c$, for a well defined z, which is the preimage of c of smallest order in D_k. (3.7)

We may also speak in an obvious way about $\hat{D}_k^{\pm}$. Then $\hat{T}(\hat{D}_k^{\pm}) = \hat{D}_{k+1}^{\pm}$ for $r < k < s$. Following this line we write for $\hat{x} \in \hat{D}_k$,

$\tilde{k}(\hat{x}) = k$ if $\hat{x} \in \hat{D}_k^+$ and $\tilde{k}(\hat{x}) = \bar{k}$ if $\hat{x} \in \hat{D}_k^-$. Observe that if $\hat{\eta}$ is a cylinder, $\hat{\eta} \subset \hat{D}_k$, $\hat{\eta} \neq \hat{D}_k$, then $\tilde{k}$ is the same for all $\hat{x} \in \hat{\eta}$ and one can talk about $\tilde{k}(\hat{\eta})$. (3.8)

Finally we define for $k > 1$,

$$\hat{x}_{-\tilde{k}} := \langle y, k - \tilde{k}\rangle \text{ with } y \in \eta_{\tilde{k}}[c^+] \cup \eta_{\tilde{k}}[c^-] \text{ such that } \hat{T}^{\tilde{k}}\langle y, k - \tilde{k}\rangle = \hat{x}\,. \tag{3.9}$$

The point y is not always unambigously defined, it may be to the left or to the right of c. If this happens, we choose $y \in \eta_{\tilde{k}}[c^-]$; but note that $\hat{T}\langle y, k - \tilde{k}\rangle$ is the same for either choice. For $\hat{x} \in \hat{D}_0$ let $\hat{x}_{\tilde{0}} = \hat{x}$, and for $\hat{x} \in \hat{D}_i$ let $\hat{x}_{\tilde{1}} = \langle y, 0\rangle$, $y \in [0, c^-]$ such that $\hat{T}\langle y, 0\rangle = \hat{x}$.

The motivation for these definitions is that we want a point to shadow the critical trajectory $\mathscr{C}$. For $\hat{x} \in \hat{D}_k$ there are two candidates in $\mathscr{C}$, namely c_k and $c_{\bar{k}}$ to be shadowed. From them we select the closer one $c_{\tilde{k}}$, closer in the dynamically defined $\hat{D}^{\pm}$ sense, so that after the next splitting $\hat{x}$ may still shadow $c_{\tilde{k}}$ as they jump or climb together.

This is important, because by the C–E condition we control only initial segments of $\mathscr{C}$. As it will be explained later on, the trajectory of $\hat{x}$ will stick to a new initial segment of $\mathscr{C}$ as soon as it can, i.e. at the first time it reaches a D^- part after some splitting level. Then it shadows $\mathscr{C}$ through consecutive D^- to the next splitting and further through all following D^+ till the next opportunity (new D^- after splitting), when it sticks to a new initial segment of $\mathscr{C}$ again.

We use the notation $\hat{x}_n = \hat{T}^n\hat{x}$, but simple arithmetic make sense only on positive indices. By $\hat{x}_{-\tilde{j}+n}$ we mean $\hat{T}^n(\hat{x}_{-\tilde{j}})$, defined only for $\hat{x} \in \hat{D}_j$. Similarly $\hat{x}_{n-\tilde{k}} = (\hat{T}^n\hat{x})_{-\tilde{k}}$ makes sense only for $\hat{x}_n \in \hat{D}_k$.

Remark 3.1. We want to point out that the sets $\hat{D}_0$ and $\hat{D}_1$ are dynamically unimportant. Namely:

$$\hat{T}\hat{D}_0 = \hat{D}_1, \quad \hat{T}(\hat{D}_1 \cap \pi^{-1}[0, c^-]) = \hat{D}_1, \quad \hat{T}(\hat{D}_1 \cap \pi^{-1}[c^+, c_1]) = \hat{D}_2, \quad \text{and}$$

$$\hat{T}\left(\bigcup_{k \geqq 2} \hat{D}_k\right) = \bigcup_{k \geqq 2} \hat{D}_k \,.$$

4. Transfer Operators and Zeta Functions

The aim of this section is a proof of slight generalizations of Theorems 2.1 and 2.2. We proceed as in [BK] and approximate iterates $\hat{\mathscr{L}}^n$ at an exponential rate by some compact operators. This yields the spectral decomposition (2.8), which depends analytically on the parameters β and t of $\hat{\mathscr{L}}$. Finally we relate the eigenvalues of the operators $\hat{\mathscr{L}}$ to the poles of corresponding zeta functions.

4.1. The Essential Spectral Radius. For a transfer function $\hat{\varphi}: \hat{X} \to \mathbf{C}$ and $n \in \mathbf{N}$ let

$$\hat{\varphi}_n(\hat{x}) = \hat{\varphi}(\hat{x}) \cdot \hat{\varphi}(\hat{T}\hat{x}) \cdot \ldots \cdot \hat{\varphi}(\hat{T}^{n-1}\hat{x})$$

and

$$\vartheta[\hat{\varphi}] = \lim_{n \to \infty} \sqrt[n]{\sup\{|\hat{\varphi}_n(\hat{x})| : \hat{x} \in \hat{X}\}} \,.$$

As in Sect. 2 denote $\hat{\psi}(\hat{x}) = 1/|\hat{T}'(\hat{x})|$. In Sect. 6 we define a weight function $\hat{w}: \hat{X} \to]0, +\infty[$ for which we prove that the cohomologous transfer function $\hat{\Psi} = \hat{\psi} \cdot \dfrac{\hat{w}}{\hat{w} \circ \hat{T}}$ satisfies:

$$\vartheta[\hat{\Psi}] < 1 \quad \text{and} \quad \sup\{\mathrm{var}_{\hat{D}_i}(\hat{\Psi}^\beta)\} < \infty \tag{4.1}$$

for any $\beta \in \mathbf{C}$ with $\Re\beta > 0$ (see Propositions 6.3 and 6.2). It follows immediately that for each $F: [0, 1] \to \mathbf{C}$ of bounded variation and each $t \in \mathbf{C}$ the transfer function $\hat{\Phi} = \hat{\Psi}^\beta \cdot \exp(t\hat{F})$, where $\hat{F} = F \circ \pi$, satisfies:

$$\sup\left\{ \mathrm{var}_{\hat{D}_i}(\hat{\Phi}) + \sup_{\hat{D}_i} |\hat{\Phi}| : \hat{D}_i \in \mathscr{D} \right\} =: V(\beta, t, \hat{F}) < \infty \;, \tag{4.2}$$

and because of Proposition 6.2 $V(\beta, t, \hat{F})$ is a locally bounded function of the parameters β and t.

Having established this property we can apply the results of [BK, Sect. 2] to the operator $\hat{\mathscr{L}}_{\hat{\Phi}}$, which we denote in this section often just by $\hat{\mathscr{L}}$. For readers who want to check the use of these results carefully we note that $V = V(\beta, t, \hat{F})$ is related to the constant M in Eq. (2.3) of [BK] by $M \leqq 2V$.

Lemma 4.1. (Corollary 2.4 in [BK]) *For each* $\Theta > \vartheta[\hat{\Phi}]$ *there exists a constant* $C > 0$ *such that*

$$\|\hat{\mathscr{L}}^n_{\hat{\Phi}} \chi_{\hat{\eta}}\|_{\widehat{BV}}, \ \mathrm{var}_{\hat{\eta}}(\hat{\Phi}_n), \ \sup_{\hat{\eta}} |\hat{\Phi}_n| \leqq C \cdot \Theta^n \quad \textit{for all } n > 0 \textit{ and } \hat{\eta} \in \hat{\mathscr{Z}}_n \,.$$

The constant C is locally uniform in the parameters β and t.

The local uniformity of C is indeed not explicitly stated in [BK], but can be checked easily by following the (simple) proofs of these estimates.

We need some further notation: For each $n > 0$ and each $\hat{\eta} \in \hat{\mathscr{Z}}_n$ fix $\hat{x}_{\hat{\eta}} \in \hat{\eta}$ in such a way that $\hat{T}^n \hat{x}_{\hat{\eta}} = \hat{x}_{\hat{\eta}}$ if $\hat{\eta} \subseteq \hat{T}^n \hat{\eta}$ and arbitrarily otherwise. (Note that the Markov property of $(\hat{X}, \hat{T})$ implies that $\hat{\eta}$ and $\hat{T}^n \hat{\eta}$ are disjoint if $\hat{\eta} \nsubseteq \hat{T}^n \hat{\eta}$.) Then

$$\hat{\eta} \subseteq \hat{T}^n \hat{\eta} \text{ for some } \hat{\eta} \in \hat{\mathscr{Z}}_n \text{ if and only if } \hat{x}_{\hat{\eta}} \text{ is the only fix point for } \hat{T}^n \text{ in } \hat{\eta} . \tag{4.3}$$

Denote by $\hat{\mathscr{A}}_n$ the set of those $\hat{\eta} \in \hat{\mathscr{Z}}_n$ which are not among the two leftmost or the two rightmost $\hat{\eta}$ in the interval $\hat{D}_i$ they belong to. Then

$$\sup_i \operatorname{card}\{\hat{\eta} \in \hat{\mathscr{Z}}_n : \hat{\eta} \subseteq \hat{D}_i, \hat{\eta} \notin \hat{\mathscr{A}}_n\} \leqq 4 \tag{4.4}$$

and

$$\text{if } \hat{\eta} \in \hat{\mathscr{A}}_n \text{ is contained in } \mathscr{D}_i, \text{ then it is separated from the endpoints of } \hat{D}_i \text{ by a distance of at least } \min\{|\hat{\eta}|: \eta' \in \mathscr{Z}_n\} \text{ which depends on } n \text{ but not on } i . \tag{4.5}$$

As $\hat{\mathscr{Z}}_n = \pi^{-1} \mathscr{Z}_n \vee \mathscr{D}$, we have

$$\pi(\hat{\eta}) \in \mathscr{Z}_n \text{ and hence } \hat{T}^n \hat{\eta} \subseteq \hat{D}_0 \cup \ldots \hat{D}_n \text{ for each } \hat{\eta} \in \hat{\mathscr{A}}_n . \tag{4.6}$$

For $\hat{f} \in \widehat{BV}$ define

$$\hat{\alpha}_n \hat{f} = \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n} \hat{f}(\hat{x}_{\hat{\eta}}) \chi_{\hat{\eta}} .$$

$\hat{\alpha}_n : \widehat{BV} \to \widehat{BV}$ is linear, $\|\hat{\alpha}_n\|_{\widehat{BV}} \leqq 1$, and

$$\sum_{\hat{\eta} \in \hat{\mathscr{A}}_n} \|\hat{f}(\hat{x}_{\hat{\eta}}) \chi_{\hat{\eta}}\|_{\widehat{BV}} = \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n} 2|\hat{f}(\hat{x}_{\hat{\eta}})| \leqq 2 \cdot \operatorname{card}(\mathscr{Z}_n) \cdot \|\hat{f}\|_{\widehat{BV}} < \infty . \tag{4.7}$$

We remark that the particular choice of the grid $\{\hat{x}_{\hat{\eta}} : \hat{\eta} \in \hat{\mathscr{A}}_n\}$ for the approximation operator $\hat{\alpha}_n$ is important for estimations on the zeta function but not for the investigation of the spectral properties of $\hat{\mathscr{L}}$.

Lemma 4.2 (Proposition 2.7 in [BK]). *For each $\Theta > \vartheta[\hat{\Phi}]$ there is a constant $C > 0$ such that*

$$\|\hat{\mathscr{L}}^n_{\hat{\Phi}} \hat{\alpha}_n - \hat{\mathscr{L}}^n_{\hat{\Phi}}\|_{\widehat{BV}} \leqq C \cdot \Theta^n \quad \textit{for all } n > 0 .$$

The constant C is locally uniform in the parameters β and t.

The local uniformity follows again from the proof in [BK].

If the operators $\hat{\alpha}_n$ had finite rank, this lemma would imply immediately that the essential spectral radius ϑ of $\hat{\mathscr{L}}^n_{\hat{\Phi}}$ does not exceed $\vartheta[\hat{\Phi}]$. (For more details see the discussion in [K2, Sect. 2.A/B].) If $\hat{\Phi} = \Phi \circ \pi$, this is actually the case as is shown in [BK, Lemma 4.2]. Unfortunately our transfer functions do not have this property, and we have to modify the approach of [BK] in order to prove

Proposition 4.1. *The operators $\hat{\mathscr{L}}^n_{\hat{\Phi}} \hat{\alpha}_n : \widehat{BV} \to \widehat{BV}$ are compact. Therefore, in view of Lemma 4.2, the essential spectral radius ϑ of $\hat{\mathscr{L}}^n_{\hat{\Phi}}$ does not exceed $\vartheta[\hat{\Phi}]$, and $\hat{\mathscr{L}}_{\hat{\Phi}}$ has* *a spectral decomposition as in* (2.8). (*If* ϑ *coincides with the spectral radius of* $\hat{\mathscr{L}}_{\hat{\Phi}}$, *then* $N(\Theta) = 0$ *in* (2.8).)

Proof. For notational convenience, we prove the proposition only for $\beta = t = 1$. However, exactly the same proof works for general $\beta, t \in \mathbf{C}$ with $\Re\beta > 0$.

For $\eta \in \mathscr{Z}_n$ denote

$$\hat{\mathscr{A}}_n(\eta) = \{\hat{\eta} \in \hat{\mathscr{A}}_n : \pi\hat{\eta} = \eta\} .$$

Suppose $\hat{T}^n\hat{\eta} = \hat{D}$ for some $\hat{\eta} \in \hat{\mathscr{A}}_n(\eta)$ and some $\hat{D} \in \mathscr{D}$. Then $T^n\eta = D$, and by (3.4), $\hat{T}^n\hat{\eta} = \hat{D}$ for each $\hat{\eta} \in \hat{\mathscr{A}}_n(\eta)$.

Let $\hat{\mathscr{L}} = \mathscr{L}[\hat{\Phi}]$. For $\hat{f} \in \widehat{BV}$

$$\hat{\mathscr{L}}^n\hat{\alpha}_n(\hat{f}) = \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n} \hat{f}(\hat{x}_{\hat{\eta}}) \cdot \hat{\mathscr{L}}^n\chi_{\hat{\eta}} = \sum_{n \in \mathscr{Z}_n} \hat{\mathscr{Q}}_\eta(\hat{f}) ,$$

where

$$\begin{aligned} \hat{\mathscr{Q}}_\eta(\hat{f}) &= \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n(\eta)} \hat{f}(\hat{x}_{\hat{\eta}}) \cdot \hat{\mathscr{L}}^n\chi_{\hat{\eta}} \\ &= \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n(\eta)} \hat{f}(\hat{x}_{\hat{\eta}}) \cdot \chi_{\hat{T}^n\hat{\eta}} \cdot (\hat{\Phi}_n \circ \hat{T}_{\hat{\eta}}^{-n}) . \end{aligned}$$

Here $\hat{T}_{\hat{\eta}}^{-n}$ denotes the inverse of $(\hat{T}^n)|_{\hat{\eta}}$. Now it suffices to prove the compactness of $\hat{\mathscr{Q}}_\eta$ for fixed $\eta \in \mathscr{Z}_n$, because $\mathscr{Z}_n$ is finite.

As $\hat{\psi} = \psi \circ \pi$ and $\hat{F} = F \circ \pi$, it follows

$$\begin{aligned} \hat{\mathscr{Q}}_\eta(\hat{f}) &= \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n(\eta)} \hat{f}(\hat{x}_{\hat{\eta}}) \cdot \chi_{\hat{D}} \cdot \left(\hat{\psi} e^{\hat{F}} \frac{\hat{w}}{\hat{w} \circ \hat{T}}\right)_n \circ \hat{T}_{\hat{\eta}}^{-n} \\ &= \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n(\eta)} \hat{f}(\hat{x}_{\hat{\eta}}) \cdot \chi_{\hat{D}} \cdot \frac{1}{\hat{w}} \cdot ((\psi e^F) \circ \pi)_n \circ \hat{T}_{\hat{\eta}}^{-n} \cdot (\hat{w} \circ \hat{T}_{\hat{\eta}}^{-n}) \\ &= \chi_{\hat{D}} \cdot \frac{1}{\hat{w}} \cdot ((\psi e^F)_n \circ T_\eta^{-n} \circ \pi) \cdot \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n(\eta)} \hat{f}(\hat{x}_{\hat{\eta}}) \cdot (\hat{w} \circ \hat{T}_{\hat{\eta}}^{-n}) . \end{aligned}$$

If $\hat{\mathscr{A}}_n(\eta) \neq \emptyset$, let $\hat{\eta}_0 = \hat{D}_0 \cap \pi^{-1}\eta$, and define for $\hat{\eta} \in \hat{\mathscr{A}}_n(\eta)$ with $\hat{\eta} \subseteq \hat{D}_i$

$$v_{\hat{\eta}} \colon \eta \to \mathbf{R}, \quad v_{\hat{\eta}}(x) = \frac{\hat{w}(\langle x, i \rangle)}{\hat{w}(\langle w, 0 \rangle)} .$$

Then

$$\begin{aligned} \hat{\mathscr{Q}}_\eta(\hat{f}) &= \chi_{\hat{D}} \cdot \frac{1}{\hat{w}} \cdot ((\psi e^F)_n \cdot \hat{w}(\langle \cdot, 0 \rangle)) \circ T_\eta^{-n} \circ \pi \cdot \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n(\eta)} \hat{f}(\hat{x}_{\hat{\eta}}) \cdot (v_{\hat{\eta}} \circ T_\eta^{-n} \circ \pi) \\ &= \hat{\mathscr{L}}^n\chi_{\hat{\eta}_0} \cdot \left(\sum_{\hat{\eta} \in \hat{\mathscr{A}}_n(\eta)} \hat{f}(\hat{x}_{\hat{\eta}}) \cdot v_{\hat{\eta}} \right) \circ T_\eta^{-n} \circ \pi . \end{aligned}$$

As $\mathrm{var}_{\hat{D}}(\hat{\mathscr{L}}^n\chi_{\hat{\eta}_0}) < \infty$ and as $T_\eta^{-n} \circ \pi : \hat{D} \to \eta$ is monotone, it suffices to prove that

$$\hat{\mathscr{R}}_\eta \colon \widehat{BV} \to BV(\eta), \quad \hat{\mathscr{R}}_\eta(\hat{f}) = \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n(\eta)} \hat{f}(\hat{x}_{\hat{\eta}}) \cdot v_{\hat{\eta}}$$

is compact, and as $\sum_{\hat\eta\in\hat{\mathscr{A}}_n(\eta)}|\hat f(\hat x_{\hat\eta})|\leqq\|\hat f\|_{\widehat{BV}}$, it is enough to show that the family $(v_{\hat\eta}:\hat\eta\in\hat{\mathscr{A}}_n(\eta))$ is relatively compact in $BV(\eta)$, see [Rud, Theorem 3.25]:

In Sect. 6 we define $\hat w(\langle x,i\rangle)$ for $i>1$ as

$$\hat w(\langle x,i\rangle)=\left(\frac{|x-c_i|\cdot|x-c_{\tilde\imath}|}{|c_i-c_{\tilde\imath}|}\right)^{-(1-\xi)}\cdot q^{-\tilde\imath}\cdot|D\hat T^{\tilde\imath-1}(\hat x_{-\tilde\imath+1})|^{-\xi},$$

where $\xi=1/l$ and $\tilde\imath$ is defined in (3.8). q can be any number in $]\lambda_E,1[$.

Consider $\hat\eta\in\hat{\mathscr{A}}_n(\eta)$ such that $\hat\eta\subseteq\hat D_i$. With the notation of Sect. 3, $D_i=(c_i,c_{\tilde\imath})\subseteq D_0=(0,1)$. Hence

$$\begin{aligned}v_{\hat\eta}(x)&=\left(\frac{|x-c_i|\cdot|x-c_{\tilde\imath}|\cdot|1-0|}{|c_i-c_{\tilde\imath}|\cdot|x-0|\cdot|x-1|}\right)^{-(1-\xi)}\cdot q^{-\tilde\imath}\cdot|D\hat T^{\tilde\imath-1}(\hat x_{-\tilde\imath+1})|^{-\xi}\\&=\rho(c_i,c_{\tilde\imath};x)\cdot q^{-\tilde\imath}\cdot|D\hat T^{\tilde\imath-1}(\hat x_{-\tilde\imath+1})|^{-\xi},\end{aligned}\tag{4.8}$$

where

$$\rho(u,v;x):=\left(\frac{|u-v|}{|x-u|\cdot|x-v|}\cdot x(1-x)\right)^{1-\xi}.$$

Observe that by (4.5),

$$\operatorname{dist}(c_i,\eta),\operatorname{dist}(c_{\tilde\imath},\eta)\geqq\delta:=\min\{\operatorname{length}(\eta'):\eta'\in\mathscr{Z}_n\},$$

δ depending on n but not on i. Let $I=[0,1]\setminus\{x:\operatorname{dist}(x,\eta)<\delta\}$. I is a compact set, and it follows easily that

$$\Gamma:I\times I\to BV(\eta),\quad(u,v)\mapsto\rho(u,v;\cdot)$$

is continuous. Hence $\{\rho(u,v;\cdot):(u,v)\in I\times I\}$ is compact in $BV(\eta)$.

As the supremum and the variation of $q^{-\tilde\imath}\cdot[D\hat T^{\tilde\imath-1}(\hat x_{-\tilde\imath+1})|^{-\xi}$ over η is exponentially decreasing in $\tilde\imath$ by Corollary 6.2, it follows that the family $(v_{\hat\eta}:\hat\eta\in\hat{\mathscr{A}}_n(\eta))$ is relatively compact in $BV(\eta)$. □

Proof of Theorem 2.1. Theorem 2.1 is the special case $\beta=1$ and $F=0$ of this proposition, for in this case $\hat\Phi=\hat\Psi$, and it follows from Proposition 6.3 that, given $\Theta\in]\lambda_E^{-1},1[$, the constant q involved in the definition of the weight function $\hat w$ can be chosen such that $\vartheta[\hat\Phi]=\vartheta[\hat\Psi]<\Theta$. $\hat{\mathscr{L}}_{\hat\Psi}:\widehat{BV}\to\widehat{BV}$ has spectral radius 1, as $\int\hat{\mathscr{L}}_{\hat\Psi}(\hat f)\hat w\,d\hat m=\int\mathscr{L}_{\hat\psi}(\hat f\cdot\hat w)\,d\hat m=\int\hat f\cdot\hat w\,d\hat m$ for all $\hat f\in\widehat{BV}$, see (2.3). □

4.2. Analytic Perturbations of the Spectrum. The operator $\hat{\mathscr{L}}_{\hat\Phi}$ depends via the function $\hat\Phi$ on the parameters β and t. In order to be able to apply analytic perturbation theory to it, we show that this dependence is holomorphic.

Lemma 4.3. *$(\hat{\mathscr{L}}[\hat\Psi^\beta e^{t\hat F}]:\beta,t\in\mathbf{C},\Re\beta>0)$ is, as a function of β and as a function of t, a holomorphic family of operators on $\widehat{BV}$ in the sense of [Ka] and*

$$\frac{d^n}{d\beta^n}\hat{\mathscr{L}}[\hat\Psi^\beta e^{t\hat F}]=\hat{\mathscr{L}}[(\log\hat\Psi)^n\cdot\hat\Psi^\beta e^{t\hat F}],\quad\frac{d^n}{dt^n}\hat{\mathscr{L}}[\hat\Psi^\beta e^{t\hat F}]=\hat{\mathscr{L}}[\hat\Psi^\beta\cdot\hat F^n e^{t\hat F}].$$

Proof. We prove the assertion for β. The proof for t is practically the same.

Fix $\hat{f} \in \widehat{BV}$ which is different from 0 only on one level $\hat{D} \in \mathscr{D}$. For $u \in \mathbf{C}$, $|u|$ small, we have:

$$\left\| \frac{1}{u}(\hat{\mathscr{L}}[\hat{\Psi}^{\beta+u} e^{t\hat{F}}](\hat{f}) - \hat{\mathscr{L}}[\hat{\Psi}^{\beta} e^{t\hat{F}}](\hat{f})) - \hat{\mathscr{L}}[\log \hat{\Psi} \cdot \hat{\Psi}^{\beta} e^{t\hat{F}}](\hat{f}) \right\|_{\widehat{BV}}$$

$$= \left\| \hat{\mathscr{L}}[\hat{\Psi}^{\beta} e^{t\hat{F}}]\left(\frac{1}{u}(\hat{f}\Psi^{u} - \hat{f}) - \hat{f} \log \hat{\Psi} \right) \right\|_{\widehat{BV}}$$

$$\leqq \| \hat{\mathscr{L}}[\hat{\Psi}^{\beta} e^{t\hat{F}}] \|_{\widehat{BV}} \cdot \left\| \hat{f} \cdot \left(\frac{1}{u}(e^{u \log \hat{\Psi}} - 1) - \log \hat{\Psi} \right) \right\|_{\widehat{BV}} .$$

Now, on a fixed $\hat{D} \in \mathscr{D}$, $\hat{\Psi}$ is bounded away from 0 and $+\infty$, and as $\hat{\Psi}|_{\hat{D}} \in BV(\hat{D})$,

$$\lim_{|u| \to 0} \left\| \hat{f} \cdot \left(\frac{1}{u}(e^{u \log \hat{\Psi}} - 1) - \log \hat{\Psi} \right) \right\|_{\widehat{BV}} = 0 .$$

As $\| \hat{\mathscr{L}}[\hat{\Psi}^{\beta} e^{t\hat{F}}] \|_{\widehat{BV}} \leqq 6 \cdot V(\beta, t, \hat{F})$ by [BK, Lemma 2.2], and as $V(\beta, t, \hat{F})$ varies continuously with β and t (see 4.2), the family $(\hat{\mathscr{L}}[\hat{\Psi}^{\beta} e^{t\hat{F}}] : \beta, t \in \mathbf{C}, \Re\beta > 0)$ is locally bounded. Therefore

$$\lim_{|u| \to 0} \left\| \frac{1}{u}(\hat{\mathscr{L}}[\hat{\Psi}^{\beta+u} e^{t\hat{F}}](\hat{f}) - \hat{\mathscr{L}}[\hat{\Psi}^{\beta} e^{t\hat{F}}](\hat{f})) - \hat{\mathscr{L}}[\log \hat{\Psi} \cdot \hat{\Psi}^{\beta} e^{t\hat{F}}](\hat{f}) \right\|_{\widehat{BV}} = 0 ,$$

and in view of [Ka, Ch. 7.1.1], this proves the analyticity of the family of operators. The formulae for the higher derivatives follow from the observation that $\hat{\mathscr{L}}[\hat{F} \cdot \hat{\Phi}](\hat{f}) = \hat{\mathscr{L}}[\hat{\Phi}](\hat{F} \cdot \hat{f})$. □

As a first consequence of Proposition 4.1 we note

Proposition 4.2. *Let $\beta_0, t_0 \in C$, $\Re\beta_0 > 0$. For each $\Theta > \vartheta(\hat{\Phi}(\beta_0, t_0, \cdot))$ there are a neighbourhood $U \subseteq \mathbf{C} \times \mathbf{C}$ around (β_0, t_0), a positive integer N, and a real constant $C > 0$ such that for each $\hat{\mathscr{L}}[\hat{\Phi}(\beta, t, \cdot)]$ with $(\beta, t) \in U$ holds:*

There is a projection $\hat{\mathscr{P}}[\hat{\Phi}(\beta, t, \cdot)]$ commuting with $\hat{\mathscr{L}}[\hat{\Phi}(\beta, t, \cdot)]$ such that

$$\operatorname{rank}(\mathrm{Id} - \hat{\mathscr{P}}[\hat{\Phi}(\beta, t, \cdot)]) = N ,$$

$$\| \hat{\mathscr{P}}[\hat{\Phi}(\beta, t, \cdot)] \hat{\mathscr{L}}^{n}[\hat{\Phi}(\beta, t, \cdot)] \|_{\widehat{BV}} \leqq C \cdot \Theta^{n}, \text{ and}$$

$$(\mathrm{Id} - \hat{\mathscr{P}}[\hat{\Phi}(\beta, t, \cdot)]) \hat{\mathscr{L}}[\hat{\Phi}(\beta, t, \cdot)] \text{ has no eigenvalues of modulus } \leqq \Theta .$$

The projections $\hat{\mathscr{P}}[\hat{\Phi}(\beta, t, \cdot)]$ and Id $-$ *$\hat{\mathscr{P}}[\hat{\Phi}(\beta, t, \cdot)]$ are analytic functions of β and t, $(\beta, t) \in U$.*

Of course, Id $- \hat{\mathscr{P}} = \sum_i \hat{\mathscr{P}}_i$ from (2.8).

Proof. Everything follows from Proposition 4.1 and [Ka, Ch. VII.1.3], except for the uniformity of the constant C. The argument to prove this is classical: Denote by $R(\beta, t, z) = (\mathrm{Id} - z\hat{\mathscr{L}}[\hat{\Phi}(\beta, t, \cdot)])^{-1}$ the resolvent of $\hat{\mathscr{L}}[\hat{\Phi}(\beta, t, \cdot)]$ and observe that

$$R(\beta, t, z) = \sum_{k=0}^{\infty} (zR(\beta_0, t_0, z)(\hat{\mathscr{L}}[\hat{\Phi}(\beta, t, \cdot)] - \hat{\mathscr{L}}[\hat{\Phi}(\beta_0, t_0, \cdot)]))^{k} R(\beta_0, t, z)$$

and

$$\hat{\mathscr{L}}[\hat{\Phi}(\beta, t, \cdot)]^n \hat{\mathscr{P}}[\hat{\Phi}(\beta, t, \cdot)] = -\frac{1}{2\pi i} \int_{|z|=\Theta} z^n R(\beta, t, z)\, dz\,.$$

As the resolvent is analytic in $(\beta, t) \in U$ (see [Ka, Ch. VII.1.3]) and analytic in z in a neighbourhood of $\{|z| = \Theta\}$, the uniform estimate for C follows. □

4.3. Zeta Functions. As our transfer function $\hat{\Phi}$ does not have the form $\hat{\Phi} = \Phi \circ \pi$, we cannot treat the zeta functions exactly the same way as in [BK]. The necessary modifications are, although crucial, of technical nature, and we can basically follow the proof from [BK].

Lemma 4.4. *For each finite rank operator $\hat{\mathscr{Q}}: \widehat{BV} \to \widehat{BV}$ and each $n > 0$, $\hat{\mathscr{Q}}\hat{\mathscr{L}}^n \hat{\alpha}_n$ is of finite rank and hence of trace class and*

$$\operatorname{tr}(\hat{\mathscr{Q}}\hat{\mathscr{L}}^n \hat{\alpha}_n) = \sum_{\hat{\eta} \in \mathscr{A}_n} (\hat{\mathscr{Q}}\hat{\mathscr{L}}^n \chi_{\hat{\eta}})(\hat{x}_{\hat{\eta}})\,,$$

where the right-hand side converges absolutely.

Proof. Because of the linearity of the trace functional, it is sufficient to prove the lemma for rank 1 operators. Then $\operatorname{rank}(\hat{\mathscr{Q}}\hat{\mathscr{L}}^n \hat{\alpha}_n) \leq \operatorname{rank}(\hat{\mathscr{Q}}) = 1$, and, if $\hat{\mathscr{Q}}$ maps $\widehat{BV}$ to the one-dimensional subspace spanned by the function $\hat{f}$, say, then

$$\operatorname{tr}(\hat{\mathscr{Q}}\hat{\mathscr{L}}^n \hat{\alpha}_n) \cdot \hat{f} = \hat{\mathscr{Q}}\hat{\mathscr{L}}^n \hat{\alpha}_n(\hat{f}) = \sum_{\hat{\eta} \in \mathscr{A}_n} \hat{f}(\hat{x}_{\hat{\eta}}) \cdot \hat{\mathscr{Q}}\hat{\mathscr{L}}^n \chi_{\hat{\eta}}$$

with an absolutely converging right-hand side, see (4.7). Define $d_{\hat{\eta}} \in \mathbf{C}$ by

$$\hat{\mathscr{Q}}\hat{\mathscr{L}}^n \chi_{\hat{\eta}} = d_{\hat{\eta}} \cdot \hat{f}\,.$$

Then

$$\operatorname{tr}(\hat{\mathscr{Q}}\hat{\mathscr{L}}^n \hat{\alpha}_n) = \sum_{\hat{\eta} \in \mathscr{A}_n} \hat{f}(\hat{x}_{\hat{\eta}}) \cdot d_{\hat{\eta}} = \sum_{\hat{\eta} \in \mathscr{A}_n} \hat{\mathscr{Q}}\hat{\mathscr{L}}^n \chi_{\hat{\eta}}(\hat{x}_{\hat{\eta}})$$

with an absolutely converging right-hand side. □

In view of Proposition 4.1, $\hat{\mathscr{L}}_{\hat{\Phi}}$ has a spectral decomposition as in (2.8), i.e. for $\Theta > \vartheta[\hat{\Phi}]$,

$$\hat{\mathscr{L}}_{\hat{\Phi}} = \sum_{i=1}^{N(\Theta)} \rho_i (\hat{\mathscr{P}}_i + \hat{\mathscr{N}}_i) + \hat{\mathscr{P}}\hat{\mathscr{L}}_{\hat{\Phi}}\,, \tag{4.9}$$

where $\hat{\mathscr{P}}_i$, for $i = 1, \ldots, N(\Theta)$, and $\hat{\mathscr{P}}$ are projections commuting with $\hat{\mathscr{L}}_{\hat{\Phi}}$ and such that $\hat{\mathscr{P}}_i \hat{\mathscr{P}}_j = \hat{\mathscr{P}}_i \hat{\mathscr{P}} = 0$ for $i \neq j$ and $\hat{\mathscr{P}} + \sum_i \hat{\mathscr{P}}_i = \mathrm{Id}$. For each $i = 1, \ldots, N(\Theta)$ we have $|\rho_i| > \Theta$, $\operatorname{rank}(\hat{\mathscr{P}}_i) < \infty$, and $\hat{\mathscr{N}}_i$ is nilpotent with $\hat{\mathscr{P}}_i \hat{\mathscr{N}}_i = \hat{\mathscr{N}}_i \hat{\mathscr{P}}_i = \hat{\mathscr{N}}_i$. Finally $\|\hat{\mathscr{P}}\hat{\mathscr{L}}^n\|_{\widehat{BV}} \leq \mathrm{const} \cdot \Theta^n$.

Proposition 4.3. *Let $\hat{\Phi}(\hat{x}) = \hat{\Psi}(\hat{x})^\beta \cdot e^{t\hat{F}(\hat{x})}$, $\beta, t \in \mathbf{C}$, $\Re\beta > 0$. For each $\Theta > \vartheta[\hat{\Phi}(\beta, t, \cdot)]$,*

$$\hat{\zeta}[\hat{\Phi}](z) \cdot \prod_{i=1}^{N(\Theta)} (1 - \rho_i z)^{\operatorname{rank}(\hat{\mathscr{P}}_i)} = \exp\left\{ \sum_{n=1}^{\infty} \frac{z^n}{n} \left(\hat{\zeta}_n[\hat{\Phi}] - \sum_{i=1}^{N(\Theta)} \operatorname{rank}(\hat{\mathscr{P}}_i) \rho_i^n \right) \right\}$$

is analytic and nonzero in $\{z: |z| < \Theta^{-1}\}$.

As the $\hat{\mathscr{P}}_i$ are projections and the $\hat{\mathscr{N}}_i = \hat{\mathscr{P}}_i\hat{\mathscr{N}}_i$ are nilpotent, $\operatorname{rank}(\hat{\mathscr{P}}_i) = \operatorname{tr}(\hat{\mathscr{P}}_i)$ and $\operatorname{tr}((\hat{\mathscr{P}}_i + \hat{\mathscr{N}}_i)^n) = \operatorname{tr}(\hat{\mathscr{P}}_i)$ for all $n > 0$. Therefore

$$\begin{aligned}\prod_{i=1}^{N(\Theta)} (1 - \rho_i z)^{\operatorname{rank}(\hat{\mathscr{P}}_i)} &= \exp \sum_{i=1}^{N(\Theta)} \sum_{n=1}^{\infty} -\frac{z^n}{n} \rho_i^n \operatorname{tr}(\hat{\mathscr{P}}_i) \\ &= \exp \sum_{n=1}^{\infty} -\frac{z^n}{n} \operatorname{tr}\left(\sum_{i=1}^{N(\Theta)} \rho_i^n (\hat{\mathscr{P}}_i + \hat{\mathscr{N}}_i)^n\right) \\ &= \exp \sum_{n=1}^{\infty} -\frac{z^n}{n} \operatorname{tr}\left(\sum_{i=1}^{N(\Theta)} \hat{\mathscr{P}}_i \hat{\mathscr{L}}^n\right).\end{aligned}$$

Hence, in view of the definition of $\hat{\zeta}[\hat{\Phi}](z)$, the proposition follows from

Lemma 4.5.

$$\left|\hat{\zeta}_n[\hat{\Phi}] - \sum_{i=1}^{N(\Theta)} \operatorname{rank}(\hat{\mathscr{P}}_i)\rho_i^n\right| = \left|\hat{\zeta}_n[\hat{\Phi}] - \operatorname{tr}\left(\sum_{i=1}^{N(\Theta)} \hat{\mathscr{P}}_i \hat{\mathscr{L}}^n\right)\right| \leqq \text{const} \cdot \Theta^n$$

with a constant uniform in n and locally uniform in the parameters β and t. (Indeed, the uniformity is not necessary for the proof of Proposition 4.3, *but will be used in Sect.* 5.)

Proof. In the course of the proof, estimates by terms of the form $n \cdot \Theta^n$ or $(1 + \varepsilon) \cdot \Theta^n$ will occur. In order to simplify our notation, we shall replace them tacitly by $\text{const} \cdot \Theta^n$. This is possible, because $\Theta > \vartheta[\hat{\Phi}]$ is arbitrary. Observe also that all constants can be chosen such that they are locally uniform in β and t.

Let

$$\hat{\zeta}_n^{(0)} = \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n, \hat{\eta} \subseteq \hat{T}^n \hat{\eta}} \hat{\Phi}_n(\hat{x}_{\hat{\eta}}), \quad \hat{\zeta}_n^{(1)} = \sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n \setminus \hat{\mathscr{A}}_n, \hat{\eta} \subseteq \hat{T}^n \hat{\eta}} \hat{\Phi}_n(\hat{x}_{\hat{\eta}}).$$

Then $\hat{\zeta}_n[\hat{\Phi}] = \hat{\zeta}_n^{(0)} + \hat{\zeta}_n^{(1)}$ by (4.3). As $\hat{\zeta}_n^{(0)}$ is a finite sum (see (4.6)) and as $\hat{\mathscr{L}}^n \chi_{\hat{\eta}}(\hat{y}) = 0$ for $\hat{y} \notin \hat{T}^n \hat{\eta}$, we have (observing of Lemma 4.4 for the last equality)

$$\begin{aligned}\hat{\zeta}_n^{(0)} &= \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n, \hat{\eta} \subseteq \hat{T}^n \hat{\eta}} (\hat{\Phi}_n \cdot \chi_{\hat{\eta}}) \circ \hat{T}_{\hat{\eta}}^{-n}(\hat{x}_{\hat{\eta}}) \\ &= \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n, \hat{\eta} \subseteq \hat{T}^n \hat{\eta}} (\hat{\mathscr{L}}^n \chi_{\hat{\eta}})(\hat{x}_{\hat{\eta}}) = \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n} (\hat{\mathscr{L}}^n \chi_{\hat{\eta}})(\hat{x}_{\hat{\eta}}) \\ &= \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n} (\hat{\mathscr{P}} \hat{\mathscr{L}}^n \chi_{\hat{\eta}})(\hat{x}_{\hat{\eta}}) + \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n} (\hat{\mathscr{P}}^{\perp} \hat{\mathscr{L}}^n \chi_{\hat{\eta}})(\hat{x}_{\hat{\eta}}) \\ &= \sum_{\hat{\eta} \in \hat{\mathscr{A}}_n} (\hat{\mathscr{P}} \hat{\mathscr{L}}^n \chi_{\hat{\eta}})(\hat{x}_{\hat{\eta}}) + \operatorname{tr}(\hat{\mathscr{P}}^{\perp} \hat{\mathscr{L}}^n \hat{\alpha}_n),\end{aligned}$$

where $\hat{\mathscr{P}}^{\perp} = \operatorname{Id} - \hat{\mathscr{P}} = \sum_{i=1}^{N(\Theta)} \hat{\mathscr{P}}_i$.

For $\hat{\zeta}_n^{(1)}$ we have the estimate (observe (3.6) and (4.4))

$$\begin{aligned}|\hat{\zeta}_n^{(1)}| &\leqq \sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n \setminus \hat{\mathscr{A}}_n, \hat{\eta} \subseteq \hat{T}^n \hat{\eta}} |\hat{\Phi}_n(\hat{x}_{\hat{\eta}})| \\ &\leqq 4 \cdot (2n + 1) \cdot \sup|\hat{\Phi}_n| \leqq \text{const} \cdot \Theta^n. \end{aligned} \tag{4.10}$$

so the lemma will follow, if we show that

$$|\mathrm{tr}(\hat{\mathscr{P}}^{\perp}(\hat{\mathscr{L}}^{n}\hat{\alpha}_{n} - \hat{\mathscr{L}}^{n})| + \left|\sum_{\hat{\eta}\in\hat{\mathscr{A}}_{n}} (\hat{\mathscr{P}}\hat{\mathscr{L}}^{n}\chi_{\hat{\eta}})(\hat{x}_{\hat{\eta}})\right| \leqq \mathrm{const}\cdot\Theta^{n} . \tag{4.11}$$

By Lemma 4.2 and Proposition 4.2,

$$|\mathrm{tr}(\hat{\mathscr{P}}^{\perp}(\hat{\mathscr{L}}^{n}\hat{\alpha}_{n} - \hat{\mathscr{L}}^{n}))| \leqq \mathrm{rank}(\hat{\mathscr{P}}^{\perp})\cdot \|\hat{\mathscr{P}}^{\perp}\|_{\widehat{BV}}\cdot \|\hat{\mathscr{L}}^{n}\hat{\alpha}_{n} - \hat{\mathscr{L}}^{n}\|_{\widehat{BV}} ,$$
$$\leqq \mathrm{const}\cdot\Theta^{n} ,$$

and the proof must be finished by showing that

$$\left|\sum_{\hat{\eta}\in\hat{\mathscr{A}}_{n}} (\hat{\mathscr{P}}\hat{\mathscr{L}}^{n}\chi_{\hat{\eta}})(\hat{x}_{\hat{\eta}})\right| \leqq \mathrm{const}\cdot\Theta^{n} \tag{4.12}$$

with a constant locally uniform in β and t.

As the proof of this estimate is very similar to the corresponding one in [BK], we defer it to Appendix A. □

5. Probability Transforms and Zeta Functions

In this section we prove Theorems 1.2 and 1.3 on asymptotic normality and large deviations both for Lebesgue typical trajectories and for typical periodic orbits.

Further Consequences from Analytic Perturbation Theory. Suppose T is a non-renormalizable C–E map, i.e. $(\hat{T}, \mu)$ is mixing, where $\mu = hm$, see Sect. 2.2. Then the spectral representation for $\mathscr{L}_{\hat{\psi}}$ reduces to

$$\hat{\mathscr{L}}^{n}_{\hat{\psi}} = \rho_1^{n}\hat{\mathscr{P}}_1 + \hat{\mathscr{P}}_1^{\perp}\hat{\mathscr{L}}^{n}_{\hat{\psi}}, \text{ where } \hat{\mathscr{P}}_1(\hat{f}) = \int \hat{f}\, d\hat{m}\cdot\hat{h} \text{ and } h(x) = \sum_{\hat{x}\in\pi^{-1}x} \hat{h}(\hat{x}) . \tag{5.1}$$

Here $\hat{\mathscr{P}}_1^{\perp} = \mathrm{Id} - \hat{\mathscr{P}}_1, \rho_1 = 1$, and

$$\|\hat{\mathscr{L}}^{n}_{\hat{\psi}}\hat{\mathscr{P}}_1^{\perp}\|_{\hat{w}} \leqq \mathrm{const}\cdot r^{n} \quad \text{for some } r < 1 .$$

This follows e.g. from (LM, Theorem 5.5.3] together with [K3, Lemma 1].

Consider now $\mathscr{L}[\hat{\Phi}]$ where $\hat{\Phi} = \hat{\Psi}^{\beta}\cdot e^{t\hat{F}}$ depends on the parameters $(\beta, t)\in U$, U a neighbourhood of $(1, 0)$ in $\mathbf{C}\times\mathbf{C}$. For $(\beta, t) = (1, 0)$ we have $\mathscr{L}_{\hat{\Phi}} = \mathscr{L}_{\hat{\Psi}}$. Because of the conjugation (2.3) between $\mathscr{L}_{\hat{\psi}}$ and $\mathscr{L}_{\hat{\Psi}}$, the operator $\mathscr{L}_{\hat{\Psi}}$ has $\rho_1 = 1$ as a simple, isolated eigenvalue with the rest of the spectrum contained in $\{|z| \leqq r\}$. Hence, as stated in Proposition 4.2, ρ_1 is an analytic function of β and t, $(\beta, t)\in U$. Kato [Ka, Ch. VII.1.5 and Ch. II.2.2] gives explicit expressions for the first and second derivatives of ρ_1 with respect to the parameters. These expressions can be evaluated explicitly using the formulas of Lemma 4.3 for the derivatives of $\hat{\mathscr{L}}[\hat{\Phi}]$. The calculations are tedious but straightforward, and we give only the results:

$$\left(\frac{d}{dt}\log\rho_1\right)\bigg|_{\beta=1,t=0} = \int \hat{F}\,d\hat{\mu} = \int F\,d\mu\,,$$

$$\left(\frac{d^2}{dt^2}\log\rho_1\right)\bigg|_{\beta=1,t=0} = \sigma_F^2\,, \tag{5.2}$$

and similarly

$$\left(\frac{d}{d\beta}\log\rho_1\right)\bigg|_{\beta=1,t=0} = \int \log\hat{\Psi}\,d\hat{\mu} = \int \log\hat{\Phi}\,d\hat{\mu} = -\int \log|T'|\,d\mu\,,$$

$$\left(\frac{d^2}{d\beta^2}\log\rho_1\right)\bigg|_{\beta=1,t=0} = \sigma^2_{\log|T'|}\,. \tag{5.3}$$

For the evaluation of the derivatives with respect to β one must use the fact that $\hat{\mu} = \hat{h}\cdot\hat{m} = \hat{w}\cdot\hat{f}\cdot\hat{m}$ for some $\hat{f}\in\widehat{BV}$ and therefore

$$\int |\log\hat{w}|\,d\hat{\mu} = \int |\hat{w}\log\hat{w}|\cdot\hat{f}\,d\hat{m} \leqq \sup_i \int_{\hat{D}_i} |\hat{w}\log\hat{w}|\,d\hat{m}|\cdot\|\hat{f}\|_{\widehat{BV}} < +\infty$$

by Proposition 6.1.

A more direct calculation of such derivatives, which does not rely on analytic perturbation theory, can be found in [Rou].

Proof of Theorem 1.2. The central limit part of Theorem 1.2 can now be proved as in [Rou], the large deviations part as in [K4, 9.6].

Proof of Theorem 1.3. In [K5] it is shown how Theorem 1.3 can be derived from our results on spectra and zeta functions. As this reference is not very well accessible, we repeat its proof here. Without loss of generality we assume that $\int F\,d\mu = 0$.

The distributions ν_n on the sets $\mathrm{Per}_n = \{x\in[0,1]: T^n x = x\}$ defined in the Introduction are related to zeta functions in the following way: For $F:[0,1]\to\mathbf{C}$, $S'_n(x) = \sum_{i=0}^{n-1} F(T^i x)$ and $\tau\in\mathbf{C}$,

$$\int e^{\tau S'_n}\,d\nu_n = \frac{\sum_{x\in\mathrm{Per}_n}\psi_n(x)e^{\tau S'_n(x)}}{\sum_{x\in\mathrm{Per}_n}\psi_n(x)} = \frac{\zeta_n[\psi e^{\tau F}]}{\zeta_n[\psi]} = \frac{\hat{\zeta}_n[\hat{\Psi}e^{\tau\hat{F}}]}{\hat{\zeta}_n[\hat{\Psi}]}\,,$$

where $\hat{F} = F\circ\pi$. In view of Lemma 4.5 and the special spectral representation (5.1) chosen in this section it follows that for small $|\tau|$,

$$\left|\int e^{\tau S'_n}\,d\nu_n - \frac{\rho_1^n[\hat{\Psi}e^{\tau\hat{F}}]}{\rho_1^n[\hat{\Psi}]}\right| \leqq \mathrm{const}\cdot\bar{r}^n \tag{5.4}$$

for some $\bar{r} < 1$ and a constant not depending on τ. Here $\rho_1[\hat{\Phi}]$ denotes the eigenvalue ρ_1 of the operator $\hat{\mathscr{L}}[\hat{\Phi}]$. (If F is of bounded variation, then we work with $\hat{\mathscr{L}}[\hat{\Phi}(1,\tau,\cdot)]$, if $F = \log|T'|$, we use $\hat{\mathscr{L}}[\hat{\Phi}(1+\tau,0,\cdot)]$.)

In particular, if τ_n is a sequence of sufficiently small complex numbers, then

$$\lim_{n\to\infty} \int e^{\tau_n S_n'} d\nu_n = \lim_{n\to\infty} \frac{\rho_1^n[\hat{\Psi} e^{\tau_n \hat{F}}]}{\rho_1^n[\hat{\Psi}]},$$

if the limit on the right hand side exists.

The Central Limit Theorem. Let $\tau_n = i\tau n^{-1/2}$, $\tau \in \mathbf{R}$. Then, in view of (5.4) and (5.2),

$$\lim_{n\to\infty} \log \int e^{i\tau n^{-1/2} S_n'} d\nu_n = \lim_{n\to\infty} n \cdot (\log \rho_1[\hat{\Psi} e^{i\tau n^{-1/2}\hat{F}}] - \log \rho_1[\hat{\Psi}])$$

$$= \lim_{n\to\infty} n \cdot \left(\log \rho_1[\hat{\Psi}] + i\tau \cdot n^{-1/2} \cdot \int F\, d\mu - \frac{\tau^2}{2n} \cdot \sigma_F^2 - \log \rho_1[\hat{\Psi}] + o\left(\frac{1}{n}\right)\right)$$

$$= -\frac{\tau^2 \cdot \sigma_F^2}{2},$$

and it follows that the characteristic functions of $n^{-1/2} S_n'$ converge to that of $\mathcal{N}(0, \sigma_F^2)$. □

The Large Deviations Estimate. Let $\tau_n = \tau \in \mathbf{R}$. Then

$$\lim_{n\to\infty} \frac{1}{n} \log \int e^{\tau \cdot S_n'} d\nu_n = \lim_{n\to\infty} (\log \rho_1[\hat{\Psi} e^{\tau F}] - \log \rho_1[\hat{\Psi}]) = \log \rho_1[\hat{\Psi} e^{\tau F}],$$

and the assertion of Theorem 1.3 follows from general large deviations theory, as the function $\tau \mapsto \log \rho_1[\hat{\Psi} e^{\tau F}]$ is strictly convex at $\tau = 0$ if $\sigma_F^2 = \left(\frac{d^2}{d\tau^2} \log \rho_1\right)\Big|_{\tau=0} > 0$. For an account of general large deviations see e.g. [PS] or [CG]. □

6. Estimates on Derivatives and Distortions

In this section we define the weight function $\hat{w}$. This function induces a new geometry on the levels of the tower by setting for $\hat{x}, \hat{y} \in \hat{D}_i$,

$$\rho(\hat{x}, \hat{y}) = \int_{\hat{x}}^{\hat{y}} \hat{w}\, d\hat{m}.$$

The new metric ρ increases the distance near the endpoints of each $\hat{D}_i$, where the singularities of $\hat{w}$ are of order $1 - 1/l$. On the other hand the new length of the levels stays bounded. In this way $\hat{T}^n$ near critical points loses nonlinearity. Due to some additional factor in $\hat{w}$ the derivative of $\hat{T}^n$ in the new metric is bounded from below by const$\cdot q^{-n}$ for some $q < 1$. The exponential bound is possible because of hyperbolic properties of C–E maps. The inverse of this derivative is the transfer function $\hat{\Psi}$ defined in (2.2).

In order to prove spectral properties of the corresponding transfer operator on an appropriate Banach space we need to verify some estimates on T, $\hat{w}$ and $\hat{\Psi}$.

Hyperbolic Properties of C–E *Maps.* If the map T fulfills C–E, then there are constants $K_c > 0$ and $\lambda_c > 1$ such that

$$|DT^n(c_1)| \geqq K_c \lambda_c^n .$$

It is known [N1, N3, NvS1] that in this case there are constants $\lambda_{\mathrm{per}} > 1$, $K_\eta > 0$, and $\lambda_\eta > 1$, such that for any n and any $p \in \mathrm{Per}_n$ and $\eta_n \in \mathscr{Z}_n$,

$$|DT^n(p)| > \lambda_{\mathrm{per}}^n, \quad \text{and} \quad \operatorname{diam} \mathscr{Z}_n = \max |\eta_n| < K_\eta \lambda_\eta^{-n} .$$

Both $\lambda_H = \min\{\lambda_c, \lambda_{\mathrm{per}}\}$ and $\lambda_E = \min\{\lambda_H^{1/l}, \lambda_\eta\}$ as defined in (2.7) are bigger than one. (Observe that the constants λ_c, λ_η, and λ_E are not exactly the same as those with the same names defined in Sect. 2 but can be chosen arbitrarily close to them.)

The relations between the different expansion coefficients (λ's) are not yet completely clear. Arguing along the lines of [N1] one can show that

$$|DT^{nj}(p)| \geqq \text{const} \cdot (\operatorname{diam} \mathscr{Z}_{jn})^{-1} \quad \text{for } j > 0 \text{ and}$$

$$\operatorname{diam} \mathscr{Z}_n \leqq \text{const} \cdot (\min\{\lambda_c^{1/l}, \lambda_0\})^{-n} ,$$

where $\lambda_0 > 1$ depends on how close the above λ_c is chosen to the one from Sect. 2.

Moreover $\limsup_{n\to\infty} |DT^n(c_1)|^{1/n} \geqq \liminf_{n\to\infty} |DT^n(p)|^{1/n}$. It is unfortunately still unknown whether a uniformly hyperbolic structure on periodic points (i.e. $\lambda_{\mathrm{per}} > 1$) implies the C–E condition.

6.1. Distortions and Variations Related to T.

Expansion Due to the Negative Schwarzian.

Lemma 6.1. (*Crossratio expansion*) [*MS*]. *Suppose that $Sg \leqq 0$ and $Dg|_J \neq 0$ on some interval J. If J is a disjoint union of three intervals L, M, R, (M is the middle one), then*

$$\frac{|gJ|}{|J|}\frac{|gM|}{|M|} \geqq \frac{|gL|}{|L|}\frac{|gR|}{|R|} .$$

In particular, when M is reduced to a point x, then

$$|Dg(x)| \geqq \frac{|gL||gR|}{|gJ|}\frac{|J|}{|L||R|} . \qquad \square$$

For $x \in J = (a, b)$ let us define

$$\partial_J(x) = \frac{|b-x||x-a|}{|b-a|} .$$

$\partial_J(x)$ describes the distance from x to the endpoints of J in the following sense:

$$\frac{1}{2}\min\{|b-x|, |x-a|\} \leqq \partial_J(x) \leqq \min\{|b-x|, |x-a|\} , \tag{6.1}$$

and if $x \in [\alpha, \beta] \subset [a, b]$, then

$$\partial_{[\alpha,\beta]}(x) \leqq \partial_{[a,b]}(x) . \tag{6.2}$$

The second inequality from Lemma 6.1 can be now written as

$$|Dg(x)| \geqq \partial_{g(J)}(gx)/\partial_J(x) \,. \tag{6.3}$$

Now we come back to the map T. We can speak about $\partial_{\hat{J}}(\hat{x})$ on the tower when $\hat{x} \in \hat{J} \subset \hat{D}_j$ and $x = \pi\hat{x} \in \pi(\hat{J}) = [a, b] \subset D_j$. Then $\partial_{\hat{J}}(\hat{x}) := \partial_J(x)$. Suppose that $\hat{x} \in \hat{D}_j$ and $\hat{T}^n\hat{x} \in \hat{D}_k$. Define

$$\delta_n(\hat{x}) := \partial_{\hat{\eta}_n[\hat{x}]}(\hat{x}) \,,$$

where $\hat{\eta}_n[\hat{x}]$, the cylinder of order n containing $\hat{x}$, is mapped by $\hat{T}^n$ onto $\hat{D}_k$. Define moreover

$$\partial(\hat{x}) := \partial_{\hat{D}_j}(\hat{x}) \,.$$

Then

$$|D\hat{T}^n(\hat{x})| \frac{\delta_n(\hat{x})}{\partial(\hat{T}^n\hat{x})} \geqq 1 \,, \tag{6.4}$$

and, by (6.2), for any $n > 0$,

$$\partial(\hat{x}) \geqq \delta_n(\hat{x}) \,. \tag{6.5}$$

Corollary 6.1. *Suppose that $\hat{x} \in \hat{D}_j$ and $\hat{T}^n\hat{x} \in \hat{D}_k$. Assume that both components of $D_j\backslash\{x\}$ and both components of $D_k\backslash\{T^n x\}$ have length bigger than ε. Then for any $\xi \in]0, 1[$,*

$$|D\hat{T}^n(\hat{x})| \left(\frac{\partial(\hat{x})}{\partial(\hat{T}^n\hat{x})}\right)^{\xi} \geqq \text{const} \frac{\varepsilon}{\operatorname{diam} \mathscr{Z}_n} \,.$$

Proof. This follows from the following decomposition of the left-hand side of the inequality: By (6.1) and (6.4),

$$\left[|D\hat{T}^n(\hat{x})| \frac{\delta_n(\hat{x})}{\partial(\hat{T}^n\hat{x})}\right] \frac{\partial(\hat{x})}{\delta_n(\hat{x})} \left(\frac{\partial(\hat{T}^n\hat{x})}{\partial(\hat{x})}\right)^{1-\xi} \geqq 1 \cdot \frac{\varepsilon}{2 \operatorname{diam} \mathscr{Z}_n} \left(\frac{\varepsilon}{2\varepsilon}\right)^{1-\xi} \,. \qquad \square$$

Lemma 6.2. (*Koebe Lemma*) [vS]. *Suppose that $Sg \leqq 0$ and $Dg|_J \neq 0$ on some interval $J = (a, b)$. Then for $x \in J$ one has*

$$\frac{|Dg(x)|}{|Dg(a)|} \geqq \left(\frac{|gx - gb|}{|ga - gb|}\right)^2 \,.$$

In particular, if $|T^n x - T^n a| \leqq |T^n x - T^n b|$, then $|DT^n(x)| \geqq |DT^n(a)|/4$, otherwise $|DT^n(x)| \geqq |DT^n(b)|/4$. $\square$

More generally one may say that a map g satisfies the Koebe Lemma if for any $\sigma > 0$ there is a $\tau > 0$ such that for any n and any interval (a, b) on which g^n is monotone holds: if $x \in (a, b)$ and $|g^n x - g^n b|/|g^n a - g^n b| > \sigma$, then $|Dg^n(x)| / |Dg^n(a)| > \tau$.

Remark 6.1 (See [NvS1]). It may be interesting to point out that general C–E maps (not necessarily S-unimodal) have also the above described properties concerning hyperbolicity, expanding the crossratio (perhaps with some constant smaller than one but uniform in n) and the Koebe Lemma.

Bounds Related to the Nonflatness of T.

Lemma 6.3. *For each $\beta_0 > 0$ there is a uniform bound on* $\sup_{(a,b)}$ *and* $\operatorname{var}_{(a,b)}$ *of*

$$\left(\frac{\partial_{(Ta,Tb)}(Tx)}{|DT(x)|\,\partial_{(a,b)}(x)}\right)^{\beta}$$

for $\beta \in [0, \beta_0]$ and for any interval (a, b) on which T is monotone.

Proof. Denote the expression under consideration by $F(x)$. Then $\sup_{(a,b)} F \leqq 1$ by (6.3). We may assume that $b \in (a, c)$. Then

$$F(x) = \left(\frac{|Tx - Ta|}{|x-a|\,|DT(a)|}\right)^{\beta} \cdot \left(\frac{|Tx - Tb|}{|x-b|\,|DT(x)|}\right)^{\beta} \cdot \left(\frac{|DT(a)|\,|a-b|}{|Ta - Tb|}\right)^{\beta}.$$

The suprema of all three factors and the variation of the second factor are bounded by M, see (1.2). The third factor is constant, and the first one has at most two monotone branches by negative Schwarzian. Hence its variation is bounded by $2M$. □

Estimations on the Shadows. In this part we prove two technical lemmas which give exponential estimations for the weight function. c may denote c^+ or c^-.

Lemma 6.4. *Suppose that $y \in \eta = \eta_d[c]$, and that there is some v such that $x = T^d y \in (v, c) \subset (c_d, c) \subset (c_d, c_{\bar{d}}) = T^d \eta$. Let $\beta_0 > 0$. Then there exists a constant K independent of d, y and v such that for all $\beta \in [0, \beta_0]$,*

$$G(x)^{\beta} := |DT^d(Ty)|^{\beta} \left(\frac{\partial_{(c_d, c_{\bar{d}})}(x)}{\partial_{(v,c)}(x)}\right)^{\beta(l-1)} \geqq K\lambda_H^{\beta d}$$

and

$$\operatorname{var}_{(v,c)} \frac{1}{G^{\beta}} \leqq K \cdot \lambda_H^{-\beta d}.$$

Proof. Assume first that $|c_d - x| \leqq |x - c_{\bar{d}}|$ and write

$$G(x) = |DT^d(c_1)| \frac{|DT^{d-1}(y_1)|}{|DT^{d-1}(c_1)|} \frac{|DT(x)|}{|DT(c_d)|} \left(\frac{\dfrac{|x - c_{\bar{d}}|\,|x - c_d|}{|c_d - c_{\bar{d}}|}}{\dfrac{|x - c|\,|x - v|}{|v - c|}}\right)^{l-1}$$

$$= |DT^d(c_1)| \underbrace{\frac{|DT^{d-1}(y_1)|}{|DT^{d-1}(c_1)|}}_{=:G_1(x)} \underbrace{\frac{|DT(x)|}{|x - c|^{l-1}} \frac{|c_d - c|^{l-1}}{|DT(c_d)|}}_{=:G_2(x)}$$

$$\times \Bigg(\underbrace{\frac{|x - c_{\bar{d}}|}{|c_d - c_{\bar{d}}|}}_{=:G_3(x)} \cdot \underbrace{\frac{|c_d - x|\,|c - v|}{|c_d - c|\,|x - v|}}_{=:G_4(x)}\Bigg)^{l-1}$$

$G_1^\beta \geqq 1/4^{\beta_0}$ by the Koebe Lemma and as $1/G_1$ has at most two monotone branches, $\operatorname{var} 1/G_1^\beta \leqq 2 \cdot 4^{\beta_0} \cdot G_2^\beta$ and $1/G_2^\beta$ are bounded by $M^{2\beta_0}$ and are also of bounded variation uniformly in d (see (1.2)). $G_3 \geqq 1/2$ by the assumption on the position of $x \in (c_d, c_{\bar{d}})$ such that $\operatorname{var} 1/G_3^\beta \leqq 2^{\beta_0}$ because of its monotonicity, and finally G_4 is monotonically decreasing and not smaller than 1, whence $\operatorname{var} 1/G_4^\beta \leqq 1$. In view of the definition of λ_H we thus obtain

$$G(x)^\beta \geqq \text{const} \cdot \lambda_H^{\beta d} \quad \text{and} \quad \operatorname{var} \frac{1}{G^\beta} \leqq \text{const} \cdot \lambda_H^{-\beta d} .$$

Assume now that $|c_d - x| > |x - c_{\bar{d}}|$. By Lemma 10 in [N2] there exist periodic points $p, p_1 = Tp$ of period d such that $p = T^d p \in (c, c_{\bar{d}})$. Write $G(x)$ as

$G(x) =$

$$|DT^d(p)| \frac{|DT^{d-1}(y_1)|}{|DT^{d-1}(p_1)|} \frac{|DT(x)|}{|x-c|^{l-1}} \frac{|p-c|^{l-1}}{|DT(p)|} \left(\frac{|x-c_d|}{|c_d - c_{\bar{d}}|} \cdot \frac{|c_{\bar{d}} - x| \, |c - v|}{|p - c| \, |x - v|} \right)^{l-1} ,$$

which can be estimated as before. □

Lemma 6.5. *There exsists a constant K such that for any $\hat{x}$,*

$$|D\hat{T}^{\tilde{k}-1}(\hat{x}_{-\tilde{k}+1})| \geqq K \lambda_H^{\tilde{k}-1} .$$

Proof. Assume first $\tilde{k} = k$, so that $\hat{x} \in \hat{D}_k^+$. We shall reduce the other case to this one later on. If $|x - c_k| \leqq |x - c_{\bar{k}}|$, then the assertion follows from the Koebe Lemma with $K = K_c/4$, because $\lambda_H \leqq \lambda_c$. So we may assume $|x - c_k| > |x - c_{\bar{k}}|$ and in particular $|D_k^+| > |D_k^-|$.

Consider $\eta = \eta_{k-\bar{k}}[c]$ such that $T^{k-\bar{k}}$ is decreasing on η. $D_{k-\bar{k}} = T^{k-\bar{k}}\eta$ is the highest splitting level below D_k. Hence there exist $\alpha \in \eta$ and $p \in (c, \alpha)$ such that $T^{k-\bar{k}}\alpha = c$ and p is periodic with period $k - \bar{k}$, cf. [N2, Lemma 10]. (Then $p_1 = Tp$ is also periodic with the same period.) Observe that $c \notin T^i(]p, \alpha[)$ for $i = 0, \ldots, 2(k - \bar{k})$ as $T^{k-\bar{k}}$ is monotone on (p, α) and on $(p, c) = T^{k-\bar{k}}(p, \alpha)$. On the other hand, as T has no sinks, there is $\beta \in (p, c)$ such that $T^{k-\bar{k}}(\beta) = \alpha$. Therefore there are $\gamma \in (c, \beta)$ and $k \leqq s \leqq 2(k - \bar{k})$ such that $T^s\gamma = c$, D_s is a splitting level, $\eta_s[c] = (c, \alpha)$, and the trajectory $(\gamma_i : i = k - \bar{k} + 1, \ldots, s)$ defines the partition of D_i's into D^+ and D^- parts, which are monotonically mapped one onto another. Therefore $D_i^+ = (c_i, \gamma_i)$ and $p_i \in D_i^- = (\gamma_i, \alpha_i)$, $(i = k - \bar{k} + 1, \ldots, s)$.

By the Koebe Lemma (or by more elementary consequences of negative Schwarzian) one has for $y \in (c, \gamma)$,

$$|DT^{i-1}(y_1)| \geqq \min\{|DT^{i-1}(c_1)|, |DT^{i-1}(p_1)|\}/4$$

for i as above. In particular this holds for $i = k$ and $y = x_{-k}$, and we have to estimate $|DT^{k-1}(p_1)|$ from below.

As p_1 is periodic with period $k - \bar{k}$, we have $|DT^{k-1}(p_1)| = |DT^{k-\bar{k}}(p_1)| \, |DT^{\bar{k}-1}(p_1)|$. The first factor can be estimated by $\lambda_H^{k-\bar{k}}$. For the second one we use the observation $D_k \subset D_{\bar{k}}$, which gives $|p_{\bar{k}} - c_{\bar{k}}| = |p_k - c_{\bar{k}}| \leqq |x - c_{\bar{k}}| \leqq |x - c_k| \leqq |p_k - c_k| \leqq |p_{\bar{k}} - c_{\bar{k}}|$. We apply the Koebe Lemma to $T^{\bar{k}-1}$ on $(p_1, c_i) \subset T\eta_{\bar{k}}[c]$, and obtain $|DT^{\bar{k}-1}(p_1)| \geqq |DT^{\bar{k}-1}(c_1)|/4$. This finishes the estimation in the case $\tilde{k} = k$.

Suppose now that $\tilde{k} = \bar{k}$, i.e. $\hat{x} \in \hat{D}_{\bar{k}}$. Remark that, as before, $D_k \subseteq D_{\bar{k}}$, but also $D_{\bar{k}}^- \subset D_{\bar{k}}^+$. This is due to the fact that the next splitting after $\bar{k}$ can appear not later than the next one after k. ($D_{\bar{k}}$ is longer, it includes more preimages of c.) If it occurs earlier, then the splitting point must lie in $D_{\bar{k}} \backslash D_k$ and in this case $D_k \subset D_{\bar{k}}^+$, otherwise $D_{\bar{k}}^+ = D_{\bar{k}}^-$. Now we can estimate the derivative at $\hat{x}$ by the identical derivative at his brother $\hat{x}' \in \hat{D}_{\bar{k}}^+$ and use the first part of the proof ($\tilde{\kappa}(\hat{x}') = \kappa(\hat{x}') = \bar{k} = \bar{\kappa}(\hat{x}) = \tilde{\kappa}(\hat{x})$). □

Corollary 6.2. *Let $k > 0$. Then*

$$\sup_{\hat{D}_k} |DT^{\tilde{k}-1}(\hat{x}_{-\tilde{k}+1})|^{-1} + \mathrm{var}_{\hat{D}_k} |D\hat{T}^{\tilde{k}-1}(\hat{x}_{-\tilde{k}+1})|^{-1} \leqq 5K^{-1}\lambda_H^{-\tilde{k}+1} ,$$

where K is the constant from Lemma 6.5.

Proof. This follows from Lemma 6.5 and the fact that by negative Schwarzian derivative $\hat{x} \mapsto |D\hat{T}^{\tilde{k}-1}(\hat{x}_{-\tilde{k}+1})|^{-1}$ has at most two monotone parts on each of $\hat{D}_k^+$ and $\hat{D}_k^-$. □

6.2. Construction of the New Metric. We want to find such $\hat{w}$ that

$$\hat{\Psi}_n = \left| \frac{\hat{w}}{\hat{w} \circ \hat{T}^n \cdot D\hat{T}^n} \right| \leqq q^n$$

for some n and $q < 1$. In other words we want to change the geometry in such a way that $\hat{T}^n$ becomes uniformly expanding. As already said, C–E transformations exhibit a lot of expanding features, and also the negative Schwarzian gives some expansion. Those two properties are sufficient away from critical points.

On the other hand the C–E condition provides the expansion near the critical trajectory. So one has to combine these two contributions taking care of the passage through neighbourhoods of the critical point, which must be visited, as the C–E condition gives expansion only along initial segments of the trajectory of c_1. Therefore one has to estimate derivatives of T^n in arbitrary points by derivatives of carefully chosen initial parts of the critical trajectory.

Let $q \in]\lambda_E^{-1}, 1[$. Define $\hat{w}: \hat{X} \to \mathbf{R}$ by

$$\hat{w}(\hat{x}) = \hat{w}_q(\hat{x}) := (q^{\tilde{i}} \cdot |D\hat{T}^{\tilde{i}-1}(\hat{x}_{-\tilde{i}+1})|^{1/l} \cdot \partial^{1-1/l}(\hat{x}))^{-1} \quad \text{for } i > 1 ,$$

where $i = \kappa(\hat{x})$ and $\tilde{i}$ was defined in (3.8). On the level $\hat{D}_0$ let $\hat{w}(\hat{x}) = 1$, and on the level $\hat{D}_1$ let $\hat{w}(\hat{x}) = (c_1 - x)^{-(1-1/l)}$. As the levels $\hat{D}_0$ and $\hat{D}_1$ are dynamically transient (see Remark 3.1), we shall skip in the sequel the details of estimates concerning these two sets.

The factor ∂ guarantees expansion near critical points, q gives artificial expansion on levels without splitting, where there is a mean natural expansion but not necessary on each step, and the derivative part allows to shadow the critical trajectory from its start at c_1.

Proposition 6.1. *For any $\delta \in \left[0, \frac{1}{l-1}\right[$ there is a constant $C = C_\delta > 0$ such that*

$$\int_{\hat{D}_k} \hat{w}_q^{1+\delta}\, d\hat{m} \leqq C \cdot (q\lambda_H^{1/l})^{-\tilde{k}\cdot(1+\delta)} \quad \textit{for all } k > 1 ,$$

where $(q\lambda_H^{1/l})^{-(1+\delta)} < 1$.

Proof. By Lemma 6.5,

$$\int_{\hat{D}_k} \hat{w}_q^{1+\delta} \, d\hat{m} \leqq \text{const} \cdot (q\lambda_H^{1/l})^{-k \cdot (1+\delta)} \int_{c_k}^{c_k} \left| \frac{1}{x - c_{\tilde{k}}} + \frac{1}{c_k - x} \right|^{(1-1/l)(1+\delta)} dx .$$

The assertion follows from $q\lambda_H^{1/l} > 1$ and $\left(1 - \frac{1}{l}\right)(1 + \delta) < 1$. □

6.3. Bounds on the Transfer Function $\hat{\Psi}$. For $\hat{x} \in \hat{D}_j$ and $\hat{x}_n = \hat{T}^n \hat{x} \in \hat{D}_k$ we have

$$\hat{\Psi}_n(\hat{x}, q) := \frac{\hat{w}_q(\hat{x})}{\hat{w}_q(\hat{T}^n \hat{x}) \cdot |D\hat{T}^n(\hat{x})|}$$

$$= q^n \frac{\partial^{1-1/l}(\hat{x}_n) \cdot |D\hat{T}^{\tilde{k}-1}((\hat{x}_n)_{-\tilde{k}+1})|^{1/l}}{q^{n+\tilde{j}-\tilde{k}} \partial^{1-1/l}(\hat{x}) \cdot |D\hat{T}^{\tilde{j}-1}(\hat{x}_{-\tilde{j}+1})|^{1/l} \cdot |D\hat{T}^n(\hat{x})|} .$$

It is clear that we want to bound the quotient.

Bounds on $\hat{\Psi}_1$.

Proposition 6.2. *For each $\beta_0 > 0$,*

$$\sup_{0 \leqq \beta \leqq \beta_0} \sup_{\hat{D}_k \in \mathscr{D}} \left(\operatorname{var}_{\hat{D}_k}(\hat{\Psi}_1^\beta) + \sup_{\hat{D}_k} \hat{\Psi}_1^\beta \right) < \infty .$$

Proof. Let $\hat{x} \in \hat{D}_j$ and $\hat{T}\hat{x} \in \hat{D}_k$. We consider only the case $j > 1$. The cases $j = 0$ and $j = 1$ can be treated similarly.

Denote $\tilde{j} = \tilde{j}(\hat{x})$ and $\tilde{k} = \tilde{k}(\hat{T}\hat{x})$. There are several possibilities:

1. $\hat{D}_j$ is not a splitting level.
   Then $k = j + 1$ and $\tilde{k} = \tilde{j} + 1$, $(\hat{T}\hat{x})_{-\tilde{k}+1} = \hat{x}_{-\tilde{j}+1}$.

2. $\hat{D}_j$ is a splitting level, $\hat{x} \in \hat{D}_j^+$ and $\hat{T}\hat{x} \in \hat{D}_k^+$.
   Then $\tilde{k} = k = j + 1 = \tilde{j} + 1$, $(\hat{T}\hat{x})_{-\tilde{k}+1} = \hat{x}_{-\tilde{j}+1}$.

3. $\hat{D}_j$ is a splitting level, $\hat{x} \in \hat{D}_j^+$ and $\hat{T}\hat{x} \in \hat{D}_k^-$.
   Then $k = j + 1$, $\tilde{k} = 1$, $\tilde{j} = j$, $(\hat{T}\hat{x})_{-\tilde{k}+1} = \hat{T}\hat{x}$.

4. $\hat{D}_j$ is a splitting level, $\hat{x} \in \hat{D}_j^-$ and $\hat{T}\hat{x} \in \hat{D}_k^+$.
   Then $\tilde{k} = k = \tilde{j} + 1 = \tilde{j} + 1$, and $(\hat{T}\hat{x})_{-\tilde{k}+1} = \hat{y}' \in \hat{D}_1$ is a brother of $\hat{y} = \hat{x}_{-\tilde{j}+1} \in \hat{D}_{j-\tilde{j}}$ (i.e. $\pi\hat{y} = \pi\hat{y}'$). Their trajectories meet at level $\hat{D}_k$ after $\tilde{k} - 1 = \tilde{j}$ steps.

5. $\hat{D}_j$ is a splitting level, $\hat{x} \in \hat{D}_j^-$ and $\hat{T}\hat{x} \in \hat{D}_k^-$.
   Then $\tilde{k} = 1$, $\tilde{j} = \tilde{j}$, and $(\hat{T}\hat{x})_{-1} = \hat{x}' \in \hat{D}_{k-1}$ is a brother of $\hat{x} \in \hat{D}_j$. They meet at level $\hat{D}_k$ after one step.

We have '

$$\hat{\Psi}_1 = \frac{\hat{w}(\hat{x})}{\hat{w}(\hat{T}\hat{x})|D\hat{T}(\hat{x})|} = \left(\frac{\partial(\hat{T}\hat{x})}{\partial(\hat{x})}\right)^{1-1/l}\left(\frac{|D\hat{T}^{\tilde{k}-1}((\hat{T}\hat{x})_{-\tilde{k}+1})|}{|D\hat{T}^{\tilde{j}-1}(\hat{x}_{-\tilde{j}+1})|}\right)^{1/l}\frac{1}{|D\hat{T}(\hat{x})|}\frac{q^{\tilde{k}}}{q^{\tilde{j}}}$$

$$= q\left(\frac{\partial(\hat{T}\hat{x})}{\delta_1(x)|D\hat{T}(\hat{x})|}\right)^{1-1/l}\left[q^{-(\tilde{j}+1-\tilde{k})}\left(\frac{\delta_1(\hat{x})}{\partial(\hat{x})}\right)^{1-1/l}\right.$$

$$\left.\times\left|\frac{D\hat{T}^{\tilde{k}-1}((\hat{T}\hat{x})_{-\tilde{k}+1})}{D\hat{T}^{\tilde{j}-1}(\hat{x}_{-\tilde{j}+1})D\hat{T}(\hat{x})}\right|^{1/l}\right].$$

We simplify the derivatives (taking brothers if necessary) and obtain in cases 1, 2, 4,

$$\hat{\Psi}_1^\beta = q^\beta\left(\frac{\partial(\hat{T}\hat{x})}{\delta_1(\hat{x})|D\hat{T}(\hat{x})|}\right)^{\beta(1-1/l)}\cdot\left(\frac{\delta_1(\hat{x})}{\partial(\hat{x})}\right)^{\beta(1-1/l)}.$$

Supremum and variation over $\hat{D}_k$ of the first factor are uniformly bounded by Lemma 6.3 (i.e. uniformly in $\beta\in[0,b]$ and $\hat{D}_k\in\mathscr{D}$). The second one is identically equal to 1 in case 1 and is monotone and bounded by 1 on $\hat{D}_j^-$ in cases 2 and 4.

In cases 3 and 5,

$$\hat{\Psi}_1^\beta = q^\beta\left(\frac{\partial(\hat{T}\hat{x})}{\delta_1(\hat{x})|D\hat{T}(\hat{x})|}\right)^{\beta(1-1/l)}\left[q^{-\tilde{j}}\left(\frac{\delta_1(\hat{x})}{\partial(\hat{x})}\right)^{1-1/l}\left|\frac{1}{D\hat{T}^{\tilde{j}}(\hat{x}_{-\tilde{j}+1})}\right|^{1/l}\right]^\beta.$$

The first factor is the same as before. The bound on the [ ] factor follows from Lemma 6.4, as $q^{-1}<\lambda_H^{1/l}$. □

Bounds on $\hat{\Psi}_n$.

Proposition 6.3. *For any* $q\in]\lambda_E^{-1},1[$ *there exists* $C>0$ *such that the estimate*

$$\hat{\Psi}_n(\hat{x},q)\leqq C\cdot q^n$$

holds uniformly in $n>0$ *and* $\hat{x}$.

In order to prove this proposition we have to decompose carefully the trajectory of $\hat{x}$ from $-\tilde{j}$ to n.

The Trajectory. We divide the trajectory of a point $\hat{x}\in\hat{D}_j$ up to $\hat{x}_n\in\hat{D}_k$ in parts corresponding to the initial segments of the critical trajectory. Let $\hat{y}=\hat{x}_{-\tilde{j}}$ and $y=\pi\hat{y}$. Set $t_0=0$ and define t_1 to be the minimal $t\geqq\tilde{j}$ such that $\hat{y}_t$ is on a splitting level (i.e. $c\in D_{\kappa(\hat{y}_t)}$), and $\hat{y}_{t+1}$ is in the $\hat{D}^-$ part. Analogously, if t_i is defined, let t_{i+1} be the minimal $t>t_i$ such that $\hat{y}$ is on a splitting level and $\hat{y}_{t+1}\in\hat{D}^-$. Let r be the maximal index i such that $t_i<\tilde{j}+n$. (Observe that r depends on n.) This defines $t_0,\ldots,t_r$. Finally let $t_{r+1}=\tilde{j}+n$.

Put $d_i:=t_{i+1}-t_i$ for $0\leqq i\leqq r$. Then $\hat{y}_{t_{i+1}}\in\hat{D}_{d_i}$ for $0<i<r$. The trajectory $\hat{T}^j\hat{y}_{t_i}$, $(0<j\leqq d_i)$, first follows one block of $\hat{D}^-$'s between two splitting levels, then jumps down and climbs through consecutive blocks of $\hat{D}^+$'s, until it reaches at $j=d_i+1$ a new $\hat{D}^-$.

Let (c,α) be an interval which is mapped by T^{d_i} monotonically onto (c_{d_i},c). (Observe that $(c,\alpha)=\eta_{d_i+1}[c^\pm]$.) Then $y_{t_i}\in(c,\alpha)$. Analogously there is an $\alpha'\in(c_{d_i},c)$ such that $T^{d_{i+1}}\alpha'=c$ and $y_{t_{i+1}}\in(c,\alpha')$. We pull back the interval (c_{d_i},α')

by T^{-d_i} into the interval (c, y_{t_i}) and we obtain a cylinder of order $d_i + d_{i+1} + 1$ on one side of y_{t_i}. On the other side of y_{t_i} (but still in D_{d_i-1}) we can pull back the analogous cylinder from near $y_{t_{i+1}}$ and obtain a cylinder of order $d_i + d_{i+1} + d_{i+2} + 1$. Hence

The point y_{t_i} divides D_{d_i-1} into two parts each of which contains one of the finitely many cylinders for T of order at most $d_i + d_{i+1} + d_{i+2} + 1$. (6.6)

Any $\hat{y}_{t_i}$ has a brother $\hat{y}'_{t_i} \in \hat{D}_0$ such that they meet after the first jump of $\hat{y}_{t_i}$ and climb together thereafter.

Remember that $\hat{x}_n \in \hat{D}_k$.

If $\tilde{k}(\hat{x}) = k$ for, then $\hat{y}_{t_r}$ is a brother of $\hat{x}_{n-k}$ and if $\tilde{k} = \bar{k}$ then $\hat{y}_{t_r} = (\hat{x}_n)_{-\bar{k}}$. In both cases $\tilde{j} + n = t_r + \tilde{k}$. (6.7)

In particular $d_r = \tilde{j} + n - t_r = \tilde{k}$.

The Estimation.

Lemma 6.6. *Suppose that $\tilde{\kappa}(\hat{x}) = 1$ and $\tilde{\kappa}(\hat{x}_m) = 1$. Then*

$$\hat{\Psi}_m(\hat{x}) \leqq \left(\frac{\partial(\hat{x})}{\partial(\hat{x}_m)}\right)^{1/l} \left(\frac{\delta_m(\hat{x})}{\partial(\hat{x}_m)}\right).$$

Proof. In this case

$$\hat{\Psi}_m(\hat{x}) = \frac{\partial^{1-1/l}(\hat{x}_m)}{\partial^{1-1/l}(\hat{x})|D\hat{T}^m(\hat{x})|} = \left[\frac{\partial(\hat{x}_m)}{\delta_m(\hat{x})|DT^m(\hat{x})|}\right]\left[\frac{\partial(\hat{x})}{\partial(\hat{x}_m)}\right]^{1/l}\left[\frac{\delta_m(\hat{x})}{\partial(\hat{x})}\right].$$

The first factor is smaller than 1 by (6.4). □

Corollary 6.3. *For any N there exists a constant $C = C(N)$ such that if in the situation of the previous lemma the three consecutive d_i's after $\hat{x}$ and the three consecutive d_i's after $\hat{x}_m$ are smaller than N, then*

$$\hat{\Psi}_m(\hat{x}) \leqq C \cdot \operatorname{diam} \mathscr{Z}_m \leqq C \cdot K_\eta \cdot \lambda_\eta^{-m} .$$

Proof. Let $\varepsilon := \min\{|\eta| : \eta \in \mathscr{Z}_{3N+1}\}$. By the assumption on the d_i's and (6.6) we find, to both sides of $\hat{x}$ and $\hat{x}_m$, cylinders of order not exceeding $3N + 1$ and hence of length not smaller than ε. The corollary follows from (6.1) taking $C = (2/\varepsilon)^{1+1/l}$. □

Lemma 6.7. *There exists a constant K such that*

$$\hat{\Psi}_n(\hat{x}) \leqq q^n \left[\prod_{i=0}^{r-1} K(\lambda_H^{1/l} q)^{d_i}\right]^{-1} .$$

Proof. Recall from the decomposition of the trajectory that $\hat{y} = \hat{x}_{-\tilde{j}}$,

$$\hat{\Psi}_n(\hat{x}) = q^n \left|\frac{D\hat{T}^{\tilde{k}-1}(\hat{x}_{n-\tilde{k}+1})}{D\hat{T}^{\tilde{j}-1}(\hat{x}_{-\tilde{j}+1})}\right|^{1/l} \left(\frac{\partial(\hat{x}_n)}{\partial(\hat{x})}\right)^{1-1/l} \frac{1}{|D\hat{T}^n(\hat{x})| q^{n+\tilde{j}-\tilde{k}}} .$$

By (6.7), $\hat{y}^{1}_{\tilde{j}+n-\tilde{k}}$ is a brother of $\hat{x}_{n-\tilde{k}}$. Therefore

$$\begin{aligned}\hat{\Psi}_n(\hat{x}) &= q^n \cdot \left(\frac{\partial(\hat{x}_n)}{\partial(\hat{x})}\right)^{1-1/l} \cdot |D\hat{T}^{\tilde{j}+n-\tilde{k}}(\hat{y}_1)|^{-1/l} \cdot |D\hat{T}^n(\hat{y}_{\tilde{j}})^{-1+1/l} \cdot q^{-(\tilde{j}+n-\tilde{k})} \\ &= q^n \cdot \left[q^{-(\tilde{j}+n-\tilde{k})} \prod_{i=0}^{r-1} |D\hat{T}^{d_i}(\hat{T}\hat{y}_{t_i})|^{-1/l} \left(\frac{\partial(\hat{y}_{t_{i+1}})}{\delta_{d_{i+1}}(\hat{y}_{t_{i+1}})} \right)^{-1+1/l} \right] \\ &\quad \cdot \left[|D\hat{T}^{t_1-\tilde{j}}(\hat{y}_{\tilde{j}})| \frac{\delta_{t_1-\tilde{j}}(\hat{y}_{\tilde{j}})}{\partial(\hat{y}_{t_1})} \frac{\partial(\hat{y}_{\tilde{j}})}{\delta_{t_1-\tilde{j}}(\hat{y}_{\tilde{j}})} \right]^{-1+1/l} \\ &\quad \cdot \left[\prod_{i=1}^{r} |D\hat{T}^{d_i}(\hat{y}_{t_i})| \frac{\delta_{d_i}(\hat{y}_{t_i})}{\partial(\hat{y}_{t_{i+1}})} \right]^{-1+1/l} .\end{aligned}$$

By (6.4) and (6.5) the two last factors are bounded by 1. For the first factor we use Lemma 6.4 with $(c, v) = \eta_{d_{i+1}}[c]$ and observe that $t_r = \sum_{i=0}^{r-1} d_i = \tilde{j} + n - \tilde{k}$. □

Proof of Proposition 6.3. Fix N such that $(\lambda_H^{1/l} q)^N > K^{-3}$, where K is the constant from Lemma 6.7. Let $t' = t_{i_1} > t_0$ be minimal such that the three consecutive d's, d_{i_1}, d_{i_1+1} and d_{i_1+2}, are smaller than N. Let $t'' = t_{i_2} < t_r$ be maximal with this property. Remembering that $\hat{y} = \hat{x}_{-\tilde{j}}$, so that $\hat{y}_{\tilde{j}+n} = \hat{x}_n$, we write

$$\hat{\Psi}_n(\hat{x}) = \hat{\Psi}_{t'-\tilde{j}+1}(\hat{y}_{\tilde{j}}) \hat{\Psi}_{t''-t'}(\hat{y}_{t'+1}) \hat{\Psi}_{n+\tilde{j}-t''-1}(\hat{y}_{t''+1}) .$$

The middle factor starts and ends just after splitting levels with three consecutive small d_i's. Therefore $\hat{y}_{t'+1}$ and $\hat{y}_{t''+1}$ are separated from the endpoints of their levels $\hat{D}$ at least by a distance depending only on N, see (6.6). In the first factor there are no three consecutive small d_i's and in the third one such d_i's appears only once, namely at the beginning.

We estimate the middle factor by Corollary 6.3 and the two other factors by Lemma 6.7. Using the definition of N we obtain

$$\hat{\Psi}_n < \text{const} \cdot q^n . \qquad \square$$

A. Appendix

We prove the estimate (4.12) for $\hat{\zeta}_n^{(2)} := \sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n} (\hat{\mathscr{P}} \hat{\mathscr{L}}^n \chi_{\hat{\eta}})(\hat{x}_{\hat{\eta}})$. The proof, which relies heavily on an estimation technique due to Haydn [Ha], is very similar to the corresponding one in Sect. 5 of [BK]. But since the changes which are necessary are crucial for our more general setting where $\hat{\Phi}$ is not of the form $\Phi \circ \pi$, we give a complete proof.

For each $\hat{D} \in \mathscr{D}$ we fix some $\hat{y}_{\hat{D}} \in \hat{D}$. If $\hat{\eta} \in \hat{\mathscr{Z}}_j$ and $\hat{T}^j \hat{\eta} = \hat{D}$, we denote by $\hat{y}_{\hat{\eta}}$ the unique $\hat{T}^j$-preimage of $\hat{y}_{\hat{D}} \in \hat{\eta}$.

For $\hat{\eta} \in \hat{\mathscr{Z}}_j$ define

$$\hat{Y}_{\hat{\eta}} = \begin{cases} \hat{\mathscr{L}}^j \chi_{\hat{\eta}} - \hat{\Phi}(\hat{y}_{\hat{\eta}}) \cdot \hat{\mathscr{L}}^{j-1} \chi_{\hat{T}\hat{\eta}} & \text{if } j \geqq 2 , \\ \hat{\mathscr{L}} \chi_{\hat{\eta}} & \text{if } j = 1 . \end{cases}$$

Observe that $\hat{T}\hat{y}_{\hat{\eta}} = \hat{y}_{\hat{T}\hat{\eta}}$. Hence, for $\hat{\eta} \in \hat{\mathscr{Z}}_j$ and $j \geqq 2$,

$$\hat{Y}_{\hat{\eta}} = \hat{\mathscr{L}}^{j-1}\chi_{\hat{T}\hat{\eta}} \cdot (\hat{\Phi} \circ \hat{T}_{\hat{\eta}}^{-j} - \hat{\Phi}(\hat{y}_{\hat{\eta}})) = \chi_{\hat{T}^j\hat{\eta}} \cdot (\hat{\Phi}_{j-1} \circ \hat{T}_{\hat{T}\hat{\eta}}^{-(j-1)}) \cdot (\hat{\Phi} \circ \hat{T}_{\hat{\eta}}^{-j} - \hat{\Phi}(\hat{y}_{\hat{\eta}})) ,$$

so that

$$\mathrm{var}_{\hat{T}^j\hat{\eta}}(\hat{Y}_{\hat{\eta}}) \leqq \mathrm{var}_{\hat{T}\hat{\eta}}(\hat{\Phi}_{j-1}) \cdot \mathrm{var}_{\hat{\eta}}(\hat{\Phi}) + \sup|\hat{\Phi}_{j-1}| \cdot \mathrm{var}_{\hat{\eta}}(\hat{\Phi})$$

and

$$\sup|\hat{Y}_{\hat{\eta}}| \leqq \sup|\hat{\Phi}_{j-1}| \cdot \mathrm{var}_{\hat{\eta}}(\hat{\Phi}) .$$

Therefore, in view of Lemma 4.1,

$$\begin{aligned} \|\hat{Y}_{\hat{\eta}}\|_{\widehat{BV}} &= \mathrm{var}_{\hat{T}^j\hat{\eta}}(\hat{Y}_{\hat{\eta}}) + \sup|\hat{Y}_{\hat{\eta}}| \\ &\leqq \mathrm{var}_{\hat{\eta}}(\hat{\Phi}) \cdot (\mathrm{var}_{\hat{T}\hat{\eta}}(\hat{\Phi}_{j-1}) + 2 \cdot \sup|\hat{\Phi}_{j-1}|) \\ &\leqq \mathrm{const} \cdot \Theta^j \cdot \mathrm{var}_{\hat{\eta}}(\hat{\Phi}) \end{aligned} \tag{A.1}$$

with a constant which is locally uniform in the parameters β and t.

In the next lemma we show that all sums occurring in the following decomposition of $\hat{\zeta}_n^{(2)}$ are absolutely convergent,

$$\hat{\zeta}_n^{(2)} = \underbrace{\sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n} (\hat{\mathscr{P}}\hat{\mathscr{L}}^n \chi_{\hat{\eta}})(\hat{x}_{\hat{\eta}})}_{=:\gamma_n^{(1)}} - \underbrace{\sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n \setminus \mathscr{A}_n} (\hat{\mathscr{P}}\hat{\mathscr{L}}^n \chi_{\hat{\eta}})(\hat{x}_{\hat{\eta}})}_{=:\gamma_n^{(2)}} , \tag{A.2}$$

$$\begin{aligned} \gamma_n^{(1)} &= \sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n} \sum_{k=0}^{n-1} \hat{\Phi}_k(\hat{y}_{\hat{\eta}}) \hat{\mathscr{P}} \hat{Y}_{\hat{T}^k\hat{\eta}}(\hat{x}_{\hat{\eta}}) \\ &= \underbrace{\sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n} \sum_{k=0}^{n-1} \hat{\Phi}_k(\hat{y}_{\hat{\eta}}) (\hat{\mathscr{P}} \hat{Y}_{\hat{T}^k\hat{\eta}}(\hat{x}_{\hat{\eta}}) - \hat{\mathscr{P}} \hat{Y}_{\hat{T}^k\hat{\eta}}(\hat{y}_{\hat{\eta}}))}_{=:\gamma_n^{(3)}} \\ &\quad + \underbrace{\sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n} \sum_{k=0}^{n-1} \hat{\Phi}_k(\hat{y}_{\hat{\eta}}) \hat{\mathscr{P}} \hat{Y}_{\hat{T}^k\hat{\eta}}(\hat{y}_{\hat{\eta}})}_{=:\gamma_n^{(4)}} , \end{aligned} \tag{A.3}$$

where we use the convention $\hat{\Phi}_0 \equiv 1$. Finally, defining $\hat{\mathscr{Z}}_n^k = \{\hat{\eta} \in \hat{\mathscr{Z}}_n : \hat{\eta} \subseteq \hat{D}_0 \cup \ldots \cup \hat{D}_k\}$, we have

$$\begin{aligned} \gamma_n^{(4)} &= \underbrace{\sum_{k=0}^{n-1} \sum_{\hat{\eta}' \in \hat{\mathscr{Z}}_{n-k}^{2n}} \sum_{\substack{\hat{\eta} \in \hat{\mathscr{Z}}_n \\ \hat{T}^k\hat{\eta} = \hat{\eta}'}} \hat{\Phi}_k(\hat{y}_{\hat{\eta}}) (\hat{\mathscr{P}} \hat{Y}_{\hat{\eta}'})(\hat{y}_{\hat{\eta}})}_{=:\gamma_n^{(5)}} \\ &\quad + \underbrace{\sum_{k=0}^{n-1} \sum_{\hat{\eta}' \in \hat{\mathscr{Z}}_{n-k} \setminus \hat{\mathscr{Z}}_{n-k}^{2n}} \sum_{\substack{\hat{\eta} \in \hat{\mathscr{Z}}_n \\ \hat{T}^k\hat{\eta} = \hat{\eta}'}} \hat{\Phi}_k(\hat{y}_{\hat{\eta}}) (\hat{\mathscr{P}} \hat{Y}_{\hat{\eta}'})(\hat{y}_{\hat{\eta}})}_{=:\gamma_n^{(6)}} . \end{aligned} \tag{A.4}$$

The estimate for $\hat{\zeta}_n^{(2)}$ follows from

Lemma A.1 (Compare Lemma 5.1 in [BK]).
1. $\gamma_n^{(4)}$ *converges absolutely.*
2. $\gamma_n^{(2)}$ *converges absolutely and* $|\gamma_n^{(2)}| \leqq \text{const} \cdot \Theta^n$.
3. $\gamma_n^{(3)}$ *converges absolutely and* $|\gamma_n^{(3)}| \leqq \text{const} \cdot \Theta^n$.
4. $|\gamma_n^{(5)}| \leqq \text{const} \cdot \Theta^n$.
5. $|\gamma_n^{(6)}| \leqq \text{const} \cdot \Theta^n$.

All constants are uniform in $n > 0$ *and locally uniform in the parameters* β *and* t *of* $\hat{\Phi}$.

Indeed, the absolute convergence of $\gamma_n^{(3)}$ and $\gamma_n^{(4)}$ implies that of $\gamma_n^{(1)}$ and shows that all equalities of (A.2)–(A.4) are correct. Thus we only need to use

$$|\hat{\zeta}_n^{(2)}| \leqq |\gamma_n^{(2)}| + |\gamma_n^{(3)}| + |\gamma_n^{(5)}| + |\gamma_n^{(6)}|$$

to obtain the desired inequality.

Proof of Lemma A.1. In the proof we will often use the decomposition $\hat{\mathscr{P}} = \text{Id} - \hat{\mathscr{P}}^{\perp}$ and the fact that there are $\hat{f}_j \in \widehat{BV}$ and linear functionals $\hat{G}_j : \widehat{BV} \to \mathbf{C}$ $(j = 1, \ldots, d := \text{rank}(\hat{\mathscr{P}}^{\perp}))$ such that $\hat{\mathscr{P}}^{\perp}\hat{f} = \sum_{j=1}^{d} \hat{G}_j(\hat{f}) \cdot \hat{f}_j$. As the projection $\hat{\mathscr{P}}$ depends analytically on the parameters β and t (see Proposition 4.2), d is locally constant in β and t, and the $\hat{G}_j$ and $\hat{f}_j$ can be chosen such that their norms are locally uniformly bounded in β and t.

We use the notations $|\hat{\mathscr{L}}|$ for $\hat{\mathscr{L}}[|\hat{\Phi}|]$ and $\hat{D}(i)$ for $\hat{D}_i$.

1.

$$\begin{aligned}
&\sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n} \sum_{k=0}^{n-1} |\hat{\Phi}_k(\hat{y}_{\hat{\eta}})| \, |(\hat{\mathscr{P}} \hat{Y}_{\hat{T}^k \hat{\eta}})(\hat{y}_{\hat{\eta}})| \\
&= \sum_{k=0}^{n-1} \sum_{\hat{\eta}' \in \hat{\mathscr{Z}}_{n-k}} \sum_{\substack{\hat{\eta} \in \hat{\mathscr{Z}}_n \\ \hat{T}^k \hat{\eta} = \hat{\eta}'}} |\hat{\Phi}_k(\hat{y}_{\hat{\eta}})| \, |\hat{\mathscr{P}} \hat{Y}_{\hat{\eta}'}|(\hat{y}_{\hat{\eta}}) \\
&= \sum_{k=0}^{n-1} \sum_{\hat{\eta}' \in \hat{\mathscr{Z}}_{n-k}} (|\hat{\mathscr{L}}|^k |\hat{\mathscr{P}} \hat{Y}_{\hat{\eta}'}|)(\hat{y}_{\hat{\eta}'}) \\
&\leqq \underbrace{\sum_{k=0}^{n-1} \sum_{\hat{\eta}' \in \hat{\mathscr{Z}}_{n-k} \setminus \hat{\mathscr{Z}}_{n-k}^{2n}} \sum_{j=1}^{d} (|\hat{\mathscr{L}}|^k |\hat{f}_j|)(\hat{y}_{\hat{\eta}'}) \|\hat{G}_j\| \, \|\hat{Y}_{\hat{\eta}'}\|_{\widehat{BV}}}_{=: \beta_n^{(1)}} \\
&\quad + \underbrace{\sum_{k=0}^{n-1} \sum_{\hat{\eta}' \in \hat{\mathscr{Z}}_{n-k}^{2n}} (|\hat{\mathscr{L}}|^k |\hat{\mathscr{P}} \hat{Y}_{\hat{\eta}'}|)(\hat{y}_{\hat{\eta}'})}_{=: \beta_n^{(2)}}
\end{aligned}$$

Remark. For the last inequality we used the fact that the support of $\hat{Y}_{\hat{\eta}'}$ is $\hat{T}^{n-k}\hat{\eta}'$, such that $(|\hat{\mathscr{L}}|^k |\hat{Y}_{\hat{\eta}'}|)|_{\hat{\eta}'} \neq 0$ if and only if $\hat{T}^n \hat{\eta}' \supseteq \hat{\eta}'$. Hence $\hat{\eta}' \in \hat{\mathscr{Z}}_{n-k}^{2n}$ by (3.6). Therefore the term corresponding to the Id part in the decomposition $\hat{\mathscr{P}} = \text{Id} - \hat{\mathscr{P}}^{\perp}$ is zero in the sum for $\beta_n^{(1)}$.

Now, by (A.1) (and here we deviate for the first time essentially from [BK]),

$$\beta_n^{(1)} \leqq \sum_{k=0}^{n-1} \sum_{j=1}^{d} \|\hat{G}_j\| \cdot \text{const} \cdot \Theta^{n-k} \sum_{i=0}^{\infty} \sum_{\substack{\hat{\eta}' \in \hat{\mathscr{Z}}_{n-k} \\ \hat{\eta}' \subseteq \hat{D}_i}} (|\hat{\mathscr{L}}|^k |\hat{f}_j|)(\hat{y}_{\hat{\eta}'}) \cdot \text{var}_{\hat{\eta}'}(\hat{\Phi})$$

$$\leqq \sum_{k=0}^{n-1} \sum_{j=1}^{d} \|\hat{G}_j\| \cdot \text{const} \cdot \Theta^{n-k} \sum_{i=0}^{\infty} \sup_{\hat{D}_i} (|\hat{\mathscr{L}}|^k |\hat{f}_j|) \cdot \underbrace{\sum_{\substack{\hat{\eta}' \in \hat{\mathscr{Z}}_{n-k} \\ \hat{\eta}' \subseteq \hat{D}_i}} \text{var}_{\hat{\eta}'}(\hat{\Phi})}_{\leqq V \text{ by } (4.2)}$$

$$\leqq \sum_{k=0}^{n-1} \sum_{j=1}^{d} \|\hat{G}_j\| \cdot \text{const} \cdot \Theta^{n-k} \| \, |\hat{\mathscr{L}}|^k |\hat{f}_j| \, \|_{\widehat{BV}} < \infty ,$$

and $\beta_n^{(2)} < \infty$ as $\text{card}(\hat{\mathscr{Z}}_{n-k}^{2n}) \leqq 2n \cdot \text{card}(\mathscr{Z}_{n-k}) < \infty$.

2. Again we use the decomposition $\hat{\mathscr{P}} = \text{Id} - \hat{\mathscr{P}}^{\perp}$:

$$|\gamma_n^{(2)}| \leqq \underbrace{\sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n \setminus \hat{\mathscr{A}}_n} |\hat{\mathscr{L}}^n \chi_{\hat{\eta}}|(\hat{x}_{\hat{\eta}})}_{=: \beta_n^{(3)}} + \underbrace{\sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n \setminus \hat{\mathscr{A}}_n} |\hat{\mathscr{P}}^{\perp} \hat{\mathscr{L}}^n \chi_{\hat{\eta}}|(\hat{x}_{\hat{\eta}})}_{=: \beta_n^{(4)}} .$$

Now, by the same reasoning as in (4.10),

$$\beta_n^{(3)} \leqq \sum_{\substack{\hat{\eta} \in \hat{\mathscr{Z}}_n \setminus \hat{\mathscr{A}}_n \\ \hat{\eta} \subseteq \hat{T}^n \hat{\eta}}} \sup_{\hat{\eta}} |\hat{\Phi}_n| \leqq \text{const} \cdot \Theta^n ,$$

and in view of Lemma 4.1 and assertion (4.4) we have

$$\beta_n^{(4)} \leqq \sum_{j=1}^{d} \sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n \setminus \hat{\mathscr{A}}_n} |\hat{G}_j(\hat{\mathscr{L}}^n \chi_{\hat{\eta}})| \cdot |\hat{f}_j(\hat{x}_{\hat{\eta}})|$$

$$\leqq \sum_{j=1}^{d} \sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n \setminus \hat{\mathscr{A}}_n} \|\hat{G}_j\| \cdot \text{const} \cdot \Theta^n \cdot |\hat{f}_j(\hat{x}_{\hat{\eta}})|$$

$$\leqq \text{const} \cdot \Theta^n \sum_{j=1}^{d} \sum_{i=0}^{\infty} 4 \cdot \sup_{\hat{D}_i} |\hat{f}_j| \leqq \text{const} \cdot \Theta^n \sum_{j=1}^{d} \|\hat{f}_j\|_{\widehat{BV}}$$

$$\leqq \text{const} \cdot \Theta^n .$$

3. Let $0 \leqq k < n$. We first study

$$\gamma_{n,k}^{(7)} := \sum_{\hat{\eta} \in \hat{\mathscr{Z}}_n} \text{var}_{\hat{\eta}}(\hat{\mathscr{P}} \hat{Y}_{\hat{T}^k \hat{\eta}})$$

$$= \underbrace{\sum_{\hat{\eta}' \in \hat{\mathscr{Z}}_{n-k}^{2n}} \sum_{\substack{\hat{\eta} \in \hat{\mathscr{Z}}_n \\ \hat{T}^k \hat{\eta} = \hat{\eta}'}} \text{var}_{\hat{\eta}}(\hat{\mathscr{P}} \hat{Y}_{\hat{\eta}'})}_{=: \beta_n^{(5)}} + \underbrace{\sum_{\hat{\eta} \in \hat{\mathscr{Z}}_{n-k} \setminus \hat{\mathscr{Z}}_{n-k}^{2n}} \sum_{\substack{\hat{\eta} \in \hat{\mathscr{Z}}_n \\ \hat{T}^k \hat{\eta} = \hat{\eta}'}} \text{var}_{\hat{\eta}} (\hat{\mathscr{P}} \hat{Y}_{\hat{\eta}'})}_{=: \beta_n^{(6)}} .$$

In view of (A.1) and (4.2), the first term is bounded by

$$\begin{aligned}
\beta_n^{(5)} &\leqq \sum_{i=0}^{\infty} \sum_{\substack{\hat\eta' \in \hat{\mathscr{Z}}_{n-k}^{2n} \\ \hat\eta' \subseteq \hat D(i)}} \operatorname{var}(\hat{\mathscr{P}} \hat Y_{\hat\eta'}) \\
&\leqq \sum_{i=0}^{2n} \|\hat{\mathscr{P}}\| \cdot \text{const} \cdot \Theta^{n-k} \cdot \sum_{\substack{\hat\eta' \in \hat{\mathscr{Z}}_{n-k}^{2n} \\ \hat\eta' \subseteq \hat D(i)}} \operatorname{var}_{\hat\eta'}(\hat\Phi) \\
&\leqq \sum_{i=0}^{2n} \|\hat{\mathscr{P}}\| \cdot \text{const} \cdot \Theta^{n-k} \cdot \operatorname{var}_{\hat D(i)}(\hat\Phi) \leqq \text{const} \cdot (2n+1) \cdot \Theta^{n-k} \cdot V \\
&\leqq \text{const} \cdot \Theta^{n-k} .
\end{aligned}$$

For the other term we use the remark of part 1 and obtain

$$\begin{aligned}
\beta_n^{(6)} &\leqq \sum_{j=1}^{d} \sum_{\hat\eta' \in \hat{\mathscr{Z}}_{n-k} \setminus \hat{\mathscr{Z}}_{n-k}^{2n}} \sum_{\substack{\hat\eta \in \hat{\mathscr{Z}}_n \\ \hat T^k \hat\eta = \hat\eta'}} |\hat G_j(\hat Y_{\hat\eta'})| \cdot \operatorname{var}_{\hat\eta}(\hat f_j) \\
&= \sum_{j=1}^{d} \|\hat G_j\| \cdot \sum_{i=0}^{\infty} \sum_{\hat\eta' \in \hat{\mathscr{Z}}_{n-k} \setminus \hat{\mathscr{Z}}_{n-k}^{2n}} \sum_{\substack{\hat\eta \in \hat{\mathscr{Z}}_n \\ \hat T^k \hat\eta = \hat\eta', \hat\eta \subseteq \hat D(i)}} \|\hat Y_{\hat\eta'}\|_{\widehat{BV}} \cdot \operatorname{var}_{\hat\eta}(\hat f_j) \\
&\leqq \sum_{j=1}^{d} \|\hat G_j\| \cdot \sum_{i=0}^{\infty} \sum_{\hat\eta' \in \hat{\mathscr{B}}(n,k,i)} \|\hat Y_{\hat\eta'}\|_{\widehat{BV}} \cdot \operatorname{var}_{\hat D(i)}(\hat f_j) \\
&\leqq \text{const} \cdot \Theta^{n-k} \sum_{j=1}^{d} \|\hat G_j\| \cdot \sum_{i=0}^{\infty} \operatorname{var}_{\hat D(i)}(\hat f_j) \cdot \sum_{\hat\eta' \in \hat{\mathscr{B}}(n,k,i)} \operatorname{var}_{\hat\eta'}(\hat\Phi) ,
\end{aligned}$$

where $\hat{\mathscr{B}}(n, k, i)$ denotes the family of those $\hat\eta' \in \hat{\mathscr{Z}}_{n-k} \setminus \hat{\mathscr{Z}}_{n-k}^{2n}$ which are contained in $\hat T^k \hat D_i$.

Now we use the following fact which is proved as Lemma 3.3 in [BK]:

Given $\eta' \in \mathscr{Z}_{n-k}$ and $\hat D_i$, there are at most two $\hat\eta \in \hat{\mathscr{Z}}_n$ such that $\hat\eta \subseteq \hat D_i$, $\hat T^k \hat\eta \in \hat{\mathscr{Z}}_{n-k} \setminus \hat{\mathscr{Z}}_{n-k}^{2n}$ and $\pi(\hat T^k \hat\eta) \subseteq \eta'$. (A.5)

Observe also:

Suppose $\hat T^k \hat\eta \subseteq \hat D_r$ for such an $\hat\eta$. Then $r > 2n$ and $\hat T^k \hat D_i \supseteq \hat D_r$, whence $r = i + k$ or $r = \bar i + k$. (Otherwise there were $0 \leqq s < t < k$ such that $\hat T^s \hat\eta$ and $\hat T^t \hat\eta$ are contained in splitting levels $\hat D_{i+s}$ and $\hat D_{\bar i + t}$ respectively, from which they jump back. In particular $\hat T^{t+1} \hat\eta \subseteq \hat D_{\overline{\bar i + t + 1}}$. As $\overline{\bar i + s + 1} = 1$ (see Sect. 3), it follows that $\overline{\bar i + t} \leqq t - s < k < n$, which contradicts $r > 2n$.) (A.6)

Now we can continue the above estimate as follows:

$$\begin{aligned}
\beta_n^{(6)} &\leqq \text{const} \cdot \Theta^{n-k} \sum_{j=1}^{d} \|\hat G_j\| \cdot \sum_{i=0}^{\infty} \operatorname{var}_{\hat D(i)}(\hat f_j) \cdot (\operatorname{var}_{\hat D(i+k)}(\hat\Phi) + \operatorname{var}_{\hat D(\bar i + k)}(\hat\Phi)) \\
&\leqq \text{const} \cdot \Theta^{n-k} \sum_{j=1}^{d} \|\hat G_j\| \cdot \sum_{i=0}^{\infty} \operatorname{var}_{\hat D(i)}(\hat f_j) \cdot 2V \\
&\leqq \text{const} \cdot \Theta^{n-k} \sum_{j=1}^{d} \|\hat G_j\| \cdot \|\hat f_j\|_{\widehat{BV}} \\
&\leqq \text{const} \cdot \Theta^{n-k} .
\end{aligned}$$

We have thus proved that $\gamma_{n,k}^{(7)} \leqq \text{const} \cdot \Theta^{n-k}$. Therefore

$$|\gamma_n^{(3)}| \leqq \sum_{\hat\eta \in \hat{\mathscr{Z}}_n} \sum_{k=0}^{n-1} |\hat\Phi_k(\hat y_{\hat\eta})| \cdot \operatorname{var}_{\hat\eta}(\hat{\mathscr{P}} \hat Y_{\hat T^k \hat\eta}) \leqq \sum_{k=0}^{n-1} \text{const} \cdot \Theta^k \cdot \gamma_{n,k}^{(7)}$$
$$\leqq \text{const} \cdot \Theta^n .$$

This proves at the same time the absolute convergence of $\gamma_n^{(3)}$.

4. The definition of $\hat{\mathscr{L}}$ yields:

$$|\gamma_n^{(5)}| = \left| \sum_{k=0}^{n-1} \sum_{\hat\eta' \in \hat{\mathscr{Z}}_{n-k}^{2n}} (\hat{\mathscr{L}}^k \hat{\mathscr{P}} \hat Y_{\hat\eta'})(\hat y_{\hat\eta'}) \right|$$
$$\leqq \sum_{k=0}^{n-1} \text{const} \cdot \Theta^k \cdot \sum_{\hat\eta' \in \hat{\mathscr{Z}}_{n-k}^{2n}} \| \hat Y_{\hat\eta'} \|_{\widehat{BV}} \quad \text{by Proposition 4.2,}$$
$$\leqq \text{const} \cdot \Theta^n \cdot \sum_{i=0}^{2n} \sum_{\substack{\hat\eta' \in \hat{\mathscr{Z}}_{n-k} \\ \hat\eta' \subseteq \hat D(i)}} \operatorname{var}_{\hat\eta'}(\hat\Phi) \quad \text{by (A.1)},$$
$$\leqq \text{const} \cdot V \cdot (2n+1) \cdot \Theta^n$$
$$\leqq \text{const} \cdot \Theta^n .$$

5. We use again the remark in 1 and obtain

$$|\gamma_n^{(6)}| = \left| \sum_{k=0}^{n-1} \sum_{i=0}^{\infty} \sum_{\hat\eta' \in \hat{\mathscr{Z}}_{n-k} \setminus \hat{\mathscr{Z}}_{n-k}^{2n}} \sum_{\substack{\hat\eta \in \hat{\mathscr{Z}}_n \\ \hat T^k \hat\eta = \hat\eta', \hat\eta \subseteq \hat D(i)}} \hat\Phi_k(\hat y_{\hat\eta}) (\hat{\mathscr{P}}^{\perp} \hat Y_{\hat\eta'})(\hat y_{\hat\eta}) \right|$$
$$\leqq \sum_{j=1}^{d} \sum_{k=0}^{n-1} \sum_{i=0}^{\infty} \sum_{\hat\eta' \in \hat{\mathscr{Z}}_{n-k} \setminus \hat{\mathscr{Z}}_{n-k}^{2n}} \sum_{\substack{\hat\eta \in \hat{\mathscr{Z}}_n \\ \hat T^k \hat\eta = \hat\eta', \hat\eta \subseteq \hat D(i)}} \text{const} \cdot \Theta^k$$
$$\cdot |\hat f_j(\hat y_{\hat\eta})| \cdot \|\hat G_j\| \cdot \|\hat Y_{\hat\eta'}\|_{\widehat{BV}}$$
$$\leqq \text{const} \cdot \Theta^n \sum_{j=1}^{d} \|\hat G_j\| \cdot \sum_{k=0}^{n-1} \sum_{i=0}^{\infty} \sum_{\hat\eta' \in \hat{\mathscr{Z}}_{n-k} \setminus \hat{\mathscr{Z}}_{n-k}^{2n}} \operatorname{var}_{\hat\eta'}(\hat\Phi)$$
$$\cdot \sum_{\substack{\hat\eta \in \hat{\mathscr{Z}}_n \\ \hat T^k \hat\eta = \hat\eta', \hat\eta \subseteq \hat D(i)}} |\hat f_j(\hat y_{\hat\eta})|$$
$$\leqq \text{const} \cdot \Theta^n \sum_{j=1}^{d} \|\hat G_j\| \cdot \sum_{k=0}^{n-1} \sum_{i=0}^{\infty} \sup_{\hat D_i} |\hat f_j| \cdot \sum_{\hat\eta' \in \hat{\mathscr{B}}(n,k,i)} 2 \cdot \operatorname{var}_{\hat\eta'}(\hat\Phi) ,$$

where $\hat{\mathscr{B}}(n, k, i)$ is defined as in the proof of 3, and we used again (A.5). Observing also (A.6) we can thus continue

$$|\gamma_n^{(6)}| \leqq \text{const} \cdot \Theta^n \sum_{j=1}^{d} \|\hat G_j\| \cdot \sum_{k=0}^{n-1} \sum_{i=0}^{\infty} \sup_{\hat D_i} |\hat f_j| \cdot 2 \cdot (\operatorname{var}_{\hat D(i+k)}(\hat\Phi) + \operatorname{var}_{\hat D(\bar i+k)}(\hat\Phi))$$
$$\leqq \text{const} \cdot \Theta^n \cdot 4nV \cdot \sum_{j=1}^{d} \|\hat G_j\| \cdot \sum_{i=0}^{\infty} \sup_{\hat D_i} |\hat f_j| \quad \text{by (4.2)}$$
$$\leqq \text{const} \cdot \Theta^n \sum_{j=1}^{d} \|\hat G_j\| \cdot \|\hat f_j\|_{\widehat{BV}}$$
$$\leqq \text{const} \cdot \Theta^n . \qquad \square$$

B. Appendix

Proof that Corollary 2.1 implies Theorem 1.1. If T is nonrenormalizable, i.e. if (T, μ) is mixing, $\mu = hm$, then the spectral representation for $\hat{\mathscr{L}}_{\hat{\psi}} : \widehat{BV}_{\hat{w}} \to \widehat{BV}_{\hat{w}}$ with norm $\|\cdot\|_{\hat{w}}$ reduces to (see the beginning of Sect. 5)

$$\hat{\mathscr{L}}_{\hat{\psi}}^n = \hat{\mathscr{P}}_1 + \hat{\mathscr{P}}_1^\perp \hat{\mathscr{L}}_{\hat{\psi}}^n, \quad \text{where } \hat{\mathscr{P}}_1(\hat{f}) = \int \hat{f}\, d\hat{m} \cdot \hat{h} \quad \text{and} \quad h(x) = \sum_{\hat{x} \in \pi^{-1}x} \hat{h}(\hat{x})\,.$$

Here $\hat{\mathscr{P}}_1^\perp = \mathrm{Id} - \hat{\mathscr{P}}_1$ and

$$\|\hat{\mathscr{L}}_{\hat{\psi}}^n \hat{\mathscr{P}}_1^\perp\|_{\hat{w}} \leqq \mathrm{const} \cdot r^n \quad \text{for some } r < 1\,.$$

Write $F_0(x) = F(x) - \int F\, dm$, $\hat{F}_0 = F_0 \circ \pi$, and analogously for G. Then

$$\frac{\hat{\mathscr{P}}_1(\hat{F}_0 \cdot \hat{h})}{\hat{h}} = \int (F_0 \circ \pi) \cdot \hat{h}\, d\hat{m} = \int F_0 \cdot h\, dm = \int F_0\, d\mu = 0\,. \tag{B.1}$$

Hence, observing that $\pi \circ \hat{T}^n = T^n \circ \pi$,

$$\begin{aligned}
&\left|\int F \cdot (G \circ T^n)\, d\mu - \int F\, d\mu \cdot \int G\, d\mu\right| = \left|\int F_0 \cdot (G_0 \circ T^n) \cdot h\, dm\right| \\
&= \left|\int \hat{F}_0 \cdot (\hat{G}_0 \circ \hat{T}^n) \cdot \hat{h}\, d\hat{m}\right| = \left|\int \hat{\mathscr{L}}_{\hat{\psi}}^n(\hat{F}_0 \cdot \hat{h}) \cdot \hat{G}_0\, d\hat{m}\right| \\
&= \left|\int \hat{\mathscr{L}}_{\hat{\psi}}^n \hat{\mathscr{P}}_1^\perp(\hat{F}_0 \cdot \hat{h}) \cdot \hat{G}_0\, d\hat{m}\right| \quad \text{by (B.1)} \\
&\leqq \sum_{i=0}^{\infty} \sup_{\hat{D}_i} \frac{|\hat{f}_n|}{\hat{w}} \cdot \int_{\hat{D}_i} |\hat{G}_0| \hat{w}\, d\hat{m}, \quad \text{where } \hat{f}_n := \hat{\mathscr{L}}_{\hat{\psi}}^n \hat{\mathscr{P}}_1^\perp(\hat{F}_0 \cdot \hat{h})\,.
\end{aligned}$$

Here $\hat{w}$ is the weight function introduced in Sect. 6. Now

$$\begin{aligned}
\int_{\hat{D}_i} |\hat{G}_0| \hat{w}\, d\hat{m} &= \int_{D_i} |G_0(x)| \cdot \hat{w}(\langle x, i \rangle)\, dx \leqq \|G_0\|_{l+\delta} \cdot \|\hat{w}(\langle \cdot, i \rangle)\|_{(l+\delta)/(l-1+\delta)} \\
&\leqq \mathrm{const} \cdot \|G_0\|_{l+\delta}
\end{aligned}$$

by Proposition 6.1, and

$$\begin{aligned}
\sum_{i=0}^{\infty} \sup_{\hat{D}_i} \frac{|\hat{f}_n|}{\hat{w}} &\leqq \|\hat{f}_n\|_{\hat{w}} \leqq \mathrm{const} \cdot r^n \cdot \|\hat{F}_0 \cdot \hat{h}\|_{\hat{w}} = \mathrm{const} \cdot r^n \cdot \left\|\frac{\hat{F}_0 \hat{h}}{\hat{w}}\right\|_{\widehat{BV}} \\
&= \mathrm{const} \cdot r^n \cdot \sum_{i=0}^{\infty} \|\hat{F}_0|_{\hat{D}_i}\|_{\widehat{BV}} \cdot \left\|\left(\frac{\hat{h}}{\hat{w}}\right)\Big|_{\hat{D}_i}\right\|_{\widehat{BV}} \\
&\leqq \mathrm{const} \cdot r^n \cdot \mathrm{var}(F_0) \cdot \left\|\frac{\hat{h}}{\hat{w}}\right\|_{\widehat{BV}} \\
&= \mathrm{const} \cdot r^n \cdot \mathrm{var}(F) \cdot \|\hat{h}\|_{\hat{w}}\,,
\end{aligned}$$

such that

$$\left|\int F \cdot (G \circ T^n)\, d\mu - \int F\, d\mu \cdot \int G\, d\mu\right| \leqq \mathrm{const} \cdot r^n \cdot \mathrm{var}(F) \cdot \|G_0\|_{l+\delta}\,. \qquad \square$$

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