**Integral Equations
and Operator Theory**

Spectral Transition for Dirac Operators with Electrostatic δ-Shell Potentials Supported on the Straight Line

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Abstract. In this note the two dimensional Dirac operator A_η with an electrostatic δ-shell interaction of strength $\eta \in \mathbb{R}$ supported on a straight line is studied. We observe a spectral transition in the sense that for the critical interaction strengths $\eta = \pm 2$ the continuous spectrum of A_η inside the spectral gap of the free Dirac operator A_0 collapses abruptly to a single point.

Mathematics Subject Classification. Primary 81Q10; Secondary 35Q40.

Keywords. Dirac operator, Singular potential, Boundary triple, Spectral transition.

1. Introduction

Differential operators that admit a spectral transition are of particular interest in mathematical analysis and its applications. Typically, one expects that the properties of a model described by a differential operator depend continuously on the parameters. However, in some cases it turns out that there is an abrupt change in the spectral properties-in other words, a spectral transition. A well known example in this regard is the Smilansky model [29,30] (see also [4–6]) or the indefinite Laplacian studied in [13,20]. A spectral transition was also observed for Dirac operators with singular potentials supported on bounded curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$. More precisely, when studying perturbations A_η of the free Dirac operator by an electrostatic δ-shell potential of strength $\eta \in \mathbb{R}$, it turned out that there is an abrupt change in the spectral properties for $\eta = \pm 2$. While for $\eta \neq \pm 2$-which is referred to as the *non-critical case*-it is shown in the three dimensional situation in [2,3,7,8] that the essential spectrum consists of two unbounded rays and finitely many eigenvalues between these rays, the *critical case* $\eta = \pm 2$ remained initially open. In the critical case it was then proved in [10,18,27] that there is a

loss of smoothness in the operator domain and that there may be one additional point in the essential spectrum; for general combinations of interaction strengths in the two dimensional setting see [12]. We note that similar effects also appear in the study of Dirac operators on bounded domains with suitable boundary conditions; cf. [11,17,18,21,26].

In this note we study the spectrum of a Dirac operator in $\mathbb{R}^2$ with an electrostatic δ-shell potential of strength $\eta \in \mathbb{R}$ supported on the straight line $\Sigma \cong \mathbb{R}$ which is formally given by

$$A_\eta = -i(\sigma_1 \partial_1 + \sigma_2 \partial_2) + m\sigma_3 + \eta\sigma_0 \delta_\Sigma;$$

here and in the following $\sigma_1, \sigma_2, \sigma_3$ are the $\mathbb{C}^{2\times2}$-valued Pauli spin matrices defined in (1.3), σ_0 is the 2×2-identity matrix, and $m \in \mathbb{R} \setminus \{0\}$. In order to define this expression rigorously, we denote by $\mathbb{R}^2_+$ and $\mathbb{R}^2_-$ the upper and lower half plane, respectively, and we use the notation $f_\pm := f \restriction \mathbb{R}^2_\pm$ for the restriction of a function f defined on $\mathbb{R}^2$. Next, let

$$H(\sigma, \mathbb{R}^2_\pm) := \big\{ f \in L^2(\mathbb{R}^2_\pm; \mathbb{C}^2) : (\sigma_1 \partial_1 + \sigma_2 \partial_2)f \in L^2(\mathbb{R}^2_\pm; \mathbb{C}^2) \big\}.$$

Then there exists a bounded Dirichlet trace map $H(\sigma, \mathbb{R}^2_\pm) \to H^{-1/2}(\Sigma; \mathbb{C}^2)$, $f_\pm \mapsto f_\pm|_\Sigma$ (one shows this in the same way as in [17, Lemma 2.1] and [17, Lemma 2.3] for bounded domains), and this allows us to define for $\eta \in \mathbb{R}$ and $m \in \mathbb{R} \setminus \{0\}$ the operator

$$A_\eta f := \big(-i(\sigma_1 \partial_1 + \sigma_2 \partial_2) + m\sigma_3 \big)f_+ \oplus \big(-i(\sigma_1 \partial_1 + \sigma_2 \partial_2) + m\sigma_3 \big)f_-,$$

$$\operatorname{dom} A_\eta := \bigg\{ f = f_+ \oplus f_- \in H(\sigma, \mathbb{R}^2_+) \oplus H(\sigma, \mathbb{R}^2_-) : \tag{1.1}$$

$$i\sigma_2(f_+|_\Sigma - f_-|_\Sigma) = \frac{\eta}{2}(f_+|_\Sigma + f_-|_\Sigma) \bigg\},$$

in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. This operator is the rigorous mathematical definition of a Dirac operator with an electrostatic δ-shell interaction of strength η and models, for $m > 0$, the propagation of a particle with mass m and spin $1/2$ under the influence of such a potential; cf. [2,8,12,21]. Observe that for $\eta = 0$ the operator in (1.1) coincides with the free Dirac operator

$$A_0 f = -i(\sigma_1 \partial_1 + \sigma_2 \partial_2)f + m\sigma_3 f, \quad \operatorname{dom} A_0 = H^1(\mathbb{R}^2; \mathbb{C}^2), \tag{1.2}$$

and recall that A_0 is self-adjoint in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ with purely (absolutely) continuous spectrum

$$\sigma(A_0) = (-\infty, -|m|] \cup [|m|, +\infty).$$

The main result of this paper is the following theorem on the self-adjointness and the spectra of the operators A_η. It turns out that the continuous spectrum grows under the influence of the δ-shell interaction and also half of the gap $(-|m|, |m|)$ is filled when η approaches the critical values ± 2 from above or below. In the critical case $\eta = \pm 2$ a spectral transition appears: the continuous spectrum inside $(-|m|, |m|)$ vanishes abruptly and 0 becomes an infinite dimensional eigenvalue.

Theorem 1.1. *The operator A_η is self-adjoint in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ and the following holds for the spectrum of A_η:*

(i) *If $\eta < -2$, then $\sigma(A_\eta) = \left(-\infty, -|m|\frac{\eta^2-4}{\eta^2+4}\right] \cup [|m|, +\infty)$.*

(ii) *If $\eta = -2$, then $\sigma(A_\eta) = (-\infty, -|m|] \cup \{0\} \cup [|m|, +\infty)$.*

(iii) *If $-2 < \eta < 0$, then $\sigma(A_\eta) = (-\infty, -|m|] \cup \left[|m|\frac{4-\eta^2}{\eta^2+4}, +\infty\right)$.*

(iv) *If $\eta = 0$, then $\sigma(A_0) = (-\infty, -|m|] \cup [|m|, +\infty)$.*

(v) *If $0 < \eta < 2$, then $\sigma(A_\eta) = \left(-\infty, -|m|\frac{4-\eta^2}{\eta^2+4}\right] \cup [|m|, +\infty)$.*

(vi) *If $\eta = 2$, then $\sigma(A_\eta) = (-\infty, -|m|] \cup \{0\} \cup [|m|, +\infty)$.*

(vii) *If $2 < \eta$, then $\sigma(A_\eta) = (-\infty, -|m|] \cup \left[|m|\frac{\eta^2-4}{\eta^2+4}, +\infty\right)$.*

For $\eta \neq \pm 2$ the spectrum of A_η is purely continuous and for $\eta = \pm 2$ the point 0 is an isolated eigenvalue of A_η with infinite multiplicity and the remaining spectrum is purely continuous.

Note that the spectrum of A_η is invariant under the transformation $\eta \mapsto -\frac{4}{\eta}$. This symmetry would also follow from the stronger fact that A_η and $A_{-4/\eta}$ are unitary equivalent; this can be shown in the same way as in [12, Proposition 4.8 (i) and Proposition 4.15 (i)].

The proof of Theorem 1.1 is based on an efficient abstract technique that was applied in a similar form also in [12,21]: We use a so-called boundary triple and its Weyl function to reduce the spectral analysis of A_η in the gap $(-|m|, |m|)$ of A_0 to a certain boundary operator in $L^2(\Sigma; \mathbb{C}^2) \simeq L^2(\mathbb{R}; \mathbb{C}^2)$; cf. [9,19,22,23] for details on boundary triples and Weyl functions in the extension theory of symmetric operators. Since Σ is a straight line the spectral properties of this boundary operator can be studied with the help of the Fourier transform. From the limit behaviour of the Weyl function and the boundary operator towards the real line we also conclude that the set $(-\infty, -|m|] \cup [|m|, +\infty)$ consists of purely continuous spectrum of A_η for all $\eta \in \mathbb{R}$. For $m = 0$ this method is also applicable, then it follows for all $\eta \in \mathbb{R}$ that $\sigma(A_\eta) = \mathbb{R}$, i.e. there is no spectral transition. We note that the method of direct integrals decomposing A_η in its fibers would yield a similar result; cf. [24] for a related problem on Dirac operators with Robin type boundary conditions. In this note we prefer to work with the boundary triple technique, since it allows generalizations for more complicated curves Σ in a natural way, which we plan to study in the near future.

Notations

Let

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (1.3)$$

be the Pauli spin matrices and denote by σ_0 the 2×2-identity matrix. Note that the Pauli matrices satisfy $\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk}\sigma_0$, $j, k \in \{1, 2, 3\}$. For $x = (x_1, x_2)$ we will often write $\sigma \cdot x = \sigma_1 x_1 + \sigma_2 x_2$ and $\sigma \cdot \nabla = \sigma_1 \partial_1 + \sigma_2 \partial_2$.

2. Proof of Theorem 1.1

Let A_0 be the free Dirac operator in (1.2) and recall that for $z \in \rho(A_0)$ the resolvent of A_0 is given by

$$\left[(A_0 - z)^{-1}f\right](x) = \int_{\mathbb{R}^2} G_z(x-y)f(y)dy, \quad f \in L^2(\mathbb{R}^2; \mathbb{C}^2), \, x \in \mathbb{R}^2,$$

where

$$G_z(x) = \frac{i\sqrt{m^2 - z^2}}{2\pi}K_1\left(\sqrt{m^2 - z^2}|x|\right)\frac{\sigma \cdot x}{|x|} + \frac{1}{2\pi}K_0\left(\sqrt{m^2 - z^2}|x|\right)(z\sigma_0 + m\sigma_3)$$

and K_j is the modified Bessel functions of the second kind and order j; cf. [1] and [31]. Here and in the following the complex square root is chosen such that it is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$ and $\operatorname{Re}\sqrt{z} > 0$. An important object in our analysis is the mapping $\mathcal{C}_z$ which is defined for $z \in \rho(A_0)$ by

$$\mathcal{C}_z\varphi := G_z * \varphi, \quad \varphi \in \mathcal{S}(\Sigma; \mathbb{C}^2),$$

where $\mathcal{S}(\Sigma; \mathbb{C}^2) \simeq \mathcal{S}(\mathbb{R}; \mathbb{C}^2)$ denotes the Schwartz space and the convolution is understood in the sense of distributions. The action of $\mathcal{C}_z$ is

$$\mathcal{C}_z\varphi(x) := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R} \setminus (x_1 - \varepsilon, x_1 + \varepsilon)} G_z(x-y)\varphi(y)dy_1, \quad x = (x_1, 0), y = (y_1, 0) \in \Sigma.$$

In the following we will denote by $\mathcal{F}$ the Fourier transform on $\Sigma \simeq \mathbb{R}$.

Lemma 2.1. *The map $\mathcal{F}\mathcal{C}_z\mathcal{F}^{-1}$ is the multiplication operator with the matrix valued function*

$$p \mapsto \begin{pmatrix} \frac{z+m}{2\sqrt{p^2+m^2-z^2}} & \frac{p}{2\sqrt{p^2+m^2-z^2}} \\ \frac{p}{2\sqrt{p^2+m^2-z^2}} & \frac{z-m}{2\sqrt{p^2+m^2-z^2}} \end{pmatrix}.$$

In particular, $\mathcal{C}_z$ gives rise to a bounded operator in $H^s(\Sigma; \mathbb{C}^2)$ for any $s \in \mathbb{R}$.

Proof. Let $\phi_z(x_1) := \frac{1}{2\pi}K_0(\sqrt{m^2 - z^2}|x_1|)$, $x_1 \in \mathbb{R} \setminus \{0\}$. Since the Fourier transform takes $K_0(\kappa|x|)$ to $\sqrt{\pi/2}(p^2 + \kappa^2)^{-1/2}$ we obtain

$$\mathcal{F}(\phi_z * \varphi)(p) = \frac{1}{2\sqrt{p^2 + m^2 - z^2}}\mathcal{F}\varphi(p), \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

Now observe that

$$G_z(x) = \left(-i\sigma_1\frac{d}{dx_1} + m\sigma_3 + z\sigma_0\right)\phi_z(x_1), \quad x = (x_1, 0) \in \Sigma,$$

and hence the calculation rules for the Fourier transform of distributions [28, Chapter IX] lead to

$$\mathcal{F}\mathcal{C}_z\mathcal{F}^{-1}\varphi(p) = (\sigma_1 p + m\sigma_3 + z\sigma_0)\frac{1}{2\sqrt{p^2 + m^2 - z^2}}\varphi(p)$$

$$= \begin{pmatrix} \frac{z+m}{2\sqrt{p^2+m^2-z^2}} & \frac{p}{2\sqrt{p^2+m^2-z^2}} \\ \frac{p}{2\sqrt{p^2+m^2-z^2}} & \frac{z-m}{2\sqrt{p^2+m^2-z^2}} \end{pmatrix}\varphi(p)$$

for $\varphi \in \mathcal{S}(\Sigma; \mathbb{R}^2)$, which yields the claimed result about the representation of $\mathcal{F}\mathcal{C}_z\mathcal{F}^{-1}$. Finally, taking the definition of the norm in $H^s(\Sigma; \mathbb{C}^2) \simeq H^s(\mathbb{R}; \mathbb{C}^2)$ with the help of the Fourier transform into account, one sees that $\mathcal{C}_z$ gives rise to a bounded operator in $H^s(\Sigma; \mathbb{C}^2)$ for all $s \in \mathbb{R}$. $\qquad\square$

In the following we shall make use of the closed symmetric restriction

$$Sf = (-i\sigma \cdot \nabla + m\sigma_3)f, \quad \text{dom}\, S = H_0^1(\mathbb{R}^2 \setminus \Sigma; \mathbb{C}^2), \tag{2.1}$$

of the free Dirac operator A_0, and its adjoint

$$S^*f = (-i\sigma \cdot \nabla + m\sigma_3)f_+ \oplus (-i\sigma \cdot \nabla + m\sigma_3)f_-,$$
$$\text{dom}\, S^* = H(\sigma, \mathbb{R}_+^2) \oplus H(\sigma, \mathbb{R}_-^2);$$

the above mentioned properties of S and S^* can be shown in the same way as in [10, Proposition 3.1], where similar operators in $\mathbb{R}^3$ with compact surfaces Σ have been studied.

Fix some $\zeta \in \rho(A_0)$. Define the maps $\Gamma_0, \Gamma_1 : \text{dom}\, S^* \to L^2(\Sigma; \mathbb{C}^2)$ by

$$\Gamma_0 f := -i\Lambda^{-1}\sigma_2(f_+|_\Sigma - f_-|_\Sigma),$$
$$\Gamma_1 f := \frac{1}{2}\Lambda\big((f_+|_\Sigma + f_-|_\Sigma) - (\mathcal{C}_\zeta + \mathcal{C}_{\overline{\zeta}})\Lambda\Gamma_0 f\big), \quad f = f_+ \oplus f_- \in \text{dom}\, S^*, \tag{2.2}$$

where $\Lambda = (-\Delta + 1)^{\frac{1}{4}} = \mathcal{F}^{-1}(p^2 + 1)^{\frac{1}{4}}\mathcal{F}$ is viewed as a bijective operator from $H^s(\Sigma)$ onto $H^{s-1/2}(\Sigma)$ for $s = 1/2$ or $s = 0$, or as a self-adjoint unbounded operator in $L^2(\Sigma)$ defined on $H^{1/2}(\Sigma)$. It follows in the same way as in [12, Proposition 3.5] or [21, Proposition 6.1] that $\{L^2(\Sigma; \mathbb{C}^2), \Gamma_0, \Gamma_1\}$ is a boundary triple for S^* with $A_0 = S^* \upharpoonright \ker \Gamma_0$ and corresponding Weyl function

$$\rho(A_0) \ni z \mapsto M(z) = \Lambda\left(\mathcal{C}_z - \frac{1}{2}(\mathcal{C}_\zeta + \mathcal{C}_{\overline{\zeta}})\right)\Lambda;$$

cf. [9,19,22,23] for details on boundary triples and Weyl functions in the extension theory of symmetric operators.

Next we describe the operators A_η with the help of the boundary triple (2.2) and also conclude their self-adjointness in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. The proof of this result follows the lines of [21, Proposition 6.3] and [12, Proposition 4.3].

Lemma 2.2. *For $\eta \in \mathbb{R} \setminus \{0\}$ the operator*

$$\Theta\varphi := -\Lambda\left[\frac{1}{\eta}\sigma_0 + \frac{1}{2}(\mathcal{C}_\zeta + \mathcal{C}_{\overline{\zeta}})\right]\Lambda\varphi,$$
$$\text{dom}\,\Theta = \left\{\varphi \in L^2(\Sigma; \mathbb{C}^2) : \left[\frac{1}{\eta}\sigma_0 + \frac{1}{2}(\mathcal{C}_\zeta + \mathcal{C}_{\overline{\zeta}})\right]\Lambda\varphi \in H^{1/2}(\Sigma; \mathbb{C}^2)\right\},$$

is self-adjoint in $L^2(\Sigma; \mathbb{C}^2)$ and we have

$$A_\eta = S^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0). \tag{2.3}$$

In particular, the operator A_η is self-adjoint in $L^2(\mathbb{R}^2; \mathbb{C}^2)$.

Proof. Define the $\mathbb{C}^{2\times2}$-valued function

$$\theta(p) := -\sqrt{p^2 + 1}\begin{pmatrix} \frac{1}{\eta} + \text{Re}\, \frac{\zeta+m}{2\sqrt{p^2+m^2-\zeta^2}} & \text{Re}\, \frac{p}{2\sqrt{p^2+m^2-\zeta^2}} \\ \text{Re}\, \frac{p}{2\sqrt{p^2+m^2-\zeta^2}} & \frac{1}{\eta} + \text{Re}\, \frac{\zeta-m}{2\sqrt{p^2+m^2-\zeta^2}} \end{pmatrix}.$$

Then, by Lemma 2.1 we have

$$\mathcal{F}\Theta\mathcal{F}^{-1}\varphi(p) = \theta(p)\varphi(p),$$
$$\text{dom}\,\mathcal{F}\Theta\mathcal{F}^{-1} = \{\varphi \in L^2(\mathbb{R}; \mathbb{C}^2) : \theta\varphi \in L^2(\mathbb{R}; \mathbb{C}^2)\}.$$

Since θ is a symmetric matrix and $\mathcal{F}$ is unitary, we conclude that Θ is self-adjoint in $L^2(\Sigma; \mathbb{C}^2) \simeq L^2(\mathbb{R}; \mathbb{C}^2)$. Using (2.2) it is not difficult to verify that (2.3) holds and hence the self-adjointness of A_η follows, see, e.g., [9, Corollary 2.1.4 (v)]. $\qquad\square$

In the next lemma we analyze, when zero belongs to the point spectrum or continuous spectrum of $\Theta - M(z)$ for $z \in (-|m|, |m|)$.

Lemma 2.3. *Let $\eta \in \mathbb{R} \setminus \{0\}$. For $z \in (-|m|, |m|)$ the following holds:*

(i) *If $\eta \neq \pm 2$, then $0 \notin \sigma_{\mathrm{p}}(\Theta - M(z))$ and $0 \in \sigma_{\mathrm{c}}(\Theta - M(z))$ if and only if*

$$\frac{z\eta}{\eta^2 - 4} > 0 \qquad (2.4)$$

and

$$z \leq -\frac{|m(\eta^2 - 4)|}{\eta^2 + 4} \quad or \quad z \geq \frac{|m(\eta^2 - 4)|}{\eta^2 + 4}.$$

(ii) *If $\eta = \pm 2$, then $0 \in \sigma_{\mathrm{p}}(\Theta - M(0))$ with $\dim \ker(\Theta - M(0)) = +\infty$, and $0 \in \rho(\Theta - M(z))$ for $z \neq 0$.*

Proof. Let $z \in (-|m|, |m|)$ and observe that

$$\Theta - M(z) = -\Lambda \left[\frac{1}{\eta} \sigma_0 + \mathcal{C}_z \right] \Lambda, \quad \mathrm{dom}\,(\Theta - M(z)) = \mathrm{dom}\,\Theta. \qquad (2.5)$$

Using Lemma 2.1 we conclude

$$\mathcal{F}(\Theta - M(z))\mathcal{F}^{-1}\varphi(p) = \theta_z(p)\varphi(p),$$
$$\mathrm{dom}\,\mathcal{F}(\Theta - M(z))\mathcal{F}^{-1} = \{\varphi \in L^2(\mathbb{R}; \mathbb{C}^2) : \theta_z \varphi \in L^2(\mathbb{R}; \mathbb{C}^2)\},$$

where

$$\theta_z(p) = -\sqrt{p^2 + 1} \begin{pmatrix} \frac{1}{\eta} + \frac{z+m}{2\sqrt{p^2+m^2-z^2}} & \frac{p}{2\sqrt{p^2+m^2-z^2}} \\ \frac{p}{2\sqrt{p^2+m^2-z^2}} & \frac{1}{\eta} + \frac{z-m}{2\sqrt{p^2+m^2-z^2}} \end{pmatrix},$$

and hence it suffices to consider the self-adjoint multiplication operator with the function θ_z in $L^2(\mathbb{R}; \mathbb{C}^2)$. In the following we discuss for which $\eta \in \mathbb{R} \setminus \{0\}$ and $z \in (-|m|, |m|)$ the multiplication operator θ_z has 0 as an eigenvalue or as a point in the continuous spectrum. Note first that

$$\det \theta_z(p) = (p^2 + 1) \left[\frac{1}{\eta^2} + \frac{z}{\eta \sqrt{p^2 + m^2 - z^2}} - \frac{1}{4} \right].$$

We verify assertion (ii). In the case $\eta = \pm 2$ we have $\det \theta_z(p) = 0$ if and only if $z = 0$, and in this situation $\det \theta_z(p) = 0$ for all $p \in \mathbb{R}$. Therefore, 0 is an eigenvalue of infinite multiplicity of the multiplication operator θ_z. Observe that for $z \in (-|m|, |m|) \setminus \{0\}$ we have $\det \theta_z(p) \neq 0$ and hence one finds that 0 is in the resolvent set of θ_z.

Now we prove (i). If $\eta \neq \pm 2$, then $\det \theta_z(p) = 0$ is equivalent to

$$\sqrt{p^2 + m^2 - z^2} = 4\frac{z\eta}{\eta^2 - 4}. \qquad (2.6)$$

The left hand side of the last equation is positive and hence there exist solutions only if also the right hand side is positive, i.e. only if (2.4) holds. Assuming this, we see by squaring the last equation that it is equivalent to

$$p^2 = \frac{(\eta^2 + 4)^2}{(\eta^2 - 4)^2} z^2 - m^2. \tag{2.7}$$

When $p \in \mathbb{R}$ varies the left hand side can be any non-negative number, and hence (2.6) has a solution, whenever the right hand side is non-negative, i.e. if and only if $z \leq z_-$ or $z \geq z_+$, where $z_\pm$ are the zeros of the polynomial on the right hand side of the last equation given by

$$z_\pm = \pm \left| m \frac{\eta^2 - 4}{\eta^2 + 4} \right|.$$

It is also clear that for a fixed $z \in (-|m|, |m|)$ with $z \leq z_-$ or $z \geq z_+$ there are only two values $p \in \mathbb{R}$ with (2.7) holds, and hence 0 is in the continuous spectrum of the multiplication operator associated with θ_z. $\qquad\square$

In order to prove Theorem 1.1, we also investigate the limiting behaviour of $(\Theta - M(z))^{-1}$, when $z \in \mathbb{C} \setminus \mathbb{R}$ approaches $\sigma(A_0) = (-\infty, -|m|] \cup [|m|, +\infty)$:

Lemma 2.4. *Let $\eta \in \mathbb{R} \setminus \{0\}$. Then, the following is true:*

(i) *For any $x \in (-\infty, -|m|] \cup [|m|, +\infty)$ and all $\varphi \in L^2(\Sigma; \mathbb{C}^2)$ one has*

$$\lim_{y \searrow 0} iy \big(\Theta - M(x + iy)\big)^{-1} \varphi = 0. \tag{2.8}$$

(ii) *For any $x \in (-\infty, -|m|) \cup (|m|, +\infty)$ there exists $\varphi \in L^2(\Sigma; \mathbb{C}^2)$ such that*

$$\lim_{y \searrow 0} \mathrm{Im} \big((\Theta - M(x + iy))^{-1} \varphi, \varphi \big) \neq 0. \tag{2.9}$$

Proof. In order to show the claims, we note first with the help of (2.5) that the operator $\mathcal{F}(\Theta - M(z))^{-1}\mathcal{F}^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, is the maximal multiplication operator associated with the matrix-valued function

$$\theta_z^{-1}(p) = -\frac{\sqrt{p^2 + 1}}{\det \theta_z(p)} \begin{pmatrix} \frac{1}{\eta} + \frac{z-m}{2\sqrt{p^2+m^2-z^2}} & -\frac{p}{2\sqrt{p^2+m^2-z^2}} \\ -\frac{p}{2\sqrt{p^2+m^2-z^2}} & \frac{1}{\eta} + \frac{z+m}{2\sqrt{p^2+m^2-z^2}} \end{pmatrix}$$

$$= -\frac{2\eta}{c_z(p)\sqrt{p^2+1}} \begin{pmatrix} 2\sqrt{p^2+m^2-z^2}+\eta(z-m) & -\eta p \\ -\eta p & 2\sqrt{p^2+m^2-z^2}+\eta(z+m) \end{pmatrix}$$

with

$$c_z(p) = (4 - \eta^2)\sqrt{p^2 + m^2 - z^2} + 4\eta z.$$

By the continuity of the complex square root one sees for a fixed $p \in \mathbb{R}$ that the limit $c_{x+i0}(p) := \lim_{y \searrow 0} c_{x+iy}(p)$ exists. One verifies for any number $x \in (-\infty, -|m|] \cup [|m|, +\infty)$ in a similar way as in (2.6) and (2.7) that $c_{x+i0}(p)$ has no zero, if $\eta = \pm 2$ or if $\frac{x\eta}{\eta^2-4} < 0$, and for $\frac{x\eta}{\eta^2-4} > 0$ the term $c_{x+i0}(p)$ has two zeros at

$$p_\pm = \pm \left(\frac{(\eta^2 + 4)^2}{(\eta^2 - 4)^2} x^2 - m^2 \right)^{1/2}. \tag{2.10}$$

Let us prove (i). Let $x \in (-\infty, -|m|] \cup [|m|, +\infty)$ be fixed. Since the map $z \mapsto M(z)$ is the Weyl function of a boundary triple it is a Nevanlinna function and the values $M(z)$ are bounded and everywhere defined operators in $L^2(\Sigma; \mathbb{C}^2)$; cf. [9, Corollary 2.3.7]. It follows that $z \mapsto (\Theta - M(z))^{-1}$ is also a Nevanlinna function and for $z \in \rho(A_\eta) \cap \rho(A_0)$ the values are also bounded and everywhere defined operators in $L^2(\Sigma; \mathbb{C}^2)$. From the operator representation of Nevanlinna functions (see, e.g., [25, Theorem 4.2]) we then conclude that there exists a constant $C_1 > 0$ such that

$$\left\| (\Theta - M(x+iy))^{-1} \right\| \leq \frac{C_1}{y}$$

for $y > 0$ sufficiently small. Since $(\Theta - M(x+iy))^{-1}$ is unitarily equivalent to the multiplication operator with the function θ_{x+iy}^{-1}, we conclude that

$$\left\| \theta_{x+iy}^{-1} \right\|_\infty \leq \frac{C_1}{y} \tag{2.11}$$

for all $y > 0$ sufficiently small. Next, we define for $y > 0$ the set

$$I_x(y) := \begin{cases} \{p \in \mathbb{R} : |p - p_\pm| \geq \sqrt{y}\}, & \text{if } c_{x+i0} \text{ has a zero,} \\ \mathbb{R}, & \text{if } c_{x+i0} \text{ has no zero,} \end{cases} \tag{2.12}$$

and prove for all sufficiently small $y > 0$ and $p \in I_x(y)$ that

$$\left| \theta_{x+iy}^{-1}(p) \right| \leq \frac{C_2}{\sqrt{y}} \tag{2.13}$$

for some $C_2 > 0$, where the absolute value is understood elementwise.

In order to show (2.13), we note first that for some $C_3 > 0$ independent of $p \in \mathbb{R}$ and $y \in [0, 1]$ one has the estimate

$$\left| c_{x+iy}(p) \theta_{x+iy}^{-1}(p) \right| \leq C_3. \tag{2.14}$$

Consider first the case when c_{x+i0} has no zero. From the definition of c_{x+iy} we conclude that there exists $P_0 > 0$ such that $\left| c_{x+iy}(p) \right| > 1$ holds for all $|p| > P_0$ and all sufficiently small $y > 0$. Moreover, the map $[-P_0, P_0] \times [0, 1] \ni (p, y) \mapsto c_{x+iy}(p)$ is uniformly continuous and hence, there exists a constant $C_4 > 0$ such that for all sufficiently small $y > 0$ and $p \in [-P_0, P_0]$ the relation $|c_{x+iy}(p)| > C_4$ holds, as c_{x+i0} has no zero. Thus, for $y > 0$ sufficiently small $|c_{x+iy}|$ is uniformly bounded from below by a positive constant, which together with (2.14) leads to the estimate in (2.13).

Assume now that c_{x+i0} has the zeros $p_\pm$, let $y > 0$ be sufficiently small, and fix $p \in \mathbb{R}$ with $|p - p_\pm| \geq \sqrt{y}$. From (2.10) we get $\sqrt{p_\pm^2 + m^2 - x^2} > 0$ and thus,

$$\left| \sqrt{p^2 + m^2 - (x+iy)^2} - \sqrt{p_\pm^2 + m^2 - x^2} \right|$$

$$= \left| \frac{p^2 - (x+iy)^2 - p_\pm^2 + x^2}{\sqrt{p^2 + m^2 - (x+iy)^2} + \sqrt{p_\pm^2 + m^2 - x^2}} \right|$$

$$\geq \left| \frac{(p - p_\pm)(p + p_\pm)}{\sqrt{p^2 + m^2 - (x+iy)^2} + \sqrt{p_\pm^2 + m^2 - x^2}} \right| - \left| \frac{x^2 - (x+iy)^2}{\sqrt{p_\pm^2 + m^2 - x^2}} \right|$$

$$\geq C_5 \sqrt{y}$$

with a constant $C_5 > 0$, where we used in the first inequality for the denominator that the real part of the complex square root is positive and in the last estimate that the functions

$$\mathbb{R}_\pm \ni p \mapsto \left| \frac{p \pm |p_\pm|}{\sqrt{p^2 + m^2 - (x+iy)^2} + \sqrt{p_\pm^2 + m^2 - x^2}} \right|$$

are uniformly bounded from below independently of $y > 0$ sufficiently small. Hence, we conclude for those $y > 0$ that

$$\left| c_{x+iy}(p) \right| = \left| c_{x+iy}(p) - c_{x+i0}(p_\pm) \right| \geq C_6 \sqrt{y}.$$

Together with (2.14) this yields (2.13).

Now, we have everything in hands to prove item (i). Let χ be the characteristic function for the set $I_x(y)$ defined in (2.12). Then (2.11) and (2.13) imply for an arbitrary $\varphi \in L^2(\Sigma; \mathbb{C}^2)$

$$\lim_{y \searrow 0} \left\| y \theta_{x+iy}^{-1} \varphi \right\|_{L^2(\mathbb{R};\mathbb{C}^2)} \leq \lim_{y \searrow 0} \left\| y(1-\chi)\theta_{x+iy}^{-1}\varphi \right\|_{L^2(\mathbb{R};\mathbb{C}^2)} + \lim_{y \searrow 0} \left\| y\chi\theta_{x+iy}^{-1}\varphi \right\|_{L^2(\mathbb{R};\mathbb{C}^2)}$$

$$\leq \lim_{y \searrow 0} C_1 \left\| (1-\chi)\varphi \right\|_{L^2(\mathbb{R};\mathbb{C}^2)} + \lim_{y \searrow 0} C_2 \sqrt{y} \|\varphi\|_{L^2(\mathbb{R};\mathbb{C}^2)} = 0.$$

Thus, assertion (i) is true.

To show item (ii), let $x \in (-\infty, -|m|) \cup (|m|, +\infty)$ and let I be a non-empty compact interval which is contained in $(-\sqrt{x^2 - m^2}, \sqrt{x^2 - m^2})$. Note that the numbers $p_\pm$ in (2.10) are not contained in I and that

$$\lim_{y \searrow 0} \mathrm{Im}\, \sqrt{p^2 + m^2 - (x+iy)^2} \neq 0$$

for $p \in I$. Let $\widetilde{\chi}$ be the characteristic function for I. Taking the form of θ_{x+iy}^{-1} into account, we get that $\mathrm{Im}\,(\widetilde{\chi}\theta_{x+iy}^{-1})$ converges uniformly to a nontrivial limit and hence, one can choose $\varphi \in L^2(\Sigma; \mathbb{C}^2)$ supported in I such that

$$\lim_{y \searrow 0} \mathrm{Im}\, \left((\Theta - M(x+iy))^{-1} \varphi, \varphi \right) \neq 0,$$

i.e. also item (ii) is true. $\qquad\square$

Proof. (Proof of Theorem 1.1) First, we note that the case $\eta = 0$ reduces to the free Dirac operator and hence, all claims are known here. So we assume from now on that $\eta \neq 0$. The self-adjointness of A_η is shown in Lemma 2.2. The claims about $\sigma(A_\eta) \cap (-|m|, |m|)$ follow from Lemma 2.3 and [9, Theorem 2.6.2].

It remains to show for $\eta \in \mathbb{R} \setminus \{0\}$ that $(-\infty, -|m|] \cup [|m|, +\infty)$ is purely continuous spectrum of A_η. For this purpose let S be the operator from (2.1)

and define the operator $T := S^* \restriction (\operatorname{dom} A_\eta + \operatorname{dom} A_0)$. Let Γ_0, Γ_1 be the mappings from (2.2) and define $\Gamma_0^\Theta, \Gamma_1^\Theta : \operatorname{dom} T \to L^2(\Sigma; \mathbb{C}^2)$ by

$$\Gamma_0^\Theta f := \Gamma_1 f - \Theta \Gamma_0 f, \quad \Gamma_1^\Theta f := -\Gamma_0 f, \quad f \in \operatorname{dom} T.$$

Similarly as in [11, Proposition 2.4] one verifies that $\{L^2(\Sigma; \mathbb{C}^2), \Gamma_0^\Theta, \Gamma_1^\Theta\}$ is a quasi boundary triple for S^* in the sense of [14] with $T \restriction \ker \Gamma_0^\Theta = A_\eta$ and Weyl function

$$\rho(A_\eta) \ni z \mapsto -\big(\Theta - M(z)\big)^{-1}.$$

Note that the operator S in (2.1) is simple; this can be shown as in [11, Proposition 3.2] using [16, Step 2 in the proof of Theorem 3.4]. Hence, Lemma 2.4 (i) and [15, Theorem 3.2] imply that $(-\infty, -|m|] \cup [|m|, +\infty) \cap \sigma_{\mathrm{p}}(A_\eta) = \emptyset$. Furthermore, Lemma 2.4 (ii) and [15, Theorem 3.5] yield that the inclusion $(-\infty, -|m|) \cup (|m|, +\infty) \subset \sigma_{\mathrm{c}}(A_\eta)$ holds. By combining these facts one shows that $(-\infty, -|m|] \cup [|m|, +\infty)$ is contained in the purely continuous spectrum of A_η. $\qquad\square$

Data Availibility Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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