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Spectrum and $L-$Spectrum of the Power Graph and its Main Supergraph for Certain Finite Groups

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Abstract. Let G be a finite group. The power graph $\mathcal{P}(G)$ and its main supergraph $\mathcal{S}(G)$ are two simple graphs with the same vertex set G. Two elements $x, y \in G$ are adjacent in the power graph if and only if one is a power of the other. They are joined in $\mathcal{S}(G)$ if and only if $o(x)|o(y)$ or $o(y)|o(x)$. The aim of this paper is to compute the characteristic polynomial of these graph for certain finite groups. As a consequence, the spectrum and Laplacian spectrum of these graphs for dihedral, semi-dihedral, cyclic and dicyclic groups were computed.

1. Basic Concepts

All groups and graphs considered here are assumed to be finite and graph means simple graph. Only basic concepts about graphs will be needed for this paper. They can be found in any book about graph theory, for example [34]. Our group theory notations are taken from [31] and we refer to [10, 11] for the algebraic graph theory concepts and notations.

Suppose Γ is a graph with edge set $E(\Gamma)$, vertex set $V(\Gamma)$, adjacency matrix $A(\Gamma)$ and Laplacian matrix $L(\Gamma)$. The cardinality of $V(\Gamma)$ is called the order of Γ and if $e \in E(\Gamma)$ has end points u and v, then we write $e = uv$. Define $N_\Gamma(u) = \{v \in V(G) : uv \in E(\Gamma)\}$. It is easy to see that the cardinality of $N_\Gamma(u)$ is the degree of u in Γ. If all degrees are equal to p then the graph Γ is called to be p-regular. The multi-sets of all eigenvalues and Laplacian eigenvalues of Γ are denoted by $\sigma(\Gamma)$ and $\sigma_L(\Gamma)$, respectively. We usually write $\sigma(\Gamma) = \{\lambda_1^{(s_1)}, \ldots, \lambda_m^{(s_m)}\}$, where $\lambda_1, \ldots, \lambda_m$ are different $\Gamma-$eigenvalues and s_j is the multiplicity of λ_j, $1 \leq j \leq m$. The polynomial $\Phi(\Gamma, x) = det(xI - A(\Gamma))$ is called the characteristic polynomial of Γ. By an undirected graph Σ, we mean a pair $(V(\Sigma), E(\Sigma))$ in which $V(\Sigma)$ is a non-empty set and $E(\Sigma)$ is a subset of all unordered pairs of distinct elements of $V(\Sigma)$. If we consider the elements of $E(\Sigma)$ to be ordered pairs, then the graph Σ will be directed.

A partition $V_1 \cup V_2 \cup \cdots \cup V_m$ of the vertex set of a graph Γ is called equitable if for each i and for all $u, v \in V_i$, $|N_\Gamma(u) \cap V_j| = |N_\Gamma(v) \cap V_j|$, for all j. The set of all positive divisors of an integer n is denoted by $D(n)$.

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Following Sabidussi [32, p. 396], the $A-$join of a set of graphs $\{\Gamma_a\}_{a\in A}$ is the graph Δ with vertex and edge sets

$$\begin{array}{rcl} V(\Delta) & = & \{(x,y) \mid x \in V(A) \ \& \ y \in V(\Gamma_x)\}, \\ E(\Delta) & = & \{(x,y)(x',y') \mid xx' \in E(A) \ or \ else \ x = x' \ \& \ yy' \in E(\Gamma_x)\}. \end{array}$$

One can easily see that this graph can be constructed from A by replacing each vertex $a \in V(A)$ by the graph Γ_a and inserting either all or none of the possible edges between vertices of Γ_a and Γ_b depending on whether or not a and b are joined by an edge in A. If A is an $p-$vertex labeled graph then the $A-$join of $\Delta_1, \Delta_2, \ldots, \Delta_p$ is denoted by $A[\Delta_1, \Delta_2, \ldots, \Delta_p]$.

Let G be a finite group. The order of $x \in G$ is denoted by $o(x)$. The set of all element orders of G is denoted by $\pi_e(G)$ and $\Omega_i(G)$ stands for the number of all elements of G of order i. The notation ϕ is used for the Eulers totient function.

2. Power Graph of Finite Groups: A Literature Review

For a group G, there are two simple graphs with the same vertex set G as follows:

- The power graph $\mathcal{P}(G)$ with edge set

$$E(\mathcal{P}(G)) = \{xy \mid x, y \in G \ \& \ (\langle x \rangle \subseteq \langle y \rangle \ or \ \langle y \rangle \subseteq \langle x \rangle)\};$$

- The main supergraph $\mathcal{S}(G)$ with edge set

$$E(\mathcal{S}(G)) = \{xy \mid x, y \in G \ \& \ (o(x) \mid o(y) \ or \ o(y) \mid o(x))\}.$$

The proper power graph $\mathcal{P}^*(G)$ [4] and its proper main supergraph $\mathcal{S}^*(G)$ are defined as graphs constructed from $\mathcal{P}(G)$ and $\mathcal{S}(G)$ by removing identity element of G, respectively.

Let G be a group. The power digraph $\vec{\mathcal{P}}(G)$ is a digraph with the group G as its vertex set. There is an arc from x to y if $x \neq y$ and $y = x^r$, for some positive integer r. This graph was introduced by Kelarev and Quinn in their seminal paper [24]. Kelarev and Quinn focused on the study of semigroups by directed graph, but a very technical description of the structure of the power digraph of all finite abelian groups can be found in [24]. The power digraphs of semigroups were also considered in [21–23]. Motivated by the work of Kelarev and Quinn, Chakrabarty et al. considered undirected power graphs (power graph for short) of semigroups [8]. In recent years, there has been considerable interest to the study of power graphs, but the second graph introduced very recently by the present authors [19, 20]. In [19], the authors focused on the relationship between power graph and its main supergraph and some basic properties of this graph are studied. In [20], the automorphism group of this graph in general are computed, but this paper devotes to the study of graph eigenvalues of main supergraph.

It is clear that the power graph of finite group is connected. Chakrabarty et al. [8] studied the completeness of $\mathcal{P}(G)$ and proved that this graph is complete if and only if G is a finite cyclic $p-$group. It were also proved that for $n \geq 3$, the power graph $\mathcal{P}(U_n)$ is not Hamiltonian, when $n = 2^m p_1 p_2 \ldots p_k$, where U_n denotes the unit group of the cyclic group Z_n, p_1, $p_2, \ldots$, p_k are distinct Fermat primes, m and k are nonnegative integers, $m \geq 2$ for $k = 0, 1$ and $k \geq 2$ for $m = 0, 1$. They conjectured that $\mathcal{P}(U_n)$ is Hamiltonian for all values of $n \geq 3$ except those listed above. Pourgholi et al. [30] presented several counterexamples for this conjecture.

Cameron and Ghosh [5] proved that abelian groups with isomorphic power graphs must be isomorphic and conjectured that two finite groups with isomorphic power graphs have the same number of elements of each order. This conjecture is affirmatively proved by Cameron in [6].

In 2012, Mirzargar et al. proved that the power graph of cyclic groups of order p^m, p is prime, has the maximum number of edges among the power graph of all finite groups with the same order and conjectured that among all finite groups of any given order, the cyclic group of that order has the maximum number of edges in its power graph [27]. They also conjectured that the clique number of the power graph of a finite group and its largest cyclic subgroup are equal. The first conjecture was the starting point of a series of papers by Curtin and Pourgholi. These authors have shown the first conjecture for both directed [15] and undirected [12] power graphs. It is merit to mention here that Amiri et al. [3] was previously shown that the directed power graph of the cyclic group of order n has the maximum number of edges, among the directed power graphs of finite groups of order n. In [13], Curtin and Pourgholi studied the proper power graph of finite groups and proved that the diameter of this graph is ≤ 2 if and only if the group is nilpotent and every Sylow subgroup is either cyclic or a generalized quaternion. They finally applied an elegant number theory discussion to present an infinite family of counterexamples for the second conjecture [14].

Moghaddamfar et al. [28] considered the proper power graph of a finite group into account. Note that the power graph will always contain at least one vertex that is connected to every other vertex. In this paper, the authors established a number of ways in which the connectivity of proper power graph depends on the structural properties of the group under consideration. They proved that the proper power graph of a finite group G is connected if and only if G has a unique minimal subgroup. Moreover, if the order of the center of G has at least two prime divisors then the proper power graph of G will be connected, and if the center is a $p-$subgroup of G and the order of G has at least two prime divisors, then the proper power graph of G is connected if and only if every non-central element of order p in G is connected to a non$-p-$element in $\mathcal{P}^*(G)$. In [29], the authors computed the number of spanning trees of power graphs of some finite groups. A survey of recent works on this topic together with some open questions can be found in [1].

In [30], the authors asked the structure of all non-abelian simple groups with 2-connected power graphs. Doostabadi et al. [16] computed the number of component in the proper power graphs of the alternating groups which shows that the power graph of these simple groups can be 2$-$connected. This result recently corrected by Bubboloni et al. [4] which again shows the existence of simple group with 2$-$connected power graph. Akbari and Ashrafi [2] proved that the power graph of some classes of finite simple groups are not 2$-$connected and conjectured that a simple group with 2$-$connected power graph is of alternating type. Doostabadi et al. [17], started the study of the automorphism group of power graph and presented a conjecture about the automorphism group of $\mathcal{P}(Z_n)$ that proved affirmatively in [25].

3. Main Results

One of the key tools to studying a graph Γ is spectrum or Laplacian spectrum ($L-$spectrum for short) of Γ. In this field of study, there are two main questions that arise. Which set of algebraic integers can occur as $Sepc(\Gamma)$ or $L - Sepc(\Gamma)$, for some graph Γ, and if there is some set X so that $X = Sepc(\Gamma)$ or $X = L - Sepc(\Gamma)$, for a graph Γ, what can be said about the structure of Γ? To aid in the study of these questions for power graph and main supergraph of a finite group we will compute the spectrum and $L-$spectrum of some important classes of finite groups.

Chattopadhyaya et al. [9] started the study of the Laplacian spectrum of the power graph of cyclic and dihedral groups and in [26], the authors investigated the spectrum of cyclic, dicyclic, dihedral groups. Our results given this section generalize some results in the mentioned papers.

Suppose G is a finite group. Define the graph Δ_G with vertex set $\pi_e(G)$, the set of all element orders of G, and two vertices x and y are adjacent if and only if $x|y$ or $y|x$. Then $\mathcal{S}(G) = \Delta_G[K_{\Omega_{a_1}(G)}, \ldots, K_{\Omega_{a_r}(G)}]$, where $\pi_e(G) = \{a_1, \ldots, a_r\}$. By definition of the main supergraph this gives an equitable partition for the $\mathcal{S}(G)$.

For the sake of completeness, we mention here some results which are crucial throughout this paper.

Theorem 3.1. *[33] Let Γ be a graph and $V_1, V_2, \cdots, V_m$ be an equitable partition for Γ. If for $v \in V_i$, $t_{ij} = |N(v) \cap V_j|$ and T is the matrix (t_{ij}), then $\Phi(T, x)$ divide $\Phi(\Gamma, x)$.*

Theorem 3.2. *[33] Let H_i, $1 \leq i \leq p$ be all r_i-regular. Then $V(H_1) \cup V(H_2) \cup \ldots \cup V(H_p)$ is an equitable partition of $\Gamma[H_1, H_2, \ldots, H_p]$. If T is the matrix associated with this partition, then the characteristic polynomial of $\Gamma[H_1, H_2, \ldots, H_p]$ is $\Phi(\Gamma[H_1, H_2, \ldots, H_p], x) = \Phi(T, x). \prod_{i=1}^{p} \Phi(H_i)/(x - r_i)$.*

Theorem 3.3. *[26] The characteristic polynomial of the power graph of Z_n can be computed as $\Phi(\mathcal{P}(Z_n), x) = \Phi(T, x)(x+1)^{n-t-1}$, where d_i, $1 \leq i \leq t$, are all non-trivial divisors of n.*

$$T = \begin{pmatrix} \phi(n) & \phi(d_1) & \phi(d_2) & \cdots & \phi(d_t) \\ \phi(n)+1 & \phi(d_1)-1 & \alpha_{d_1d_2} & \cdots & \alpha_{d_1d_t} \\ \phi(n)+1 & \alpha_{d_2d_1} & \phi(d_2)-1 & \cdots & \alpha_{d_2d_t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi(n)+1 & \alpha_{d_td_1} & \alpha_{d_td_2} & \cdots & \phi(d_t)-1 \end{pmatrix}$$

and

$$\alpha_{d_id_j} = \begin{cases} \phi(d_j) & d_i|d_j \text{ or } d_j|d_i \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 3.4. *[26] With notation of Theorem 3.3, the characteristic polynomial of $\mathcal{P}^*(Z_n)$ is as follows:*

$$\Phi(\mathcal{P}^*(Z_n), x) = \Phi(T, x)(x+1)^{n-t-2},$$

where

$$T = \begin{pmatrix} \phi(n)-1 & \phi(d_1) & \phi(d_2) & \cdots & \phi(d_t) \\ \phi(n) & \phi(d_1)-1 & \alpha_{d_1d_2} & \cdots & \alpha_{d_1d_t} \\ \phi(n) & \alpha_{d_2d_1} & \phi(d_2)-1 & \cdots & \alpha_{d_2d_t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi(n) & \alpha_{d_td_1} & \alpha_{d_td_2} & \cdots & \phi(d_t)-1 \end{pmatrix}.$$

In the next theorem, characteristic polynomial of the main supergraph $\mathcal{S}(G)$ is obtained.

Theorem 3.5. *Let G be a group of order n with $\pi_e(G) = \{a_1, \ldots, a_k\}$. Then the characteristic polynomial of $\mathcal{S}(G)$ is as follows:*

$$\Phi(\mathcal{S}(G), x) = \Phi(T, x)(x+1)^{(n-k)},$$

where

$$T = \begin{pmatrix} \Omega_{a_1}(G)-1 & \alpha_{a_1a_2} & \alpha_{a_1a_3} & \cdots & \alpha_{a_1a_k} \\ \alpha_{a_2a_1} & \Omega_{a_2}(G)-1 & \alpha_{a_2a_3} & \cdots & \alpha_{a_2a_k} \\ \alpha_{a_3a_1} & \alpha_{a_3a_2} & \Omega_{a_3}(G)-1 & \cdots & \alpha_{a_3a_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{a_ka_1} & \alpha_{a_ka_2} & \alpha_{a_ka_3} & \cdots & \Omega_{a_k}(G)-1 \end{pmatrix}$$

and

$$\alpha_{a_ia_j} = \begin{cases} \Omega_{a_j}(G) & a_i|a_j \text{ or } a_j|a_i \\ 0 & \text{otherwise} \end{cases}.$$

Proof. By Theorem 3.2 and the structure of $\mathcal{S}(G)$,

$$\begin{aligned} \Phi(\mathcal{S}(G), x) &= \Phi(T, x)\left(\prod_{i=1}^{k} \frac{(x-(\Omega_{a_i}(G)-1))(x+1)^{\Omega_{a_i}(G)-1}}{(x-(\Omega_{a_i}(G)-1))}\right) \\ &= \Phi(T, x)\prod_{i=1}^{k}(x+1)^{\Omega_{a_i}(G)-1} \\ &= \Phi(T, x)(x+1)^{\sum_{i=1}^{k}(\Omega_{a_i}(G)-1)} \\ &= \Phi(T, x)(x+1)^{(n-k)}, \end{aligned}$$

proving the result. □

The dihedral, semi-dihedral and dicyclic groups can be presented as follows:

$$\begin{aligned}
D_{2n} &= < a, b | a^n = b^2 = 1, bab = a^{-1} >, \\
SD_{8n} &= < a, b | a^{4n} = b^2 = 1, bab = a^{2n-1} >, \\
T_{4n} &= < a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} > .
\end{aligned}$$

Example 3.6. *Consider the dihedral group D_{2n}. If $n = 2^k$, then the main supergraph of D_{2n} is isomorphic to the complete graph K_{2n}. So, the characteristic polynomial of D_{2n} is $\Phi(\mathcal{S}(D_{2n}), x) = (x+1)^{2n-1}(x-2n+1)$. We now assume that n is an odd number. If n is a prime power power then $\mathcal{S}(D_{2n}) = P_3[K_{n-1}, K_1, K_n]$. Apply Theorem 3.2 to calculate the characteristic polynomial of $\mathcal{S}(D_{2n})$ as follows:*

$$\Phi(\mathcal{S}(D_{2n}), x) = (x+1)^{2n-3}[x^3 + (-2n+3)x^2 + (n^2 - 5n + 3)x + (2n^2 - 4n + 1)].$$

In the case that n is an even number which is not a power of 2 *or n is an odd number which is not a prime power, it is easy to calculate the matrix T in Theorem 3.4.*

Definition 3.7. *If G and H are two rooted graphs with roots r and s, then a coalescence of these graphs is another graph G.H obtained from G and H by identifying their roots.*

In [26], the characteristic polynomial of $\mathcal{P}(D_{2n})$, $\mathcal{P}(T_{4n})$ and $\mathcal{P}(SD_{8n})$ in some special cases are calculated. In what follows, we will compute these polynomials in general. To do this, we state here a useful result of [33].

Theorem 3.8. *[33] If G and H are two rooted graphs with roots r and s, then the characteristic polynomial of coalescence G.H can be computed as follows:*

$$\Phi(G.H, x) = \Phi(G, x)\Phi(H - s, x) + \Phi(G - r, x)\Phi(H, x) - x\Phi(G - r, x)\Phi(H - s, x).$$

Theorem 3.9. *The characteristic polynomial of $\mathcal{P}(D_{2n})$ can be computed as follows:*

$$\Phi(\mathcal{P}(D_{2n}), x) = x^{n-1}(x+1)^{n-t-2}[x(x+1)\Phi(T, x) - n\Phi(T', x)].$$

Where d_i, $1 \le i \le t$, are all non-trivial divisors of n,

$$T = \begin{pmatrix} \phi(n) & \phi(d_1) & \phi(d_2) & \cdots & \phi(d_t) \\ \phi(n)+1 & \phi(d_1)-1 & \alpha_{d_1d_2} & \cdots & \alpha_{d_1d_t} \\ \phi(n)+1 & \alpha_{d_2d_1} & \phi(d_2)-1 & \cdots & \alpha_{d_2d_t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi(n)+1 & \alpha_{d_td_1} & \alpha_{d_td_2} & \cdots & \phi(d_t)-1 \end{pmatrix},$$

$$T' = \begin{pmatrix} \phi(n)-1 & \phi(d_1) & \phi(d_2) & \cdots & \phi(d_t) \\ \phi(n) & \phi(d_1)-1 & \alpha_{d_1d_2} & \cdots & \alpha_{d_1d_t} \\ \phi(n) & \alpha_{d_2d_1} & \phi(d_2)-1 & \cdots & \alpha_{d_2d_t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi(n) & \alpha_{d_td_1} & \alpha_{d_td_2} & \cdots & \phi(d_t)-1 \end{pmatrix},$$

and

$$\alpha_{d_id_j} = \begin{cases} \phi(d_j) & d_i | d_j \text{ or } d_j | d_i \\ 0 & otherwise \end{cases}.$$

Proof. By the structure of $\mathcal{P}(D_{2n})$ and Definition 3.7, $\mathcal{P}(D_{2n}) = S_n.\mathcal{P}(Z_n)$, where S_n is the star graph with root vertex of degree $n-1$ and $\mathcal{P}(Z_n)$ is an induced subgraph of $\mathcal{P}(D_{2n})$ obtained from $\langle a \rangle$. Moreover, $\Phi(S_n, x) = (x^2 - n)x^{n-1}$ and $\Phi(\overline{K_n}, x) = x^n$. Hence by Theorems 3.3, 3.4 and 3.8,

$$\begin{aligned}\Phi(\mathcal{P}(D_{2n}), x) &= \Phi(S_n.\mathcal{P}(Z_n), x) = \Phi(S_n, x)\Phi(\mathcal{P}^*(Z_n), x) + \Phi(\bar{K}_n, x)\Phi(\mathcal{P}(Z_n), x)\\ &- x\Phi(\bar{K}_n, x)\Phi(\mathcal{P}^*(Z_n), x)\\ &= x^{n-1}(x+1)^{n-t-2}[x(x+1)\Phi(T, x) - n\Phi(T', x)],\end{aligned}$$

which completes the proof. □

We are now ready to compute the characteristic polynomial of $\mathcal{P}^*(SD_{8n})$.

Theorem 3.10. *The characteristic polynomial of $\mathcal{P}^*(SD_{8n})$ is computed as follows:*

$$\Phi(\mathcal{P}^*(SD_{8n}), x) = x^{2n}(x+1)^{5n-t-2}[\Phi(T, x)(x-1)^n + \Phi(T'', x)\Phi(T', x) - x(x-1)^n\Phi(T'', x)],$$

where d_i, $1 \le i \le t$, are all non-trivial divisors of $4n$,

$$T = \begin{pmatrix} \phi(4n)-1 & \phi(d_1) & \phi(d_2) & \cdots & \phi(d_t)\\ \phi(4n) & \phi(d_1)-1 & \alpha_{d_1d_2} & \cdots & \alpha_{d_1d_t}\\ \phi(4n) & \alpha_{d_2d_1} & \phi(d_2)-1 & \cdots & \alpha_{d_2d_t}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \phi(4n) & \alpha_{d_td_1} & \alpha_{d_td_2} & \cdots & \phi(d_t)-1 \end{pmatrix},$$

$$T'' = \begin{pmatrix} \phi(4n)-2 & \phi(d_1) & \phi(d_2) & \cdots & \phi(d_t)\\ \phi(4n)-1 & \phi(d_1)-1 & \alpha_{d_1d_2} & \cdots & \alpha_{d_1d_t}\\ \phi(4n)-1 & \alpha_{d_2d_1} & \phi(d_2)-1 & \cdots & \alpha_{d_2d_t}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \phi(4n)-1 & \alpha_{d_td_1} & \alpha_{d_td_2} & \cdots & \phi(d_t)-1 \end{pmatrix},$$

$$T' = \begin{pmatrix} 0 & 2 & 2 & \cdots & 2\\ 1 & 1 & 0 & \cdots & 0\\ 1 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

and

$$\alpha_{d_id_j} = \begin{cases} \phi(d_j) & d_i|d_j \ \ or \ \ d_j|d_i\\ 0 & otherwise \end{cases}.$$

Proof. Define the rooted graph B as $B = K_1 + (\bigcup_{i=1}^{n} K_2)$ with root vertex r, where $V(K_1) = \{r\}$. We also consider $\mathcal{P}^*(Z_{4n})$ as a rooted graph with root vertex a connected to all other vertices of $\mathcal{P}^*(Z_{4n})$. To apply Theorem 3.8, we construct a graph A by identifying the vertex a in $\mathcal{P}^*(Z_{4n})$ and the vertex r in B, i.e. $A = \mathcal{P}^*(Z_{4n}).B$. By the structure of $\mathcal{P}^*(SD_{8n})$,

$$\mathcal{P}^*(SD_{8n}) = A \bigcup \overline{K_{2n}}.$$

Thus, $\Phi(\mathcal{P}^*(SD_{8n}), x) = \Phi(A, x)\Phi(\overline{K_{2n}}, x)$ and $\Phi(\overline{K_{2n}}, x) = x^{2n}$. So for computing the characteristic polynomial of $\mathcal{P}^*(SD_{8n})$, it is enough to compute the characteristic polynomial of graph A. By Theorems 3.3, 3.4 and 3.8,

$$\begin{aligned}
\Phi(A,x) &= \Phi(\mathcal{P}^*(Z_{4n}).B,x)\\
&= \Phi(\mathcal{P}^*(Z_{4n}),x)\Phi(B-r,x)+\Phi(\mathcal{P}^*(Z_{4n})-a,x)\Phi(B,x)\\
&- x\Phi(\mathcal{P}^*(Z_{4n})-a,x)\Phi(B-r,x)\\
&= \Phi(T,x)(x+1)^{4n-t-2}(x-1)^n(x+1)^n+\Phi(T'',x)(x+1)^{4n-t-2}\Phi(T',x)(x+1)^n\\
&- x\Phi(T'',x)(x+1)^{4n-t-2}(x-1)^n(x+1)^n\\
&= (x+1)^{5n-t-2}[\Phi(T,x)(x-1)^n+\Phi(T'',x)\Phi(T',x)-x(x-1)^n\Phi(T'',x)].
\end{aligned}$$

Therefore,

$$\Phi(\mathcal{P}^*(SD_{8n}),x)=x^{2n}(x+1)^{5n-t-2}[\Phi(T,x)(x-1)^n+\Phi(T'',x)\Phi(T',x)-x(x-1)^n\Phi(T'',x)],$$

proving the result. □

The characteristic polynomial of $\mathcal{P}^*(T_{4n})$ is the subject of our next result. We have:

Theorem 3.11. *The characteristic polynomial of $\mathcal{P}^*(T_{4n})$ can be computed as follows:*

$$\Phi(\mathcal{P}^*(T_{4n}),x)=(x+1)^{3n-t-2}[\Phi(T,x)(x-1)^n+\Phi(T'',x)\Phi(T',x)-x(x-1)^n\Phi(T'',x)],$$

where d_i, $1\le i\le t$, are all non-trivial divisors of $2n$,

$$T=\begin{pmatrix}
\phi(2n)-1 & \phi(d_1) & \phi(d_2) & \cdots & \phi(d_t)\\
\phi(2n) & \phi(d_1)-1 & \alpha_{d_1d_2} & \cdots & \alpha_{d_1d_t}\\
\phi(2n) & \alpha_{d_2d_1} & \phi(d_2)-1 & \cdots & \alpha_{d_2d_t}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\phi(2n) & \alpha_{d_td_1} & \alpha_{d_td_2} & \cdots & \phi(d_t)-1
\end{pmatrix},$$

$$T''=\begin{pmatrix}
\phi(2n)-2 & \phi(d_1) & \phi(d_2) & \cdots & \phi(d_t)\\
\phi(2n)-1 & \phi(d_1)-1 & \alpha_{d_1d_2} & \cdots & \alpha_{d_1d_t}\\
\phi(2n)-1 & \alpha_{d_2d_1} & \phi(d_2)-1 & \cdots & \alpha_{d_2d_t}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\phi(2n)-1 & \alpha_{d_td_1} & \alpha_{d_td_2} & \cdots & \phi(d_t)-1
\end{pmatrix},$$

$$T'=\begin{pmatrix}
0 & 2 & 2 & \cdots & 2\\
1 & 1 & 0 & \cdots & 0\\
1 & 0 & 1 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
1 & 0 & 0 & \cdots & 1
\end{pmatrix},$$

and

$$\alpha_{d_id_j}=\begin{cases}\phi(d_j) & d_i|d_j \ \text{or}\ d_j|d_i\\ 0 & \text{otherwise}\end{cases}.$$

Proof. Define the rooted graph B as $B=K_1+(\bigcup_{i=1}^n K_2)$ with root r, where $V(K_1)=\{r\}$. We consider $\mathcal{P}^*(Z_{2n})$ as a rooted graph with root vertex a such that a is adjacent with all vertices of this graph and construct $\mathcal{P}^*(T_{4n})$ by identifying the vertex a in $\mathcal{P}^*(Z_{2n})$ and the vertex r in B, i.e. $\mathcal{P}^*(T_{4n})=\mathcal{P}^*(Z_{2n}).B$. By Theorems 3.3, 3.4

and 3.8,

$$\begin{aligned}
\Phi(\mathcal{P}^*(T_{4n}),x) &= \Phi(\mathcal{P}^*(Z_{2n}).B,x)\\
&= \Phi(\mathcal{P}^*(Z_{2n}),x)\Phi(B-r,x)+\Phi(\mathcal{P}^*(Z_{2n})-a,x)\Phi(B,x)\\
&- x\Phi(\mathcal{P}^*(Z_{2n})-a,x)\Phi(B-r,x)\\
&= \Phi(T,x)(x+1)^{2n-t-2}(x-1)^n(x+1)^n\\
&+ \Phi(T'',x)(x+1)^{2n-t-2}\Phi(T',x)(x+1)^n\\
&- x\Phi(T'',x)(x+1)^{2n-t-2}(x-1)^n(x+1)^n\\
&= (x+1)^{3n-t-2}[\Phi(T,x)(x-1)^n+\Phi(T'',x)\Phi(T',x)\\
&- x(x-1)^n\Phi(T'',x)].
\end{aligned}$$

This completes the proof. □

We now state a result of [7] which is important in our next result.

Theorem 3.12. *[7] Let G_j's be graphs of order n_j, with $j\in\{1,\ldots,k\}$, with Laplacian spectrum $\sigma_L(G_j)$. If H is a graph such that $V(H)=\{1,\ldots,k\}$, then the Laplacian spectrum of $H[G_1,G_2,\ldots,G_k]$ can be computed as follows:*

$$\sigma_L(H[G_1,G_2,\ldots,G_k])=\left(\bigcup_{j=1}^{k}(N_j+(\sigma_L(G_j)\backslash\{0\}))\right)\bigcup\sigma(C),$$

where

$$N_j=\begin{cases}\sum_{i\in N_H(j)}n_i & N_H(j)\neq\emptyset\\ 0 & otherwise\end{cases}.$$

$$\rho_{l,q}=\rho_{q,l}=\begin{cases}\sqrt{n_ln_q} & if\ \ lq\in E(H)\\ 0 & otherwise\end{cases}.$$

and

$$C=\begin{pmatrix}N_1 & -\rho_{1,2} & \cdots & -\rho_{1,k}\\ -\rho_{2,1} & N_2 & \cdots & -\rho_{2,k}\\ \vdots & \vdots & \ddots & -\rho_{k-1,k}\\ -\rho_{1,k} & -\rho_{2,k} & \cdots & N_k\end{pmatrix}.$$

The following result is an immediate consequence of Theorem 3.12 and the fact that $\sigma_L(K_n)=\{0,n^{(n-1)}\}$.

Corollary 3.13. *Suppose $\mathcal{S}(G)=\Delta[K_{\Omega_{a_1}(G)},\ldots,K_{\Omega_{a_k}(G)}]$. Then the Laplacian spectrum of the main supergraph is computed as follows:*

$$\sigma_L(\mathcal{S}(G))=\left(\bigcup_{j=1}^{k}(N_j+\Omega_{a_j}(G))^{(\Omega_{a_j}(G)-1)}\right)\bigcup\sigma(C),$$

where

$$N_j = \begin{cases}\sum_{a_i\in N_\Delta(a_j)}\Omega_{a_i}(G), & N_\Delta(a_j)\neq\emptyset\\ 0, & otherwise\end{cases},$$

$$\rho_{l,q} = \rho_{q,l}=\begin{cases}\sqrt{\Omega_{a_l}(G)\Omega_{a_q}(G)} & a_l|a_q\ \ or\ \ a_q|a_l\\ 0 & otherwise\end{cases},$$

and

$$C = \begin{pmatrix} N_1 & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & N_2 & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & -\rho_{k-1,k} \\ -\rho_{1,k} & -\rho_{2,k} & \cdots & N_k \end{pmatrix}.$$

In [9], the authors computed Laplacian spectrum of the power graph of cyclic and dihedral groups. In the following example, the Laplacian spectrum of the power graph and main supergraph of T_{4n} and SD_{8n} are computed in some special cases.

Example 3.14. *In this example, the Laplacian eigenvalues of $\mathcal{S}(D_{2n})$ in the case that n is a prime power is computed. Suppose n is odd. Since $\mathcal{S}(D_{2n}) = P_3[K_{n-1}, K_1, K_n]$, by Theorem 3.12,*

$$C = \begin{pmatrix} 1 & -\sqrt{n-1} & 0 \\ -\sqrt{n-1} & 2n-1 & -\sqrt{n} \\ 0 & -\sqrt{n} & 1 \end{pmatrix}.$$

Now by computing eigenvalues of the matrix C, it follows that $\sigma(C) = \{0, 1, 2n\}$. Therefore, $\sigma_L(\mathcal{S}(D_{2n})) = \{0, 1, 2n, n^{(n-2)}, n+1^{(n-1)}\}$. If $n = 2^k$, then $\mathcal{S}(D_{2n}) = K_{2n}$ and $\sigma_L(\mathcal{S}(D_{2n})) = \{0, 2n^{(2n-1)}\}$.

In the next two examples, the Laplacian spectrum of the power graphs of T_{4n} and SD_{8n} are computed.

Example 3.15. *Consider the dicyclic group T_{4n}. If n is power of 2, then*

$$\mathcal{P}(T_{4n}) = W[K_{2n-2}, K_2, \underbrace{K_2, K_2, \cdots, K_2}_{n}],$$

where the graph W is depicted in Figure 1.

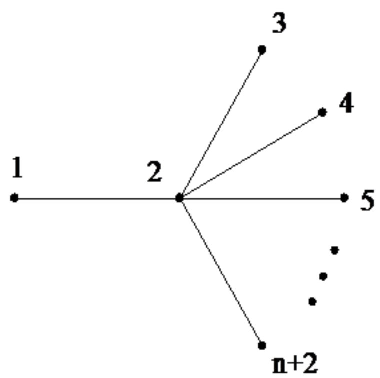

Figure 1: The Graph W in the Power Graph of T_{4n}

.

By Theorem 3.12,

$$\sigma_L(\mathcal{P}(T_{4n})) = \{4n, 4^{(n)}, 2n^{(2n-3)}, \sigma(C)\},$$

where elements of the matrix C are $N_1 = 2, N_2 = 4n-2, N_3 = \ldots = N_{n+2} = 2$, $\rho_{1,2} = \rho_{2,1} = 2\sqrt{n-1}$, $\rho_{1,3} = \rho_{3,1} = \rho_{1,4} = \rho_{4,1} = \ldots = \rho_{n+2,1} = 0$, $\rho_{2,3} = \rho_{3,2} = 2$, $\rho_{2,4} = \rho_{4,2} = 2$, $\rho_{2,5} = \rho_{5,2} = \rho_{2,6} = \rho_{6,2} = \ldots = \rho_{n+2,2} = 2$, $\rho_{3,4} = \rho_{4,3} = \ldots = \rho_{n+2,3} = 0$, $\rho_{4,5} = \rho_{5,4} = \ldots = \rho_{n+2,4} = 0, \ldots, \rho_{n+1,n+2} = 0$.

Example 3.16. *Consider the semi-dihedral group SD_{8n}. If n is power of 2, then*

$$\mathcal{P}(SD_{8n}) = U[K_{4n-2}, K_1, K_1, \overline{K_{2n}}, \underbrace{K_2, K_2, \cdots, K_2}_{n}],$$

where U is depicted in Figure 2.

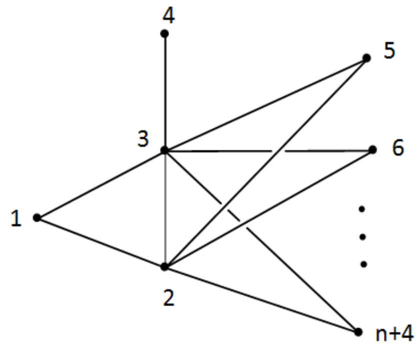

Figure 2: The Labeled Graph U in the Power Graph of SD_{8n}.

By Theorem 3.12,

$$\sigma_L(\mathcal{P}(SD_{8n})) = \{4^{(n)}, 4n^{(4n-3)}, 1^{(2n-1)}, \sigma(C)\},$$

where elements of the matrix C are $N_1 = 2, N_2 = 6n-1, N_3 = 8n-1, N_4 = 1, N_5 = \ldots = N_{n+4} = 2$, $\rho_{1,2} = \rho_{2,1} = \sqrt{4n-2}$, $\rho_{1,3} = \rho_{3,1} = \sqrt{4n-2}$, $\rho_{1,4} = \rho_{4,1} = \ldots = \rho_{n+4,1} = 0$, $\rho_{2,3} = \rho_{3,2} = 1$, $\rho_{2,4} = \rho_{4,2} = 0$, $\rho_{2,5} = \rho_{5,2} = \rho_{2,6} = \rho_{6,2} = \ldots = \rho_{n+4,2} = \sqrt{2}$, $\rho_{3,4} = \rho_{4,3} = \sqrt{2n}$, $\rho_{3,5} = \rho_{5,3} = \ldots = \rho_{n+4,3} = \sqrt{2}$, $\rho_{4,5} = \rho_{5,4} = \ldots = \rho_{n+4,4} = 0$, $\rho_{5,6} = \rho_{6,5} = \ldots = \rho_{n+4,5} = 0, \ldots \rho_{n+3,n+4} = 0$.

Let G be a group and $C(G) = \{C_1, \ldots, C_k\}$ be the set of all cyclic subgroups of G. Set L_G to be the graph with vertex set $C(G)$, and two cyclic subgroups are adjacent if one is contained in the other. Let K_{a_i} be the complete graph of order $a_i = \phi(|C_i|)$. If $K_G = \{K_{a_i} | a_i = \phi(|C_i|), C_i \in C(G)\}$, then the power graph $\mathcal{P}(G)$ is isomorphic to L_G−join of $K_{a_1}, K_{a_2}, \ldots, K_{a_k}$ ($L_G[K_{a_1}, K_{a_2}, \ldots, K_{a_k}]$), see [18] for details.

Theorem 3.17. *The Laplacian spectrum of* $\mathcal{P}(G) = L_G[K_{a_1}, K_{a_2}, \ldots, K_{a_k}]$ *can be calculated as follows:*

$$\sigma_L(\mathcal{P}(G)) = \left(\bigcup_{j=1}^{k} (N_j + a_j)^{(a_j - 1)} \right) \bigcup \sigma(C),$$

where

$$N_j = \begin{cases} \sum_{C_i \in N_{L_G}(C_j)} a_i & N_{L_G}(C_j) \neq \emptyset \\ 0 & otherwise \end{cases},$$

$$\rho_{l,q} = \rho_{q,l} = \begin{cases} \sqrt{a_l a_q} & C_l \subseteq C_q \ or \ C_q \subseteq C_l \\ 0 & otherwise \end{cases},$$

and

$$C = \begin{pmatrix} N_1 & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & N_2 & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & -\rho_{k-1,k} \\ -\rho_{1,k} & -\rho_{2,k} & \cdots & N_k \end{pmatrix}.$$

Proof. The proof follows from the Theorem 3.12, the graph structure of $\mathcal{P}(G)$ and the fact that $\sigma_L(K_n) = \{0, n^{(n-1)}\}$. $\square$

The Laplacian polynomial of $\mathcal{P}(Z_n)$ is obtained in [9]. We now apply the previous theorem to find a complete description of the Laplacian spectrum of $\mathcal{P}(Z_n)$.

Corollary 3.18. *Suppose $\{d_1, \ldots, d_k\}$ is the set of all divisors of n, $d_1 = 1$ and $d_2 = n$. Then $C(G) = \{C_1, \ldots, C_k\}$, $\mathcal{P}(Z_n) = L_G[K_1, K_{\phi(n)}, K_{\phi(d_3)}, \ldots, K_{\phi(d_k)}]$, $N_1 = \phi(n) + \phi(d_3) + \cdots + \phi(d_k)$, $N_2 = 1 + \phi(d_3) + \cdots + \phi(d_k)$, $N_j = \sum_{d_i|d_j} \phi(d_i) + \sum_{d_j|d_r} \phi(d_r) - 2\phi(d_j), 3 \le j \le k, 1 \le i \le k, 1 \le r \le k$, $\rho_{1,2} = \rho_{2,1} = \sqrt{\phi(n)}$, $\rho_{1,j} = \rho_{j,1} = \sqrt{\phi(d_j)}, 3 \le j \le k$, $\rho_{2,j} = \rho_{j,2} = \sqrt{\phi(d_j)\phi(n)}, 3 \le j \le k$. If $4 \le j \le k$ then*

$$\rho_{3,j} = \rho_{j,3} = \begin{cases} \sqrt{\phi(d_j)\phi(d_3)} & d_3|d_j \ \ or \ \ d_j|d_3 \\ 0 & otherwise \end{cases}.$$

For $5 \le j \le k$,

$$\rho_{4,j} = \rho_{j,4} = \begin{cases} \sqrt{\phi(d_j)\phi(d_4)} & d_4|d_j \ \ or \ \ d_j|d_4 \\ 0 & otherwise \end{cases}.$$

Finally,

$$\rho_{k-1,k} = \rho_{k,k-1} = \begin{cases} \sqrt{\phi(d_k)\phi(d_{k-1})} & d_k|d_{k-1} \ \ or \ \ d_{k-1}|d_k \\ 0 & otherwise \end{cases},$$

and for $1 \le i \le k$ and $1 \le r \le k$,

$$\sigma_L(\mathcal{P}(Z_n)) = n^{(\phi(n)-1)} \bigcup \{\bigcup_{j=3}^{k} (\sum_{d_i|d_j} \phi(d_i) + \sum_{d_j|d_r} \phi(d_r) - \phi(d_j))^{(\phi(d_j)-1)}\} \bigcup \sigma(C).$$

Corollary 3.19. (*See [9, Corollary 2.3]*) *If n is a prime power then the Laplacian spectrum of $\mathcal{P}(Z_n)$ is $\sigma_L(\mathcal{P}(Z_n)) = \{0, n^{(n-1)}\}$.*

Proof. By Corollary 3.18, $\sigma_L(\mathcal{P}(Z_n)) = \{n^{(n-2)}\} \bigcup \sigma(C)$, where the matrix C is as follows:

$$C = \begin{pmatrix} n-1 & -\sqrt{(n-1)} \\ -\sqrt{(n-1)} & 1 \end{pmatrix}.$$

The proof now follows from the fact that $\sigma(C) = \{n, 0\}$. $\square$

Corollary 3.20. (*See [9, Theorem 2.5]*) *If $n = pq$, p and q are distinct primes, then the Laplacian spectrum of $\mathcal{P}(Z_n)$ is as follows:*

$$\sigma_L(\mathcal{P}(Z_n)) = \{0, \phi(n) + 1, n - p + 1^{(q-2)}, n - q + 1^{(p-2)}, n^{\phi(n)+1}\}.$$

Proof. By Corollary 3.18, $\sigma_L(\mathcal{P}(Z_n)) = \{n - p + 1^{(q-2)}, n - q + 1^{(p-2)}, n^{\phi(n)-1}\} \bigcup \sigma(C)$, where

$$C = \begin{pmatrix} n-1 & -\sqrt{n-p-q+1} & -\sqrt{p-1} & -\sqrt{q-1} \\ -\sqrt{n-p-q+1} & p+q-1 & -(p-1)\sqrt{q-1} & -(q-1)\sqrt{p-1} \\ -\sqrt{p-1} & -(p-1)\sqrt{q-1} & n-p-q+2 & 0 \\ -sqrtq-1 & -(q-1)\sqrt{p-1} & 0 & n-p-q+2 \end{pmatrix}.$$

Now the proof follows from the fact that $\sigma(C) = \{0, \phi(n) + 1, n^{(2)}\}$. $\square$

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References

[1] J. Abawajy, A. Kelarev, M Chowdhury, Power graphs: a survey, Electronic Journal of Graph Theory and Applications (EJGTA) 1 (2) (2013) 125–147.
[2] N. Akbari, A. R. Ashrafi, Note on the power graph of finite simple groups, Quasigroups and Related Systems 23 (2) (2015) 165–173.
[3] H. Amiri, S. M. Jafarian Amiri, I. M. Isaacs, Sums of element orders in finite groups, Communications in Algebra 37 (9) (2009) 2978–2980.
[4] D. Bubboloni, M. A. Iranmanesh, S. M. Shaker, On some graphs associated with the finite alternating groups, Communications in Algebra DOI: 10.1080/00927872.2017.1307381.
[5] P. J. Cameron, S. Ghosh, The power graph of a finite group, Discrete Mathematics 311 (2011) 1220–1222.
[6] P. J. Cameron, The power graph of a finite group, II, Journal of Group Theory 13 (2010) 779–783.
[7] D. M. Cardoso, M. A. de Freitas, E. A. Martins, M. Robbiano, Spectra of graphs obtained by a generalization of the join graph operation, Discrete Mathematics 313 (2013) 733–741.
[8] I. Chakrabarty, S. Ghosh, M. K. Sen, Undirected power graphs of semigroups, Semigroup Forum 78 (2009) 410–426.
[9] S. Chattopadhyaya, P. Panigrahi, On Laplacian spectrum of power graphs of finite cyclic and dihedral groups, Linear and Multilinear Algebra 63 (7) (2015) 1345–1355.
[10] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs: Theory and Application, Academic Press, New York, 1980.
[11] D. Cvetković, P. Rowlinson, S. Simić, An Introduction to the Theory of Graph Spectra, Cambridge University Press, 2010.
[12] B. Curtin, G. R. Pourgholi, Edge-maximality of power graphs of finite cyclic groups, Journal of Algebraic Combinatorics 40 (2014) 313–330.
[13] B. Curtin, G. R. Pourgholi, H. Yousefi-Azari, On the punctured power graph of a finite group, Australasian Journal of Combinatorics 62 (2015) 1–7.
[14] B. Curtin, G. R. Pourgholi, An Euler totient sum inequality, Journal of Number Theory 163 (2016) 101–113.
[15] B. Curtin, G. R. Pourgholi, A group sum inequality and its application to power graphs, Bulletin of the Australian Mathematical Society 90 (2014) (3) 418–426.
[16] A. Doostabadi, M. D. G. Farrokhi, On the connectivity of proper power graphs of finite groups, Communications in Algebra **43** (2015) 4305–4319.
[17] A. Doostabadi, A. Erfanian, A. Jafarzadeh, Some results on the power graphs of finite groups, Science Asia 41 (2015) 73–78.
[18] M. Feng, X. Ma, K. Wang, The structure and metric dimension of the power graph of a finite group, European Journal of Combinatorics 43 (2015) 82–97.
[19] A. Hamzeh, A. R. Ashrafi, The main supergraph of the power graph of a finite group, in press.
[20] A. Hamzeh, A. R. Ashrafi, Automorphism group of supergraphs of the power graph of a finite group, European Journal of Combinatorics 60 (2017) 82–88.
[21] A. V. Kelarev, S. J. Quinn, A combinatorial property and power graphs of semigroups, Commentationes Mathematicae Univaersitatis Carolinae 45 (1) (2004) 1–7.
[22] A. V. Kelarev, S. J. Quinn, Directed graphs and combinatorial properties of semigroups, Journal of Algebra 251 (1) (2002) 16–26.
[23] A. V. Kelarev, S. J. Quinn, R. Smolíková, Power graphs and semigroups of matrices, Bulletin of the Australian Mathematical Society 63 (2) (2001) 341–344.
[24] A. V. Kelarev, S. J. Quinn, A combinatorial property and power graphs of groups, Contributions to General Algebra 12 (Vienna, 1999), 229–235, Heyn, Klagenfurt, 2000.
[25] Z. Mehranian, A. Gholami, A. R. Ashrafi, A note on the power graph of a finite group, International Journal of Group Theory 5 (1) (2016) 1–10.
[26] Z. Mehranian, A. Gholami, A. R. Ashrafi, The Spectra of power graphs of certain finite groups, Linear and Multilinear Algebra 65 (5) (2017) 1003-1010.
[27] M. Mirzargar, A. R. Ashrafi, M. J. Nadjafi-Arani, On the power graph of a finite group, Filomat 26 (2012) 1201–1208.
[28] A. R. Moghaddamfar, S. Rahbariyan, W. J. Shi, Certain properties of the power graph associated with a finite group, Journal of Algebra and its Applications 13 (7) (2014) 1450040, 18 pp.
[29] A. R. Moghaddamfar, S. Rahbariyan, S. Navid Salehy, S. Nima Salehy, The number of spanning trees of power graphs associated with specific groups and some applications, Ars Combinatoria, in press.
[30] G. R. Pourgholi, H. Yousefi-Azari, A. R. Ashrafi, The undirected power graph of a finite group, Bulletin of the Malaysian Mathematical Sciences Society **38** (4) (2015) 1517–1525.
[31] J. S. Rose, A Course on Group Theory, Cambridge University Prees, Cambridge, New York- Melbourne, 1978.
[32] G. Sabidussi, Graph Derivatives, Mathematische Zeitschrift 76 (1961) 385–401.
[33] A. J. Schwenk, Computing the characteristic polynomial of a graph, in: R. Bary, F. Harary (Eds.), Graphs and Combinatorics, Lecture Notes in Mathematics 406, Springer-Verlag, Berlin-Heidelberg-New York, (1974) 153–172.
[34] D. B. West, Introduction to Graph Theory, (2nd edition), Prentice Hall, Inc., Upper Saddle River, NJ, 2001.