

## SPECTRUM OF THE LAPLACIAN OF AN ASYMMETRIC FRACTAL GRAPH

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(Received 8 July 2004)

*Abstract* We consider a simple self-similar sequence of graphs that does not satisfy the symmetry conditions that imply the existence of a spectral decimation property for the eigenvalues of the graph Laplacians. We show that, for this particular sequence, a very similar property to spectral decimation exists, and we obtain a complete description of the spectra of the graphs in the sequence.

*Keywords:* fractal graph; Laplacian; spectral decimation

*2000 Mathematics subject classification:* Primary 28A80

### 1. Introduction and definitions

#### 1.1. Introduction

Many self-similar graphs, and related fractals, display a property known as *spectral decimation*: that the spectrum of the Laplacian can be described in terms of the iteration of a rational function  $f$ . Eigenvalues  $\lambda$  of the Laplacian at a given stage of the construction are related to eigenvalues  $\mu$  of the Laplacian at the following stage of the construction by the relationship

$$\lambda = f(\mu), \quad (1.1)$$

where  $f$  is a rational function on  $\mathbb{R}$ , unless  $\mu$  is a member of a small *exceptional set*,  $\mathcal{E}$ . This was first observed for the specific case of the Sierpiński gasket graph in [8], and this was given a rigorous mathematical treatment in [4, 11, 12]. In the case of the Sierpiński gasket, using our definition of the Laplacian (see § 1.3), the function  $f(\mu) = \mu(5 - 4\mu)$  and the exceptional set is  $\{\frac{1}{2}, \frac{5}{4}, \frac{3}{2}\}$ .

A generalization of spectral decimation to a much larger class of self-similar graphs, including the Vicsek set graph, appears in [7], in which a symmetry condition is developed which, if satisfied, ensures that spectral decimation applies to the graph. Each self-similar graph in this class has a function  $f$  and exceptional set  $\mathcal{E}$  associated with it.

In this paper we consider a simple asymmetric self-similar graph which does not satisfy the symmetry condition of [7]. In § 1.2 we define a sequence of graphs  $(G_n)_{n \in \mathbb{N}}$ , which can be used to define a self-similar graph  $G_\infty$  as for the self-similar graphs in [7].

In § 2 we show that, for this example, a property similar to spectral decimation exists, in which (1.1) is replaced by

$$2(1 - \lambda)^2 = f(\mu), \quad (1.2)$$

where  $f(\mu)$  is a quartic polynomial. Again there is an exceptional set of values of  $\mu$  for which the relationship does not necessarily hold. This is proved in Theorems 2.1 and 2.2.

Another common spectral property of self-similar graphs and related fractals is that there are many eigenvalues of the Laplacian with high multiplicity and Dirichlet–Neumann eigenfunctions, i.e. eigenfunctions which are zero on the boundary. In [9] it is shown that the eigenvalues with Dirichlet–Neumann eigenfunctions dominate the spectrum in a large class of cases, that of the nested fractals introduced in [5]. In [6] a similar result is shown for two-point self-similar graphs, a class which includes the example in this paper.

In § 3 we calculate the number of linearly independent eigenfunctions of the Laplacian which are Dirichlet–Neumann or non-Dirichlet–Neumann. In § 4, we use the results of §§ 2 and 3 to describe the spectra of the graphs in the self-similar sequence of finite graphs used in the construction of our graph. This is stated in Theorem 4.1, which gives a complete description of the spectrum, including the multiplicity of the eigenvalues and which eigenvalues are associated with Dirichlet–Neumann and non-Dirichlet–Neumann eigenfunctions.

An example of a self-similar graph which does not satisfy the symmetry conditions of [7], and for which spectral decimation appears not to apply, is associated with the pentagasket, as described in [1], in which numerical approximations for eigenvalues and eigenvectors are obtained, and some theoretical results are obtained that show how to construct eigenspaces of high multiplicity.

A more complicated method, using a rational map on a projective variety rather than on  $\mathbb{R}$ , which works for a larger class of self-similar graphs than that in [7] including some for which spectral decimation does not apply, is described in [10]. However, our graph does not meet all the conditions described in § 1.1.1 of [10].

## 1.2. The graph

In this section we define a self-similar sequence of finite graphs  $(G_n)_{n \in \mathbb{N}}$ .

We label the vertex sets and edge sets of a graph  $G$  by  $V(G)$  and  $E(G)$ , respectively, and, for a vertex  $i \in V(G_n)$ , we define  $E_i^{(n)}$  to be the set of edges of  $G_n$  connected to  $i$ .

We start with  $G_0$ , a single edge between two vertices 1 and 2, and proceed inductively, constructing  $G_{n+1}$  from  $G_n$ . Our graphs will be defined in such a way that  $V(G_{n-1}) \subseteq V(G_n)$ .

To construct  $G_{n+1}$ , we assume as an induction hypothesis that, if  $n \geq 1$ , the graph  $G_n$  is bipartite with the two parts being  $V(G_{n-1})$  and  $V(G_n) \setminus V(G_{n-1})$ , and hence that each edge  $e \in E(G_n)$  can be thought of as connecting two vertices  $i(e)$  and  $j(e)$ , defined so that  $i(e) \in V(G_{n-1})$  and  $j(e) \in V(G_n) \setminus V(G_{n-1})$ . To deal with the special case  $G_0$ , we set  $i(e_0) = 1$  and  $j(e_0) = 2$  for its single edge  $e_0$ .

For each  $e \in E(G_n)$  we introduce a new vertex, which we label  $k(e)$ , and we let the vertex set  $V(G_{n+1})$  of  $G_{n+1}$  be the union of  $V(G_n)$  with the set of new vertices

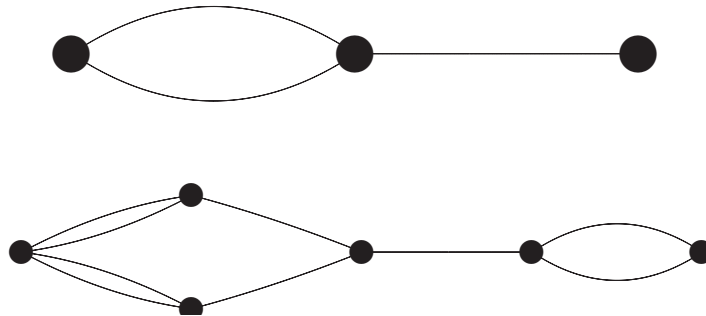
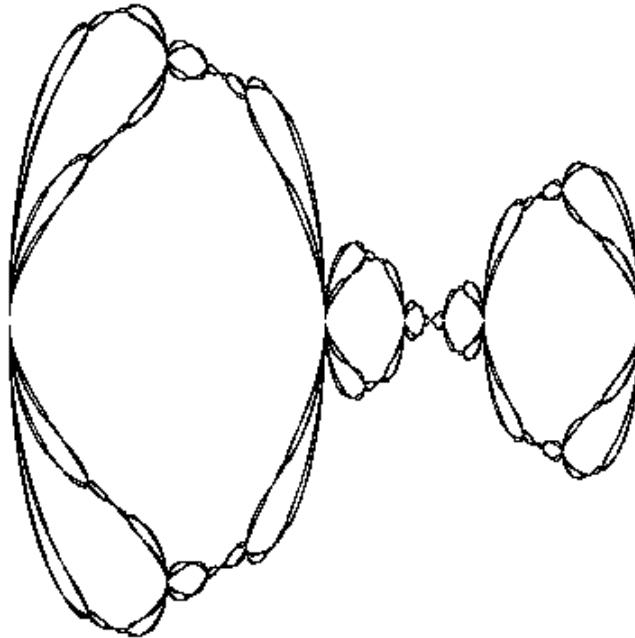
Figure 1. The graphs  $G_1$  and  $G_2$ .

Figure 2. A later stage of the construction.

$\{k(e), e \in E(G_n)\}$ . We then define  $E(G_{n+1})$  to consist of, for each  $e \in E(G_n)$ , two edges connecting  $k(e)$  with  $i(e)$  and one edge connecting  $k(e)$  with  $j(e)$ . This ensures that the new graph  $G_{n+1}$  is bipartite with the two parts being  $V(G_n)$  and  $V(G_{n+1}) \setminus V(G_{n-1})$ , so that we can continue the construction inductively.

Figure 1 shows  $G_1$  and  $G_2$ , and Figure 2 shows a later stage of the construction, generated using MAPLE.

The same sequence of graphs can be obtained by using the framework of Definition 5.2 of [7], with the model graph being identical to  $G_1$  above, but with conditions on the orientation to deal with the asymmetry.

It can also be obtained by a variation on the framework of § 1.1.1 of [10]. In that

framework, the sequence of graphs is obtained from a basic cell  $F = \{1, \dots, N_0\}$ , for some  $N_0$ , and an equivalence relation  $\mathcal{R}$  defined on  $\{1, \dots, N\} \times F$ , where  $N$  is the number of cells and satisfies  $N \geq N_0$ . In our case,  $N_0 = 2$  and  $N = 3$ , and our model graph  $G_1$  can be defined by an equivalence relation  $\mathcal{R}$  on  $\{1, 2, 3\} \times \{1, 2\}$  with three equivalence classes  $\{(1, 1), (3, 1)\}$ ,  $\{(1, 2), (2, 2), (3, 2)\}$  and the singleton  $\{(2, 1)\}$ . However, in [10] the equivalence relation  $\mathcal{R}$  is required to satisfy three conditions, one of which is that the equivalence class of  $(i, i)$  for  $1 \leq i \leq N_0$  should be a singleton, and our equivalence relation does not satisfy this condition, although it does satisfy the other two. As a result of this, the definitions of the equivalence relations  $\mathcal{R}_{\langle \infty \rangle}$  and  $\mathcal{R}_{\langle n \rangle}$ , used in [10] to define the infinite graph and its subsets, need to be modified to deal with the more complicated behaviour of the boundary points.

When  $1 \leq m < n$ , the graph  $G_n$  contains  $3^{n-m}$  subgraphs isomorphic to  $G_m$ . We will call these subgraphs *m-cells*. Using this, we can define a sequence  $(\tilde{G}_n)_{n \in \mathbb{N}}$  such that  $\tilde{G}_n$  is isomorphic to  $G_n$  and  $\tilde{G}_m$  is a subgraph of  $\tilde{G}_n$  for  $m < n$ . We then define the infinite graph  $G_\infty = \bigcup_{n=0}^\infty \tilde{G}_n$ . This is analogous to Definition 5.5 of [7].

We define maps  $f_i : V(G_{n-1}) \rightarrow V(G_n)$ ,  $i = 1, 2, 3$ , mapping each vertex of  $G_{n-1}$  to the corresponding vertex in each  $(n-1)$ -cell. We will label these so that  $f_1$  and  $f_2$  correspond to the two parallel cells.

We note that this graph is similar to that described in [3], although in the context of that paper the orientation of the cells is not important.

### 1.3. The Laplacian

There are a number of different definitions of the Laplacian of a graph. The definition of the graph Laplacian used in [7] is the generator matrix of a continuous-time random walk on the graph, while in [2] a related symmetric matrix is used. However, the eigenvalues of the different definitions differ by at most a simple transformation.

For convenience in describing the eigenfunctions, we use the following definition of the Laplacian: the Laplacian  $\mathcal{L}_G$  of a graph  $G$  (which may have multiple edges but with no loops) is a  $|V(G)| \times |V(G)|$  matrix with, for a vertex  $i \in V(G)$ ,  $\mathcal{L}_G(i, i) = 1$ , and, for  $i, j \in V(G)$  with  $i \neq j$ ,  $\mathcal{L}_G(i, j) = -e_{i,j}/\delta_i$ , where  $e_{i,j}$  is the number of edges linking  $i$  and  $j$  in  $G$  and  $\delta_i$  is the degree of vertex  $i$  in  $G$ . This gives the same eigenvalues as the symmetric Laplacian described in [2], and the eigenfunctions are the ‘harmonic eigenfunctions’ described in [2]. Our definition of the graph Laplacian differs from that in [7] only in that the sign of each entry (and hence of the eigenvalues) is reversed.

## 2. The relationship between the eigenvalues

We set  $f(\mu) = 9(\mu - 1)^4 - 9(\mu - 1)^2 + 2$ , so that (1.2) becomes

$$2(1 - \lambda)^2 = 9(\mu - 1)^4 - 9(\mu - 1)^2 + 2. \quad (2.1)$$

We first show how to construct eigenvalues  $\mu$  of  $\mathcal{L}_{G_{n+1}}$  from eigenvalues  $\lambda$  of  $\mathcal{L}_{G_n}$  when  $\lambda \notin \{0, 1, 2\}$ .

Given  $\lambda$  and  $\mu$ , we set

$$\gamma = \frac{3(\mu - 1)^2 - 2}{1 - \lambda}. \quad (2.2)$$

**Theorem 2.1.** Given an eigenfunction  $x$  of  $\mathcal{L}_{G_n}$  with eigenvalue  $\lambda \notin \{0, 1, 2\}$ , we can do the following.

- (i) Solve (2.1) for  $\mu$  to obtain four roots.
- (ii) For each possible  $\mu$ , set  $\gamma$  using (2.2).
- (iii) Then define  $x'$  by

$$x'_i = \begin{cases} x_i & i \in V(G_{n-1}), \\ \gamma x_i & i \in V(G_n) \setminus V(G_{n-1}), \end{cases} \quad (2.3)$$

and, for a vertex  $k = k(e) \in V(G_{n+1}) \setminus V(G_n)$ , we set

$$x'_k = \frac{2x_{i(e)}}{3(1 - \mu)} + \frac{\gamma x_{j(e)}}{3(1 - \mu)}. \quad (2.4)$$

Then  $x'$  is an eigenfunction of  $\mathcal{L}_{G_{n+1}}$  with eigenvalue  $\mu$ .

**Proof.** To check this, we just calculate  $\mathcal{L}_{G_{n+1}}x'$ . For  $i \in V(G_{n-1})$ ,

$$\begin{aligned} (\mathcal{L}_{G_{n+1}}x')_i &= x_i + \frac{1}{\delta_i^{(n)}} \sum_{e \in E_i^{(n)}} \left( \frac{2x_i}{3(\mu - 1)} + \frac{\gamma x_{j(e)}}{3(\mu - 1)} \right) \\ &= x_i + \frac{2x_i}{3(\mu - 1)} + \frac{\gamma x_i(1 - \lambda)}{3(\mu - 1)} \\ &= x_i \left( \frac{3(\mu - 1) + 2 + 3(\mu - 1)^2 - 2}{3(\mu - 1)} \right) \\ &= \mu x_i = \mu x'_i. \end{aligned}$$

For  $j \in V(G_n) \setminus V(G_{n-1})$ ,

$$\begin{aligned} (\mathcal{L}_{G_{n+1}}x')_j &= \gamma x_j + \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} \left( \frac{2x_{i(e)}}{3(\mu - 1)} + \frac{\gamma x_j}{3(\mu - 1)} \right) \\ &= \gamma x_j + \frac{\gamma x_j}{3(\mu - 1)} + \frac{2x_j(1 - \lambda)}{3(\mu - 1)} \\ &= x_j \left( \frac{3(\mu - 1)\gamma + \gamma + 2(1 - \lambda)}{3(\mu - 1)} \right). \end{aligned}$$

Using (2.1) and (2.2),

$$\begin{aligned} 2(1 - \lambda) &= \frac{9(\mu - 1)^4 - 9(\mu - 1)^2 + 2}{1 - \lambda} \\ &= \frac{(3(\mu - 1)^2 - 1)(3(\mu - 1)^2 - 2)}{1 - \lambda} \\ &= (3(\mu - 1)^2 - 1)\gamma, \end{aligned}$$

and so

$$\begin{aligned} (\mathcal{L}_{G_{n+1}}x')_j &= x_j\gamma\left(\frac{3(\mu-1)+1+3(\mu-1)^2-1}{3(\mu-1)}\right) \\ &= \mu\gamma x_j \\ &= \mu x'_j, \end{aligned}$$

and, finally, for  $k \in j \in V(G_n) \setminus V(G_{n-1})$ , which satisfies  $k = k(e)$  for some edge  $e$  of  $G_n$ , we have

$$\begin{aligned} (\mathcal{L}_{G_{n+1}}x')_k &= x'_k - \frac{2}{3}x'_i(e) - \frac{1}{3}x'_j(e) \\ &= \left(\frac{1}{1-\mu} - 1\right)\left(\frac{2}{3}x_i + \frac{1}{3}\gamma x_j\right) \\ &= \mu\left(\frac{2x_i}{3(1-\mu)} + \frac{\gamma x_j}{3(1-\mu)}\right) \\ &= \mu x'_k, \end{aligned}$$

so  $x'$  is indeed an eigenfunction of  $\mathcal{L}_{G_{n+1}}$  with eigenvalue  $\mu$ .  $\square$

**Theorem 2.2.** *If*

$$\mu \notin \left\{1, 1 + \sqrt{\frac{2}{3}}, 1 - \sqrt{\frac{2}{3}}, 1 + \sqrt{\frac{1}{3}}, 1 - \sqrt{\frac{1}{3}}\right\}$$

and  $\lambda$  and  $\mu$  satisfy (2.1), then  $\mu$  is an eigenvalue of  $G_{n+1}$  if and only if  $\lambda$  is an eigenvalue of  $G_n$ , with the same multiplicity.

**Proof.** If we have an eigenfunction  $x'$  of  $\mathcal{L}_{G_{n+1}}$  with eigenvalue  $\mu \neq 1$ , then, for each edge  $e$  of  $G_{n+1}$ ,

$$x'_{k(e)} = \frac{2x'_{i(e)}}{3(1-\mu)} + \frac{x'_{j(e)}}{3(1-\mu)},$$

so that, for each  $i \in V(G_{n-1})$ ,

$$\begin{aligned} x'_i(1-\mu) &= \frac{1}{\delta_i^{(n)}} \sum_{e \in E_i^{(n)}} \left(\frac{2x'_i}{3(1-\mu)} + \frac{x'_{j(e)}}{3(1-\mu)}\right) \\ &= \frac{2x'_i}{3(1-\mu)} + \frac{1}{\delta_i^{(n)}} \sum_{e \in E_i^{(n)}} \frac{x'_{j(e)}}{3(1-\mu)}, \end{aligned}$$

giving

$$x'_i(3(1-\mu)^2 - 2) = \frac{1}{\delta_i^{(n)}} \sum_{e \in E_i^{(n)}} x'_{j(e)}. \quad (2.5)$$

Similarly, for  $j \in V(G_n) \setminus V(G_{n-1})$ ,

$$\begin{aligned} x'_j(1-\mu) &= \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} \left( \frac{x'_j}{3(1-\mu)} + \frac{2x'_{i(e)}}{3(1-\mu)} \right) \\ &= \frac{x'_j}{3(1-\mu)} + \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} \frac{2x'_{i(e)}}{3(1-\mu)}, \end{aligned}$$

giving

$$x'_j(3(1-\mu)^2 - 1) = \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} x'_{i(e)}. \quad (2.6)$$

By the conditions of the theorem,  $(1-\mu)^2 \neq \frac{2}{3}$ . Then (2.5) implies that, for any  $\lambda$ ,

$$x'_i(1-\lambda) = \frac{1}{\delta_i^{(n)}} \sum_{e \in E_i^{(n)}} \frac{1-\lambda}{3(1-\mu)^2 - 2} x'_{j(e)}, \quad (2.7)$$

while (2.6) gives, if  $\lambda \neq 1$ ,

$$\frac{1-\lambda}{3(1-\mu)^2 - 2x'_j} \frac{(3(1-\mu)^2 - 2)(3(1-\mu)^2 - 1)}{2(1-\lambda)} = \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} x'_{i(e)}. \quad (2.8)$$

Then, if we set  $x_i = x'_i$  for  $i \in V(G_{n-1})$  and

$$x_j = x'_j \frac{1-\lambda}{3(1-\mu)^2 - 2},$$

(2.7) and (2.8) become

$$x_i(1-\lambda) = \frac{1}{\delta_i^{(n)}} \sum_{e \in E_i^{(n)}} x_{j(e)}$$

and

$$x_j \frac{(3(1-\mu)^2 - 2)(3(1-\mu)^2 - 1)}{2(1-\lambda)} = \frac{1}{\delta_j^{(n)}} \sum_{e \in E_j^{(n)}} x_{i(e)},$$

which imply that  $x$  is an eigenfunction of  $\mathcal{L}_{G_n}$  with eigenvalue  $\lambda$  if

$$(1-\lambda) = \frac{(3(1-\mu)^2 - 2)(3(1-\mu)^2 - 1)}{2(1-\lambda)},$$

which is equivalent to the quartic (2.1).

This eigenfunction can be degenerate only if  $1-\lambda = 0$ , i.e. if either  $(1-\mu)^2 = \frac{1}{3}$  or  $(1-\mu)^2 = \frac{2}{3}$ .  $\square$

The set

$$\left\{1, 1 + \sqrt{\frac{2}{3}}, 1 - \sqrt{\frac{2}{3}}, 1 + \sqrt{\frac{1}{3}}, 1 - \sqrt{\frac{1}{3}}\right\}$$

of values of  $\mu$  where Theorem 2.2 does not apply plays a similar role to that of the *exceptional set* in [7].

We note that the eigenvalues  $\lambda$  and  $2 - \lambda$  produce the same values of  $\mu$ , with the same eigenfunctions. This is related to the bipartite nature of the graph; in fact, if  $x$  is an eigenfunction with eigenvalue  $\lambda$ , then, following [2], we can obtain an eigenfunction with eigenvalue  $2 - \lambda$  by simply changing the sign of  $x$  on  $V(G_n) \setminus V(G_{n-1})$ . These two eigenfunctions will then produce the same new eigenfunction using the above construction.

We now consider the cases when Theorems 2.1 and 2.2 do not apply, i.e. when  $\lambda \in \{0, 1, 2\}$  or

$$\mu \in \left\{1, 1 + \sqrt{\frac{2}{3}}, 1 - \sqrt{\frac{2}{3}}, 1 + \sqrt{\frac{1}{3}}, 1 - \sqrt{\frac{1}{3}}\right\}.$$

We note that if  $\mu = 1$  and  $\lambda$  and  $\mu$  satisfy (2.1), then  $\lambda = 0$  or  $2$ .

When  $\lambda = 1$  and  $x_i \neq 0$  for some  $i \in V(G_{n-1})$ , we use the same method, but with  $\gamma = 0$  and the quartic (2.1) replaced by

$$(\mu - 1)^2 = \frac{2}{3}. \quad (2.9)$$

We cannot use this method if  $x_i = 0$  for all  $i \in V(G_{n-1})$ , because the constructed eigenfunction would be zero everywhere.

However, in the case where  $\lambda = 1$  and  $x_i = 0$  for all  $i \in V(G_{n-1})$ , we can construct an eigenfunction  $x'$  by setting  $x'_i = 0$  for  $i \in V(G_{n-1})$ , and  $x'_i = x_i$  for  $i \in V(G_n) \setminus V(G_{n-1})$ . This gives eigenvalues  $\mu$  with

$$(\mu - 1)^2 = \frac{1}{3}, \quad (2.10)$$

using similar methods to those above.

### 3. Dirichlet–Neumann and non-Dirichlet–Neumann eigenfunctions

The graph  $G_n$  has  $v_n$  vertices and  $e_n$  edges where  $v_0 = 2$ ,  $e_0 = 1$  and  $e_n = 3e_{n-1}$ ,  $v_n = v_{n-1} + e_{n-1}$ . Hence  $e_n = 3^n$  and  $v_n = \frac{1}{2}(3^n + 3)$ .

The following lemma provides a means of constructing Dirichlet–Neumann eigenfunctions, which are zero on the two boundary vertices 1 and 2.

**Lemma 3.1.** *Let  $\Gamma_0$  be a connected graph with  $m$  vertices, including distinguished endpoints 1 and 2, and let  $\Gamma$  be the graph formed by defining  $\Gamma_1$  and  $\Gamma_2$  to be two identical copies of  $\Gamma_0$  and connecting them in parallel by identifying their endpoints. Then the Laplacian of  $\Gamma$  has  $m - 2$  linearly independent eigenfunctions which are zero on the endpoints.*

*The associated eigenvalues are the eigenvalues of the Laplacian  $\mathcal{L}_\Gamma$  restricted to the set  $\{2j : 2 \leq j \leq m - 1\}$  of vertices in  $\Gamma_2$ .*

**Proof.** We label the vertices of  $\Gamma_0$ ,  $1, 2, \dots, m$ . Then we label the vertices in  $\Gamma$  so that, for  $j \geq 3$ , vertex  $j$  in  $\Gamma_0$  corresponds to vertex  $2j - 3$  in  $\Gamma_1$  and vertex  $2j - 2$  in  $\Gamma_2$ .



Now consider the Laplacian  $\mathcal{L}_\Gamma$ . For  $2 \leq j, k \leq m-1$  we have

$$\begin{aligned}\mathcal{L}_\Gamma(1, 2j-1) &= \mathcal{L}_\Gamma(1, 2j), \\ \mathcal{L}_\Gamma(2, 2j-1) &= \mathcal{L}_\Gamma(2, 2j), \\ \mathcal{L}_\Gamma(2j-1, 2k-1) &= \mathcal{L}_\Gamma(2j, 2k), \\ \mathcal{L}_\Gamma(2j-1, 2k) &= \mathcal{L}_\Gamma(2j, 2k-1) = 0\end{aligned}$$

and consider functions  $x$  satisfying

$$\begin{aligned}x(1) &= x(2) = 0, \\ x(2j-1) &= -x(2j) \quad \text{for } 2 \leq j \leq m-1.\end{aligned}$$

Now

$$(\mathcal{L}_\Gamma x)(1) = \sum_{j=2}^{m-1} (\mathcal{L}_\Gamma(1, 2j-1)x(2j-1) + \mathcal{L}_\Gamma(1, 2j)x(2j)) = 0$$

and, similarly,  $(\mathcal{L}_\Gamma x)(2) = 0$ , while

$$\begin{aligned}(\mathcal{L}_\Gamma x)(2j-1) &= \sum_{k=2}^{m-1} \mathcal{L}_\Gamma(2j-1, 2k-1)x(2k-1) \\ &= - \sum_{k=2}^{m-1} \mathcal{L}_\Gamma(2j, 2k)x(2k) = (\mathcal{L}_\Gamma x)(2j).\end{aligned}$$

So the Laplacian  $\mathcal{L}_\Gamma$  preserves vectors of this form, which form a vector space of dimension  $m-2$ , and it acts on them in a similar way to the Laplacian restricted to the interior vertices of  $\Gamma_0$ . As the Laplacian is symmetric, there are  $m-2$  linearly independent Dirichlet–Neumann eigenfunctions of the Laplacian.  $\square$

If  $G_n$  contains a subgraph  $\Gamma$  of this form, where the vertices of  $\Gamma$  other than the endpoints have no edges linking them to  $G_n \setminus \Gamma$ , then we can take one of the eigenfunctions  $x$  on  $\Gamma$  constructed by the above lemma and extend it to an eigenfunction  $\tilde{x}$  on  $G_n$  by setting

$$\tilde{x}(v) = \begin{cases} x(v) & \text{for } v \in V(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

If neither of the endpoints 1 or 2 is in the interior of the subgraph  $\Gamma$ , then this  $\tilde{x}$  will be Dirichlet–Neumann.

**Proposition 3.2.** *The graph  $G_n$  has at least  $\frac{1}{2}(3^n + 3) - 2^n - 1$  linearly independent Dirichlet–Neumann eigenvalues.*

**Proof.** The model graph contains parallel edges, so  $G_n$  contains a subgraph consisting of two copies of  $G_{n-1}$  with their boundary points identified as in Lemma 3.1. This gives  $v_{n-1} - 2$  eigenfunctions. For each eigenfunction  $x$  obtained thus, we have  $x(f_1(v)) = -x(f_2(v))$  and  $x(f_3(v)) = 0$  for each  $v \in V(G_{n-1})$ .

Furthermore, given a Dirichlet–Neumann eigenfunction  $x$  of  $G_{n-1}$ , we can obtain three Dirichlet–Neumann eigenfunctions  $x_1, x_2, x_3$  of  $G_n$  by extending them from  $(n - 1)$ -cells to the whole graph, i.e.

$$x_i(f_j(v)) = \delta_{ij}x(v) \quad \text{for all } v \in V(G_{n-1}), \quad i, j = 1, 2, 3.$$

However, we only obtain two linearly independent eigenfunctions, because the linear combination  $x_1 - x_2$  is of the form obtained using Lemma 3.1.

Hence, if  $l_n$  is the number of Dirichlet–Neumann eigenvalues constructed by these methods, it satisfies

$$l_n = 2l_{n-1} + \frac{1}{2}(3^{n-1} + 3) - 2$$

and  $l_2 = 1$ , which gives the result. □

Let the total number of Dirichlet–Neumann eigenfunctions of  $G_n$  be  $l_n + \hat{l}_n$ , so that  $\hat{l}_n$  is the number of eigenfunctions that are not constructed by the methods of Proposition 3.2.

For this graph, we can also describe a set of non-Dirichlet–Neumann eigenfunctions.

**Proposition 3.3.** *The graph  $G_n$  has  $2^n + 1$  linearly independent eigenfunctions which are not Dirichlet–Neumann.*

**Proof.** We consider the set of functions  $x$  on  $V(G_1)$  which are zero on the central vertex of  $V(G_1)$ , i.e. they satisfy  $x(3) = 0$  and, for each  $v \in V(G_{n-1})$ ,  $x(f_1(v)) = x(f_2(v)) = \frac{1}{2}x(f_3(v))$ ; these form a  $(v_{n-1} - 1)$ -dimensional subspace. This is preserved by the Laplacian of  $G_n$ , so we can find  $v_{n-1} - 1 = \frac{1}{2}(3^{n-1} + 1)$  linearly independent eigenfunctions satisfying these properties.

To exclude those which are Dirichlet–Neumann, this is equivalent to the condition that  $x(1) = x(2) = 0$ . So a Dirichlet–Neumann eigenfunction satisfying the above conditions reduces to a Dirichlet–Neumann eigenfunction on each  $(n - 1)$ -cell. Hence there are  $\frac{1}{2}(3^{n-1} + 1) - l_{n-1} - \hat{l}_{n-1} = 2^{n-1} - \hat{l}_{n-1}$  non-Dirichlet–Neumann eigenfunctions satisfying the above conditions.

Now, given any eigenfunction  $x$  of  $G_{n-1}$ , we can extend it to an eigenfunction  $x'$  of  $G_n$  by setting  $x'(1) = \sqrt{2}x(1)$ ,  $x'(2) = x(1)$ ,  $x'(3) = \sqrt{3}x(2)$ ,  $x'(f_1(v)) = x'(f_2(v)) = x'(f_3(v)) = x(v)$  for  $v \geq 3$ , from the structure of the graph. This means that each of the eigenfunctions constructed in Proposition 3.3 for  $G_{n-1}$  can be extended to a non-Dirichlet–Neumann eigenfunction of  $G_n$ , which will be linearly independent of those already found (because  $x'(3) \neq 0$ ). Inductively, this also applies to those constructed for  $G_{n-m}$ ,  $m > 1$ .

The graphs are bipartite, so eigenfunctions (which are zero nowhere) for eigenvalues 0 and 2 exist as described in [2].

Hence the total number of linearly independent non-Dirichlet–Neumann eigenfunctions is at least  $2^n + 1 - \sum_{m=0}^{n-1} \hat{l}_m$ . But we know that there are exactly  $\frac{1}{2}(3^n + 3)$  linearly independent eigenfunctions. Hence

$$\frac{1}{2}(3^n + 3) \geq 2^n + 1 - \sum_{m=0}^{n-1} \hat{l}_m + \frac{1}{2}(3^n + 3) - 2^n - 1 + \hat{l}_n$$

and so  $\hat{l}_n \leq \sum_{m=0}^{n-1} \hat{l}_m$ . But  $\hat{l}_m = 0$  for  $m \leq 2$ , and hence for all  $m$ .

Hence the eigenfunctions constructed are all that exist, and there are  $2^n + 1$  linearly independent non-Dirichlet–Neumann ones, the remainder being Dirichlet–Neumann.  $\square$

#### 4. The spectra of the graphs

We can now use the relationships between eigenvalues and the information on Dirichlet–Neumann and non-Dirichlet–Neumann eigenfunctions to obtain a complete description of the spectra of the graphs  $G_n$ .

**Theorem 4.1.** *Set*

$$\alpha_1^{(1)} = 1, \quad \alpha_1^{(2)} = 1 - \sqrt{\frac{2}{3}} \quad \text{and} \quad \alpha_2^{(2)} = 1 + \sqrt{\frac{2}{3}}.$$

We extend this to define

$$\{\alpha_i^{(n)}; 1 \leq i \leq 2^{n-1}\}$$

to be the  $2^{n-1}$  values  $\mu$  satisfying the quartic (2.1), with  $\lambda = \alpha_j^{(n-1)}$  for some  $j$ .

Similarly, set

$$\beta_1^{(1)} = 1, \quad \beta_1^{(2)} = 1 - \sqrt{\frac{1}{3}} \quad \text{and} \quad \beta_2^{(2)} = 1 + \sqrt{\frac{1}{3}}.$$

We extend this to define

$$\{\beta_i^{(n)}; 1 \leq i \leq 2^{n-1}\}$$

to be the  $2^{n-1}$  values  $\mu$  satisfying the quartic (2.1), with  $\lambda = \beta_j^{(n-1)}$  for some  $j$ .

Then we have the following.

- (a) If  $n \geq m$ , then  $\mathcal{L}_{G_n}$  has a non-Dirichlet–Neumann eigenfunction with eigenvalue  $\alpha_i^{(m)}$ ,  $1 \leq i \leq 2^{m-1}$ . Together with eigenfunctions with eigenvalues 0 and 2, this describes the non-Dirichlet–Neumann spectrum of  $\mathcal{L}_{G_n}$ .
- (b) If  $n \geq m + 1$ , then  $\mathcal{L}_{G_n}$  has  $\frac{1}{2}(3^{n-m} - 1)$  linearly independent Dirichlet–Neumann eigenvalues with eigenvalue  $\beta_i^{(m)}$ , for  $1 \leq i \leq 2^{m-1}$ .

**Proof.** We note that  $G_1$  has a non-Dirichlet–Neumann eigenfunction with eigenvalue 1. As described in the proof of Proposition 3.3, this can then be extended to give a non-Dirichlet–Neumann eigenfunction  $x$  with eigenvalue 1 for each  $G_n$ ,  $n \geq 1$ .

Because this eigenfunction is non-Dirichlet–Neumann, it has  $x_i \neq 0$  for at least some  $i \in V(G_{n-1})$ . Hence the construction of eigenfunctions of  $G_{n+1}$  with eigenvalues  $\mu$  satisfying  $(1 - \mu)^2 = \frac{2}{3}$  produces non-degenerate eigenfunctions, which are also non-Dirichlet–Neumann. So  $G_{n+1}$  has non-Dirichlet–Neumann eigenfunctions with eigenvalues

$$1 \pm \sqrt{\frac{2}{3}}.$$

We have already shown that  $G_n$  has a non-Dirichlet–Neumann eigenfunction with eigenvalue  $\alpha_i^{(2)}$ ,  $1 \leq i \leq 2^{n-1}$ , for  $n \geq 2$ . Now, if  $G_{n-1}$  has a non-Dirichlet–Neumann

eigenfunction with eigenvalue  $\alpha_i^{(m-1)}$  for each  $1 \leq i \leq 2^{m-2}$ , then the construction of eigenfunctions gives us a non-Dirichlet–Neumann eigenfunction of  $G_n$  with eigenvalue  $\alpha_i^{(m)}$  for each  $1 \leq i \leq 2^{m-1}$ . Using this inductively, we find that  $G_n$  has a non-Dirichlet–Neumann eigenfunction with eigenvalue  $\alpha_i^{(m)}$ ,  $1 \leq i \leq 2^{m-1}$ , for  $n \geq m$ .

Along with the eigenfunctions with eigenvalues 0 and 2, this gives us all  $2^n + 1$  non-Dirichlet–Neumann eigenfunctions from Proposition 3.3, and hence completes the proof of (a).

We now consider the Dirichlet–Neumann eigenfunctions. We note that  $G_2$  has a Dirichlet–Neumann eigenfunction with eigenvalue 1. Such an eigenfunction is zero on  $V(G_1)$ , so our main construction produces a degenerate eigenfunction. However, the alternative construction with  $(1 - \mu)^2 = \frac{1}{3}$  does produce two eigenfunctions of  $G_3$ . Hence, as there are  $\frac{1}{2}(3^n + 3) - 2^n - 1$  Dirichlet–Neumann eigenfunctions of  $G_n$ , we can use the constructions to obtain  $3^n + 3 - 2^{n+1} - 2$  Dirichlet–Neumann eigenfunctions, with eigenvalues other than 1, of  $G_{n+1}$ .

We now show that, for  $n \geq 2$ ,  $G_n$  has  $\frac{1}{2}(3^{n-1} - 1)$  linearly independent Dirichlet–Neumann eigenfunctions with eigenvalue 1. This is the case for  $n = 2$ . For each such eigenfunction of  $G_n$ , we can construct three eigenfunctions of  $G_{n+1}$  using the methods in the proof of Proposition 3.2.

Assuming that  $G_n$  has  $\frac{1}{2}(3^n + 3) - 2^n - 1$  linearly independent Dirichlet–Neumann eigenfunctions of which  $\frac{1}{2}(3^{n-1} - 1)$  have eigenvalue 1, we have  $\frac{1}{2}(3^n - 3)$  linearly independent Dirichlet–Neumann eigenfunctions of  $G_{n+1}$  with eigenvalue 1, and  $3^n + 3 - 2^{n+1} - 2$  with other eigenvalues. However, we know from Proposition 3.2 that there are  $\frac{1}{2}(3^{n+1} + 3) - 2^{n+1} - 1$  in total, and

$$\frac{1}{2}(3^n - 3) + 3^n + 3 - 2^{n+1} - 2 = \frac{1}{2}(3^{n+1} + 3) - 2^{n+1} - 2.$$

The one unexplained eigenfunction must also have eigenvalue 1 because, if it had eigenvalue  $\lambda \neq 1$ , we would also have an unexplained eigenfunction with eigenvalue  $2 - \lambda$ . Hence we have  $\frac{1}{2}(3^n - 1)$  linearly independent Dirichlet–Neumann eigenfunctions of  $G_{n+1}$  with eigenvalue 1, giving the result by induction.

We know that, for  $n \geq 2$ ,  $G_n$  has  $\frac{1}{2}(3^{n-1} - 1)$  linearly independent Dirichlet–Neumann eigenfunctions with eigenvalue 1. We now use our constructions  $m$  times to show that, for  $n \geq m + 1$ ,  $G_n$  has  $\frac{1}{2}(3^{n-m} - 1)$  linearly independent Dirichlet–Neumann eigenfunctions with eigenvalue  $\beta_i^{(m)}$ , for  $1 \leq i \leq 2^{m-1}$ . This completes the proof of (b).  $\square$

## 5. Reversing the orientation

We remark that very similar results can be obtained if we reverse the orientation of the model graphs in the definitions of § 1.2. Because of the asymmetry this gives a different self-similar sequence of graphs. Eigenvalues  $\lambda$  and  $\mu$  of Laplacians of successive members of the sequence are related by the same equation (2.1) when  $\lambda \notin \{0, 1, 2\}$ , but the two equations (2.9) and (2.10) for  $\mu$  when  $\lambda = 1$  are reversed. This has the effect that, in Theorem 4.1, the roles of  $\alpha_i^{(n)}$  and  $\beta_i^{(n)}$  are reversed.

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