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Spectrum of the Laplacian on a complete Riemannian manifold with nonnegative Ricci curvature which possess a pole

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Let M be a complete noncompact Riemannian manifold of dimension n. The Laplacian of M is denoted by Δ. The spectrum of Δ has been studied by many authors. Especially Donnelly [**D**] showed that the Laplacian of a constant negative curvature $(-k^2)$ space form has purely continuous spectrum $\sigma_c(\Delta)=(-\infty, 0]$. Donnelly and Li [**D-L**] and Chen and author [**C-L**] observed that changing the metric of a constant negative curvature space form on a compact domain we can obtain a complete manifold with a rotational invariant metric of strongly negatively sectional curvature on which Δ has discrete spectrum. Escobar [**E**] proved that if M is a complete manifold with a rotational invariant metric of nonnegative curvature then $\sigma_c(\Delta)=(-\infty, 0]$.

In this paper we prove the following theorem.

THEOREM. *Let M be an n-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature. If M possess a pole then Δ has essential spectrum $\sigma_{ess}(\Delta)=(-\infty, 0]$.*

A pole is by definition [**G-W**] a point $x_0 \in M$ such that the exponential map $\exp_{x_0}: M_{x_0} \to M$ is a diffeomorphism.

If x_0 is a pole in M, $r(x)=\mathrm{dist}(x, x_0)$ which is the geodesic distance from x to x_0 is clearly differentiable. Let $C_t=\{x \in M \mid r(x)=t\}$. We claim that $\eta(x)=\lim_{t\to\infty}(t-\mathrm{dist}(x, C_t))=r(x)$.

In fact, for $t>r(x)$ we have $r(x)+\mathrm{dist}(x, C_t) \geqq \mathrm{dist}(x_0, C_t)=t$, and so $t-\mathrm{dist}(x, C_t) \leqq r(x)$. On the other hand since x_0 is a pole $q=\exp_{x_0} s(\exp_{x_0}^{-1}x)/r(x)|_{s=t} \in C_t$ and thus $r(x)+\mathrm{dist}(x, q)=t$, we therefore have $t-\mathrm{dist}(x, C_t)=r(x)$ for $t>r(x)$. The claim follows.

By Theorem A in [**W**] we know that if the Ricci curvature of M is nonnegative then $\Delta r(x) \geqq 0$. The laplacian comparison theorem [**G-W**] yields the following lemma.

LEMMA. *Suppose M satisfies the hypotheses in Theorem. If x_0 is a pole in M then*

$$0 \leqq \Delta r(x) \leqq \frac{n-1}{r(x)}. \tag{1}$$

PROOF OF THEOREM. Let x_0 be a pole in M. $r(x)=\mathrm{dist}(x, x_0)$, $B_{x_0}(R)=\{x\in M | r(x)<R\}$, $V_{x_0}(R)=\mathrm{Vol}(B_{x_0}(R))$ which is the volume of $B_{x_0}(R)$. By comparison theorem [**C-G-T**] one has

$$\frac{V_{x_0}(R)}{V_{x_0}(r)} \leqq \left(\frac{R}{r}\right)^n \quad \text{for } R \geqq r > 0 \tag{2}$$

$$\frac{V_{x_0}(R)}{V_{x_0}(r)} \geqq \frac{(R-2r)^n}{R^n-(R-2r)^n} \quad \text{for } R \geqq 2r > 0. \tag{3}$$

We choose $\psi(r)\in C_0^\infty(M)$ satisfying

$$\psi(r) = \begin{cases} 1 & a_n \leqq r \leqq b_n \\ 0 & r \leqq c_n, \quad r \geqq d_n \end{cases} \tag{4}$$

$0\leqq\psi(r)\leqq 1$, $|\psi'(r)|\leqq A_n$, $|\psi''(r)|\leqq A_n$ where $c_n<a_n<b_n<d_n$ are constants to be fixed, A_n depends on a_n, b_n, c_n and d_n.

For any $\lambda\geqq 0$, we consider the sequence of function

$$\phi_k^\lambda(x) = \eta_k^{1/2}\psi(\varepsilon_k r(x)) e^{i\sqrt{\lambda}r(x)} \in C_0^\infty(M) \tag{5}$$

where ε_k is a monotone sequence which tends to zero, η_k is a sequence to be fixed.

Computing directly we have

$$\nabla\phi_k^\lambda(x) = \eta_k^{1/2}\psi'(\varepsilon_k r)e^{i\sqrt{\lambda}r}\varepsilon_k \nabla r + i\eta_k^{1/2}\sqrt{\lambda}\psi(\varepsilon_k r)e^{i\sqrt{\lambda}r}\nabla r \tag{6}$$

$$\begin{aligned}\Delta\phi_k^\lambda(x) = {} & \eta_k^{1/2}\varepsilon_k^2\psi''(\varepsilon_k r)e^{i\sqrt{\lambda}r} + 2i\eta_k^{1/2}\sqrt{\lambda}\varepsilon_k\psi'(\varepsilon_k r)e^{i\sqrt{\lambda}r} \\ & + \eta_k^{1/2}\varepsilon_k\psi'(\varepsilon_k r)e^{i\sqrt{\lambda}r}\Delta r + i\eta_k^{1/2}\sqrt{\lambda}\psi(\varepsilon_k r)e^{i\sqrt{\lambda}r}\Delta r - \lambda\phi_k^\lambda(x).\end{aligned} \tag{7}$$

So

$$\begin{aligned}|\Delta\phi_k^\lambda(x)+\lambda\phi_k^\lambda(x)| \leqq {} & \eta_k^{1/2}\varepsilon_k^2|\psi''(\varepsilon_k r)| + 2\eta_k^{1/2}\varepsilon_k\sqrt{\lambda}|\psi'(\varepsilon_k r)| \\ & + \eta_k^{1/2}\varepsilon_k|\psi'(\varepsilon_k r)|\Delta r + \eta_k^{1/2}\sqrt{\lambda}|\psi(\varepsilon_k r)|\Delta r.\end{aligned}$$

By Lemma one has

$$\begin{aligned}&\frac{1}{32}\int_M |\Delta\phi_k^\lambda(x)+\lambda\phi_k^\lambda(x)|^2 dx \\ &\leqq \eta_k\varepsilon_k^4\int_M |\psi''(\varepsilon_k r)|^2 dx + \eta_k\varepsilon_k^4\int_M |\psi'(\varepsilon_k r)|^2\frac{(n-1)^2}{(\varepsilon_k r)^2}dx \\ &\quad + \eta_k\varepsilon_k^2\lambda\int_M |\psi'(\varepsilon_k r)|^2 dx + \eta_k\varepsilon_k^2\lambda\int_M |\psi(\varepsilon_k r)|^2\frac{(n-1)^2}{(\varepsilon_k r)^2}dx\end{aligned}$$

$$\leqq A_n^2\eta_k\varepsilon_k^4\int_{c_n\leq\varepsilon_k r\leq d_n}dx+\frac{A_n^2}{c_n^2}\eta_k\varepsilon_k^4(n-1)^2\int_{c_n\leq\varepsilon_k r\leq d_n}dx$$

$$+A_n^2\eta_k\varepsilon_k^2\lambda\int_{c_n\leq\varepsilon_k r\leq d_n}dx+\frac{(n-1)^2}{c_n^2}\eta_k\varepsilon_k^2\lambda\int_{c_n\leq\varepsilon_k r\leq d_n}dx$$

$$\leqq A_n^2\Big(1+\frac{(n-1)^2}{c_n^2}\Big)\eta_k\varepsilon_k^4\Big[V_{x_0}\Big(\frac{d_n}{\varepsilon_k}\Big)-V_{x_0}\Big(\frac{c_n}{\varepsilon_k}\Big)\Big]$$

$$+\Big(A_n^2+\frac{(n-1)^2}{c_n^2}\Big)\eta_k\varepsilon_k^2\lambda\Big[V_{x_0}\Big(\frac{d_n}{\varepsilon_k}\Big)-V_{x_0}\Big(\frac{c_n}{\varepsilon_k}\Big)\Big].$$

We choose

$$\eta_k=\Big[V_{x_0}\Big(\frac{d_n}{\varepsilon_k}\Big)-V_{x_0}\Big(\frac{c_n}{\varepsilon_k}\Big)\Big]^{-1} \tag{8}$$

then

$$\lim_{k\to\infty}\int_M|\Delta\phi_k^\lambda(x)+\lambda\phi_k^\lambda(x)|^2dx=0. \tag{9}$$

Obviously

$$\int_M|\phi_k^\lambda(x)|^2dx\geqq\frac{V_{x_0}(b_n/\varepsilon_k)-V_{x_0}(a_n/\varepsilon_k)}{V_{x_0}(d_n/\varepsilon_k)-V_{x_0}(c_n/\varepsilon_k)}\geqq\frac{V_{x_0}(b_n/\varepsilon_k)}{V_{x_0}(d_n/\varepsilon_k)}-\frac{V_{x_0}(a_n/\varepsilon_k)}{V_{x_0}(d_n/\varepsilon_k)}.$$

Using (2) and (3) we have

$$\int_M|\phi_k^\lambda(x)|^2dx\geqq\Big(\frac{b_n}{d_n}\Big)^n-\frac{d_n^n-(d_n-2a_n)^n}{(d_n-2a_n)^n}\geqq1+\Big(\frac{b_n}{d_n}\Big)^n-\Big(\frac{d_n}{d_n-2a_n}\Big)^n.$$

Choosing a_n sufficiently small we have

$$\int_M|\phi_k^\lambda(x)|^2dx\geqq\alpha_n>0.$$

So

$$\liminf_{k\to\infty}\|\phi_k^\lambda(x)\|_{L^2(M)}\geqq\alpha_n>0. \tag{10}$$

(3) implies

$$\frac{V_{x_0}(d_n/\varepsilon_k)}{V_{x_0}(c_n/\varepsilon_k)}\geqq\frac{(d_n-2c_n)^n}{d_n^n-(d_n-2c_n)^n}.$$

By choosing c_n sufficiently small we have

$$\frac{V_{x_0}(d_n/\varepsilon_k)}{V_{x_0}(c_n/\varepsilon_k)}\geqq\beta_n>1$$

which implies $V_{x_0}(d_n/\varepsilon_k)-V_{x_0}(c_n/\varepsilon_k)\geqq(\beta_n-1)V_{x_0}(c_n/\varepsilon_k)\to\infty$ as $k\to\infty$. We therefore have

$$\phi_k^\lambda(x)\longrightarrow0\quad\text{weakly in } L^2(M) \text{ as } k\to\infty. \tag{11}$$

(9), (10), (11) imply $-\lambda\in\sigma_{ess}(\Delta)$. This completes the proof of Theorem.

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