Spectrum of the Laplacian on Asymptotically Euclidean Spaces

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1. Introduction

The Laplacian Δ for Euclidean space R^n has the following properties: (a) the essential spectrum of $-\Delta$ is $[0, \infty)$; (b) Δ has no point spectrum; and (c) Δ has no singular continuous spectrum. If $(x_1, x_2, \dots, x_n)$ are the standard global coordinates on R^n, then the *exhaustion function* $b(x) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ satisfies (i) $|\nabla b| = 1$ for $x \neq 0$ and (ii) $\operatorname{Hess} b^2 = 2g$. Here g denotes the Euclidean metric.

Let M be a complete Riemannian manifold that admits a proper exhaustion function b. If (i) and (ii) above are satisfied in a weak or approximate sense, then we would like to show that the Laplacian Δ of M has properties similar to those of the Euclidean Laplacian. This program was started in our earlier paper [6]. Under general averaged L_2 conditions on $|\Delta b|$ and $||\nabla b| - 1|$, we showed that the essential spectrum of $-\Delta$ is $[0, \infty)$. More stringent pointwise decay conditions for $|\operatorname{Hess} b^2 - 2g|$ and $||\nabla b| - 1|$ were needed to eliminate the possibility of a point spectrum for Δ. The singular continuous spectrum was not discussed in [6].

The present paper extends the earlier work concerning the point spectrum and provides new results about the singular continuous spectrum. If M admits an exhaustion function b having Properties 2.1, then Theorem 2.3 states that Δ has no square integrable eigenfunctions. The analogous result in [6] required the stronger hypotheses $||\nabla b| - 1| \le cb^{-\varepsilon}$ and $|\operatorname{Hess} b^2 - 2g| \le cb^{-\varepsilon}$ for some $\varepsilon > 0$, whereas Properties 2.1 impose no specific decay rate on these quantities. However, Property 2.1(iv) restricts the third derivatives of b, whereas no such condition was imposed in [6]. For manifolds with nonnegative Ricci curvature, Euclidean volume growth, and quadratic curvature decay, Cheeger and Colding [3] and Colding and Minicozzi [4] constructed an exhaustion function with Properties 2.1.

The singular continuous spectrum is studied in Section 3. If b satisfies Properties 3.1 (which are more restrictive than 2.1) then Theorem 3.5 states that $-\Delta$ has no singular continuous spectrum. The asymptotically Euclidean spaces of [1] support exhaustion functions with Properties 3.1. For these spaces, the curvature may have variable sign but the curvature decay is faster than quadratic. Our treatment of the singular continuous spectrum is an application of the abstract Mourre

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theory in [2]. The Mourre theory of [5] also inspired our treatment of the point spectrum, although the work in Section 2 is logically self-contained.

Our main theorems generalize readily from $-\Delta$ to certain Schrödinger operators $-\Delta + V$ on manifolds. We prove all our results in this more general context. For the Schrödinger operator on R^n, these theorems are well-known (see [2; 5]).

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2. Absence of Point Spectrum

Let M be a connected complete Riemannian manifold. The symbol Δ will denote the Laplacian acting on functions defined on M. Assume that V is a bounded and continuously differentiable function. The Schrödinger operator $-\Delta + V$ is essentially self-adjoint [7] on $C_0^\infty M$. Suppose that $u \in L^2M \cap C^2M$ satisfies $-\Delta u + Vu = \lambda u$ ($\lambda > 0$). Then u lies in the domain of $-\Delta + V$, considered as an unbounded operator on L^2M. It follows [7] that $|\nabla u| \in L^2M$ and

$$\int_M |\nabla u|^2 = -\int_M u\Delta u.$$

We assume that M admits a proper C^2 exhaustion function b with certain properties. Suppose that $r(x)$ denotes the geodesic distance from $x \in M$ to a fixed basepoint $p \in M$. Let g denote the metric tensor of M. The symbol $\varepsilon(r)$ will signify a function satisfying $\varepsilon(r) \to 0$ as $r \to \infty$. The following properties will be required for our exhaustion function b in the complement of a compact set K.

PROPERTIES 2.1.

(i) $c_1 r \le b \le c_2 r$ for some positive constants c_1 and c_2.
(ii) $1 - \varepsilon(r) \le |\nabla b| \le 1 + \varepsilon(r)$.
(iii) $|\mathrm{Hess}\, b^2 - 2g| \le \varepsilon(r)$.
(iv) $|d\Delta b^2| \le \varepsilon(r)$.

Here d denotes the exterior derivative and $|T|$ is the pointwise norm of the tensor T.

Suitable conditions must also be imposed upon our potential function $V \in L^\infty M \cap C^1 M$. Let X signify the vector field $2b\nabla b$. We assume that V satisfies the following properties in $M - K$, where K is compact.

PROPERTIES 2.2.

(i) $|V| \le \varepsilon(r)$.
(ii) $XV \le \varepsilon(r)$.

The main result of this section is the following theorem.

THEOREM 2.3. *Suppose $u \in L^2M \cap C^2M$ satisfies $-\Delta u + Vu = \lambda u$ on M ($\lambda > 0$). Assume that M admits an exhaustion function b satisfying Properties 2.1 and the potential function V satisfies Properties 2.2. Then $u \equiv 0$.*

The proof of Theorem 2.3 will be presented in a sequence of lemmas. The overall strategy is essentially a unique continuation back from infinity. The function u is originally assumed only to be in L^2M. By employing the Mourre theory and Rellich identities, we progressively show that u lies in more and more restrictive weighted L^2 spaces. Intuitively, u vanishes to infinite order at infinity. A Carleman-type argument then shows that u must be identically zero.

The first step was already taken in [6] as follows.

LEMMA 2.4. *For all positive integers* k,

$$\int_M b^k[u^2 + |\nabla u|^2] < \infty.$$

Proof. This was proved as Proposition 3.6 of [6]. Property 2.1(iv) is not needed and Property 2.1(ii) is only used in the weaker form $c_1 \le |\nabla b| \le c_2$. □

Let $H = -\Delta + V$. If $\phi \in C^2M$ then define $A = \nabla\phi + \frac{1}{2}\Delta\phi$. One verifies that $A\colon C_0^\infty M \to C_0^\infty M$ is a first-order skew symmetric operator. The symbol T_i will denote a component of the covariant derivative ∇T of the tensor T. Repeated indices denote a sum of contractions with respect to the metric g. Our main tool will be the following (Mourre-type) estimate.

LEMMA 2.5. *If* $f \in C_0^\infty M$ *then*

$$\langle [H, A]f, f\rangle = 2\int_M \operatorname{Hess}\phi(\nabla f, \nabla f) + \int_M (\Delta\phi)_j f_j f - \int_M \phi_i V_i f^2.$$

Proof. One computes the commutator $[H, A] = HA - AH$,

$$[H, A]f = -2\phi_{jk} f_{jk} - 2\phi_{jkk} f_j - \tfrac{1}{2}(\Delta^2\phi) f - \phi_i V_i f.$$

The lemma then follows by partial integration. □

We take $\phi = b^2$ and invoke Properties 2.1 and 2.2. For any $\varepsilon > 0$, there exist a constant $c > 0$ and a compact set K such that, for $f \in C_0^\infty M$,

$$\langle Af, Hf\rangle + \langle Hf, Af\rangle \ge 2\int_M fHf - \varepsilon\int_M f^2 + |\nabla f|^2 - c\int_K f^2 + |\nabla f|^2. \quad (2.6)$$

Let $F = F(\phi) \in C^2M$ be an increasing function of ϕ. Assume that $F \le c_3$ and $|\nabla F| + |\operatorname{Hess} F| \le c_4 b^k$ for some $k > 0$. We will apply (2.6) with $f = e^F u$, where u is the eigenfunction of Theorem 2.3. Although f is no longer compactly supported, the more general use of (2.6) is justified by a standard cutoff function method (the cutoff function depends upon b). Properties 2.1 and Lemma 2.4 are used to remove the error terms in the limit.

Since u is an eigenfunction of H, an elementary calculation gives

$$Hf = \lambda f - 2F_i f_i + |\nabla F|^2 f - (\Delta F) f.$$

We write $\nabla F = w\nabla\phi$ with $w = F'(\phi) > 0$. Assume that $w \in C^2M$ and $|\nabla w| + |\operatorname{Hess} w| \le c_4 b^k$ for some $k > 0$. One observes that

$$Hf = \lambda f + |\nabla F|^2 f - 2wAf - (\nabla\phi \cdot \nabla w) f.$$

Setting $Bf = 2wAf + (\nabla\phi \cdot \nabla w) f$, we have

$$Hf = \lambda f + |\nabla F|^2 f - Bf.$$

Observe that the first order operators A and B are skew adjoint on $C_0^\infty M$.

Using Lemma 2.4 and cutoff functions defined in terms of b (to justify the partial integrations) one finds that

$$\langle Af, Hf\rangle + \langle Hf, Af\rangle = \langle Af, |\nabla F|^2 f - 2wAf - (\nabla\phi \cdot \nabla w) f\rangle$$
$$+ \langle |\nabla F|^2 f - 2wAf - (\nabla\phi \cdot \nabla w) f, Af\rangle.$$

Since $w > 0$, we deduce

$$\langle Af, Hf\rangle + \langle Hf, Af\rangle \leq \langle Af, |\nabla F|^2 f - (\nabla\phi \cdot \nabla w) f\rangle$$
$$+ \langle |\nabla F|^2 f - (\nabla\phi \cdot \nabla w) f, Af\rangle.$$

Using the skew symmetry and definition of A yields

$$\langle Af, Hf\rangle + \langle Hf, Af\rangle \leq \langle f, \nabla\phi(\nabla\phi \cdot \nabla w - |\nabla F|^2) f\rangle.$$

Moreover, since B is skew-symmetric,

$$\langle f, Hf\rangle = \lambda\langle f, f\rangle + \langle |\nabla F|^2 f, f\rangle.$$

Substitution of the last two formulas into (2.6) gives

$$\int_M \nabla\phi(\nabla\phi \cdot \nabla w - |\nabla F|^2) f^2 \geq \lambda \int_M f^2 + \int_M f^2 |\nabla F|^2 - c \int_K f^2 + |\nabla f|^2. \quad (2.7)$$

Here $f = e^F u$ and $Hu = \lambda u$.

To proceed further, we make specific choices for F. These choices are motivated by the proofs required in the rigorous justification of the virial theorem in quantum mechanics [8]. Some care is needed to justify the convergence of the integrals at each stage. Suppose that s and γ are positive constants. We define a function of the real variable t by

$$\chi_s(t) = \int_0^t (1 + s^2 x^2)^{-1}\, dx.$$

Observe that $\chi_s(t) \leq c_s$, where c_s depends only upon s. Moreover, with a constant c independent of s, $|\chi_s'(t)| \leq 1$, $|\chi_s''(t)| \leq c/t$, and $|\chi_s'''(t)| \leq c/t^2$. We apply (2.7) with $F = F_s = \gamma\chi_s((1 + b^2)^{1/2})$.

An elementary calculation yields the formulas

$$\nabla\phi(|\nabla F|^2) = \frac{\gamma^2}{4}\left[\chi_s'\chi_s''(1 + b^2)^{-3/2} - (\chi_s')^2(1 + b^2)^{-2}\right]|\nabla\phi|^4$$
$$+ \frac{\gamma^2}{2}(\chi_s')^2(1 + b^2)^{-1}\,\mathrm{Hess}\,\phi(\nabla\phi, \nabla\phi)$$

and

$$\nabla\phi(\nabla\phi\cdot\nabla w) = \frac{\gamma}{4}\left[\frac{3}{2}\chi_s'(1+b^2)^{-5/2} - \frac{3}{2}\chi_s''(1+b^2)^{-2} + \frac{1}{2}\chi_s'''(1+b^2)^{-3/2}\right]|\nabla\phi|^4 + \frac{\gamma}{2}\left[\chi_s''(1+b^2)^{-1} - \chi_s'(1+b^2)^{-3/2}\right]\operatorname{Hess}\phi(\nabla\phi,\nabla\phi).$$

Consequently, with a constant c_5 independent of s, we have

$$|\nabla\phi(|\nabla F|^2)| \le c_5\gamma^2 \quad \text{and} \quad |\nabla\phi(\nabla\phi\cdot\nabla w)| \le c_5\gamma(1+b^2)^{-1/2}.$$

Substitution in (2.7) gives, with a constant c independent of s and for a compact set K,

$$c(\gamma^2+\gamma)\int_M f^2 \ge \lambda\int_M f^2 - c\int_K f^2 + |\nabla f|^2.$$

If γ is sufficiently small, we get

$$\lambda\int_M f^2 \le 2c\int_K f^2 + |\nabla f|^2. \tag{2.8}$$

We may now deduce the following lemma.

LEMMA 2.9. *If $\gamma > 0$ is sufficiently small, then*

$$\int_M u^2\exp[2\gamma(1+b^2)^{1/2}] < \infty.$$

Proof. For each fixed t, $\lim_{s\to 0}\chi_s(t) = t$. Thus $\lim_{s\to 0} F_s = \gamma(1+b^2)^{1/2}$. The lemma follows because the constant c in (2.8) is independent of s and since $f = \exp[F_s]u$. □

By analogy with Lemma 2.4, we want to improve Lemma 2.9 by showing that $|\nabla u|$ also lies in an exponentially weighted L^2 space. This improvement is provided by our next lemma.

LEMMA 2.10. *If $Hu = \lambda u$ and $u\exp[\alpha(1+b^2)^{1/2}] \in L^2M$ for some $\alpha > 0$, then $|\nabla u|\exp[\alpha(1+b^2)^{1/2}] \in L^2M$.*

Proof. Let $f = e^F u$ with $F = \alpha(1+b^2)^{1/2}$. As before, one verifies that $Hf = \lambda f + |\nabla F|^2 f - Bf$. Here B is the skew-symmetric operator given by $Bf = 2F_i f_i + (\Delta F)f$. If $\omega = \omega(b)$ is a standard cutoff function, then

$$\langle\nabla f, \nabla(\omega^2 f)\rangle + \langle Vf, \omega^2 f\rangle = \langle Hf, \omega^2 f\rangle = \lambda\langle f, \omega^2 f\rangle + \langle f, \omega^2|\nabla F|^2 f\rangle - \langle Bf, \omega^2 f\rangle.$$

However, $\langle Bf, \omega^2 f\rangle = -\langle f, \omega^2 Bf\rangle - 2\langle f, F_i(\omega^2)_i f\rangle$. Thus,

$$\langle\nabla f, \omega^2\nabla f\rangle + \langle\nabla f, f\nabla\omega^2\rangle + \langle Vf, \omega^2 f\rangle = \lambda\langle f, \omega^2 f\rangle + \langle f, \omega^2|\nabla F|^2 f\rangle + \langle f, F_i(\omega^2)_i f\rangle.$$

Since $|\nabla F|$ is bounded, the lemma now follows by letting $\omega \uparrow 1$. □

Now let $\alpha_0 = \sup\{\alpha \mid \exp[\alpha(1+b^2)^{1/2}]u \in L^2M\}$. We plan to show that $\alpha_0 = \infty$. In order to argue by contradiction, we suppose that $\alpha_0 < \infty$. Choose $\alpha_1 > 0$ and $\gamma > 0$ with $\alpha_1 < \alpha_0 < \alpha_1 + \gamma$. Our strategy is to rework the argument leading to Lemma 2.9, starting from (2.7), but with a different choice for F. Let $F = \alpha_1(1+b^2)^{1/2} + \gamma\chi_s((1+b^2)^{1/2})$. Although F is now unbounded, the definition of α_1 and Lemma 2.10 suffice to justify the partial integrations.

Straightforward calculations give

$$\begin{aligned}\nabla\phi(|\nabla F|^2) = {}& \frac{\gamma}{4}(\alpha_1 + \gamma\chi')\chi''(1+b^2)^{-3/2}|\nabla\phi|^4 \\ & - \frac{1}{4}(\alpha_1+\gamma\chi')^2(1+b^2)^{-2}|\nabla\phi|^4 \\ & + \frac{1}{2}(\alpha_1+\gamma\chi')^2(1+b^2)^{-1}\operatorname{Hess}\phi(\nabla\phi,\nabla\phi)\end{aligned}$$

and

$$\begin{aligned}&\nabla\phi(\nabla\phi\cdot\nabla w) \\ &\quad = \frac{1}{4}\Big[\frac{3}{2}(\alpha_1+\gamma\chi')(1+b^2)^{-5/2}|\nabla\phi|^4 \\ &\qquad - \frac{3\gamma}{2}\chi''(1+b^2)^{-2}|\nabla\phi|^4 + \frac{1}{2}\gamma\chi'''(1+b^2)^{-3/2}|\nabla\phi|^4\Big] \\ &\qquad + \frac{1}{2}\big[\gamma\chi''(1+b^2)^{-1} - (\alpha_1+\gamma\chi')(1+b^2)^{-3/2}\big]\operatorname{Hess}\phi(\nabla\phi,\nabla\phi).\end{aligned}$$

If γ is sufficiently small and with c_6 independent of s, by Properties 2.1, we thus have

$$|\nabla\phi(|\nabla F|^2)| \le \varepsilon(b)\alpha_1^2 + c_6\alpha_1\gamma,$$

$$|\nabla\phi(\nabla\phi\cdot\nabla w)| \le c_6\alpha_1(1+b^2)^{-1/2}.$$

Here $\varepsilon(b) \to 0$ as $b \to \infty$.

Using (2.7), we deduce that there is a constant c (independent of s) and a compact set K such that

$$\int_M f^2 \le c\int_K f^2 + |\nabla f|^2.$$

Letting $s \downarrow 0$, one deduces that $u\exp((\alpha_1+\gamma)(1+b^2)^{1/2}) \in L^2M$. This contradiction shows that α_0 is infinite. We have established the following.

LEMMA 2.11. *For all* $\alpha > 0$,

$$\int_M [u^2 + |\nabla u|^2]\exp[2\alpha(1+b^2)^{1/2}] < \infty.$$

One more application of formula (2.7) is needed. This time we choose $F = \alpha(1+b^2)^{1/2}$. Observe that

$$|\nabla F|^2 = \tfrac{1}{4}\alpha^2(1+b^2)^{-1}|\nabla\phi|^2 \ge (1-\varepsilon(b))\alpha^2$$

with $\varepsilon(b) \to 0$ as $b \to \infty$. The estimates before Lemma 2.11 hold with $\alpha = \alpha_1$ and $\gamma = 0$.

In our previous argument, the second term on the right-hand side of the inequality (2.7) was dropped. This term is now used to strengthen our result. One has

$$\int_M \varepsilon(b)(\alpha^2 + c\alpha) f^2 \geq \int_M \lambda f^2 + \int_M (1 - \varepsilon(b))\alpha^2 f^2 - c \int_K f^2 + |\nabla f|^2.$$

Moreover, if α is sufficiently large and K is a sufficiently large compact set, then

$$\lambda \int_M f^2 \leq c(1 + \alpha^2) \int_K f^2 + |\nabla f|^2$$

with c and K independent of α. By Lemma 2.11, we may take α to be arbitrarily large. Since $f = u \exp[\alpha(1 + b^2)^{1/2}]$, this forces $u \equiv 0$ outside a compact set. By unique continuation, for second-order elliptic equations we have $u \equiv 0$ on all of M. This completes the proof of Theorem 2.3. □

The following corollary concerns an interesting class of examples of manifolds M where Theorem 2.3 is applicable.

COROLLARY 2.12. *Suppose that M^n is a complete connected Riemannian manifold satisfying, for $n \geq 3$:*

(i) $\mathrm{Ricci}(M) \geq 0$, *the Ricci curvature of M is nonnegative*;

(ii) $\mathrm{Vol}\, B_p(t) \geq ct^n$, *geodesic balls have Euclidean volume growth*; *and*

(iii) $|K| \leq cr^{-2}$, *sectional curvature decays quadratically.*

If V satisfies Properties 2.2, then $-\Delta + V$ has no positive eigenvalues.

Proof. The required exhaustion function b was constructed by Cheeger and Colding [3] and Colding and Minicozzi [4]. Properties 2.1(i)–(iii) are stated explicitly in [4, p. 28]. For (iv), recall that $b\Delta b = (n-1)|\nabla b|^2$. Consequently,

$$\Delta b^2 = 2n|\nabla b|^2 = \tfrac{1}{2} n b^{-2} |\nabla b^2|^2$$

and

$$\nabla \Delta b^2 = -nb^{-3}|\nabla b^2|^2 \nabla b + nb^{-2}\, \mathrm{Hess}\, b^2 \cdot \nabla b^2.$$

Thus $|\nabla \Delta b^2| \leq cb^{-1}$ and Property 2.1(iv) holds. □

REMARK. The proof of Theorem 2.3 may readily be modified to yield a more general result. Suppose only that $-\Delta u + Vu = \lambda u$ holds in the complement $M - K$ of a compact set K. If $M - K$ has no bounded components, then we conclude that $u \equiv 0$ in $M - K$.

3. Absence of Singular Continuous Spectrum

We proceed to establish the absence of a singular continuous spectrum for certain asymptotically Euclidean spaces. The result will follow by application of the abstract Mourre theory of [2]. Our argument requires the following strengthened version of Properties 2.1 concerning our proper C^2 exhaustion function b. Suppose there exists an $\varepsilon > 0$ such that, in the complement of a compact set, one has the following.

PROPERTIES 3.1.

(i) $c_1 r \le b \le c_2 r$ for some positive constants c_1 and c_2.
(ii) $||\nabla b| - 1| \le cb^{-\varepsilon}$.
(iii) $|\text{Hess}\, b^2 - 2g| \le cb^{-\varepsilon}$.
(iv) $|(b^2)_{kks}| + |(b^2)_{skk}| \le cb^{-\varepsilon}$.

Here c is a positive constant. We sum over the repeated index k.

Set $\phi = b^2$ and $X = \nabla\phi$. The operator $\bar{A} = -i(X + \frac{1}{2}\operatorname{div} X)$ is symmetric on $C_0^\infty M$. Moreover, if T_t is the one-parameter group generated by X then $i\bar{A}$ is the infinitesmal generator of the unitary one-parameter group on $L^2 M$,

$$U_t f(x) = \exp\left[\int_0^t \frac{\operatorname{div} X}{2}(T_s x)\, ds\right] f(T_t x).$$

By Stone's theorem (see [8]), A is essentially self-adjoint.

Suppose $H = -\Delta + V$ is the Schrödinger operator on M, and assume that V is bounded and smooth. The domain of H is the second Sobolev space $H_2 = \{ f \in L^2 M \mid \Delta f \in L^2 M \}$ (see [7]). The potential function V will be required to obey the following stronger version of Properties 2.2.

PROPERTIES 3.2.

(i) $|V| \le cb^{-\varepsilon}$.
(ii) $|XV| \le cb^{-\varepsilon}$.

Let the first Sobolev space be denoted by $H_1 = \{ f \in L^2 M \mid |\nabla f| \in L^2 M \}$. The symbol S will stand for the commutator $S = [H, i\bar{A}]$. A prerequisite for the Mourre theory is the next lemma.

LEMMA 3.3. *S is a bounded operator from H_1 to H_{-1}.*

Proof. One computes the bracket

$$Sf = [H, i\bar{A}]f = -2\phi_{kj} f_{kj} - 2\phi_{jkk} f_j - \tfrac{1}{2}(\Delta^2\phi) f - XVf,$$

where the subscripts denote covariant derivatives and repeated indices are contracted. If $f \in C^2 M$ then these are classical derivatives, but for $f \in H_1$ the derivatives may be interpreted in the distribution sense.

Suppose that $f, g \in H_1$. Let $\|f\|$ denote the norm of f in H_1, that is,

$$\|f\|^2 = \int_M f^2 + |\nabla f|^2.$$

We consider the four terms in the pairing $\langle Sf, g\rangle$. The second and fourth terms are clearly bounded. Moreover,

$$\int \phi_{kj} f_{kj} g = -\int \phi_{kjj} f_k g - \int \phi_{kj} f_k g_j,$$

$$\int \phi_{iijj} fg = -\int \phi_{iij} f_j g - \int \phi_{iij} f g_j.$$

Hence the bracket S extends from $C_0^\infty M$ to a bounded operator $S\colon H_1 \to H_{-1}$. □

Let $\psi \in C^\infty(R^+)$, with $\psi(x) = 0$ in a neighborhood of $x = 0$ and with $\psi(x) = 1$ in a neighborhood of $x = \infty$. Let $\|S\|$ denote the norm of S as an operator from H_1 to H_{-1}. Lemma 3.3 may be improved as follows.

LEMMA 3.4. *We may write $S = S_1 + S_2$, where the decomposition satisfies*

(i) $\|S_1\| + \|S_2\| \le c$,
(ii) $\|[S_1, i\bar{A}]\| \le c$, *and*
(iii) $\|\psi(b/t)S_2\| \le ct^{-\varepsilon}$

for some $\varepsilon > 0$ and sufficiently large $t \in R^+$.

Proof. Let $S_1 = -4\Delta$ and $S_2 = S + 4\Delta$. Then (i) is immediate from Lemma 3.3. For (ii), we note that $\frac{1}{4}[S_1, i\bar{A}] = [H, i\bar{A}] + XV = S + XV$, where XV is considered as a multiplication operator. Thus (ii) follows from Lemma 3.3 and the boundedness of $|XV|$. The operator S_2 is given by

$$S_2 f = -2(\phi_{kj} - 2g_{jk}) f_{jk} - 2\phi_{jkk} f_j - \tfrac{1}{2}(\Delta^2\phi) f - XVf.$$

Then (iii) follows from Properties 3.1 and 3.2, using the method of Lemma 3.3. □

The main result of this section is our next theorem.

THEOREM 3.5. *Suppose that $H = -\Delta + V$ is a Schrödinger operator for the complete Riemannian manifold M. Assume that M admits an exhaustion function b with the Properties 3.1. If the potential V satisfies Properties 3.2, then H has no singular continuous spectrum.*

Proof. Given the foregoing preliminaries and the abstract theory of [2], it suffices to establish a Mourre inequality. Let χ denote the characteristic function of a closed bounded interval on the positive real line. We need to show that, for some $\alpha > 0$ and compact operator C, one has

$$\chi(H)S\chi(H) \ge \alpha\chi^2(H) + C. \tag{3.6}$$

If $f \in C_0^\infty M$, then partial integration gives

$$\int_M fSf = 2\int_M \operatorname{Hess}\phi(\nabla f, \nabla f) + \int_M (\Delta\phi)_j f f_j - \int_M XVf^2.$$

By a standard cutoff function argument, the same formula holds for

$$f \in \chi(H)L_2M.$$

If K is a sufficiently large compact set, then Properties 3.1 and 3.2 yield

$$\int_M fSf \geq 2\int_M |\nabla f|^2 - \varepsilon \int_M f^2 + |\nabla f|^2 - c\int_K f^2 + |\nabla f|^2.$$

Assume that $2\alpha = \inf \operatorname{support}(\chi) > 0$ and $f \in \chi(H)L_2M$. Then

$$\int_M fHf = \int_M |\nabla f|^2 + \int_M Vf^2,$$

$$\int_M fHf \geq 2\alpha \int_M f^2.$$

Combining these formulas gives

$$\int_M fSf \geq \alpha \int_M f^2 - c\int_K f^2 + |\nabla f|^2.$$

The required estimate (3.6) now follows from the Rellich embedding lemma for Sobolev spaces on compact sets. □

The next corollary gives an interesting class of examples where Theorem 3.5 is applicable.

COROLLARY 3.7. *Suppose that M^n is a complete Riemannian manifold satisfying, for $n \geq 3$:*

(i) $\operatorname{Vol} B_p(t) \geq ct^n$, *geodesic balls have Euclidean volume growth*; *and*
(ii) $|K| \leq cr^{-2-\varepsilon}$, *the sectional curvature decays faster than quadratically.*

Here r is the geodesic distance from the basepoint p.

Assume that V obeys Properties 3.2. Then the Schrödinger operator $-\Delta + V$ has no singular continuous spectrum.

Proof. For these spaces, there is a compact set K such that $M - K$ is diffeomorphic to a quotient of $R^n - B_0(t)$ by a finite subgroup of $O(n)$. Moreover, there exist harmonic coordinates on a neighborhood of infinity satisfying the estimates

$$g_{ij} = \delta_{ij} + O(|x|^{-\varepsilon}) \quad \text{and} \quad |x|\frac{\partial g_{ij}}{\partial x_k} = O(|x|^{-\varepsilon})$$

for some $\varepsilon > 0$ (see [1]).

We take $\phi = b^2 = \sum x_k^2$. It suffices to verify Properties 3.1. Parts (i) and (ii) are immediate. For (iii), one calculates

$$d\phi = 2\sum x_k dx_k \quad \text{and} \quad \operatorname{Hess}\phi = 2\sum dx_k dx_k + 2\sum x_k \nabla dx_k.$$

The result follows because the Christoffel symbols satisfy $|\Gamma_{ij}^k| = O(|x|^{-1-\varepsilon})$. To establish (iv), we use the harmonicity of the coordinates x_k. Taking the trace of $\operatorname{Hess}\phi$ gives $\Delta\phi = 2g^{kk}$. Thus $|\phi_{kks}| = |d\Delta\phi| = O(|x|^{-1-\varepsilon})$. Since the curvature decay is faster than quadratic, we also have $|\phi_{skk}| = O(|x|^{-1-\varepsilon})$. □

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