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We show that the eigenvalue density of a product $X=X_1X_2\cdots X_M$ of M independent $N\times N$ Gaussian random matrices in the limit $N\to\infty$ is rotationally symmetric in the complex plane and is given by a simple expression $\rho(z,\bar{z})=\frac{1}{M\pi}\sigma^{-2/M}|z|^{-2+(2/M)}$ for $|z|\le\sigma$, and is zero for $|z|>\sigma$. The parameter σ corresponds to the radius of the circular support and is related to the amplitude of the Gaussian fluctuations. This form of the eigenvalue density is highly universal. It is identical for products of Gaussian Hermitian, non-Hermitian, and real or complex random matrices. It does not change even if the matrices in the product are taken from different Gaussian ensembles. We present a self-contained derivation of this result using a planar diagrammatic technique. Additionally, we conjecture that this distribution also holds for any matrices whose elements are independent centered random variables with a finite variance or even more generally for matrices which fulfill Pastur-Lindeberg's condition. We provide a numerical evidence supporting this conjecture.

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I. INTRODUCTION

Initiated by Wigner more than 50 years ago and developed by Dyson, Mehta, and others, random matrix theory (RMT) has been successfully applied to various problems ranging from fundamental physics (for a comprehensive review, see [1]) to engineering and financial applications [2]. One of the reasons of such a wide applicability is the universality of many results predicted by RMT. Let us take as an example the problem addressed by Wigner, which is how to determine the energy spectrum and level spacing distribution of a many-body quantum system. Due to many degrees of freedom and sophisticated nature of interactions one has to turn to a statistical description. However, in contrast to statistical mechanics where one fixes the Hamiltonian and averages over possible states of the system, Wigner proposed to treat the very Hamiltonian as a random operator, which in turn can be represented as a large random matrix. Relevant properties of such a matrix are determined by symmetries of the problem. The great discovery of RMT is that many observables are the same for various statistical ensembles of random matrices.

To illustrate this, let us cite two classical results of RMT. The eigenvalue density of a real symmetric or complex Hermitian $N\times N$ matrix, whose entries in the upper or lower triangle are independent, identically distributed random variables with a finite variance equal to σ^2/N, converges for $N\to\infty$ to a limiting distribution

$$\rho(\lambda)=\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-\lambda^2},\quad \text{for } \lambda\in[-2\sigma,2\sigma], \tag{1}$$

known as Wigner's semicircle distribution, which is one of the best known results of the classical RMT. The class of matrices whose spectrum converges to the limit law (1) is actually much broader and embraces matrices with entries being independent random variables which fulfill Pastur-Lindeberg's condition [3]. This is an example of macroscopic universality of random matrices. In this paper we concentrate on macroscopic properties and do not discuss microscopic properties of eigenvalue statistics.

An analogous formula for a non-Hermitian random matrix, which is another example of a macroscopic law, reads

$$\rho(z,\bar{z})=\begin{cases}\dfrac{1}{\pi\sigma^2} & \text{for } |z|\le\sigma\\ 0 & \text{for } |z|>\sigma,\end{cases} \tag{2}$$

where $z=x+iy$ is a complex number. Distribution (2) is called Girko-Ginibre's distribution. The eigenvalue density has a rotational symmetry in the complex plane and is uniform inside the circle of radius σ. More generally, if a matrix has independent but not identically distributed Hermitian and anti-Hermitian degrees of freedom [4], the limit law (2) assumes an elliptic form,

$$\rho(z,\bar{z})=\begin{cases}\dfrac{1}{(1-\tau^2)\pi\sigma^2} & \text{for } \dfrac{x^2}{\sigma^2(1+\tau)^2}+\dfrac{y^2}{\sigma^2(1-\tau)^2}\le 1\\ 0, & \text{otherwise},\end{cases} \tag{3}$$

where $\sigma^2>0$ is an effective scale parameter and $\tau\in[-1,1]$ is a flatness of the ellipse. For $\tau=0$ one recovers the circular law (2). For $\tau\to\pm 1$ the support of distribution (3) reduces to a cut $[-2\sigma,2\sigma]$ on the real (for $\tau\to 1$) or imaginary (for $\tau\to -1$) axis and the distribution itself reduces to a Wigner law (1), as one can see by projecting the elliptic distribution (3) onto the real (imaginary) axis before taking the limit $\tau\to\pm 1$.

It might be striking that the derivation of the (apparently simple) functional form of $\rho(z,\bar{z})$ for the Girko-Ginibre ensemble is less straightforward than the one for the (more complex) Wigner semicircle law. The reason is that there are many powerful methods invented for Hermitian random matrices: via orthogonal polynomials or Selberg's integral [5],

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supersymmetric method [6], diagrammatic expansion [7], Dyson gas [8], and free random variables [9].

In this paper we would like to present a result for non-Hermitian random matrices which is to a large extent universal, similar to the two classical examples cited above. We shall show that the eigenvalue density $\rho_X(z,\bar{z})$ of a product

$$X = X_1 X_2 \cdots X_M \tag{4}$$

of $M \geq 2$ independent $N \times N$ Gaussian matrices for which $\langle X_{1,ij} \rangle = \cdots = \langle X_{M,ij} \rangle = 0$ and $\langle |X_{1,ij}|^2 \rangle = \sigma_1^2/N, \ldots, \langle |X_{M,ij}|^2 \rangle = \sigma_M^2/N$ for all i,j assumes in the limit of $N \to \infty$ the following form:

$$\rho_X(z,\bar{z}) = \begin{cases} \dfrac{1}{M\pi} \sigma^{-2/M} |z|^{-2+(2/M)} & \text{for } |z| \leq \sigma \\ 0 & \text{for } |z| > \sigma, \end{cases} \tag{5}$$

where the effective scale parameter $\sigma = \sigma_1 \sigma_2 \cdots \sigma_M$. This surprisingly simple formula is the main result of our paper. What is even more surprising is that this formula holds for a product of independent but not identically distributed Gaussian matrices. This means that the individual matrices X_i's in the product may come from different Gaussian ensembles [unitary (GUE), orthogonal (GOE) or various elliptic non-Hermitain matrices] and the eigenvalue density will always be given by Eq. (5). In other words, even if $X_1, \ldots, X_M$ have oblate eigenvalue spectra, with $\tau_1 \neq 0, \ldots, \tau_M \neq 0$, their product will have a rotationally symmetric one. We shall derivate this result with the help of a diagrammatic technique appropriately tailored to non-Hermitian random matrices [10,11] and to products of random matrices [12]. In order to make the paper self-contained we will also give an introduction to the diagrammatic methods (for a brief review, see also [13]).

It is tempting to conjecture that the limit law for the product (5) holds also for a wider class of matrices, including Wigner matrices whose elements are independent identically distributed random variables with a finite variance or, more generally, for matrices which fulfill Pastur-Lindeberg's condition [3]. We will present a numerical support for this conjecture.

The second objective of this paper is to use Eq. (5) in order to verify an interesting conjecture made in Ref. [14] saying that if the eigenvalue density $\rho(x,y)$ of a non-Hermitian matrix X is rotationally symmetric on the complex plane $z=x+iy$, then the marginal distribution $\rho_*(x) = \int dy \rho(x,y)$ obtained by its projection onto the real axis or a projection $\rho_*(y) = \int dx \rho(x,y)$ onto the imaginary axis must be equal to the eigenvalue density of the matrix $(X+X^\dagger)/\sqrt{8}$ or $i(X-X^\dagger)/\sqrt{8}$, respectively, both being Hermitian matrices. If true, this would allow one to calculate $\rho(x,y)$ from $\rho_*(x)$ via the inverse Abel transform. In particular, if one projects the Girko-Ginibre distribution (2) onto the real (or imaginary) axis, one indeed obtains the Wigner semicircle law, $\rho_*(x) = [2/(\pi\sigma^2)]\sqrt{\sigma^2 - x^2}$, which is the same as the eigenvalue density of the matrix $(X+X^\dagger)/\sqrt{8}$ [or $i(X-X^\dagger)/\sqrt{8}$]. In [14] it was checked numerically that the relation seemed to apply also to more complicated ensembles. Here, we shall present a counterexample by showing that the projection of the eigenvalue density of a product AB of two Hermitian matrices A and B which is rotationally symmetric [Eq. (5)] is different from the eigenvalue density of the rescaled anticommutator $(AB+BA)/\sqrt{8}$ and the commutator $i(AB-BA)/\sqrt{8}$; so the conjecture is not true.

II. GENERALITIES

A. Eigenvalue density and the measure

We are interested in the eigenvalue distribution of a random matrix X [Eq. (4)] being a product of M independent $N \times N$ real or complex Gaussian matrices. The eigenvalues $\{\lambda_i\}$ of X are complex since X may in general be non-Hermitian. The eigenvalue distribution is defined by

$$\rho_X(z,\bar{z}) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta^{(2)}(z - \lambda_i) \right\rangle, \tag{6}$$

where $\bar{z}$ denotes complex conjugate of z. The averaging $\langle \cdots \rangle = \int \cdots d\mu(X_1, \ldots, X_M)$ is done with a factorized probability measure, which in the simplest case of identically distributed matrices takes the form

$$d\mu(X_1, \ldots, X_M) \propto \prod_{\mu=1}^{M} e^{-(N\alpha/4)\mathrm{Tr}\, X_\mu X_\mu^\dagger} DX_\mu, \tag{7}$$

where DX_μ denotes a flat measure. This formula applies to four generic cases of X_μ being (a) complex, (b) complex Hermitian, (c) real, and (d) real symmetric matrices. The parameter α is defined as $\alpha = \lim_{N\to\infty} 2N_{dof}/N^2$, where N_{dof} is the number of real degrees of freedom of the matrix X. For case (a) the flat measure is given by $DX_\mu = \Pi_{ij} \mathrm{d}X_{\mu,ij} \mathrm{d}\bar{X}_{\mu,ij}$ or equivalently by $DX_\mu = \Pi_{ij} \mathrm{d}(\mathrm{Re}\, X_{\mu,ij}) \mathrm{d}(\mathrm{Im}\, X_{\mu,ij})$ and $\alpha = 4$; for case (b) $DX_\mu = \Pi_i \mathrm{d}X_{ii} \Pi_{i>j} \mathrm{d}(\mathrm{Re}\, X_{\mu,ij}) \mathrm{d}(\mathrm{Im}\, X_{\mu,ij})$, $\alpha = 2$; for case (c) $DX_\mu = \Pi_{ij} \mathrm{d}X_{\mu,ij}$, $\alpha = 2$; and finally for case (d) $DX_\mu = \Pi_{i \geq j} \mathrm{d}X_{\mu,ij}$, $\alpha = 1$. For cases (c) and (d) the Hermitian conjugate $X_\mu^\dagger$ reduces to the transpose X_μ^T. The proportionality symbol in Eq. (7) means that the measure is displayed without a normalization constant which is fixed by the condition $\int d\mu(X_1, \ldots, X_M) = 1$.

With this choice of α the variance of individual elements $\langle |X_{\mu,ij}|^2 \rangle = 1/N$, so that the scaling parameters $\sigma_1 = \cdots = \sigma_M = 1$ and hence $\sigma = 1$ in Eq. (5). This means that the eigenvalue density of individual matrices X_μ is given by the Girko-Ginibre law (2) for cases (a) and (c) and the Wigner law (1) for cases (b) and (d), in both cases with $\sigma = 1$. For the sake of simplicity we stick to this choice in the rest of the paper. The spectrum for arbitrary $\sigma_1, \ldots, \sigma_M$ can be obtained by a trivial rescaling.

Later on we will also consider a general case of matrices from the elliptic ensemble with the eigenvalue distribution (3) and a product of nonidentically distributed matrices, where $X_1, \ldots, X_M$ belong to different elliptic ensembles.

B. Green's function

We shall follow here the standard strategy of calculating the eigenvalue density of a random matrix by first calculating the Green's function $g(z,\bar{z})$ and then using an exact relation between the eigenvalue density and the Green's function. Let

us recall this relation. Using the following representation of the two-dimensional δ function:

$$\delta^{(2)}(z-\lambda)=\lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon^2}{(|z-\lambda|^2+\epsilon^2)^2}=\lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\partial}{\partial\bar{z}}\left[\frac{\bar{z}-\bar{\lambda}}{|z-\lambda|^2+\epsilon^2}\right], \tag{8}$$

one finds that [4,15–17]

$$\rho_X(z,\bar{z})=\frac{1}{\pi}\frac{\partial g(z,\bar{z})}{\partial\bar{z}}, \tag{9}$$

where

$$g(z,\bar{z})=\lim_{\epsilon\to 0}\left\langle\frac{1}{N}\sum_i^N\frac{\bar{z}-\bar{\lambda}_i}{|z-\lambda_i|^2+\epsilon^2}\right\rangle$$
$$=\lim_{\epsilon\to 0}\left\langle\frac{1}{N}\mathrm{Tr}\frac{\bar{z}\mathbb{1}_N-X^\dagger}{(\bar{z}\mathbb{1}_N-X^\dagger)(z\mathbb{1}_N-X)+\epsilon^2\mathbb{1}_N}\right\rangle, \tag{10}$$

and $\mathbb{1}_N$ is an $N\times N$ identity matrix. As we shall see later, the Green's function can be calculated in the limit $N\to\infty$ using a summation method for planar Feynman diagrams. It is convenient to think of $g(z,\bar{z})$ as a part of a larger object [18], a $2N\times 2N$ matrix G with four $N\times N$ blocks [10,11],

$$G=\begin{pmatrix}G_{zz} & G_{z\bar{z}}\\ G_{\bar{z}z} & G_{\bar{z}\bar{z}}\end{pmatrix}=\lim_{\epsilon\to 0}\left\langle\begin{pmatrix}z\mathbb{1}_N-X & i\epsilon\mathbb{1}_N\\ i\epsilon\mathbb{1}_N & \bar{z}\mathbb{1}_N-X^\dagger\end{pmatrix}^{-1}\right\rangle. \tag{11}$$

Before we continue let us shortly comment on the notation used in the last formula since we will also use it in the remaining part of the paper. The subscripts zz, $z\bar{z}$, $\bar{z}z$, and $\bar{z}\bar{z}$ refer to the position of the $N\times N$ blocks in the corresponding $2N\times 2N$ matrix. In the shorthand notation the arguments $(z,\bar{z})$ of a function defined on the complex plane are skipped, so the correct reading of, for instance, G_{zz} is $G_{zz}=G_{zz}(z,\bar{z})$. We will also use a convention that the normalized trace of an $N\times N$ matrix denoted by a capital letter will be denoted by the corresponding small letter; for instance, $g_{z\bar{z}}=\frac{1}{N}\mathrm{Tr}\,G_{z\bar{z}}$.

Now coming back to the problem, by inverting the matrix in the brackets on the right-hand side in the last equation we can see that the Green's function $g(z,\bar{z})$ is equal to the normalized trace of the upper-left submatrix,

$$g(z,\bar{z})\equiv g_{zz}(z,\bar{z})=\frac{1}{N}\mathrm{Tr}\,G_{zz}(z,\bar{z}). \tag{12}$$

When one calculates the Green's function (10) or the matrix G [Eq. (11)], one has to take the limit $N\to\infty$ first, and only then allow for $\epsilon\to 0$. This comes from the following reasoning. If $\epsilon=0$, for finite N the function in the brackets $\langle\cdots\rangle$ on the right-hand side of Eq. (10) has isolated poles on the complex plane. However, in the limit $N\to\infty$ the poles coalesce and the function becomes nonholomorphic. One cannot then make an analytic continuation of the function from holomorphic to nonholomorphic region, as it is done when calculating G by diagrammatic method which utilizes $O(1/z)$ expansion. A small $\epsilon>0$ is necessary to make G analytical everywhere. If one naively first took the limit $\epsilon\to 0$, and only then the limit $N\to\infty$, the matrix G would become block diagonal: $G_{zz}=\langle(z-X)^{-1}\rangle$, $G_{\bar{z}\bar{z}}^\dagger=\langle(\bar{z}-X^\dagger)^{-1}\rangle$, and $G_{z\bar{z}}=G_{\bar{z}z}=0$. However, we shall see that

$$g_{z\bar{z}}(z,\bar{z})=\lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle\frac{1}{N}\mathrm{Tr}\frac{-i\epsilon\mathbb{1}_N}{(\bar{z}\mathbb{1}_N-X^\dagger)(z\mathbb{1}_N-X)+\epsilon^2\mathbb{1}_N}\right\rangle \tag{13}$$

and $g_{\bar{z}z}(z,\bar{z})$ differ from zero in the nonholomorphic region. In Ref. [10] it was shown that these quantities are related to the statistics of left and right eigenvectors of the non-Hermitian random matrix ensemble.

The quantities $g_{z\bar{z}}=g_{\bar{z}z}$ are purely imaginary, and $\gamma=-g_{z\bar{z}}g_{\bar{z}z}$ is a sort of order parameter for nonholomorphic behavior, which is positive in a region of the complex plane where the Green's function is nonholomorphic. The effect of pole coalescence and the emergence of a nonholomorphic behavior are very similar to the spontaneous breaking of a global symmetry in statistical models. In such systems the symmetry is preserved as long as the system size N is finite. It may, however, get spontaneously broken in the limit $N\to\infty$. Let us take the Ising model as an example. Its Hamiltonian is invariant under a global transformation flipping all spins and hence it has a Z_2 symmetry. As long as the number of spins is finite, the system is Z_2 symmetric and the average magnetization, which is an order parameter, is equal to zero. However, in the thermodynamic limit, which is when the system size becomes infinite, the Z_2 symmetry gets spontaneously broken below a critical temperature and the average magnetization is nonzero. If one first calculated the average magnetization for a finite system and only then took the limit $N\to\infty$, the magnetization would be zero in this limit for all temperatures. To avoid the problem one can introduce a tiny external magnetic field h which weakly breaks the symmetry for finite-size systems. Now, if one first takes the limit $N\to\infty$ and only then $h\to 0$, one will obtain the correct result. In our case, the small parameter ϵ plays an analogous role to h and it guarantees that nonholomorphic contributions will be correctly picked up for $N\to\infty$.

C. Linearization

Let us have a closer look at the function in the brackets in the definition of the Green's function (10). In our original problem the matrix X is a product $X=X_1\cdots X_M$ of random matrices, so it is a nonlinear object from the point of view of the degrees of freedom that one has to average over. As a consequence the diagrammatic method would become very complicated. One can, however, linearize the problem by a trick used in [12] which relies on substituting X by a matrix Y of dimensions $MN\times MN$ which is linear in X_k's and has eigenvalues closely related to those of X. The matrix Y is constructed from X_μ's which are placed in a cyclic positions of a sparse $MN\times MN$ matrix,

$$Y = \begin{pmatrix} 0 & X_1 & & & 0 \\ 0 & 0 & X_2 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & 0 & X_{M-1} \\ X_M & & & & 0 \end{pmatrix}. \tag{14}$$

One can immediately discover a relation between eigenvalues of Y and those of $X=X_1\cdots X_M$ if one calculates the Mth power Y, which gives a block-diagonal matrix

$$Y^M = \begin{pmatrix} Y_1 & & & 0 \\ & Y_2 & & \\ & & \ddots & \\ 0 & & & Y_M \end{pmatrix}, \tag{15}$$

with Y_μ being cyclic permutations of X_μ's, $Y_\mu = X_\mu X_{\mu+1}\cdots X_{\mu+M-1}$ (in the cyclic convention $X_{\mu+M}\equiv X_\mu$, and $X_0\equiv X_M$). It is easy to see that all blocks Y_μ have the same eigenvalues. Indeed, if λ is an eigenvalue of Y_μ to an eigenvector $\vec{v}_\mu$, $Y_\mu\vec{v}_\mu=\lambda\vec{v}_\mu$, it is also an eigenvalue of $Y_{\mu-1}$ to the eigenvector $\vec{v}_{\mu-1}=X_{\mu-1}\vec{v}_\mu$. One can see this by multiplying both sides of $Y_\mu\vec{v}_\mu=\lambda\vec{v}_\mu$ by $X_{\mu-1}$, obtaining $X_{\mu-1}Y_\mu\vec{v}_\mu = \lambda X_{\mu-1}\vec{v}_\mu$, which is equivalent to $Y_{\mu-1}\vec{v}_{\mu-1}=\lambda\vec{v}_{\mu-1}$. In other words, the matrix Y^M has exactly the same eigenvalues as X and each eigenvalue is M-fold degenerated. Eigenvalues of X are thus related to those of Y as $\lambda_X=\lambda_Y^M$. The eigenvalue density $\rho_X(z,\bar{z})$ can be calculated from $\rho_Y(w,\bar{w})$ of Y by changing the variables $z=w^M$,

$$\rho_X(z,\bar{z}) = M\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}}\rho_Y(w,\bar{w}) = \frac{1}{M}|z|^{-2+(2/M)}\rho_Y(w(z),\bar{w}(\bar{z})). \tag{16}$$

The factor M in front of the Jacobian is related to the fact that the transformation $z=w^M$ maps the complex plane M times onto itself. The problem is thus reduced to finding the spectral density of Y, which is linear with respect to $X_1,\ldots,X_M$. The density $\rho_Y(w,\bar{w})$ can be found from the appropriate Green's function. We will show below that $\rho_Y(w,\bar{w})$ is given by a Girko-Ginibre distribution (2), irrespective of M and of $\tau_1,\tau_2,\ldots,\tau_M$. This is a general result. In particular, for $M=2$ the matrix Y [Eq. (14)] has an antidiagonal block structure as chiral Gaussian matrices which have been intensively studied in the context of spectral properties of the Dirac operator in QCD [19]. In this case, the form of the eigenvalue density of Y for circular case ($\tau_1=\tau_2=0$) can be inferred from results presented in [20–22] for complex, quaternion real, and real matrices, respectively.

III. GREEN'S FUNCTION AND PLANAR DIAGRAMS

In this section we recall the diagrammatic technique of calculating the Green's function. We begin with Hermitian matrices and later generalize the method to non-Hermitian ones and eventually to matrices which additionally have a block structure like the matrix Y from the previous section.

Let us make a general comment before we proceed. The diagrammatic method is based on the observation that the Green's function G can be interpreted as a generating function for connected two-point Feynman diagrams. In the limit $N\to\infty$ only planar diagrams contribute to G since nonplanar ones are suppressed by at least a factor $O(1/N)$ [23,24]. In this limit one can write a set of two self-consistent algebraic matrix equations which relate G to a generating function Σ for one-line irreducible diagrams. The equations are shown schematically in Fig. 1 and will be explained later. They can be solved for G. We want to stress that these equations have exactly the same form for Hermitian complex matrices and for matrices with a block structure. They only differ by an algebraic structure reflecting the indexing of the matrices G and Σ.

We finish with a remark that these equations hold for $N\to\infty$. In the context of the discussion about the order of taking the limits in Eq. (13), this means that one can safely set $\epsilon=0$ since the limit $N\to\infty$ has already been taken.

A. Hermitian matrices

We will first demonstrate the diagrammatic technique on the example of Hermitian matrices and derive the Wigner semicircle law (1). Let us assume that $A=A^\dagger$, $A=\{A_{ab}\}$, where $a=1,\ldots,N$ and $b=1,\ldots,N$, are drawn from an ensemble with a probability measure

$$d\mu(A) \propto e^{-(N/2)\mathrm{Tr}\,A^2}DA, \tag{17}$$

where $DA=\Pi_a dA_{aa}\Pi_{a>b}d(\mathrm{Re}\,A_{ab})d(\mathrm{Im}\,A_{ab})$. The normalization constant, which is implicit in the above formula, is fixed by the condition $\int d\mu(A)=1$. The eigenvalues λ_i of the matrix A are real. This makes the situation simpler than the one for general non-Hermitian matrices discussed in Sec. II. The eigenvalue density can be expressed as [1]

$$\rho(\lambda) = \left\langle \frac{1}{N}\sum_i^N \delta(\lambda-\lambda_i)\right\rangle, \tag{18}$$

where now the δ function is one dimensional. Also the Green's function G matrix takes a simpler form,

$$G = \langle (Z-A)^{-1}\rangle \equiv \int (Z-A)^{-1}d\mu(A). \tag{19}$$

Here, $Z=z\mathbb{1}_N$, where z is a complex number. The Green's function $g(z)\equiv\frac{1}{N}\mathrm{Tr}\,G(z)$ is obtained by the Stieltjes transform of the eigenvalue density,

$$g(z) = \int d\lambda\frac{\rho(\lambda)}{z-\lambda}. \tag{20}$$

The last equation yields

$$\rho(\lambda) = -\frac{1}{\pi}\mathrm{Im}\,g(\lambda+i\epsilon), \tag{21}$$

for $\epsilon\to 0$, as follows from a standard representation of the one-dimensional δ function $\delta(x)=-\frac{1}{\pi}\mathrm{Im}(x+i\epsilon)^{-1}$. The above Green's function can be calculated analytically in the large N limit, expanding Eq. (19) in terms of powers of Z^{-1} as follows:

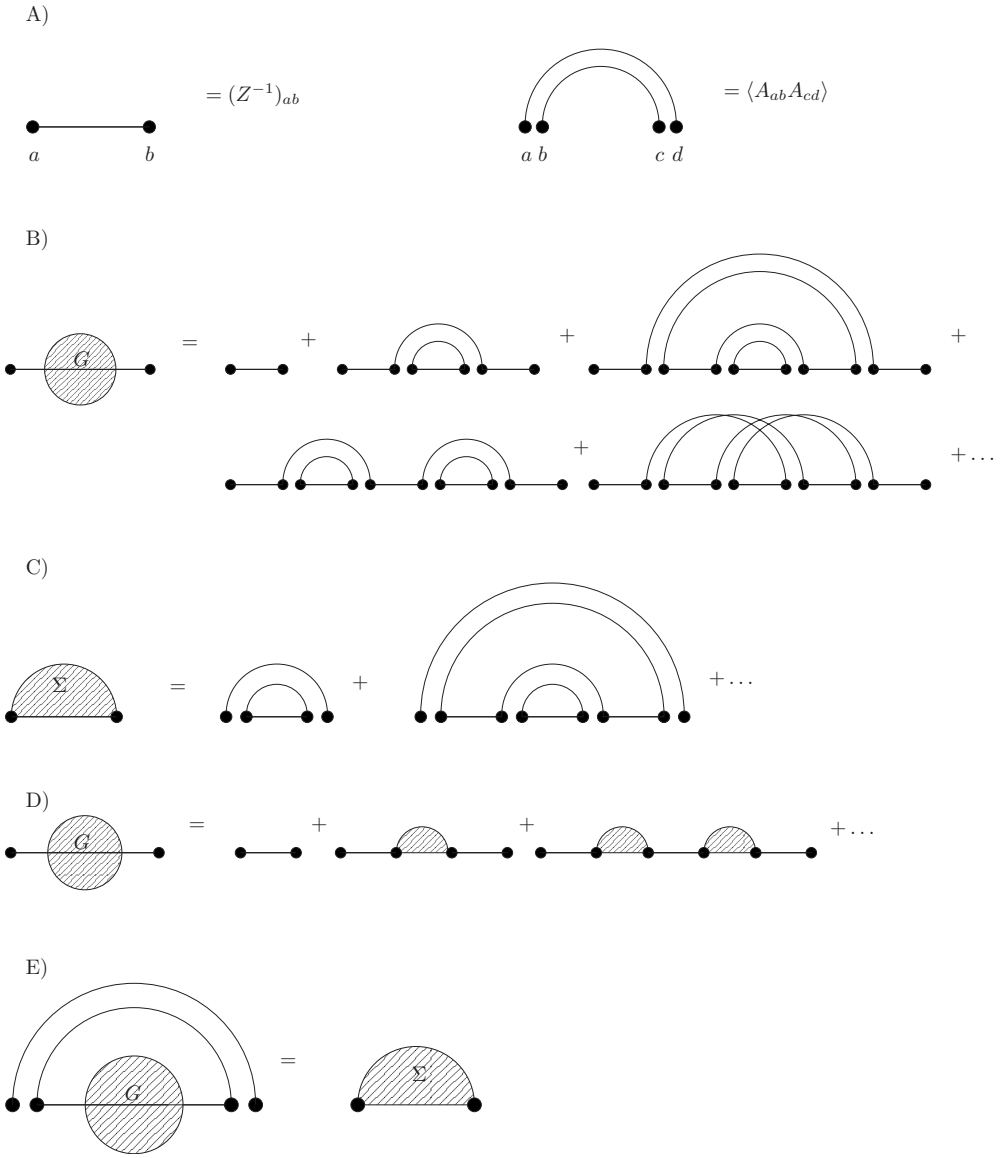

FIG. 1. (A) Feynman rules. $(Z^{-1})_{ab}$ is drawn as a line between a and b, and the propagator $\langle A_{ab}A_{cd}\rangle$ is drawn as a double arc joining a with d and b with c, respectively. (B) Graphical representation of Eq. (22). The last three displayed graphs correspond to the third term in Eq. (22). The contribution of the last diagram can be neglected in the large N limit since it is nonplanar and has a suppressing factor $1/N^2$. (C) Definition of self-energy Σ. (D) The first Dyson-Schwinger equation which relates G to Σ. (E) The second Dyson-Schwinger equation.

$$G(z) = Z^{-1} + \langle Z^{-1}AZ^{-1}AZ^{-1}\rangle + \langle Z^{-1}AZ^{-1}AZ^{-1}AZ^{-1}AZ^{-1}\rangle + \cdots . \tag{22}$$

The factors Z^{-1}'s are independent of A's and thus can be pulled out of the average brackets. What remains are correlation functions of the type $\langle A_{i_1 i_2}\cdots A_{i_{2n-1}i_{2n}}\rangle$ which by virtue of the Wick theorem can be expressed as products of two-point correlation functions (propagators),

$$\langle A_{ab}A_{cd}\rangle = \frac{1}{N}\delta_{ad}\delta_{bc}. \tag{23}$$

This observation allows one to graphically represent Eq. (22) as a sum over Feynman diagrams (see, for instance, [25]), as shown in Fig. 1(B). Each propagator is represented as a double arc joining two pairs of matrix indices, while Z^1_{ab} is drawn as a horizontal line joining indices a and b [Fig. 1(A)]. In order to calculate G_{ab} one has to sum up contributions of all connected diagrams with two external points a,b. For finite N this is not an easy task because there are infinitely many diagrams. The problem enormously simplifies in the limit $N\to\infty$ since in this limit only planar diagrams contribute to the leading term of $1/N$ expansion and all nonplanar diagrams can be neglected [23,24]. It turns out that all planar diagrams can be summed up using an old trick known from field theory which reduces the problem to a closed set of equations for G. These equations are known as Dyson-Schwinger equations and we will discuss them now.

First, we introduce a generating function Σ for one-line irreducible diagrams, that is, diagrams which cannot be split by cutting a single horizontal line [see Fig. 1(C)]. Σ_{ab} generates all one-line irreducible diagrams with vertices a and b.

The two generating functions are related to each other because any diagram from G can be constructed as a sandwich of horizontal lines and one-line irreducible diagrams [Fig. 1(D)],

$$G = Z^{-1} + Z^{-1}\Sigma Z^{-1} + Z^{-1}\Sigma Z^{-1}\Sigma Z^{-1} + \cdots = (Z-\Sigma)^{-1}. \tag{24}$$

This matrix equation can be viewed as a definition of Σ. The introduction of Σ itself does not help one to solve the problem. However, one can write down an independent equation for Σ and G. It follows from the observation that any one-line irreducible diagram can be obtained from a diagram from G by adding an arc (a propagator) to it [Fig. 1(E)]. This gives

$$\Sigma_{ab} = \sum_{c,d} G_{cd}\frac{1}{N}\delta_{cd}\delta_{ab} = g\,\delta_{ab}, \tag{25}$$

or, in matrix notation, $\Sigma = g\mathbb{1}_N$. Taking the trace of both sides we obtain $\sigma = g$, where $\sigma \equiv \frac{1}{N}\mathrm{Tr}\,\Sigma$ is the normalized trace of Σ. The two Eqs. (24) and (25) form a closed set of equations which can be solved for the Green's function $g(z)$. Inserting the last equation to Eq. (24) with $Z = z\mathbb{1}_N$ we have $g(z-g) = 1$ and hence $g(z) = \frac{1}{2}(z - \sqrt{z^2-4})$ and $\rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}$, as follows from Eq. (21).

B. Complex matrices

Let us now discuss how to calculate the Green's function in case of non-Hermitian Gaussian random matrices with complex entries (see, for instance, [13]). The probability measure is now

$$d\mu(A) \propto e^{-N\,\mathrm{Tr}\,AA^\dagger}\prod_{i,j}\mathrm{d}(\mathrm{Re}\,A_{ij})\mathrm{d}(\mathrm{Im}\,A_{ij}), \tag{26}$$

which corresponds to $\alpha = 4$ in Eq. (7). The propagators are

$$\langle A_{ab}A_{cd}\rangle = 0, \quad \langle A_{ab}A^\dagger_{cd}\rangle = \frac{1}{N}\delta_{ad}\delta_{bc},$$

$$\langle A^\dagger_{ab}A_{cd}\rangle = \frac{1}{N}\delta_{ad}\delta_{bc}, \quad \langle A^\dagger_{ab}A^\dagger_{cd}\rangle = 0. \tag{27}$$

It is convenient to think of A and $A^\dagger$ as $N\times N$ submatrices of a $2N\times 2N$ matrix,

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{zz} & \mathcal{A}_{z\bar{z}} \\ \mathcal{A}_{\bar{z}z} & \mathcal{A}_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^\dagger \end{pmatrix}. \tag{28}$$

The off-diagonal blocks are equal to zero for this particular matrix. We use a convention discussed in Sec. II: the position of an $N\times N$ submatrix is denoted by subscripts $z,\bar{z}$. We apply the same notation to other $2N\times 2N$ matrices: the Green's function, the self-energy Σ, and the matrix Z,

$$G = \begin{pmatrix} G_{zz} & G_{z\bar{z}} \\ G_{\bar{z}z} & G_{\bar{z}\bar{z}} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{zz} & \Sigma_{z\bar{z}} \\ \Sigma_{\bar{z}z} & \Sigma_{\bar{z}\bar{z}} \end{pmatrix}, \quad Z = \begin{pmatrix} Z_{zz} & Z_{z\bar{z}} \\ Z_{\bar{z}z} & Z_{\bar{z}\bar{z}} \end{pmatrix}. \tag{29}$$

Matrix elements of the block G_{zz} of G will be denoted by G_{ab}, elements of $G_{z\bar{z}}$ by $G_{a\bar{b}}$, etc. In other words, the subscripts z and $\bar{z}$ serve also as templates for the corresponding barred or unbarred indices. For completeness let us rewrite the propagators (27) using this notation,

$$\langle \mathcal{A}_{ab}\mathcal{A}_{cd}\rangle = 0, \quad \langle \mathcal{A}_{ab}\mathcal{A}_{\bar{c}\bar{d}}\rangle = \frac{1}{N}\delta_{a\bar{d}}\delta_{b\bar{c}},$$

$$\langle \mathcal{A}_{\bar{a}\bar{b}}\mathcal{A}_{cd}\rangle = \frac{1}{N}\delta_{\bar{a}d}\delta_{\bar{b}c}, \quad \langle \mathcal{A}_{\bar{a}\bar{b}}\mathcal{A}_{\bar{c}\bar{d}}\rangle = 0. \tag{30}$$

Now we are ready to write down the Dyson-Schwinger equations for complex matrices. The first equation is identical to Eq. (24), except that now G, Σ, and Z have dimensions $2N\times 2N$,

$$\begin{pmatrix} G_{zz} & G_{z\bar{z}} \\ G_{\bar{z}z} & G_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} Z_{zz}-\Sigma_{zz} & Z_{z\bar{z}}-\Sigma_{z\bar{z}} \\ Z_{\bar{z}z}-\Sigma_{\bar{z}z} & Z_{\bar{z}\bar{z}}-\Sigma_{\bar{z}\bar{z}} \end{pmatrix}^{-1}. \tag{31}$$

This equation is general, but later we will write it for a specific form of Z relevant for the calculation of the eigenvalue density. The second equation, which corresponds to Eq. (25), can be derived using the propagators defined in Eq. (30). It can be done separately in each of sectors zz, $z\bar{z}$, $\bar{z}z$, and $\bar{z}\bar{z}$ as follows:

$$\Sigma_{ad} = 0, \quad \Sigma_{a\bar{d}} = \frac{1}{N}\delta_{a\bar{d}}\delta_{b\bar{c}}G_{b\bar{c}} = \delta_{a\bar{d}}g_{z\bar{z}},$$

$$\Sigma_{\bar{a}d} = \frac{1}{N}\delta_{\bar{a}d}\delta_{\bar{b}c}G_{\bar{b}c} = \delta_{\bar{a}d}g_{\bar{z}z}, \quad \Sigma_{\bar{a}\bar{d}} = 0, \tag{32}$$

where $g_{z\bar{z}} = \frac{1}{N}\mathrm{Tr}\,G_{z\bar{z}}$ and $g_{\bar{z}z} = \frac{1}{N}\mathrm{Tr}\,G_{\bar{z}z}$. In matrix notation the last equation can be written as

$$\begin{pmatrix} \Sigma_{zz} & \Sigma_{z\bar{z}} \\ \Sigma_{\bar{z}z} & \Sigma_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & g_{z\bar{z}}\mathbb{1}_N \\ g_{\bar{z}z}\mathbb{1}_N & 0 \end{pmatrix}. \tag{33}$$

One should note that the form of this equation is independent of Z, while the form of the first Dyson-Schwinger equation (31) is independent of the propagator structure. If we insert now

$$Z = \lim_{\epsilon\to 0}\begin{pmatrix} z\mathbb{1}_N & i\epsilon\mathbb{1}_N \\ i\epsilon\mathbb{1}_N & \bar{z}\mathbb{1}_N \end{pmatrix} = \begin{pmatrix} z\mathbb{1}_N & 0 \\ 0 & \bar{z}\mathbb{1}_N \end{pmatrix} \tag{34}$$

to Eq. (31), remembering that we are allowed to take $\epsilon\to 0$ since all above equations are derived for large N and hence the limit $N\to\infty$ has been taken, we eventually obtain a matrix equation

$$\begin{pmatrix} G_{zz} & G_{z\bar{z}} \\ G_{\bar{z}z} & G_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} z\mathbb{1}_N-\Sigma_{zz} & -\Sigma_{z\bar{z}} \\ -\Sigma_{\bar{z}z} & \bar{z}\mathbb{1}_N-\Sigma_{\bar{z}\bar{z}} \end{pmatrix}^{-1}, \tag{35}$$

which together with Eq. (33) forms a closed set of algebraic equations for $G(z,\bar{z})$.

We will now solve this set of equations and then determine $\rho(z,\bar{z})$ using Eq. (9). We first notice that Eq. (33) reduces to a 2×2 matrix equation,

$$\begin{pmatrix} \sigma_{zz} & \sigma_{z\bar{z}} \\ \sigma_{\bar{z}z} & \sigma_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & g_{z\bar{z}} \\ g_{\bar{z}z} & 0 \end{pmatrix}, \tag{36}$$

where, as before, small letters denote the normalized traces of the corresponding blocks; for instance, $\sigma_{zz}=\frac{1}{N}\mathrm{Tr}\,\Sigma_{zz}$. Similarly, Eq. (35) reduces to

$$\begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} z-\sigma_{zz} & -\sigma_{z\bar{z}} \\ -\sigma_{\bar{z}z} & \bar{z}-\sigma_{\bar{z}\bar{z}} \end{pmatrix}^{-1}, \tag{37}$$

which, after eliminating σ's with the help of Eq. (36), leads to

$$\begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} z & -g_{z\bar{z}} \\ -g_{\bar{z}z} & \bar{z} \end{pmatrix}^{-1} = \frac{1}{|z|^2 - g_{z\bar{z}}g_{\bar{z}z}} \begin{pmatrix} \bar{z} & g_{z\bar{z}} \\ g_{\bar{z}z} & z \end{pmatrix}. \tag{38}$$

This equation has two solutions. The first one corresponds to $g_{\bar{z}z}=g_{z\bar{z}}=0$ which gives $g_{zz}=z^{-1}$ and is equivalent to the trivial holomorphic solution and hence must be true for large $|z|$. The second solution corresponds to $|z|^2-g_{z\bar{z}}g_{\bar{z}z}=1$. In this case the off-diagonal blocks are different from zero and $g_{zz}=\bar{z}$. The two solutions match for $|z|^2=1$. Therefore, the first solution holds outside the unit circle and the second one holds inside the circle. Using the Gauss law (9) one finds

$$\rho(z,\bar{z}) = \begin{cases} \dfrac{1}{\pi} & \text{for } |z| \le 1 \\ 0 & \text{for } |z| > 1, \end{cases} \tag{39}$$

which is the celebrated Girko-Ginibre distribution [15,16].

To summarize this part, one can write the closed set of algebraic equations for G and Σ in the large N limit using diagrammatic relations between the generating function for connected two-point planar diagrams (given by G) and the generating function for one-line irreducible two-point planar diagrams (given by the free energy Σ). One can set $\epsilon=0$ in these equations since they are derived already in the limit $N\to\infty$.

C. Complex matrices with a block structure

We are now ready to calculate the Green's function $g_Y(w,\bar{w})$ for the matrix Y [Eq. (14)] which has blocks X_μ being independent complex non-Hermitian Gaussian matrices [12]. The matrix G will be now a $2NM\times 2NM$ matrix having four $NM\times NM$ blocks G_{ww}, $G_{w\bar{w}}$, $G_{\bar{w}w}$, and $G_{\bar{w}\bar{w}}$ which themselves consists of M^2 blocks of size $N\times N$ which we shall denote by $G_{\mu\nu}$, $G_{\mu\bar{\nu}}$, $G_{\bar{\mu}\nu}$, and $G_{\bar{\mu}\bar{\nu}}$, respectively; for instance,

$$G_{w\bar{w}} = \begin{pmatrix} G_{1\bar{1}} & \cdots & G_{1\bar{M}} \\ & \cdots & \\ G_{M\bar{1}} & \cdots & G_{M\bar{M}} \end{pmatrix}. \tag{40}$$

There is an analogous block structure for the matrix Σ. One should distinguish Greek subscripts from Latin subscripts giving the position of the matrix elements within the block. For instance, $\Sigma_{\mu\bar{\nu}}$ is an $N\times N$ submatrix of the block $\Sigma_{w\bar{w}}$ and $(\Sigma_{\mu\bar{\nu}})_{a\bar{b}}$ is an element of this submatrix. In this convention the normalized trace of a block is $\sigma_{\mu\bar{\nu}}=\frac{1}{N}\mathrm{Tr}\,\Sigma_{\mu\bar{\nu}}=\frac{1}{N}\sum_{a=1}^{N}(\Sigma_{\mu\bar{\nu}})_{a\bar{a}}$. One can now repeat the same procedure which we applied to the matrix having a single block and derive exact relations between the generating function G and Σ in the planar limit. The first Dyson-Schwinger equation,

$$\begin{pmatrix} G_{ww} & G_{w\bar{w}} \\ G_{\bar{w}w} & G_{\bar{w}\bar{w}} \end{pmatrix} = \begin{pmatrix} w\mathbb{1}_{NM}-\Sigma_{ww} & -\Sigma_{w\bar{w}} \\ -\Sigma_{\bar{w}w} & \bar{w}\mathbb{1}_{NM}-\Sigma_{\bar{w}\bar{w}} \end{pmatrix}^{-1}, \tag{41}$$

is almost identical to Eq. (35), except that the blocks and the identity matrices are now of dimensions $NM\times NM$. To write the second equation, we need to know the propagators. Let us first define a $2NM\times 2NM$ matrix $\mathcal{Y}$, a counterpart of $\mathcal{A}$ from Eq. (28),

$$\mathcal{Y} = \begin{pmatrix} \mathcal{Y}_{ww} & \mathcal{Y}_{w\bar{w}} \\ \mathcal{Y}_{\bar{w}w} & \mathcal{Y}_{\bar{w}\bar{w}} \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & Y^\dagger \end{pmatrix}, \tag{42}$$

where Y is cyclic as defined in Eq. (14) and $Y^\dagger$ is anticyclic,

$$Y^\dagger = \begin{pmatrix} 0 & & & & X_M^\dagger \\ X_1^\dagger & 0 & & & 0 \\ & \ddots & & & \\ & & X_{M-2}^\dagger & 0 & \\ 0 & & & X_{M-1}^\dagger & 0 \end{pmatrix}. \tag{43}$$

Since the block matrices $\mathcal{Y}_{\mu\mu+1}=X_\mu$ are assumed to be independent of each other, the only nonzero propagators are

$$\langle \mathcal{Y}_{12,ab}\mathcal{Y}_{\bar{2}\bar{1},\bar{c}\bar{d}}\rangle = \langle \mathcal{Y}_{23,ab}\mathcal{Y}_{\bar{3}\bar{2},\bar{c}\bar{d}}\rangle = \cdots = \langle \mathcal{Y}_{M1,ab}\mathcal{Y}_{\bar{1}\bar{M},\bar{c}\bar{d}}\rangle = \frac{1}{N}\delta_{a\bar{d}}\delta_{b\bar{c}}, \tag{44}$$

or in short

$$\langle \mathcal{Y}_{12}\mathcal{Y}_{\bar{2}\bar{1}}\rangle = \langle \mathcal{Y}_{23}\mathcal{Y}_{\bar{3}\bar{2}}\rangle = \cdots = \langle \mathcal{Y}_{M1}\mathcal{Y}_{\bar{1}\bar{M}}\rangle = \mathbb{T}, \tag{45}$$

where the tensor $\mathbb{T}$ has elements $T_{abcd}=\frac{1}{N}\delta_{ab}\delta_{cd}$, with indices corresponding to those of the matrices on the left-hand side. If we now insert these propagators to the second Dyson-Schwinger equation, we obtain

$$\Sigma_{\mu\bar{\mu}} = g_{\mu+1\overline{\mu+1}}\mathbb{1}_N, \tag{46}$$

and $\Sigma_{\mu\bar{\nu}}=\Sigma_{\bar{\mu}\nu}=0$ for $\mu\neq\nu$. The problem is symmetric with respect to permutation of the matrices X_μ, so $g_{1\bar{1}}=\cdots=g_{M\bar{M}}\equiv g_{w\bar{w}}$ in the whole $w\bar{w}$ block and similarly in the $\bar{w}w$ block. Thus, the last equation can be compactly written as

$$\Sigma_{w\bar{w}} = g_{w\bar{w}}\mathbb{1}_{NM}, \quad \Sigma_{\bar{w}w} = g_{\bar{w}w}\mathbb{1}_{NM}, \tag{47}$$

where $\mathbb{1}_{NM}$ is now the identity $NM\times NM$ matrix for the whole block, $g_{w\bar{w}}=\frac{1}{NM}\mathrm{Tr}\,G_{w\bar{w}}$, and $g_{\bar{w}w}=\frac{1}{NM}\mathrm{Tr}\,G_{\bar{w}w}$. Inserting $\Sigma_{ww}=\Sigma_{\bar{w}\bar{w}}=0$ and Eq. (47) to Eq. (41) we see that each block on the right-hand side of Eq. (41) is proportional to the identity matrix. Thus, Eq. (41) reduces to a 2×2 matrix equation for the normalized traces which play the role of proportionality coefficients at the identity matrices,

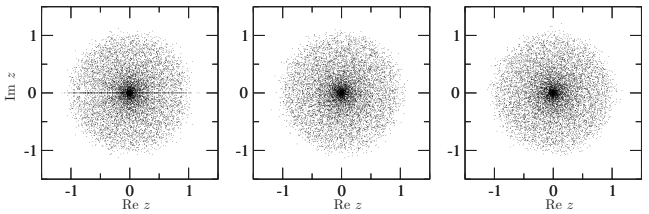

FIG. 2. Plots of $\rho_X(z,\bar{z})$ for X_1, X_2 being two Hermitian matrices (left), two complex matrices (middle), and for X_1 being a Hermitian and X_2 an elliptic random matrix with $\phi=\pi/3$ (right). For each case 100 matrices of size $N=100$ were generated.

$$\begin{pmatrix} g_{ww} & g_{w\bar{w}} \\ g_{\bar{w}w} & g_{\bar{w}\bar{w}} \end{pmatrix} = \begin{pmatrix} w & -g_{w\bar{w}} \\ -g_{\bar{w}w} & \bar{w} \end{pmatrix}^{-1}. \tag{48}$$

This is identical to Eq. (38) for a complex matrix with a single block discussed in the previous section. In other words, the Green's function, and hence also the eigenvalue density of the matrix Y, does not depend on the number of blocks in Y and is given by the Girko-Ginibre law [15,16],

$$\rho_Y(w,\bar{w}) = \begin{cases} \dfrac{1}{\pi} & \text{for } |w| \le 1 \\ 0 & \text{for } |w| > 1. \end{cases} \tag{49}$$

This result is valid also for other matrices considered in Eq. (7), that is, for real nonsymmetric and Hermitian complex matrices, as long as $M>1$. It is so because what matters is the structure of propagators only, which is the same for all mentioned ensembles. In particular, for $M=2$ one can deduce this formula from considerations of chiral ensembles [20–22]. In the next section we shall show how to derive the above result for the product of M elliptic complex and/or real matrices with different oblateness parameters $\tau_1 \neq \cdots \neq \tau_M$. Now we will only observe that by inserting the Girko-Ginibre spectrum into Eq. (16) we finally obtain

$$\rho_X(z,\bar{z}) = \rho_X(|z|) = \begin{cases} \dfrac{1}{M\pi}|z|^{-2+(2/M)} & \text{for } |z| \le 1 \\ 0 & \text{for } |z| > 1, \end{cases} \tag{50}$$

which completes the derivation of our main result. In Figs. 2 and 3 we show a comparison between the above formula and the spectrum of X obtained numerically by the diagonalization of finite matrices. The agreement is very good. For the spectrum of the product of two Hermitian matrices (GUE) shown in the left panel of Fig. 2 we observe a small deviation from rotational symmetry manifesting as an accumulation of eigenvalues along the real axis and a depletion of eigenvalues in a narrow strip close to this axis. The number of eigenvalues on the axis grows as $\sqrt{N}$ and the width of the strip decreases as $1/\sqrt{N}$ when $N\to\infty$. This effect is almost identical to the one known for real Girko-Ginibre matrices [26,27]. If one multiplies three or more GUE matrices the effect disappears. A difference between the product of two and the product of more than two GUE matrices is that for two the trace $\mathrm{Tr}\, X_1X_2$ is real, whereas for three (or more) it is not. In other words, the constraint of the trace to be real

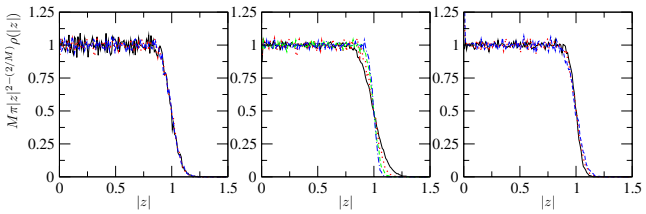

FIG. 3. (Color online) Plots of $M\pi|z|^{2-(2/M)}\rho_X(|z|)$ obtained from simulations for various M and matrix sizes N. The theoretical distribution (not shown in the figure) which corresponds to Eq. (5) is a step function $f(|z|)=1$ for $0<|z|<1$ and zero, otherwise. Left: $X=X_1X_2$ $(M=2)$ for $N=100$ and X_1, X_2 taken from the same ensembles as in Fig. 2: black solid line for Hermitian, red dotted line for complex, and blue dashed line for Hermitian elliptic matrices. Middle: $M=2$, complex matrices of sizes $N=50,100,200,400$ (black solid, red dotted, green dashed, and blue dotted-dashed lines, respectively). To obtain these plots, we averaged spectra of 10 000,1000,1000, and 500 matrices and constructed histograms of absolute values of their eigenvalues. Right: $N=200$ and $M=2,3,4$ (black solid, red dotted, and blue dashed lines). For each M, 1000 matrices were generated.

introduces a weak spherical symmetry breaking of the eigenvalue spectrum.

IV. PRODUCT OF ARBITRARY GAUSSIAN MATRICES (ELLIPTIC ENSEMBLES)

Let us now consider a general class of non-Hermitian random matrices which include as special cases the well-known examples of Hermitian (GUE), Girko-Ginibre, and anti-Hermitian ensembles. These "elliptic" ensembles were first introduced in [4] and can be defined as follows. A complex elliptic matrix X is obtained as a linear combination of two identical independent Hermitian Gaussian matrices A, B: $X=\cos(\phi)A+i\sin(\phi)B$, mixed with an arbitrary real mixing parameter ϕ. Since A and B are independent, the corresponding propagators are $\langle A_{ab}A_{cd}\rangle = \frac{1}{N}\delta_{ad}\delta_{bc}$, $\langle B_{ab}B_{cd}\rangle = \frac{1}{N}\delta_{ad}\delta_{bc}$, and $\langle A_{ab}B_{cd}\rangle = 0$. When one changes variables from A and B to X and $X^\dagger$ one finds

$$\langle X_{ab}X_{cd}\rangle = \langle X^\dagger_{ab}X^\dagger_{cd}\rangle = \tau\frac{1}{N}\delta_{ad}\delta_{bc}, \quad \langle X_{ab}X^\dagger_{cd}\rangle = \langle X^\dagger_{ab}X_{cd}\rangle = \frac{1}{N}\delta_{ad}\delta_{bc}, \tag{51}$$

where $\tau=\cos(2\phi)$. The corresponding integration measure for X reads

$$d\mu(X) \propto \exp\left\{-N\frac{1}{1-\tau^2}\left(\mathrm{Tr}\, XX^\dagger - \tau\frac{1}{2}\mathrm{Tr}(XX + X^\dagger X^\dagger)\right)\right\}\prod_{ij}\mathrm{d}(\mathrm{Re}\, X_{ij})\mathrm{d}(\mathrm{Im}\, X_{ij}). \tag{52}$$

For $\phi=0$ $(\tau=1)$ the matrix X is Hermitian, for $\phi=\pi/2$ $(\tau=-1)$ it is anti-Hermitian, while for $\phi=\pi/4$ $(\tau=0)$ it is isotropic complex.

A. Eigenvalue distribution of a single elliptic random matrix

One can determine the eigenvalue distribution of X using the same methods as in Sec. III B. The only difference is that the propagators $\langle X_{ab}X_{cd}\rangle = \langle X^{\dagger}_{ab}X^{\dagger}_{cd}\rangle$ [Eq. (51)] do not vanish but are proportional to τ. This leads to the following modification of the first Dyson-Schwinger equation (36):

$$\begin{pmatrix} \sigma_{zz} & \sigma_{z\bar{z}} \\ \sigma_{\bar{z}z} & \sigma_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} \tau g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & \tau g_{\bar{z}\bar{z}} \end{pmatrix}, \tag{53}$$

while the second one [Eq. (37)] stays intact,

$$\begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix} = \begin{pmatrix} z - \sigma_{zz} & -\sigma_{z\bar{z}} \\ -\sigma_{\bar{z}z} & \bar{z} - \sigma_{\bar{z}\bar{z}} \end{pmatrix}^{-1}. \tag{54}$$

These equations can be solved for g_{zz}. The solution reads

$$g_{zz} = \begin{cases} \dfrac{\bar{z} - \tau z}{1 - \tau^2} & \text{for } \dfrac{x^2}{(1+\tau)^2} + \dfrac{y^2}{(1-\tau)^2} \le 1 \\ \dfrac{z - \sqrt{z^2 - 4\tau}}{2\tau}, & \text{otherwise,} \end{cases} \tag{55}$$

where $z = x + iy$. The nonholomorphic solution matches the holomorphic one on the ellipse. The eigenvalue density is [4]

$$\rho(z,\bar{z}) = \frac{1}{\pi}\frac{\partial g_{zz}}{\partial \bar{z}} = \begin{cases} \dfrac{1}{\pi(1-\tau^2)} & \text{for } \dfrac{x^2}{(1+\tau)^2} + \dfrac{y^2}{(1-\tau)^2} \le 1 \\ 0, & \text{otherwise.} \end{cases} \tag{56}$$

The parameter τ is a measure of flattening of the ellipse on which $\rho(z,\bar{z}) > 0$. For $\tau = 0$ the last equation reproduces the result for non-Hermitian complex matrices. For $\tau \to 1$, the ellipse reduces to a cut on the real axis. In order to determine the eigenvalue density in this case one should first project the density for $\tau < 1$ onto the real axis, $\rho_*(x) = \int dy \rho(x,y)$, and then take the limit $\tau \to 1$. One recovers the Wigner semicircle law $\rho_*(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$, as expected.

B. Eigenvalue distribution of a product of two or more elliptic random matrices

We are now interested in the eigenvalue density of the product (4) where X_μ's are drawn from a Gaussian ensemble with the measure (52). We shall show that the result is again given by Eq. (5) and hence exhibits a large degree of universality: it does not depend on τ and is exactly the same even if each of the matrices X_μ is drawn from a Gaussian ensemble with a different flattening parameter τ_μ. We will derive Eq. (5) for $X = X_1 X_2$ and then make a comment on the generalization to $M > 2$.

We will use the linearization and calculate first the eigenvalue density of the matrix Y [Eq. (14)] constructed from X_1 and X_2, having the only nonvanishing propagators given by Eq. (51) with two parameters τ_1 and τ_2. As before, first we have to determine the propagator structure for the block matrix $\mathcal{Y}$ [Eq. (42)] and then apply it to derive the Dyson-Schwinger equation. The matrix $\mathcal{Y}$ reads

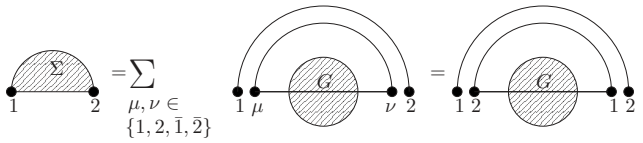

FIG. 4. Example of calculation of σ_{12} in Eq. (61). We write the second Dyson-Schwinger equation for Σ_{12}. The only nonvanishing propagator is the one between indices 1,2 and 1,2. Taking the trace of both sides of the equation we arrive at $\sigma_{12} = \tau_1 g_{21}$.

$$\mathcal{Y} = \begin{pmatrix} Y & 0 \\ 0 & Y^{\dagger} \end{pmatrix} = \begin{pmatrix} 0 & X_1 & 0 & 0 \\ X_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_2^{\dagger} \\ 0 & 0 & X_1^{\dagger} & 0 \end{pmatrix}. \tag{57}$$

The first nonvanishing propagator comes from the correlations between X_μ's and $X_\mu^{\dagger}$'s, exactly as in Eq. (45),

$$\langle \mathcal{Y}_{12}\mathcal{Y}_{\bar{2}\bar{1}}\rangle = \langle \mathcal{Y}_{21}\mathcal{Y}_{\bar{1}\bar{2}}\rangle = \mathbb{T}. \tag{58}$$

The next one comes from autocorrelations of X_μ's [Eq. (51)] which are proportional to τ,

$$\langle \mathcal{Y}_{12}\mathcal{Y}_{12}\rangle = \tau_1 \mathbb{T}, \quad \langle \mathcal{Y}_{21}\mathcal{Y}_{21}\rangle = \tau_2 \mathbb{T}, \tag{59}$$

and the last one comes from autocorrelations of $X_\mu^{\dagger}$'s,

$$\langle \mathcal{Y}_{\bar{1}\bar{2}}\mathcal{Y}_{\bar{1}\bar{2}}\rangle = \tau_1 \mathbb{T}, \quad \langle \mathcal{Y}_{\bar{2}\bar{1}}\mathcal{Y}_{\bar{2}\bar{1}}\rangle = \tau_2 \mathbb{T}. \tag{60}$$

Here, $\mathbb{T}$ denotes again a tensor with elements $T_{abcd} = \frac{1}{N}\delta_{ad}\delta_{bc}$, where a,b are indices of the first matrix and c,d are indices of the second one on the right-hand sides of the above equations. All other correlations between the blocks of $\mathcal{Y}$ vanish. We can now write two Dyson-Schwinger equations:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{1\bar{1}} & \sigma_{1\bar{2}} \\ \sigma_{21} & \sigma_{22} & \sigma_{2\bar{1}} & \sigma_{2\bar{2}} \\ \sigma_{\bar{1}1} & \sigma_{\bar{1}2} & \sigma_{\bar{1}\bar{1}} & \sigma_{\bar{1}\bar{2}} \\ \sigma_{\bar{2}1} & \sigma_{\bar{2}2} & \sigma_{\bar{2}\bar{1}} & \sigma_{\bar{2}\bar{2}} \end{pmatrix} = \begin{pmatrix} 0 & \tau_1 g_{21} & g_{2\bar{2}} & 0 \\ \tau_2 g_{12} & 0 & 0 & g_{1\bar{1}} \\ g_{\bar{2}2} & 0 & 0 & \tau_1 g_{\bar{2}\bar{1}} \\ 0 & g_{\bar{1}1} & \tau_2 g_{\bar{1}2} & 0 \end{pmatrix}, \tag{61}$$

$$\begin{pmatrix} g_{11} & g_{12} & g_{1\bar{1}} & g_{1\bar{2}} \\ g_{21} & g_{22} & g_{2\bar{1}} & g_{2\bar{2}} \\ g_{\bar{1}1} & g_{\bar{1}2} & g_{\bar{1}\bar{1}} & g_{\bar{1}\bar{2}} \\ g_{\bar{2}1} & g_{\bar{2}2} & g_{\bar{2}\bar{1}} & g_{\bar{2}\bar{2}} \end{pmatrix} = \begin{pmatrix} w - \sigma_{11} & -\sigma_{12} & -\sigma_{1\bar{1}} & -\sigma_{1\bar{2}} \\ -\sigma_{21} & w - \sigma_{22} & -\sigma_{2\bar{1}} & -\sigma_{2\bar{2}} \\ -\sigma_{\bar{1}1} & -\sigma_{\bar{1}2} & \bar{w} - \sigma_{\bar{1}\bar{1}} & -\sigma_{\bar{1}\bar{2}} \\ -\sigma_{\bar{2}1} & -\sigma_{\bar{2}2} & -\sigma_{\bar{2}\bar{1}} & \bar{w} - \sigma_{\bar{2}\bar{2}} \end{pmatrix}^{-1}. \tag{62}$$

In the first equation the off-diagonal blocks are the same as in the previous section [Eq. (46)]. The diagonal blocks $\sigma_{ww}, \sigma_{\bar{w}\bar{w}}$ now depend on τ_μ's. As an illustration we show in Fig. 4 a graphical representation of the equation for $\sigma_{12} = \tau_1 g_{21}$ which explains the flip of indices. Let us first look for

a holomorphic solution, so assume that off-diagonal blocks of g vanish: $g_{w\bar{w}}=g_{\bar{w}w}=0$. In this case the above equations reduce to

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} 0 & \tau_1 g_{21} \\ \tau_2 g_{12} & 0 \end{pmatrix}, \quad \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} w-\sigma_{11} & -\sigma_{12} \\ -\sigma_{21} & w-\sigma_{22} \end{pmatrix}^{-1}, \tag{63}$$

with the corresponding equations for $\sigma_{\bar{w}\bar{w}}$ and $g_{\bar{w}\bar{w}}$ being complex conjugate of those above. This gives

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} w & -\tau_1 g_{21} \\ -\tau_2 g_{12} & w \end{pmatrix}^{-1}, \tag{64}$$

which has two solutions: one with $g_{11}=\frac{1}{w}$ and the other one with $g_{11}=w/\sqrt{\tau_1\tau_2}$. We take the first one because it has the correct asymptotic behavior for large w. For this solution we have $g_{22}=\frac{1}{w}$ and $g_{12}=g_{21}=0$. The holomorphic solution has to be sewed with the nonholomorphic, one so that at the boundary $g_{12}=g_{21}=0$. If we assume that these elements vanish also inside the nonholomorphic region (and correspondingly $g_{\bar{1}\bar{2}}=g_{\bar{2}\bar{1}}=0$), then the Eq. (61) reduces to

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{1\bar{1}} & \sigma_{1\bar{2}} \\ \sigma_{21} & \sigma_{22} & \sigma_{2\bar{1}} & \sigma_{2\bar{2}} \\ \sigma_{\bar{1}1} & \sigma_{\bar{1}2} & \sigma_{\bar{1}\bar{1}} & \sigma_{\bar{1}\bar{2}} \\ \sigma_{\bar{2}1} & \sigma_{\bar{2}2} & \sigma_{\bar{2}\bar{1}} & \sigma_{\bar{2}\bar{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & g_{2\bar{2}} & 0 \\ 0 & 0 & 0 & g_{1\bar{1}} \\ g_{\bar{2}2} & 0 & 0 & 0 \\ 0 & g_{\bar{1}1} & 0 & 0 \end{pmatrix}, \tag{65}$$

with vanishing diagonal blocks. This equation is identical to the equation with $\tau_1=\tau_2=0$ and was discussed in the previous section. As we know it gives the Girko-Ginibre distribution for the matrix Y and hence we obtain Eq. (5) for $X=X_1X_2$.

One can repeat the whole reasoning for a product of more than two matrices. One finds again that the solution $1/w$ valid outside the nonholomorphic region corresponds to vanishing blocks $g_{\mu\nu}=g_{\bar{\mu}\bar{\nu}}=0$ for $\mu\neq\nu$, and that it can be sewed with the nonholomorphic solution for which the blocks also vanish. This gives $\sigma_{ww}=\sigma_{\bar{w}\bar{w}}$ and one obtains exactly the same equations as for $\tau_1=\cdots=\tau_M=0$. Therefore, for $M>2$ the eigenvalue distribution of Y is also given by the Girko-Ginibre law. This result is universal: the spectrum of X is given by Eq. (5), independent of whether we multiply two Hermitian matrices, or Hermitian by generic complex, or Hermitian by anti-Hermitian, etc. The limiting spectrum is always the same and differs only by finite-size effects.

One can also extend this result to purely real matrices generated from the ensemble with a measure [4]

$$d\mu(X) \sim \exp\left\{-\frac{N}{2}\frac{1}{1-\tau^2}(\mathrm{Tr}\, XX^T - \tau \mathrm{Tr}\, XX)\right\}\prod_{ij} dX_{ij}. \tag{66}$$

The case $\tau=1$ corresponds to symmetric real matrices, $\tau=-1$ corresponds to antisymmetric ones, and $\tau=0$ corresponds to isotropic real matrices. The diagrammatic equations in the limit $N\to\infty$ are exactly the same as before, because the propagators have the same structure.

V. PROJECTION OF THE SPECTRUM OF A COMMUTATOR OF GUE MATRICES

In this section we show that the conjecture made in [14] is not true. Let us consider a matrix $X=X_1X_2$ which is a product of two Hermitian GUE matrices X_1, X_2. According to the formula (5), the eigenvalue density of X is $\rho_X(z,\bar{z})=\frac{1}{2\pi|z|}$ for $|z|<1$ and zero, otherwise. The projection of this function on the real (or imaginary) axis gives

$$\rho_*(x) = \frac{1}{\pi}\ln\frac{1+\sqrt{1-x^2}}{|x|}, \tag{67}$$

for $-1\le x\le 1$. According to [14], this result should be equal to the eigenvalue density $\rho_+(x)$ of $(X_1X_2+X_2^\dagger X_1^\dagger)/\sqrt{8}$ or $\rho_-(x)$ of $i(X_1X_2-X_2^\dagger X_1^\dagger)/\sqrt{8}$. Up to a scaling factor of $\sqrt{8}$, these spectral densities are equal to the spectra of the anticommutator $\{X_1,X_2\}$ or the commutator $i[X_1,X_2]$, because $X_1=X_1^\dagger$, $X_2=X_2^\dagger$. Moreover, $\rho_-(x)=\rho_+(x)$ as follows from the observation that in the limit $N\to\infty$ all the moments of the commutator and the anticommutator are the same: $\mathrm{Tr}\langle[X_1,X_2]^k\rangle=\mathrm{Tr}\langle\{X_1,X_2\}^k\rangle$ for all $k=1,2,\ldots$.

We calculate now the eigenvalue density $\rho_+(x)$ of the rescaled anticommutator $\{X_1,X_2\}/\sqrt{8}$. We define two matrices $A=(X_1+X_2)/\sqrt{2}$ and $B=(X_1-X_2)/\sqrt{2}$ which are also mutually independent Hermitian matrices with a factorized probability measure

$$d\mu(A,B) \propto e^{-N/2\ \mathrm{Tr}\, A^2} e^{-N/2\ \mathrm{Tr}\, B^2} DADB. \tag{68}$$

We have $\{X_1,X_2\}=A^2-B^2$. One can use the technique of free random variables [28] to calculate the eigenvalue density of A^2-B^2 since in the limit $N\to\infty$ the matrices A^2 and B^2 represent free random variables. The addition law for a sum of free variables is expressed in terms of an R transform or equivalently in terms of a Blue's function $B(z)$, which is a functional inverse of the Green's function $G(B(z))=z$ and takes a simple form $B_{a+b}(z)=B_a(z)+B_b(z)-z^{-1}$, where a and b are free random variables. In our case $a=A^2$, $b=-B^2$. The Green's function G_a of A^2 is a special case of the Green's function for the Wishart distribution, while G_b for $-B^2$ corresponds to a reflected Wishart spectrum $\lambda\to-\lambda$, and hence

$$G_a(z) = \frac{1-\sqrt{1-4/z}}{2}, \quad G_b(z) = -G_a(-z) = \frac{-1+\sqrt{1+4/z}}{2}. \tag{69}$$

The Blue functions for both cases read

$$B_a(z) = \frac{1}{z(1-z)}, \quad B_b(z) = \frac{1}{z(1+z)}, \tag{70}$$

and thus

$$B_{a+b}(z) = B_a + B_b - \frac{1}{z} = \frac{1+z^2}{z(1-z^2)}. \tag{71}$$

This equation has to be inverted for $G_{a+b}(z)$, which is the Green's function for the anticommutator

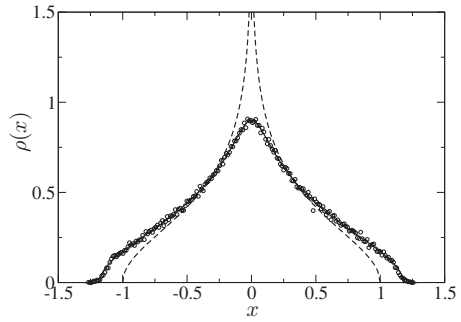

FIG. 5. Comparison between $\rho_+(x)$ from Eq. (74) (solid line), $\rho_*(x)$ from Eq. (67) (dashed line), and numerical simulations (circles) for $N=100$ (1000 matrices were generated).

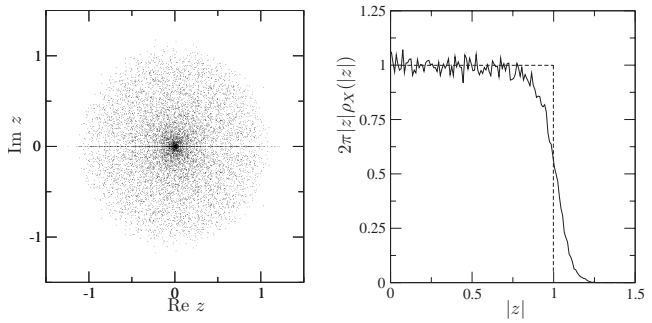

FIG. 6. Plots of numerically obtained $\rho_X(|z|)$ for X_1, X_2 being two symmetric matrices which entries (upper triangle) are taken from uniform distribution $[-\sqrt{3/N}, \sqrt{3/N}]$, for $N=200$ and for 1000 matrices generated. Dashed line shows the theoretical distribution in the limit $N\to\infty$.

$$z = \frac{1 + G_{a+b}(z)^2}{G_{a+b}(z)[1 - G_{a+b}(z)^2]}, \tag{72}$$

which leads to a cubic equation for $G_{a+b}(z)$. The solution which has the correct behavior $G_{a+b}(z)\to 1/z$ for large z reads

$$G_{a+b}(z) = \frac{1 + 3z^2 + (-1 - 18z^2 + 3\sqrt{3}\sqrt{z^2 + 11z^4 - z^6})^{2/3}}{3z(-1 - 18z^2 + 3\sqrt{3}\sqrt{z^2 + 11z^4 - z^6})^{1/3}}. \tag{73}$$

Taking into account the scaling factor of $\sqrt{8}$ we finally arrive at

$$\rho_+(x) = -\frac{\sqrt{8}}{\pi}\mathrm{Im}\, G_{a+b}(x\sqrt{8} + i0+)$$
$$= \frac{\sqrt{3}}{6\pi}\frac{1 + 24x^2 - (1 + 144x^2 - 6\sqrt{6}\sqrt{x^2 + 88x^4 - 64x^6})^{2/3}}{|x|(1 + 144x^2 - 6\sqrt{6}\sqrt{x^2 + 88x^4 - 64x^6})^{1/3}}. \tag{74}$$

This is different from $\rho_*(x)$ of Eq. (67). In Fig. 5 we compare both spectral densities and show also results of numerical simulations which perfectly agree with Eq. (74). This falsifies the conjecture that, if the spectrum of a non-Hermitian matrix is rotationally symmetric, it can be found by solving the symmetrized or antisymmetrized Hermitian problem.

VI. CONCLUSIONS

The main result of this paper is that the eigenvalue density of a product of large centered (with zero mean) Gaussian matrices assumes a very universal form (5) with a single scaling parameter σ representing the radius of a circular support in the complex plane and related to the amplitude of fluctuations of matrix entries. The matrices in the product do not have to be identical and each of them may belong to a different elliptic ensemble.

Taking into account the universality of the Wigner's semicircle law or the Girko-Ginibre distribution for matrices having their entries drawn from independent distributions, it is tempting to conjecture that our result will also hold in this setting. Namely, we suppose that the same asymptotic result holds for products of Wigner matrices having independent elements drawn from any centered distribution which fulfills Pastur-Lindeberg's condition [3]. To assess the validity of this conjecture we performed numerical simulations, assuming various distributions of elements of the matrices. The only requirement was that the variance of the distribution was equal to $1/N$. We did not observe any deviations from Eq. (5) for short-tailed distributions. In Fig. 6 we show an example for a uniform distribution with zero mean and variance $1/N$.

As far as future projects are concerned, it would be interesting to generalize the discussion to the Gaussian symplectic ensemble [21] and to study microscopic properties of eigenvalues of the product of various types of Gaussian matrices from different invariant ensembles [20–22]. It would also be interesting to analytically derive the formula for the eigenvalue distribution of the product of M matrices of finite size N (see Fig. 2 in the middle). For the Girko-Ginibre ensemble [29] it is given by $\rho(z)\cong \mathrm{erfc}[\sqrt{2}(|z| - 1)\sqrt{N}]/(2\pi)$. We expect a qualitatively similar behavior also for the product of matrices.

The discussion presented in this paper holds for Gaussian matrices for which the first moment has zero mean, $\langle \mathrm{Tr}\, X_\mu\rangle = 0$. It would be interesting to check how it changes when $\langle \mathrm{Tr}\, X_\mu\rangle \neq 0$. This could be a step toward a generalization of Voiculescu's S-transform composition rule [30] for calculating the eigenvalue density of asymptotically large matrices representing free random variables, to the case when their product has complex eigenvalues.

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