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Spectrum of the Torsional and Longitudinal Natural Frequencies of Cantilever with an Arbitrary Number of Layers with the Piece wise Constant Mechanical Properties, Densities and Thicknesses

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Abstract

Here, we derived analytically a transcendental equation, which positive roots are the spectrum of torsional and longitudinal natural frequencies of cantilever composed of an arbitrary number of layers with the piecewise constant mechanical properties, densities and thicknesses. This transcendental equation has been obtained in the Laplace image space and the one connects the mechanical properties, density and layer thickness. The present solution can be particularly useful for either a non-destructive material testing or design of the cantilever structures.

Keywords: Torsional and Longitudinal resonant frequencies, Material patterning, Atomic Force Microscopy.

Introduction

The rapid development of new materials allowed researchers to create heterogeneous structures with specific functional properties. Such structures often consist of layers with different thicknesses, cross sectional areas and mechanical properties. To ensure that the prepared material meets required functional properties or to check the structural degradation of the object in adverse environmental conditions, it is necessary to determine the basic mechanical properties such as elastic moduli or thicknesses of individual layers.

The material characterization is performed by using either destructive (DTs) [1] or non-destructive techniques (NDTs) [2]. DTs usually provide a more reliable assessment of the properties of the structure than NDTs. However, the destruction or damage caused by DTs strongly affects the functional quality of the tested structure itself. NDTs allow one to test or to inspect an object without impairing its function. Among NDTs, ultrasonic methods are commonly employed to characterize multilayered structures [3-17]. In NDTs, the required information about the object's elastic moduli and layers' thicknesses is recovered from the knowledge of the spectrum of either resonance peaks or natural frequencies of an inspected object [18-20].

The spectrum of the resonance peaks contains information about the quality factor and the resonance frequencies [21]. The resonance frequencies are used to determine the elastic moduli and layer thicknesses while the quality factors yield information about the energy-attenuating properties of the object. The analysis of experimental data becomes simpler if a sample has the shape of a rod with its length L substantially larger than its radius R ($R/L << 1$). Nevertheless, the interpretation of experimental data still remains quite a complicated problem and is usually based on the simple model of the damped harmonic oscillator [8]. Despite the fact that the harmonic oscillator model correctly catches the qualitative behavior of the resonance and attenuation, the quantitative results are far from the actual values and, as to the spectrum of the natural frequencies it cannot give in principle.

The natural frequencies of a rod with homogeneous and discontinuous properties can be found by solving the partial differential equation and the corresponding Sturm-Liouville problem [21]. However, the solution of the partial differential equations is often a very complicated problem and therefore it is usually obtained by using numerical methods [22,23]. Such methods, of course, can give quite accurate results, but they may not provide an adequate insight into the physics of the problem. Hence, the exact solutions of problems are desirable, even though they are often difficult to obtain.

Throughout the decades, plenty of theoretical works concerning vibrating non-uniform beams or rods have been published [24-32]. A couple of years ago, a study of the vibrating mechanical systems with $N-$ stepwise constant properties has been performed [33,34]. It has been shown that natural frequencies are the roots of the transcendental equation obtained directly from the Laplace imaginary space.

In this paper, we carry out the systematic investigation of longitudinal and torsional vibrations of the rod with homogeneous and discontinuous material or geometric properties. Particular attention is given to a cantilever rod that undergoes an action of periodical external force, where the obtained results can be directly used in many problems ranging from civil or mechanical engineering to material testing and micro/nano-fabrication.

Theoretical ground

Following the approach of Fedorchenko et al. [33,34], the longitudinally oscillating rod with an arbitrary number of material and geometric discontinuities (N) can be described by the following system of the partial differential equations

$$\rho_i u_{itt} - E_i u_{ixx} = 0,\ h_{i-1} < x < h_i,\ i = 1, 2,\ldots, N,\ h_0 = 0,\ h_N = l, \tag{1a}$$

with the following matching conditions

$$u_i(h_i, t) = u_{i+1}(h_{i+1}, t), \tag{1b}$$

$$k_i u_{ix}(h_i, t) = k_{(i+1)} u_{(i+1)x}(h_{i+1}, t), \tag{1c}$$

where $k_i = E_i A_i$, E, ρ and A are the elastic modulus, density and the cross sectional area of rod, respectively. For torsional oscillations of rod with an arbitrary number of discontinuous properties the equation of motion reads

$$\rho_i I_{pi} \theta_{itt} - G_i K_i \theta_{ixx} = 0 \tag{2a}$$

and the following matching conditions are imposed

$$\theta_i(h_i, t) = \theta_{i+1}(h_{i+1}, t), \tag{2b}$$

$$b_i \theta_{ix}(h_i, t) = b_{(i+1)} \theta_{(i+1)x}(h_{i+1}, t), \tag{2c}$$

where $b_i = G_i K_i$, G is shear modulus, K is a geometric function that depends on the cantilever cross section [35] and I_p is the polar moment of inertia of beam.

Longitudinally oscillating cantilever with one discontinuity

The method of solution and further analysis are going to be illustrated on the longitudinally oscillating rod with one discontinuity (see Figure 1). The motion of the one is described by Eq. (1) with $i = 1, 2$. To close the problem, the following initial and boundary conditions are imposed

$$u(x, 0) = u_t(x, 0) = 0, \tag{3a}$$

$$u_1(0, t) = 0,\ u_{2x}(l, t) = \eta_2(t), \tag{3b}$$

where $\eta_2(t) = F/(E_2A_2) \sin pt = I_2 \sin pt$, F is applied external force and p is a simple frequency. Now, applying the Laplace transform to Eqs. (1) and (3) (for method of solution see Refs. [33,34,36], the solutions of $U_i(x, s)$ in the imaginary space yield

$$U_1(x, s) = I_2 p k_2 \sinh(\xi_1 x)/D(s),\ 0 < x < h, \tag{4a}$$

$$U_2(x, s) = (I_2 p/\xi_2)\{k_1\xi_1 \cosh(\xi_1 h) \sinh[\xi_2(x - h)] + k_2\xi_2 \sinh(\xi_1 h) \cosh[\xi_2(x - h)]\}/D(s),\ h < x < l, \tag{4b}$$

where $\xi_i = s/c_i$, $c_i = (E_i/\rho_i)^{1/2}$, $D(s) = (p^2 + s^2)d(s)$ and $d(s) = \{k_1\xi_1 \cosh(\xi_1 h) \cosh[\xi_2(x - h)] + k_2\xi_2 \sinh(\xi_1 h) \sinh[\xi_2(x - h)]\}$.

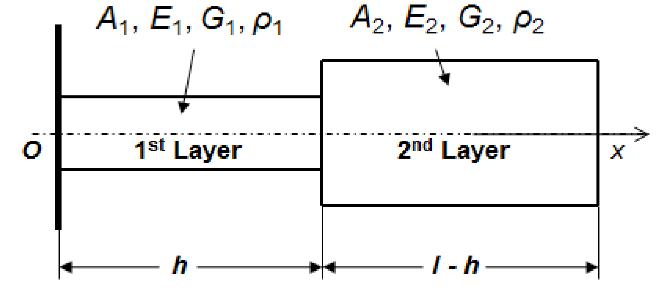

Figure 1: A sketch of the cantilever with one mechanical and geometric discontinuity

It is evident from structure of Eq. 4 that both $U_i(x, s)$ have the same common denominator $D(s)$ with simple poles at $\pm ip$ and a countable set of poles given by the equation $d(s) = 0$. The simple poles at $\pm ip$ represent a pure periodical vibration responsible for the resonance. Whereas for the poles given by equation $d(s) = 0$ correspond to the natural frequencies of the system, i.e. when external force coincides with a roots s_n of equation $d(s) = 0$, then function U_i has poles of the second order and, consequently, the solution contains resonance term [37]. Introducing a new variable $\gamma = is$ and using trigonometric identities $\sin(ix) = i \sinh x$ and $\cos(ix) = \cosh x$, equation $d(s) = 0$ yields

$$k_2c_1 \tan[\gamma(l - h)/c_2] = k_1c_2 \cot(\gamma h/c_1). \tag{5}$$

Now by using the relationships for multiplications of the goniometric functions [38], Eq. (5) can be written in the following form

$$(k_2c_1 + k_1c_2) \cos\{\gamma[hc_2 + (l - h)c_1]/(c_1c_2)\} = (k_2c_1 - k_1c_2) \cos\{\gamma[hc_2 - (l - h)c_1]/(c_1c_2)\}. \tag{6}$$

Evidently, letting $h = 0$ or l in Eq. (6) yields the well-known equation for the natural frequencies of the homogeneous cantilever. Since in the real applications, the cantilevers consist of multiple discontinuities, therefore it is necessary to derive general expression for natural frequencies.

Natural frequencies of the cantilever with an arbitrary number of mechanical and geometric discontinuities (torsional and longitudinal)

The general form of the transcendental equation for an arbitrary number of discontinuous properties can be obtained directly from a Laplace image space of the solution of the problem given by Eqs. (1) and (2). The transcendental equation for an arbitrary number of discontinuities can be derived by following the approach previously applied for a longitudinally oscillating rod with one discontinuity. Solving the system of the N – algebraic equations with given boundary and matching conditions, the complex space solution and correspondingly the transcendental equation $d(s)= 0$ is found.

Omitting the bulky and time consuming intermediate algebraic manipulations, the transcendental equation for an arbitrary number of discontinuities yields

$$\sum_{m=1}^{\left[2^{(N-1)}\right]} \left\{ \left[\cos \sum_{i=1}^{N} \left((-1)^{(q_i+1)} \gamma\varphi_i \right) \right] \prod_{i=1}^{[N-1]} Q_i \right\} = 0, \tag{7}$$

here $q_i = \left\lceil \dfrac{m}{2^{i-2}} \right\rceil$ with $q_1 \equiv 1$, $Q_i = [\,P_i + (-1)^{(q_i+q_{i+1})} P_{i+1}\,]$ and for longitudinal case: $P_i = k_i/c_i$, $\varphi_i = h_i/c_i$, while for torsional case $P_i = b_i/v_i$, $\varphi_i = h_i/v_i$, and $v_i = \sqrt{\dfrac{G_iK_i}{\rho_c I_{pi}}}$.

For instance, the transcendental equation for the four layered cantilever ($N = 4$) reads

$(P_1 + P_2)(P_2 + P_3)(P_3 + P_4) \cos[\gamma(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)] + (P_1 - P_2)(P_2 - P_3)(P_3 + P_4) \cos[\gamma(\varphi_1 - \varphi_2 + \varphi_3 + \varphi_4)] + (P_1 + P_2)(P_2 - P_3)(P_3 - P_4) \cos[\Lambda(\varphi_1 + \varphi_2 - \varphi_3 + \varphi_4)] + (P_1 - P_2)(P_2 + P_3)(P_3 - P_4) \cos[\gamma(\varphi_1 - \varphi_2 - \varphi_3 + \varphi_4)] + (P_1 + P_2)(P_2 + P_3)(P_3 - P_4) \cos[\gamma(\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4)] + (P_1 - P_2)(P_2 - P_3)(P_3 - P_4) \cos[\gamma(\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4)] +$

$$(P_1 + P_2)(P_2 - P_3)(P_3 + P_4) \cos [\gamma(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)] + (P_1 - P_2)(P_2 + P_3)(P_3 + P_4) \cos [\gamma(\varphi_1 - \varphi_2 - \varphi_3 - \varphi_4)]. \quad (8)$$

Conclusions

In this paper, theoretical investigation of the cantilever longitudinal and torsional oscillations with homogeneous and discontinuous properties has been carried out. The fundamental solution of the problem has been obtained. We show that the natural frequencies are the positive roots of the transcendental equation obtained directly from the Laplace image of the solution of the considered problem. The general form of the transcendental equation for an arbitrary number of discontinuous material (elastic/shear moduli and density) and geometric (diameter and layer length) properties has been derived for a cantilever. It allows one to use these results for determination of material properties in vibrational analysis of various cantilever based systems.

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