

Speeding Scalar Multiplication of Elliptic Curve Over $GF(2^{mn})$

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Abstract

Lee et al. proposed two methods to speed up the computation of scalar multiplication of elliptic curve defined over $GF(2^{mn})$ with a medium size of m in the range $10 \leq m \leq 20$. In these methods, Frobenius map is utilized to expand the integer k and each coefficient of the expansion is represented as a binary string. In this paper, with the application of joint sparse form (JSF) to the coefficients, some variations of Lee et al.'s methods are proposed to achieve a better performance at a lower storage requirement.

Keywords: Elliptic curve cryptography, frobenius map, joint sparse form, scalar multiplication

1 Introduction

Elliptic curve cryptosystem (ECC) was first proposed by Koblitz [17] and Miller [25] independently, and has been widely studied in recent years. It has the advantage of shorter key length and higher efficiency with the same security level over RSA, which make it more and more popular in applications especially in wireless communication system. Table 1 is listed below to illustrate the key length comparison of ECC and RSA and Table 3 to the speed comparison [25]. As for those desired properties, it had been adopted by many standards and commercial systems.

In most applications of ECC, such as signature, public key encryption, Diffie-Hellman key exchange and so on, Scalar Multiplication (SM) is the basic and most time-consuming operation. It is the k -time addition of point P , where k is a large positive integer and P is a base point on the curve. The computational speed of SM is affected by three aspects: Finite field operations, of which inversion is the most key one [1, 6, 12, 14, 16, 31]; Curve point operation, including double, addition, double and addition etc. [3]; Representation of the scalar k . An overview

Table 1: Key length comparison of RSA and ECC

MIPS Year	RSA Key Length	ECC Key Length	Ratio
10^4	512	106	5:1
10^8	768	132	6:1
10^{11}	1024	160	7:1
10^{20}	2048	210	10:1
10^{78}	21000	600	35:1

of the research results on SM of ECC can be taken from [10, 23].

Of the above three aspects, decomposition of scalar k gains most of the recent research interests. For the purpose of speeding up the computation, there are three factors in decomposition need to be considered. Firstly, the base should be selected such that the base time of points is efficient. Secondly, the representation length should be short. Finally, the representation Hamming density, which is the ratio of the non-zero digits to all digits, should be small.

According to the expansion base, scalar expansion methods can be classified as two kinds: integer base methods and endomorphism base methods. Integer 2 was firstly used as a base to decompose k . After that, NAF (Non Adjacent Form), which can assure that there are no two contiguous non-zero digits, was proposed to lower the Hamming density [2]. Furthermore, w window techniques were applied on NAF to get NAF_w , which make the density more and more sparse [5]. Besides 2, any a positive integer bigger than 2 can also be used as base to decompose k . w -based expansions have shorter length [8]. In 2006, double base chain, which uses the integer pair (2,3) as base, was given. It achieves shorter length and lower density [7, 30]. Endomorphism is a map from an ECC group to itself. Selecting an efficient endomorphism as

base is another improvement of method in SM computation. For example, endomorphism τ of Koblitz curve only needs a simple shift-row operation which is almost free when underlying finite field adopts normal basis. τ based decomposition methods has a much faster computational speed [18, 27]. Following τ , many efficient endomorphisms were given [4, 9, 19, 24].

In particular, the Frobenius map which is an endomorphism was used to obtain the Frobenius expansion of k for the curve over $\text{GF}(2^n)$ with a small m [26]. Recently, Lee et al. extended the idea to the curve over $\text{GF}(2^{mn})$ with a medium size of m ($10 \leq 20$) and proposed two algorithms to accelerate the computation of kP [20]. The joint sparse form (JSF) [28] was a variation of signed binary representation of a pair of integers, which could lead to more double zero positions and thus a reduction in computational complexity. In Lee et al.'s methods, each coefficient of Frobenius expansion is represented as a binary string and all these bits build up the bits coefficient matrix. Finally, the matrix is dealt with longitudinal w -bit combination and transverse w -bit combination respectively to build two methods.

Based on their idea, we have a little modification on the generation of the bits coefficients matrix. After Frobenius expansion of k is obtained, which is $(C_0, C_1, \dots, C_{l-1})$. If $l \bmod 2 = 1$, set $C_1 = 0$. Let $s = \lfloor \frac{l-1}{2} \rfloor$, and JSF is applied on every coefficient pair of $(c_{2x}, c_{2x+1})(x = 0, 1, \dots, s)$ to obtain the coefficient matrix

$$c = \begin{bmatrix} c_{0,m-1} & \dots & c_{0,0} \\ \vdots & \vdots & \vdots \\ c_{2s+1,m-1} & \dots & c_{2s+1,0} \end{bmatrix}$$

After the matrix is generated, we use almost the same idea to compute SM. That is to say, our main idea is combination of JSF and Frobenius map inspired by Lee et al.

The rest of this paper is organized as follows. In Section 2, the joint sparse form is briefly introduced. Lee et al.'s methods, as well as the Frobenius map are described in Section 3. Then we will propose our modified algorithms in Section 4, together with a detailed analysis of the computational complexity and the storage requirement. Finally, a conclusion is drawn in Section 5.

2 Joint Sparse Form

Joint sparse form is a variation of the signed binary representation of a pair of integers that leads to more double zero positions. Algorithm 1 is adopted to obtain the JSF of a given pair of integers a and b . The notation $c = a \bmod b$ means that $c \equiv a \pmod{b}$ and $-b/2 \leq c \leq b/2$.

- (JSF-1) Of any three consecutive positions, at least one is a double zero. In other words, for any positions i and j , we have $u_{i,j+k} = u_{1-i,j+k}$ for $k = 0$ or $u_{i,j+k} = u_{1-i,j+k} = 0$ for $k = 0$ or ± 1 .

Table 2: Speed comparison of RSA and ECC

Functions	Time(ms) ECC-163bit Security Builder 1.2	Time(ms) RSA-1024bit BSAFE 3.0
Key Generation	3.8	4,708.3
Sign	2.1(ECNRA) 3.0(ECDSA)	228.4
Verification	9.9(ECNRA) 10.7(ECDSA)	12.7
DH Exchange	7.3	1,654.0

Algorithm 1 Algorithm 1 (JSF)

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1: Input integer pair  $a$  and  $b$  Output
2: JSF of  $a$  and  $b$ 
3: Process
4: Set  $k_0 = a$  and  $k_1 = b$ , Set  $j = 0$ 
5: while  $k_0 \neq 0$  or  $k_1 \neq 0$  do
6:   For  $i = 0$  to 1 do
7:     if  $k_i$  is even then
8:        $u = 0$ 
9:     else
10:       $u = k_i \bmod 4$ 
11:      if  $k_i \equiv \pm 3 \pmod{8}$  and  $k_i \equiv 2 \pmod{4}$  then
12:         $u = -u$ 
13:        Set  $u_{i,j} = u$ 
14:      end if
15:    next  $i$ 
16:    For  $i = 0$  to 1 do
17:       $k_i = (k_i - u_{i,j})/2$ 
18:    next  $i$ 
19:     $j = j + 1$ 
20:  end if
21:  Periodically refresh the observations storage
22: end while
23: End
24: The JSF possesses the following properties that had
    been proved in [21], namely, JSF-1, JSF-2, JSF-3 and
    JSF-4.

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- (JSF-2) Adjacent terms do not have opposite signs. In other words, there is never the case that $u_{i,j}u_{1-i,j+k} = -1$.
- (JSF-3) If $u_{i,j+k} \neq 0$ then $u_{1-i,j+k} = \pm 1$, and $u_{1-i,j} = 0$.
- (JSF-4) The probability of occurrence of double zero, which satisfies $u_{i,j+k} = u_{1-i,j+k} = 0$ for position j , is $1/2$.

3 Lee et al.'s Methods

For elliptic curve over finite field $\text{GF}(2^{mn})$, where m is a middle size integer generally bounded in the area of [22, 28]. With the utilization of Frobenius map, Lee et

el. Proposed two methods to speed up the scalar multiplication.

1) **Frobenius Expansion**

For a non-supersingular curve over $GF(q^n)$, where $q = 2^m$, given by the Weierstrass equation of the form

$$E(GF(q^n)) : y^2 + xy = x^3 + ax^2 + b,$$

where $a, b \in GF(q)$ and $b \neq 0$, Frobenius map is an endomorphism of the elliptic curve group $E(GF(q^n))$. It is defined from $E(GF(q^n))$ to itself as:

$$\phi : E \mapsto E(x, y) \mapsto (x^q, y^q).$$

Let $\#E(GF(q^n))$ denote the number of $GF(q^n)$ -rational point of $E(GF(q^n))$ and $t = q + 1 - \#E(GF(q^n))$, then the characteristic polynomial of ϕ is $\phi^2 - t\phi + q = 0$ that is $\phi^2(P) - t\phi(P) + qP = 0$ for all $P \in E(GF(q^n))$. Since there is a natural homomorphism from the ring $Z[\phi]$ to the ring $Z[\phi]$, which maps $\alpha = (t + \sqrt{t^2 - 4q}/2)$ to ϕ , the integer k can be expressed as the sum of $k = \sum c_i \phi^i$. This expression is called the Frobenius expansion of k . For the map, there are following theorems that had been proved in [29].

Theorem 1. [29] For any given positive integer k , the Frobenius expansion $k = \sum_{i=0}^l -1_{i=0} C_i \phi^i (-q/2 < c_i \leq q/2, q \geq 64)$ exists. It is unique and has length $l \leq n + 3$.

Theorem 2. [29] For any given positive integer k , the Frobenius expansion $k = \sum_{i=0}^l -1_{i=0} c_i \phi^i (0 \leq c_i < q, q \geq 64)$. It is unique and has length $l \leq n + 5$.

2) **Method 1**

Firstly, the Frobenius expansion of $k = \sum_{i=0}^l -1_{i=0} c_i \phi^i (0 \leq c_i < q)$ is obtained. Then, each coefficient c_i is represented as a binary string $(c_{i,m-1} c_{i,m-2}, \dots, c_{i,1} c_{i,0})$. Finally kP can be computed via the formula:

$$\begin{aligned} kP &= \sum_{i=0}^{l-1} c_i \phi^i(P) \\ &= \sum_{i=0}^{l-1} 2^j \sum_{j=0}^{m-1} c_{i,j} \phi^i(P) \\ &= \sum_{j=0}^{m-1} 2^j \sum_{i=0}^{l-1} c_{i,j} \phi^i(P). \end{aligned} \quad (1)$$

Let bit string $a = (a_{w-1}, a_{w-2}, \dots, a_1, a_0)$ and $S_a = a_{w-1} \phi^{w-1}(P) + a_{w-2} \phi^{w-2}(P) + \dots + a_1 \phi(P) + a_0 P$, Equation (1) can be varied to Equation (2) as :

$$kP = \sum_{j=0}^{m-1} 2^j \sum_{i=0}^{l-1} c_{i,j} \phi^i(P), \quad (2)$$

where

$$\begin{aligned} T_j &= \sum_{i=0}^{l-1} c_{i,j} \phi^i(P) \\ &= \sum_{i=0}^{\lceil m/w \rceil - 1} \phi^{wi} S_{(c_{wi+w-1,j}, c_{wi+w-2,j}, \dots, c_{wi+1,j}, c_{wi,j})}. \end{aligned}$$

Algorithm 2 is the programming description of Method 1.

Algorithm 2 Algorithm 2 (Method 1)

- 1: Input
 - 2: Integer k , point P
 - 3: Output
 - 4: kP
-

Pre-computation and Storages:

- Frobenius expansion of $k = \sum_{i=0}^{l-1} c_i \phi^i (0 \leq c_i < q)$
- Binary string $(c_{i,m-1}, c_{i,m-2}, \dots, c_{i,1}, c_{i,0})$ of each coefficient c_i .
- $S_a = a_{w-1} \phi^{w-1}(P) + a_{w-2} \phi^{w-2}(P) + \dots + a_1 \phi(P) + a_0 P$ for all $(a_{w-1}, a_{w-2}, \dots, a_1, a_0) \in \{0, 1\}^w$, where w is a chosen window size.

Process:

- a. $Q = O$;
- b. for $j = m-1$ downto 0 do
 - i. for $i = \lceil l/w \rceil - 1$ downto 0 do
 - A. $T = \phi^w(T)$;
 - B. $a = (c_{wi+w-1,j}, c_{wi+w-2,j}, \dots, c_{wi+1,j}, c_{wi,j})$;
 - C. $T = T + S_a$
 - ii. $Q = Q + T$;
- c. return Q ;

From the algorithm definition, we have the following theorems.

Theorem 3. Let A stand for point addition, D for Doubling and Φ for Frobenius map ϕ , the total number of operations of Algorithm 2 is $TO \approx mD + (1 - 1/2^w)(mn/w)A + (mn)\Phi$.

Proof. One by one counting in Algorithm 2, it can be easily obtained that the total number of operations in Algorithm 2.

$$\begin{aligned} TO &= (m-1)D + (1 - 1/2^w) \\ &\quad + (l - 1/2^w)(\lceil l/w \rceil m - 1)A \\ &\quad + ((\lceil l/w \rceil - 1)wm)\Phi. \end{aligned}$$

Since $l \mid n + 5$ following Theorem 2, then

$$\begin{aligned} TO &= (m-1)D + (1-1/2^w)(\lceil(n+5)/w\rceil m - 1)A \\ &\quad + (\lceil(n+5)/w\rceil - 1)wm\Phi \\ &\approx mD + (l-1/2^w)(mn/w)A + (mn)\Phi. \end{aligned}$$

□

Theorem 4. *The number of storages required by Algorithm 2 is $2^w - w - 1$.*

Proof. The number of possible values of S_a is 2^w . However, as the map ϕ is almost free (only twice m -bit left-shift) when normal basis representation is utilized on the finite field, then $\phi^i (i = 0, 1, \dots, w-1)$ and O needn't be stored. Hence the theorem holds. □

3) Method 2

In fact, we may take all the binary strings of the Frobenius expansion coefficients as a binary matrix. Method 1 can be looked as a column combination. On the other hand, when row combination is applied on the matrix, Method 2 is proposed. In Method 2, as in Method 1, firstly, the Frobenius expansion of $k = \sum_{i=0}^{l-1} c_i \phi^i (-q/x < c_i \leq q/2)$ is obtained. Then, each coefficient $|c_i|$ is represented as a binary string $(c_{i,m}c_{i,m-3}, \dots, c_{i,1}c_{i,0})$. Finally kp can be computed via the formula:

$$\begin{aligned} kP &= \sum_{i=0}^{l-2} c_i \phi^j(P) \\ &= \sum_{i=0}^{l-1} \sum_{j=0}^{m-1} \varepsilon_i c_{i,j} 2^j \phi^i(P), \end{aligned} \quad (3)$$

where $\varepsilon_i = c_i / |c_i|$.

Let bit string $a = (a_{w-1}, a_{w-2}, \dots, a_1, a_0)$ and $S_a = a_{w-1}2^{q-1}P + a_{w-2}x^{w-2}P + \dots + a_12P + a_0P$, then Equation (3) will be modified to Equation (4):

$$\begin{aligned} kP &= \sum_{i=0}^{l-1} \sum_{j=0}^{m-1} \varepsilon_i c_{i,j} 2^j \phi^i(P) \\ &= \sum_{r=0}^{\lceil m/w \rceil - 1} 2^r T'_r \end{aligned} \quad (4)$$

where

$$\begin{aligned} T_r &= \sum_{j=rw}^{rw-1} 2^{j-rw} \sum_{i=0}^{l-1} \varepsilon_i c_{i,j} \phi^i(P) \\ &= \sum_{i=0}^{l-1} \varepsilon_i \phi^i S_{(c_{i,wr+w-1}, c_{i,wr+w-2}, \dots, c_{i,wr+1}, c_{i,wr})} \end{aligned}$$

Algorithm 3 is the programming description of Method 2.

Algorithm 3 Algorithm 3 (Method 2)

- 1: Input
 - 2: Integer k , point P
 - 3: Output
 - 4: kP
-

Pre-computation and Storages:

- a. Frobenius expansion of

$$k = \sum_{i=0}^{l-1} c_i \phi^i (-q/2 \leq c_i \leq q/2);$$

- b. Binary string $(c_{i,m-1}, c_{i,m-2}, \dots, c_{i,1}c_{i,0})$ of each coefficient c_i .
- c. $S_a = a_{w-1}2^{q-1}P + a_{w-2}x^{w-2}P + \dots + a_12P + a_0P$ for all $a = (a_{w-1}, a_{w-2}, \dots, a_1, a_0) \in \{0, 1\}^w$, where w is a chosen window size.

Process:

- a. $Q = O$;
- b. for $j = \lceil m/w \rceil - 1$ downto 0 do
 - i. for $Q = 2^w Q, T = O$;
 - ii. for $i = l-1$ downto 0 do
 - A. $T = \phi(T); \varepsilon_i = c_i / |c_i|$;
 - B. $a = (c_{i,jw+w-1}, c_{i,jw+w-2}, \dots, c_{i,jw+1}, c_{i,jw})$;
 - C. $T = T + \varepsilon_i S_a$.
 - iii. $Q = Q + T$;
- c. return Q ;

Based on Method 2, we have the following theorems.

Theorem 5. *Let A stand for point addition, D for Doubling and Φ for Frobenius map ϕ , the total number of operations of Algorithm 3 is*

$$TO \approx mD + (l-1/2^w)(mn/w)A + (mn/w)\Phi.$$

Proof. From the procedures of Algorithm 3, it can be easily counted one by one that the total number of operations

$$\begin{aligned} TO &= ((\lceil m/w \rceil - 1)w)D \\ &\quad + ((1-1/2^w)(l-1)\lceil m/w \rceil)A \\ &\quad + (l-1)\lceil m/w \rceil\Phi. \end{aligned}$$

From Theorem 1, we have $l \leq n + 3$, then

$$\begin{aligned} TO &= ((\lceil m/w \rceil - 1)w)D \\ &\quad + ((1-1/2^w)(l-1)(n+3-1)\lceil m/w \rceil)A \\ &\quad + ((n+3-1)\lceil m/w \rceil)\Phi \approx mD \\ &\quad + (1-1/2^w)(mn/w)\Phi. \end{aligned}$$

□

Theorem 6. *Theorem 6 The number of storages required by Algorithm 3 is $2^w - 2$.*

Proof. The number of possible values of S_a is 2^w . It is obvious that O and P needn't be stored. Thus the theorem holds. \square

If normal basis representation is applied in the finite field, which makes the map ϕ almost free, Algorithm 2 is more efficient. Otherwise, Algorithm 3 is more efficient, especially when the improvement of [3] is utilized to compute $2^w P$.

4 The Proposed Method

The idea of Lee et al.'s methods is like this. First, the Frobenius expansion of k is obtained. Then each coefficient of the expansion is represented as a binary string and all these bits build up the coefficient matrix. Finally, the matrix is deal with longitudinal w-bit combination by Algorithm 2, and transverse w-bit combination by Algorithm 3. In Algorithm 2, if $S_a = 0$, then the addition can be saved. In situation that the amount of storage is limited, the value of w needs to be very small. In particular, if $w = 2$, JSF of c_i and c_{i+1} can be applied to increase the probability of getting $s_a = 0$ and thus reduce the point additions in the computation of kP . This forms the idea of our proposed Method 1. On the other hand, if the storage resource is rich, a combination of the transverse w positions of the JSF of c_i and c_{i+1} is used in our proposed Method 2. We not only give a detailed description of our methods, but also have an accurate evolution on the number of atomic operations and storages. Furthermore, we do a comparison on both methods between ours and Lee et al.'s.

4.1 The Proposed Method 1

Let $k = \sum_{i=0}^{l-1} c_i \phi^i(-q/2 < c_i \leq q/2)$ be the Frobenius expansion of k as before. If $l \bmod 2 = 1$, set $c_l = 0$. Let $s = \lceil (l-1)/2 \rceil$, and JSF is applied on every coefficient pair of $(c_{2x}, c_{2x+1})(x = 0, 1, \dots, s)$ to obtain the coefficient matrix

$$c = \begin{bmatrix} c_{0,m-1} & \dots & c_{0,0} \\ \vdots & \vdots & \vdots \\ c_{2s+1,m-1} & \dots & c_{2s+1,0} \end{bmatrix}$$

Let $a = (a_1, a_0)$, and $s_a = a_1 \phi(P) + a_0 P$, then we have the Equation (5):

$$kP = \sum_{j=0}^{m-1} \sum_{i=0}^{l-1} c_{i,j} \phi^i(P) = \sum_{j=0}^{l-1} 2^j T_j \quad (5)$$

The Equation (5) can be programmed as Algorithm 4, which is our proposed Method 1.

Pre-computation and Storages:

Algorithm 4 Algorithm 4 (proposed Method 1)

1: Input
2: Integer k , point P
3: Output
4: kP

- Frobenius expansion of

$$k = \sum_{i=0}^{l-1} c_i \phi^i(-q/2 < c_i \leq q/2).$$

- JSF of (c_{2x}, c_{2x+1}) for $(x = 0, 1, \dots, s)$.

- Coefficient matrix $c = \begin{bmatrix} c_{0,m-1} & \dots & c_{0,0} \\ \vdots & \vdots & \vdots \\ c_{2s+1,m-1} & \dots & c_{2s+1,0} \end{bmatrix}$.

- $S_a = a_1 \phi(P) + a_0 P$ for $(a_1, a_0) \in \{0, \pm 1\}^2$ (only $\phi(P) \pm P$ needs to be stored).

Process:

- 1) $Q = O$;
- 2) for $j = m-1$ downto 0 do
 - a. for $Q = 2Q, T = O$;
 - b. for $i = s$ downto 0 do
 - i. $T = \phi^2(T)$;
 - ii. $a = (c_{2i+1,j}, c_{2i,j})$;
 - iii. $T = T + S_a$
 - c. $Q = Q + T$;
- 3) return Q ;

In Algorithm 4, we have the following theorems about the computing speed and the storages.

Theorem 7. *Let A stand for point addition, D for Doubling and Φ for Frobenius map ϕ , the total number of operations in Algorithm 4 is $TO \approx mD + (mn/4)A + (mn)\Phi$.*

Proof. From property JSF-4, we have $P(S_a = O) = 1/2$. From the procedure, the total number of atomic operations in Algorithm 4 is

$$TO = (m-1)D + (1-1/2)(\lceil l/2 \rceil)A + ((\lceil l/2 \rceil - 1)2m)\Phi.$$

Following Theorem 1, we have $l \leq n+3$, thus

$$TO = (m-1)D + (1-1/2)(\lceil (n+3)/2 \rceil m - 1)A + ((\lceil (n+3)/2 \rceil m - 1)A).$$

\square

Theorem 8. *The amount of storage required by Algorithm 4 is 2.*

Table 3: Comparison of our method with Lee's on Method 1

	TO	Storages
Lee's Scheme (w=2)	$mD + (3nm/8)$ $A + (mn)\Phi$	1
The Proposed Scheme	$mD + (mn/4)$ $A + (mn)\Phi$	2

Proof. Only $\phi(P) \pm P$ need to be stored. If $a \in \{(0,0), (\pm 1,0), (0,\pm 1)\}$, it is obviously that S_a need not be stored. If $a = (-1,1)$ then $S_a = -(\phi(P) - P)$. Similarly, if $a = (-1,-1)$, $S_a = -(\phi(P) + P)$. \square

Comparing with Algorithm 2 for the case $w = 2$, about $mn/8$ additions are reduced with the cost of an additional storages.

4.2 The Proposed Method 2

If the amount of storage is not too restricted, one can choose Method 2. Let the coefficient matrix

$$c = \begin{bmatrix} c_{0,m-1} & \cdots & c_{0,0} \\ \vdots & \vdots & \vdots \\ c_{2s+1,m-1} & \cdots & c_{2s+1,0} \end{bmatrix}$$

be the same as the proposed Method 1.

Let

$$a = \begin{bmatrix} a_{0,w} & \cdots & a_{0,0} \\ a_{1,w-1} & \cdots & a_{1,w-1} \end{bmatrix}$$

and

$$S_a = \sum_{j=0}^{w-1} s^j \sum_{i=0}^l a_{i,j} \phi^i(P),$$

then Equation (4) will be modified as Equation (6):

$$kP = \sum_{i=0}^{l-1} \sum_{j=0}^{m-1} c_{i,j} x^j \phi^i(P) = \sum_{r=0}^{\lceil m/w \rceil} 2^r T_r' \quad (6)$$

where

$$T_r = \sum_{i=0}^{l-1} \phi^{2i} S_{(c_{i,2r+1}, c_{i,2r})}$$

Equation (6) is also used in our Method 2, which can be programmed as Algorithm 5.

Algorithm 5 Algorithm 5 (Proposed Method 2)

- 1: Input
 - 2: Integer k , point P
 - 3: Output
 - 4: kP
-

Pre-computation and Storages:

- Frobenius expansion of

$$k = \sum_{i=0}^{l-1} c_i \phi^i(-q/2 < c_i \leq q/2).$$

- JSF of (c_{2x}, c_{2x+1}) for $(x = 0, 1, \dots, s)$.

- Coefficient matrix $c = \begin{bmatrix} c_{0,m-1} & \cdots & c_{0,0} \\ \vdots & \vdots & \vdots \\ c_{2s+1,m-1} & \cdots & c_{2s+1,0} \end{bmatrix}$.

- Compute

$$S_a = \sum_{j=0}^{w-1} 2^j \sum_{i=0}^l a_{i,j} \phi^i(P)$$

$$a = \begin{bmatrix} a_{0,w-1} & \cdots & a_{0,0} \\ a_{1,w-1} & \cdots & a_{1,w-1} \end{bmatrix}$$

and $a_{i,j} \in \{0, \pm 1\}$ -. If $b = -a$ then S_b need not to be stored as $S_b = -S_a$.

Process:

- 1) $Q = O$;
- 2) for $j = \lceil m \rceil - 1$ downto 0 do
 - a. for $Q = 2^w Q, T = O$;
 - b. for $i = s$ downto 0 do
 - i. $T = \phi^2(T)$;
 - ii. $a = a = \begin{bmatrix} a_{2i,jw+w-1} & \cdots & a_{2i,jw} \\ a_{2i+1,jw+w-1} & \cdots & a_{2i+1,jw} \end{bmatrix}$;
 - iii. $T = T + S_a$
 - c. $Q = Q + T$;
- 3) return Q ;

Based on Algorithm 5, we have the theorems below.

Theorem 9. Let A stand for point addition, D for Doubling and Φ for Frobenius map ϕ , the total number of operations required by Algorithm 5 is $TO \approx mD + (1 - 1/2^w)(mn/(2w)) + (mn/w)\Phi$.

Proof. From property JSF-4, we have $P(S_a = O) = 1/2^w$. From the procedures of Algorithm 5, it can be easily obtained that the total number of operations is $TO = ((\lceil m/w \rceil - 1)w)D + ((1 - 1/2^w)(\lfloor l - 1/2 \rfloor) \lceil m/w \rceil)A + (\lfloor (l - 1/2) \rfloor) \lceil m/w \rceil \Phi$. By Theorem 1, we have $l \leq n + 3$. Hence

$$TO = ((\lceil m/w \rceil)w)D + ((1 - 1/2^w)(\lfloor (n + 3 - 1)/2 \rfloor) \lceil m/w \rceil)A + ((\lfloor (n + 3 - 1)/2 \rfloor) \lceil m/w \rceil) \Phi \approx mD + (1 - 1/2^w)(mn/w)\Phi.$$

\square

Table 4: Comparison of our method with Lee's on Method 2

	Total Operations	Storages	
Our	$mD+(1-1/2^w)(mn/2w)$ $A+(mn/w)\Phi$	16	w=2
		62	w=3
		608	w=4
Lee's	$mD+(1-1/2^w)(mn/w)A+(mn/w)\Phi$	2	w=2
		6	w=3
		14	w=4

Theorem 10. *The number of storages required by Algorithm 5 is as below:*

- 1) If $w = 2$, it is 16;
- 2) If $w = 3$, it is 62;
- 3) If $w = 4$, it is 308.

Proof. The number of possible values of S_a is 3^{2w} . With the restriction of JSF-1, JSF-2 and JSF-3, the number decreases a lot. Since only either one of a and $-a$ needs to be stored as well as $\phi(P)$ and P needn't be stored. Thus it can be checked one by one when $w = 2, 3, 4$ respectively to get result of the theorem. \square

When the improvement of [3] is utilized to compute $2^w P$, Algorithm 5 can be further accelerated. Comparing with Algorithm 3, about $(1-1/2^w)(mn/2w)$ additions are saved. However, the amount of storage required increases substantially. Thus it is suitable for small w . For example, Algorithm 5 with $w = 2$ has the same computational complexity as Algorithm 3 with $w = 5$ with a lower amount of storages. The same result can be obtained for $w = 3$ of Algorithm 5 to $w = 7$ of Algorithm 3, also $w = 4$ of Algorithm 5 to $w = 9$ of Algorithm 3. However, for $w > 4$, the number of storages is too large and our proposed Method 2 has no practical meaning. For this reason, the storages number needed for $w > 4$ needn't to be cared.

5 Conclusion

Based on the coefficient matrix of the Frobenius expansion of the integer k when computing the scalar multiplication kP of elliptic curve cryptography over $GF(2^{mn})$ with a medium size m , the idea of longitudinal w -bit combination and transverse w -bit combination are separately proposed by Lee *et al.* to form Algorithms 2 and 3 to speed up the computation. For Algorithm 2, we apply the JSF form on the coefficients to obtain the coefficient matrix and propose our Method 1. It is suitable for limited storage situation and saves $mn/8$ point additions with an additional storage when compared with Algorithm 2 of $w = 2$. Based on proposed Method 1, the idea of transverse w -position combination is utilized to form our

Method 2. It saves about $(1-1/2^w)(mn/2w)$ point additions when compared with Algorithm 3. The value of w can be selected to balance the computational speed and the storage requirements. As the amount of storage increases substantially with w , it is practical for relatively small w . For example this algorithm with $w = 2$ has the same result as Algorithm 3 with $w = 5$, but the storage required is fewer. The value of w can also be selected as $w = 3$ or $w = 4$ if the storages resource is rich. Nevertheless, it has no any practical meaning for $w > 4$ as the storages number is too large.

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