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SPHERES IN INFINITE-DIMENSIONAL NORMED SPACES ARE LIPSCHITZ CONTRACTIBLE

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ABSTRACT. Let X be an infinite-dimensional normed space. We prove the following:
(i) The unit sphere $\{x \in X: \|x\| = 1\}$ is Lipschitz contractible.
(ii) There is a Lipschitz retraction from the unit ball of X onto the unit sphere.
(iii) There is a Lipschitz map T of the unit ball into itself without an approximate fixed point, i.e. $\inf\{\|x - Tx\|: \|x\| \leq 1\} > 0$.

Introduction. Let X be a normed space, and let $B_X = \{x \in X: \|x\| \leq 1\}$ and $S_X = \{x \in X: \|x\| = 1\}$ be its unit ball and unit sphere, respectively.

Brouwer's fixed point theorem states that when X is finite dimensional, every continuous self-map of B_X admits a fixed point. Two equivalent formulations of this theorem are the following.

1. There is no continuous retraction from B_X onto S_X.

2. S_X is not contractible, i.e., the identity map on S_X is not homotopic to a constant map.

It is well known that none of these three theorems hold in infinite-dimensional spaces (see e.g. [**1**]). The natural generalization to infinite-dimensional spaces, however, would seem to require the maps to be *uniformly-continuous* and not merely continuous. Indeed in the finite-dimensional case this condition is automatically satisfied.

In this article we show that the above three theorems fail, in the infinite-dimensional case, even under the strongest uniform-continuity condition, namely, for maps satisfying a Lipschitz condition. More precisely, we prove

THEOREM. *Let X be an infinite-dimensional normed space. Then*

(1) *The unit sphere S_X is Lipschitz contractible.*

(2) *There is a Lipschitz retraction from B_X onto S_X.*

(3) *There is a Lipschitz map T: $B_X \to B_X$ without an approximate fixed point, i.e.* $\inf\{\|x - Tx\|: x \in B_X\} = d > 0$.

The first study of Lipschitz maps without approximate fixed points, and Lipschitz retractions from B_X onto S_X, was done by K. Goebel [**3**]. B. Nowak [**5**] proved the theorem for several classical Banach spaces. Our work was greatly influenced by the work of Nowak. Actually, the general scheme of the proof as well as two of the three

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main steps in the proof (Definition 1 and Propositions 1 and 3) are slight modifications of results in [**5**] which we reproduce for the sake of completeness and because of the many misprints in [**5**].

In [**4**] the authors study fixed point properties of mappings whose iterates satisfy a uniform Lipschitz condition. In this respect it is interesting to note that the map constructed in part (3) of the Theorem has this property (as follows immediately from its definition in the next paragraph).

Note that parts (2) and (3) of the Theorem follow immediately from (1). Indeed if $H(t, x)$ is a Lipschitz homotopy joining the constant $x_0 \in S_X$ to the identity on S_X, then

$$r(x) = \begin{cases} H(2\|x\| - 1, x/\|x\|), & 1/2 \leqslant \|x\| \leqslant 1, \\ x_0, & 0 \leqslant \|x\| \leqslant 1/2, \end{cases}$$

is a Lipschitz retraction from B_X onto S_X. It is also easy to check that the map $f(x) = -r(x)$ is then a Lipschitz self-map of B_X without an approximate fixed point.

In the next section we shall formulate three propositions and deduce the theorem from them. These three propositions will be proved in §§2–4, respectively.

We use standard terminology and notation. The reader is referred to [**2**] for basic facts on normed spaces.

We end this introduction with a very useful observation which we shall use later without further notice. If $\| \|_1$ and $\| \|_2$ are two equivalent norms on a linear space X, then the pair (B_1, S_1) is Lipschitz equivalent to the pair (B_2, S_2) under the map $x \to \|x\|_1 x/\|x\|_2$ $(0 \to 0)$. (Here B_i (resp. S_i) is the unit ball (resp. sphere) of X with respect to the norm $\| \|_i$, $i = 1, 2$.) It follows that any Lipschitz property of B_X and S_X—and, in particular, our Theorem—can be proved under any norm equivalent to the original given norm.

1. In this section we give a definition and three propositions and then deduce the Theorem. The propositions will be proved in the subsequent sections.

DEFINITION. Let (S, d) be a metric space, $y_0 \in S$ and $\varepsilon > 0$. The point y_0 is said to be an *ε-escaping point* if there exists a Lipschitz mapping $T: S \to S$ satisfying:

(1.1) T is Lipschitz homotopic to the identity on S.

(1.2) $\inf\{d(T^n y_0, T^m y_0): n > m \geqslant 0\} \geqslant 5\varepsilon$.

(1.3) For all $n \geqslant 0$, T maps $B_S(T^n y_0, \varepsilon)$ *isometrically onto* $B_S(T^{n+1} y_0, \varepsilon)$ (where $B_S(y, \varepsilon) = \{x \in S: d(x, y) \leqslant \varepsilon\}$).

(1.4) For all $n \geqslant 0$, $T^{-1}(B_S(T^{n+1} y_0, \varepsilon)) = B_S(T^n y_0, \varepsilon)$.

PROPOSITION 1. *Let y_0 be an ε-escaping point in a metric space S, and let Z be another metric space. Let g: $[-1, 1] \times S \to Z$ be a Lipschitz map which constantly attains the value $z_0 \in Z$ outside the set $[\frac{1}{4}, \frac{3}{4}] \times B_S(y_0, \varepsilon)$. Then g is Lipschitz homotopic to the constant function z_0 in $[-1, 1] \times S$ by a Lipschitz homotopy $H_\tau(t, x)$ $(0 \leqslant \tau \leqslant 1, (t, x) \in [-1, 1] \times S)$, for which $H_\tau(t, x) = z_0$ whenever $|t| \geqslant \frac{3}{4}$.*

REMARK. The fact that g is Lipschitz homotopic to z_0 is, of course, obvious and does not require any assumptions on y_0. Indeed, the homotopy $h_\tau(t, x) = g(t\tau, x)$ does the job. The assumption that y_0 is ε-escaping is used to construct a homotopy H_τ with the additional property that $H_\tau(t, x) = z_0$ whenever $|t| \geqslant \frac{3}{4}$.

PROPOSITION 2. *Let X be an infinite-dimensional normed space and $\varepsilon \leqslant 1/500$. Then X admits an equivalent norm with respect to which S_X has an ε-escaping point.*

PROPOSITION 3. *Let X be a normed space, and let $x_0 \in S_X$ and $\varepsilon > 0$. Then the identity map on S_X is Lipschitz homotopic to a mapping f: $S_X \to S_X$, which constantly attains the value $-x_0$ outside the set $\{x \in S_X: \|x - x_0\| < \varepsilon\}$.*

PROOF OF THE THEOREM. Let X be an infinite-dimensional normed space, and let Y be a closed subspace of X of codimension one. By Proposition 2 there is an equivalent norm $|||\ |||$ on Y so that the unit sphere S_Y with respect to this new norm admits an ε-escaping point, y_0, for some $\varepsilon \leqslant 1/500$. We now identify X with $R \oplus Y$ under the norm $\|(t, y)\| = \max(|||y|||, |t|)$. This gives a norm on X, equivalent to the original one, and we shall prove the Theorem for this norm. To save notation we assume this is the given norm on X, and we then have $S_X = ([-1, 1] \times S_Y) \cup (\{-1, 1\} \times B_Y)$.

Set $x_0 = (\frac{1}{2}, y_0) \in S_X$ and $z_0 = -x_0$. By Proposition 3 there is a Lipschitz map f: $S_X \to S_X$, Lipschitz homotopic to the identity on S_X, so that $f(x) = z_0 = -x_0$ whenever $\|x - x_0\| \geqslant \varepsilon$. If $x = (t, y) \in S_X$ satisfies $\|x - x_0\| < \varepsilon$, then, since $\varepsilon < \frac{1}{4}$, $(t, y) \in [\frac{1}{4}, \frac{3}{4}] \times B_{S_Y}(y_0, \varepsilon) \subset [-1, 1] \times S_Y$. It follows that $g = f_{|[-1,1]\times S_Y}$ satisfies the conditions of Proposition 1 (with $S = S_Y$). As y_0 is an ε-escaping point in S_Y, it follows that g is Lipschitz homotopic, as a map from $[-1, 1] \times S_Y$ into S_X, to the constant $z_0 = -x_0 \in S_X$, by a homotopy $H_\tau(t, y)$ satisfying $H_\tau(t, y) = z_0$ whenever $|t| \geqslant \frac{3}{4}$.

Now extend H_τ to a homotopy F_τ in S_X by defining $F_\tau(x) = z_0$ for $x \in S_X \backslash [-1, 1] \times S_Y$ and all τ. It is easy to see that F_τ is a Lipschitz homotopy in S_X joining f to the constant z_0. Since f is Lipschitz homotopic to the identity on S_X, it follows that S_X is Lipschitz contractible.

REMARKS. (1) The definition of ε-escaping point, Propositions 1 and 3, and the general scheme of the proof are slight modifications of the results of Nowak [**5**], where the same terminology is also used. Notice, however, that our definition of an ε-escaping point is more general than the one used in [**5**].

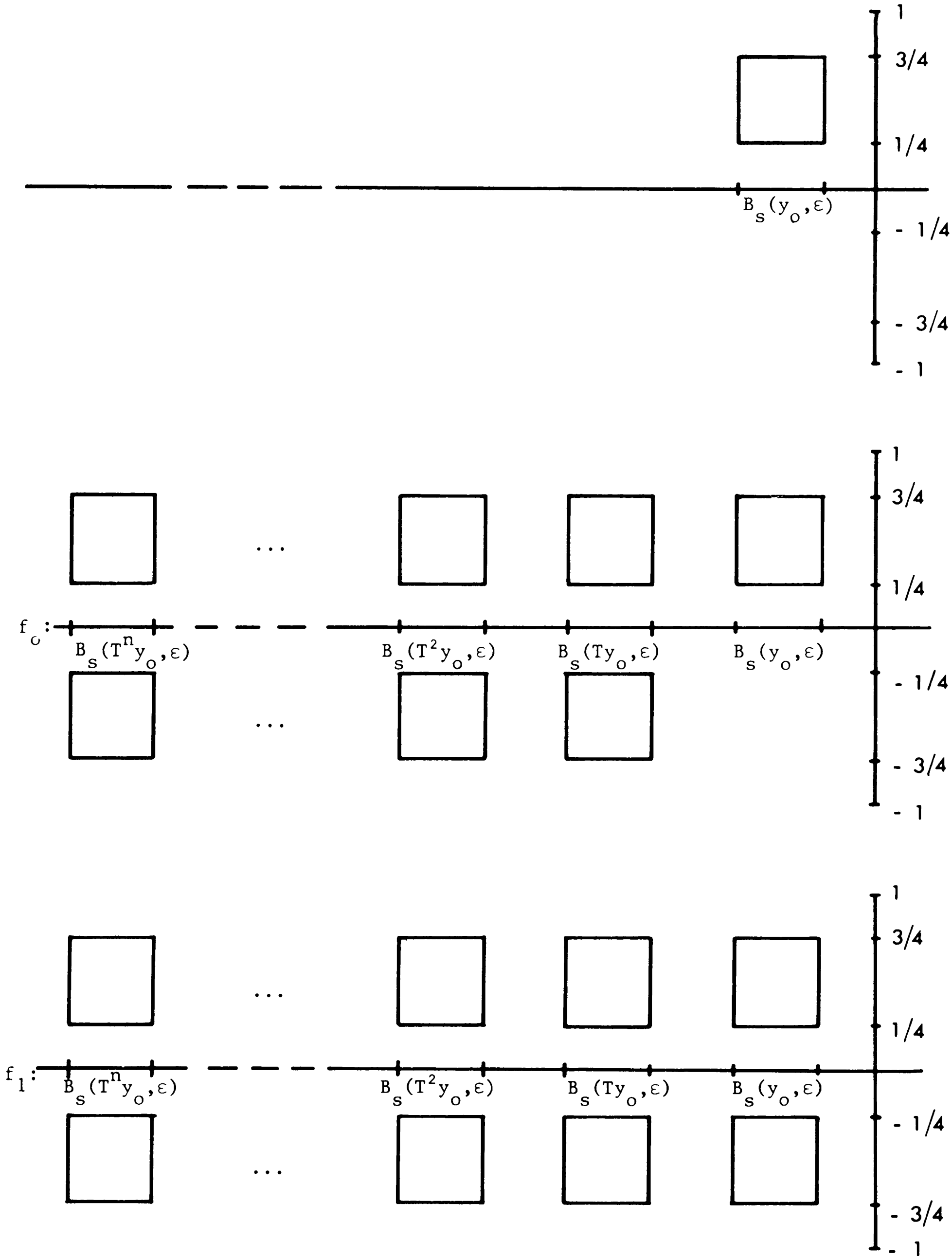
(2) Analyzing the proof of the Theorem and the propositions, one can see that there is, in fact, an absolute constant $K < \infty$, independent of the given infinite-dimensional X, so that the identity on S_X is contractible to a constant map by a homotopy satisfying a Lipschitz condition with constant at most K. To see this one should only check that all the renormings in the proofs can be made up to some absolute constants, and that ε can be chosen independent of X. A similar remark holds concerning (2) and (3) of the Theorem. We leave the details to the interested reader.

2. Proof of Proposition 1. Let T be the map associated to y_0 by the definition of an ε-escaping point. Define two maps f_i: $[-1, 1] \times S \to Z$, $i = 0, 1$, by

$$f_0(t, x) = \begin{cases} g(t, T^{-n}x), & t \geqslant 0 \text{ and } x \in B_S(T^n y_0, \varepsilon), n \geqslant 0, \\ g(-t, T^{-n}x), & t \leqslant 0 \text{ and } x \in B_S(T^n y_0, \varepsilon), n \geqslant 1, \\ z_0, & \text{otherwise;} \end{cases}$$

$$f_1(t, x) = \begin{cases} g(|t|, T^{-n}x), & x \in B_S(T^n y_0, \varepsilon), n \geqslant 0, \\ z_0, & \text{otherwise.} \end{cases}$$

The following figure illustrates the nature of g, f_0 and f_1:



Here S is realized as the horizontal line, the maps are constant (and equal to z_0) outside the "rectangles", and inside each "rectangle" they are defined by the corresponding value (with respect to T^{-n}) of g in the "rectangle" $[\frac{1}{4}, \frac{3}{4}] \times B_S(y_0, \varepsilon)$. Note that by the definition of an ε-escaping point, the "rectangles" are indeed disjoint (in fact the distance between two "rectangles" is at least 3ε), and also that T^{-n} is an isometry of $B_S(T^n y_0, \varepsilon)$ onto $B_S(y_0, \varepsilon)$. Thus f_i are indeed Lipschitz maps. By (1.1) T is Lipschitz homotopic to the identity, and let G_τ be the Lipschitz homotopy, in S, joining the identity to T. Then

$$F_\tau(t, x) = \begin{cases} f_0(t, x), & t \geq 0, \\ f_0(t, G_\tau(x)), & t \leq 0, \end{cases}$$

is a Lipschitz homotopy in $[-1, 1] \times S$ joining f_0 to f_1.

The map $F_\tau^1(t, x) = f_1(|t|(1-\tau) + \tau, x)$ gives a Lipschitz homotopy joining f_1 to the constant z_0, and the map

$$F_\tau^0(t, x) = \begin{cases} f_0(|t|\tau + (1-\tau), x), & x \notin B_S(y_0, \varepsilon), \\ g(t, x), & x \in B_S(y_0, \varepsilon), \end{cases}$$

is a Lipschitz homotopy joining g to f_0. The desired homotopy H_τ is now obtained by applying successively F_τ^0, F_τ and F_τ^1. Since all three have the constant value z_0 for $|t| \geq \frac{3}{4}$, the same holds for H_τ.

3. To prove Proposition 2, we shall need two lemmas.

LEMMA 1. *Let X be a normed space and $\frac{1}{8} > \varepsilon > 0$. Let a, b be two points in X so that $a \neq b$ and $\|a\| = \|b\| = \frac{1}{4}$. There exists a map $U = U_{a,b}$: $B_X \to B_X$ satisfying*:

(3.1) *U satisfies a Lipschitz condition with constant at most* $1 + 1/2\varepsilon$.

(3.2) *U maps $B_X(a, \varepsilon)$ isometrically onto $B_X(b, \varepsilon)$, and $Ua = b$.*

(3.3) $U^{-1}(B_X(b, \varepsilon)) = B_X(a, \varepsilon)$.

(3.4) *$Ux = x$ whenever $d(x, [a, b]) \geq 2\varepsilon$ (where $[a, b] = \{ta + (1-t)b\colon 0 \leq t \leq 1\}$). In particular $Ux = x$ for $x \in S_X$.*

(3.5) *U maps lines parallel to $[a, b]$ into themselves.*

PROOF. Define

$$\alpha(x) = \begin{cases} 1, & d(x, [a, b]) \leq \varepsilon, \\ 2 - \varepsilon^{-1} d(x, [a, b]), & \varepsilon \leq d(x, [a, b]) \leq 2\varepsilon, \\ 0, & d(x, [a, b]) > 2\varepsilon, \end{cases}$$

Then $\alpha\colon B_X \to [0, 1]$ satisfies a Lipschitz condition with constant $1/\varepsilon$.

Now set $Ux = x + \alpha(x)(b - a)$. The conditions on $\|a\|$, $\|b\|$ and ε immediately imply that U maps B_X into itself, and that it satisfies (3.1)–(3.5). We only check (3.3): If $Ux \in B(b, \varepsilon)$, we have

$$\varepsilon \geq \|b - Ux\| = \|b - x - \alpha(x)(b - a)\| = \|[\alpha(x)a + (1 - \alpha(x))b] - x\|.$$

But $0 \leq \alpha(x) \leq 1$, so that $\alpha(x)a + (1 - \alpha(x))b \in [a, b]$. Thus $d(x, [a, b]) \leq \varepsilon$ and $\alpha(x) = 1$, i.e. $\varepsilon \geq \|b - Ux\| = \|b - x - (b - a)\| = \|a - x\|$, and $x \in B(a, \varepsilon)$.

LEMMA 2. *Let X be an infinite-dimensional normed space and* $0 < \varepsilon \leqslant 1/500$. *Then there exists a point* $x_0 \in B_X$ *which is an* ε*-escaping point in* B_X *with respect to a map* T: $B_X \to B_X$ *which, in addition to* (1.1)–(1.4), *also satisfies*

$$(3.6) \qquad Tx = x \quad \text{for } x \in S_X.$$

PROOF. Note that the homotopy condition (1.1) is trivially satisfied here because B_X is convex, hence Lipschitz contractible. Let $\{w_n\}_{n=1}^{\infty}$ be a normalized basic sequence in X with biorthogonal functionals $\{\varphi_n\} \subset X^*$ satisfying $\|\varphi_n\| \leqslant 4$, and set $z_n = w_n/4$. (See [**2**, p. 93].)

Denote by $L_{n,k}$ $(n \neq k)$ the straight line $\{tz_n + (1-t)z_k : t \in R\}$ passing through z_n and z_k. If $\{n, k\} \cap \{m, l\} = \varnothing$, then

$$(3.7) \qquad d(L_{n,k}, L_{m,l}) \geqslant 1/32 > 10\varepsilon.$$

Indeed, if $x = tz_n + (1-t)z_k$, $y = \zeta z_m + (1-\zeta)z_l$, assume $|t| \geqslant \frac{1}{2}$ (otherwise $|1-t| \geqslant \frac{1}{2}$, and then

$$\|x - y\| \geqslant \tfrac{1}{4}\left|\varphi_n\left(tz_n + (1-t)z_k - \zeta z_m - (1-\zeta)z_l\right)\right| \geqslant |t|/16 \geqslant 1/32.$$

Denote by $U_{n,m}$ the map constructed in Lemma 1 for $a = z_n$, $b = z_m$ and the given ε. Note that by (3.7), if $\{n, k\} \cap \{m, l\} = \varnothing$, then $U_{n,k}(x) = x$ whenever $d(x, L_{m,l}) \leqslant 2\varepsilon$, and, in particular, when $U_{m,l}(x) \neq x$, or $x = U_{m,l}(y)$ for some $y \neq x$. Thus the infinite composition $V_1(x) = (\cdots \circ U_{2n-1,2n} \circ \cdots \circ U_{3,4} \circ U_{1,2})(x)$ is well defined and satisfies:

(3.8) V_1 is a Lipschitz map with constant at most $1 + 1/2\varepsilon$.

(3.9) V_1 maps $B_X(z_{2n-1}, \varepsilon)$ isometrically onto $B_X(z_{2n}, \varepsilon)$.

(3.10) $V_1^{-1}(B_X(z_{2n}, \varepsilon)) = B_X(z_{2n-1}, \varepsilon)$.

(3.11) $V_1 x = x$ for $x \in S_X$.

Defining similarly $V_2(X) = (\cdots \circ U_{2n,2n+1} \circ \cdots \circ U_{4,5} \circ U_{2,3})(x)$, V_2 also satisfies (3.8), (3.11) and:

(3.12) V_2 maps $B_X(z_{2n}, \varepsilon)$ isometrically onto $B_X(z_{2n+1}, \varepsilon)$.

(3.13) $V_2^{-1}(B_X(z_{2n+1}, \varepsilon)) = B_X(z_{2n}, \varepsilon)$.

We now define $T = V_2 V_1$. T is a Lipschitz function, and z_1 is an ε-escaping point with an associated map T. Indeed, as observed before, (1.1) is trivially satisfied. Also $T^n z_1 = z_{2n+1}$, and (1.2) follows from (3.7). Condition (1.3) follows from (3.9) and (3.12), and (1.4) from (3.10) and (3.13).

PROOF OF PROPOSITION 2. Let Y be a closed subspace of X of codimension one. Identify X with $R \oplus Y$ and, by equivalently renorming X, if necessary, assume that $\|(t, y)\| = \max(|t|, \|y\|)$.

By Lemma 2 there is a point $y_0 \in B_Y$ which is ε-escaping in B_Y with respect to a Lipschitz map V: $B_Y \to B_Y$ satisfying $Vy = y$ whenever $\|y\| = 1$.

We have $S_X = (\{1\} \times B_Y) \cup ([-1, 1] \times S_Y) \cup (\{-1\} \times B_Y)$, and define T: $S_X \to S_X$ by

$$T(t, y) = \begin{cases} (1, Vy), & t = 1, \\ (t, y), & t \neq 1. \end{cases}$$

Then T is a well-defined Lipschitz map, $T\colon S_X \to S_X$, and $(1, y_0)$ is an ε-escaping point of S_X associated with this T. Indeed (1.2)–(1.4) follow immediately from the corresponding properties of V, and T is Lipschitz homotopic to the identity by the homotopy $H_\tau(x) = \tau Tx + (1-\tau)x$. By the special structure of S_X and the fact that $Tx \neq x$ only for points $x = (t, y)$ with $t = 1, y \in B_Y$, it follows that $H_\tau(x)$ is indeed a point of S_X whenever x is.

4. Proof of Proposition 3. The proposition is trivial when X is one dimensional, so assume $\dim X \geqslant 2$. Fix $x_0 \in S_X$ and let $\varphi \in X^*$ satisfy $\|\varphi\| = \varphi(x) = 1$. Renorming X by $|||x||| = |\varphi(x)| + \|x - \varphi(x)x_0\|$, we obtain a representation of X as $R \oplus Y$, with $Y = \operatorname{Ker}(\varphi)$, and with norm $\|(t, y)\| = |t| + \|y\|$. The point x_0 is identified with $(1, 0)$. To save notation we assume this is the given norm on X.

Now define, for $\varepsilon/2 \leqslant \tau \leqslant 2$, a function $\varphi_\tau\colon [-1, 1] \to [-1, 1]$ by

$$\varphi_\tau(t) = \begin{cases} 2\tau^{-1}t + 1 - 2\tau^{-1}, & 1 - \tau \leqslant t \leqslant 1, \\ -1, & -1 \leqslant t \leqslant 1 - \tau, \end{cases}$$

and let

$$F_\tau(t, y) = \left(\varphi_\tau(t), \frac{1 - |\varphi_\tau(t)|}{1 - |t|} y\right) \qquad (F_\tau(\pm 1, 0) = (\pm 1, 0)).$$

If $(t, y) \in S_X$, i.e. $|t| + \|y\| = 1$, then also $F_\tau(t, y) \in S_X$, and F_τ, $\varepsilon/2 \leqslant \tau \leqslant 2$, is a Lipschitz homotopy of S_X, with Lipschitz constant c/ε for some $c < \infty$. (We leave the straightforward verification to the reader.)

For $\tau = 2$, $F_2(t, y) = (t, y)$, i.e. F_2 is the identity, while $f(t, y) = F_{\varepsilon/2}(t, y)$ satisfies $f(x) = (-1, 0) = -x_0$ whenever $\|x - x_0\| \geqslant \varepsilon$.

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