SPHERICAL ABERRATION IN THIN LENSES.

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ABSTRACT.

It is proposed in this article to present an elementary theory of the spherical aberration of thin lenses, to give means for determining quickly the aberration of any thin lens for any position of the object, and to formulate a statement of the conditions under which the spherical aberrations of two thin lenses will compensate one another. This last is confined to the simplest case, in which the lenses are close together. The treatment is in part analytical, in part graphical.

In addition there is included in this paper a graphical solution of the problem as to the conditions under which a two-piece lens may be achromatic, free from axial spherical aberration, cemented, and free from coma, and the shapes of the lenses necessary to satisfy these different conditions are shown. The effect of a slight change in the shape of the lenses is also indicated.

It is not expected that any of the material is really new, but the author knows of no place where the information given may be readily obtained, even piecemeal.

CONTENTS.

I. LENS LAW FOR PARAXIAL RAYS.

For a thin lens of very small aperture, and for objects very near the axis of the lens, the relation between object and image distances may be expressed as

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f} \qquad (1)$$

The convention of signs here used is as follows: u and v are positive if the image is real, negative if the image is virtual. f is positive for a converging lens. Radii are positive when the surface involved has the effect of increasing the power of the lens positively. For a double-convex lens both radii are +, for a double-concave −.

II. SPHERICAL ABERRATION—THIN LENS.

For apertures of any considerable diameter the equation (1) is insufficient, for here the rays which pass through the outer zones of the lens do not, after refraction, intersect the axis at the same point as the rays which pass through zones which are close to the axis of the lens. Equation (1) is based on the assumption that all the angles involved are so small that the sine of the angle

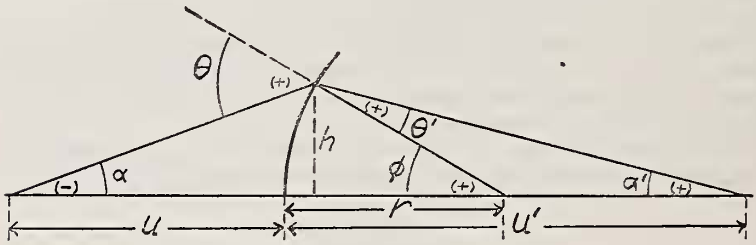

FIG. 1.—*Refraction at a spherical surface.*

may be taken as equal to the angle. A somewhat closer approximation is obtained by writing (consult Fig. 1),

$$\sin\alpha = -\frac{h}{u}\left[1-\frac{1}{2}\frac{h^2}{u}\left(\frac{1}{u}+\frac{1}{r}\right)\right]$$
$$\sin\alpha' = +\frac{h}{u'}\left[1-\frac{1}{2}\frac{h^2}{u'}\left(\frac{1}{u'}+\frac{1}{r}\right)\right] \qquad (2)$$

The equations which connect u and u' are

$$\frac{u+r}{\sin\theta}=\frac{r}{-\sin\alpha}$$

$$\frac{\sin\theta}{\sin\theta'}=n$$

$$\frac{u'-r}{\sin\theta'}=\frac{r}{+\sin\alpha'}$$

or, eliminating θ and θ'

$$\frac{u+r}{u'-r}=-n\frac{\sin\alpha'}{\sin\alpha} \qquad (3)$$

If, now, the values for α and α' from (2) be substituted in (3), a somewhat tedious but not difficult reduction gives

$$\frac{n}{u'}-\frac{n}{u'_0}=+\frac{h^2}{2}\frac{n-1}{n^2}\left(\frac{1}{u}+\frac{1}{r}\right)^2\left(\frac{1}{r}+\frac{n+1}{u}\right) \tag{4}$$

where u_0 indicates the distance from the refracting surface to the point at which the paraxial rays cross the axis.

A similar expression is obtained for the refraction at the second surface of a lens. The two terms may be readily combined into a single expression in the case of a thin lens. For a thick lens this combination can not be made simply.

The expression for the thin lens is

$$\begin{aligned}\frac{1}{v}-\frac{1}{v_0}=\Delta\left(\frac{1}{v}\right)=\frac{h^2}{2}\frac{n-1}{n^2}\Bigg[&\left(\frac{1}{r_1}+\frac{1}{u}\right)^2\left(\frac{1}{r_1}+\frac{n+1}{u}\right)\\ &+\left(\frac{1}{r_2}+\frac{1}{v}\right)^2\left(\frac{1}{r_2}+\frac{n+1}{v}\right)\Bigg]\end{aligned} \tag{5}$$

where u is the distance from the lens to the object and v is the distance from the lens to the image.

III. THE CODDINGTON NOTATION.

The expression for the aberration (5) may be put in a form, which is more readily handled, by using a notation which is due to Coddington.[1]

In order to express the spherical aberration of a thin lens, one needs to specify: (*a*) The shape of the lens, and (*b*) the position of the object in terms of the focal length. The two factors which Coddington used for this purpose are:

A. Shape factor (s).

$$s=\frac{2\rho}{r_1}-1=1-\frac{2\rho}{r_2}=\frac{1-r_1/r_2}{1+r_1/r_2}$$

where as a notation

$$\frac{1}{\rho}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{(n-1)f}$$

B. Position factor (p),

$$p=\frac{2f}{u}-1=1-\frac{2f}{v}=\frac{1-u/v}{1+u/v}$$

[1] Taylor, System of Applied Optics, p. 66, 1906.

These expressions may be solved for r_1, r_2, u, and v. These values when substituted in (5) reduce the expression for the aberration to

$$\Delta\left(\frac{1}{v}\right)=\frac{1}{n\,(n-1)}\,\frac{h^2}{8f^3}\left[\frac{n+2}{n-1}\,s^2+2(2n+2)s\cdot p+(3n+2)(n-1)^2p^2+\frac{n^3}{n-1}\right]=\frac{h^2}{f^3}\left[As^2+B\,s\cdot p+Cp^2+D\right]=\frac{h^2}{f^3}\cdot S \qquad (6)$$

where A, B, C, and D involve the indices of refraction only, and are consequently constant for a given type of optical glass. Beck[2] has published in the Proceedings of the Optical Convention for 1912 an extensive table from which the constants A, B, C, D may

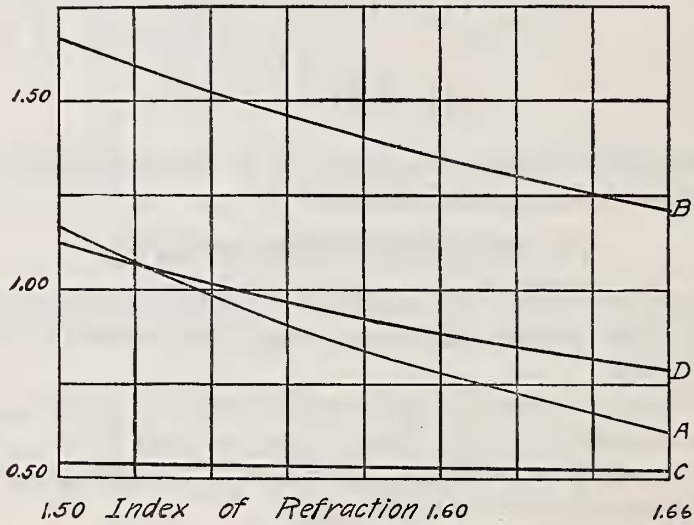

FIG. 2.—*Constants involved in the algebraic expression for spherical aberration.*

The plot is a graphical representation of the variation of the constants of equation (6) with the index of refraction.

be readily obtained. To give an idea of the manner in which these constants vary, their values are shown graphically in Figure 2 in terms of the index of refraction.

IV. SHAPE AND POSITION FACTORS.

With the aid of the shape and position factors of Coddington the simplified and rather easily manipulated expression of equation (6) was obtained. Before proceeding with the application of this

[2] Beck tabulates $8A$, $2B$, $8C$, and $8D$.

quadratic in the calculation of the spherical aberration it is desirable to give some interpretation of these two factors.

The shape factor is expressed in Figure 3 in terms of the ratio of the radii of the lens, and one may read from the graph the shape of the lens corresponding to various values of *s*. Several examples are given in Table 1.

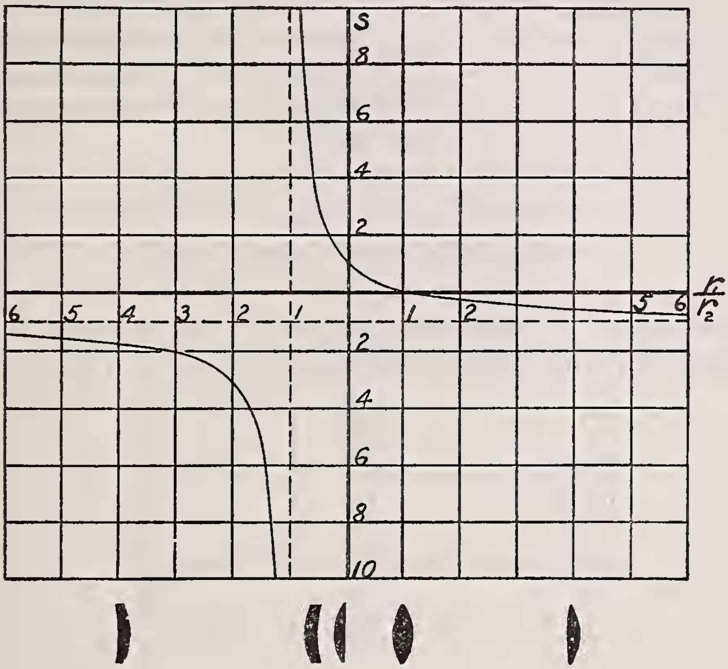

FIG. 3.—*Coddington's shape factor as a function of the radii.*

The shape factor for a plano-convex lens is +1 or −1, according as the convex side ($r_1/r_2=0$) or the plane side ($r_1/r_2=\infty$) faces the object. For an equiconvex lens ($r_1/r_2=1$) the shape factor is zero. The range from −1 to +1 for the shape factor includes all save meniscus lenses. The sketches at the bottom of the figure indicate the types of lenses corresponding to several values of the shape factor. Obviously, if both radii change sign, the ratio r_1/r_2 and the shape factor are both unchanged.

TABLE 1.

	s.	r_1/r_2.	Type of lens.
a.....	0	1	The lens is equi convex, either double convex or double concave.
b.....	+1	0	The lens is plano-convex, or plano-concave, with the curved side toward the incident beam.
c.....	−1	∞	The lens is plano-convex, or plano-concave, with the flat side toward the incident beam.
d.....	±∞	−1	Watch glass.
e.....	+1 to ∞	0 to −1	A concave-convex converging lens with the convex side toward the incident light or a like diverging lens with the concave side toward the incident beam.
f.....	−1 to ∞	∞ to −1	The lenses of e, reversed with reference to the direction of the incident beam.

The position factor is represented in Figure 4 in terms of the object distance. A short table of values follows which shows the values of *u* and *v* corresponding to a number of values of *p*.

u	v	p
∞	f	−1
f	∞	+1
2f	2f	0
<f	Negative.	Positive and greater than +1.
Negative.	<f	Negative and less than −1.

V. ABERRATION OF SIMPLE, THIN LENSES.

Equation (6) may be used to calculate the spherical aberration of any thin lens for any position of the object. The values of the constants A, B, C, and D of equation (6) for two common glasses, $n=1.520$ and $n=1.620$, are given below:

n	A	B	C	D
1.520	1.07054	1.59413	0.53947	1.06805
1.620	0.72664	1.30426	0.52932	0.85341

And in Figure 2 is represented graphically the variation of these constants for indices within the range from 1.50 to 1.65.

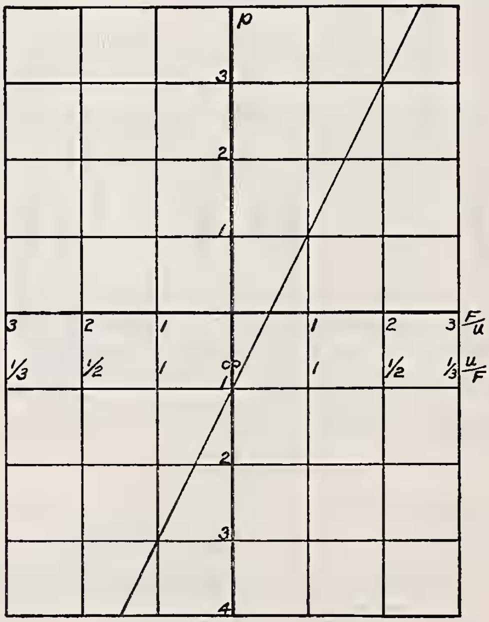

FIG. 4.—*Coddington's position factor as a function of the object distance.*

For a distant object, $p=-1$; for an object located at the principal focus of the lens, $p=+1$; for the symmetrical case where object and image are equally distant from the lens, $p=0$. With a converging lens, large positive values of p correspond to virtual images; large negative values correspond to virtual objects. With a diverging lens "images" and "objects" should be interchanged in the statement just made.

The values of the coefficient of h^2/f^3 for a number of indices and for various values of p and s were calculated by Arthur F. Eckel, of the optical instruments section of the Bureau of Standards, and the values so obtained are given in Tables 1 to 7. To obtain $\Delta\left(\frac{1}{v}\right)$ these figures need to be multiplied by h^2/f^3, and to obtain the longitudinal aberration by $(h^2/f^3)v^2$. If the values entered in the tables be indicated by S, then the spherical aberration is given by

$$\Delta\left(\frac{1}{v}\right)=\frac{h^2}{f^3}\cdot S$$

These same figures have been represented graphically by Mr. Eckel and the results are shown in Plates I to VII. The curves are plotted for a given lens, the position of the object

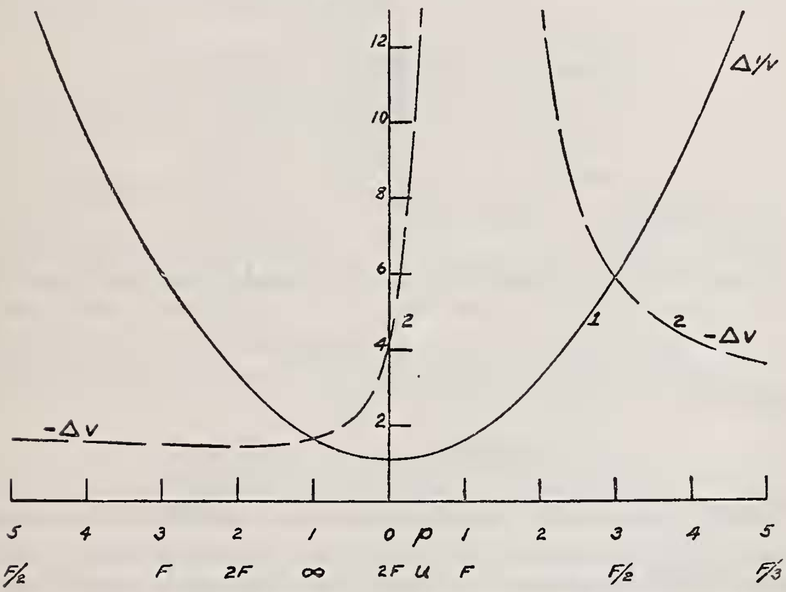

FIG. 5.—*Comparison of the longitudinal aberration* ($\triangle v$) *and the reciprocal aberration* ($\triangle\frac{1}{v}$).

The curve marked $\triangle\frac{1}{v}$ represents the values of S of equation (6) plotted as a function of the position factor p. The curve is a parabola. The longitudinal aberration is $-\frac{h^2}{f^3}Sv^2$, and the curve marked $\triangle v$ gives the value of $S\frac{v^2}{f^2}$ as a function of the position factor p.

being assumed to vary. The curves are, of course, all parabolas in p and the representation is comparatively simple.

To show the advantage of Coddington's notation in such representation, and to show likewise the simplicity of dealing with $\Delta\left(\frac{1}{v}\right)$ instead of Δv, Figures 5 and 6 have been drawn. Figure 5 shows, with p as the independent variable, the aberration for an equi-convex lens of index 1.52. Curve *1* represents the value of $\Delta\left(\frac{1}{v}\right)$ and the curves *2* the values of Δv. Figure 6 shows the same quantities plotted with the object distance (u) as the independent variable. The representation in Figure 5, curve *1*, is much the simplest.

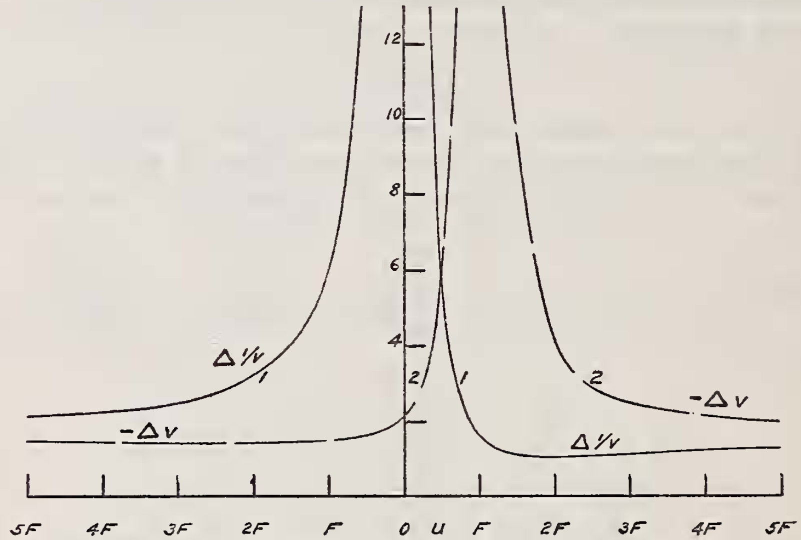

FIG. 6.—*The aberrations as a function of the object distance.*

The values of Figure 5 are replotted as a function of the object distance, u. These curves are appreciably less simple than curve *1* of Figure 5.

VI. COMBINATION OF LENSES.

With the data now in hand, one can now proceed to obtain approximately the spherical aberrations for any combination of thin lenses. If the lenses are in contact, one needs to add the spherical aberrations $\left(\Delta\left(\frac{1}{v}\right)\right)$ for the separate lenses in order to obtain the resultant reciprocal aberration. It is, therefore, possible to find the aberration of any combination of thin lenses or at pleasure to determine the combinations of lenses which will have any desired aberration. In particular, it is possible to calculate all of the combinations of two given glasses which shall be achromatic (to the first approximation) and which shall likewise be free from axial spherical aberration. A method of carrying out such a calculation is given herewith.

1. ELIMINATION OF CHROMATIC ABERRATION.

It is well known[3] that for two thin lenses in contact the chromatic aberration will be corrected if the focal lengths of the two lenses are in the ratio of the dispersive powers of the two glasses. It is, however, customary to use the reciprocal of the dispersive power in lens calculations. This constant, designated as ν, is $\frac{n_D - 1}{n_F - n_C}$. If the focal lengths of the two lenses be f_1 and f_2, and if the dispersion constants of the two glasses be ν_1 and ν_2, then

$$\frac{f_1}{f_2} = -\frac{\nu_2}{\nu_1} = -k \tag{8}$$

where k indicates the ratio $\frac{\nu_2}{\nu_1}$

2. ELIMINATION OF SPHERICAL ABERRATION.

The condition to be satisfied here is

$$\Delta\left(\frac{1}{v}\right) = O = \Delta\left(\frac{1}{v_1}\right) + \Delta\left(\frac{1}{v_2}\right) \tag{9}$$

where $\Delta\left(\frac{1}{v_1}\right)$ and $\Delta\left(\frac{1}{v_2}\right)$ are to be calculated from equation (6), or its equivalent. We have, therefore,

$$\Delta\left(\frac{1}{v}\right) = \frac{h_1^2}{f_1^3} S_1 + \frac{h_2^2}{f_2^3} S_2 \tag{10}$$

where h_1 and h_2 are, of course, equal for two thin lenses in contact, and where the relation between f_1 and f_2 is that given in equation (8).

As a matter of convenience, this relation may be incorporated in (10) by substituting f_1 for f_2, which gives

$$\Delta\left(\frac{1}{v}\right) = \frac{h^2}{f_1^3} [S_1 - k^3 \cdot S_2] \tag{11}$$

and the condition for eliminating spherical aberration is that the aberration constant, S_1, of the leading lens shall be equal to k^3 times the aberration constant, S_2, of the second lens, for the particular shapes of lenses used and for the position of the object or of the apparent object in either case.

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[3] Southall, Geometrical Optics, 2d ed., p. 519; Whittaker, Theory of Optical Instruments, p. 50; Houstoun, Treatise on Light, p. 63.

3. POSITION OF THE APPARENT OBJECT FOR THE SECOND LENS.

The values of S_1 and S_2 depend upon both the shape of the lens and the position of the apparent object for the lens. The shape of one lens of a combination is entirely independent of the shape of the other, unless one interposes the condition that the spherical aberration shall vanish or some other limiting condition. Between the two position factors, however, a relation has already been assumed, when the condition was laid down that the combination should be achromatic.

From the definition of p (see sec. 3 above) one may write

$$p_1 = 1 - \frac{2f_1}{v_1} \quad p_2 = \frac{2f_2}{u_2} - 1 \tag{12}$$

and because the lenses are to be in contact,

$$v_1 = -u_2 \tag{13}$$

Substituting in (13) from (12) gives the result

$$p_1 - 1 = -k(p_2 + 1) \tag{14}$$

Equations (11) and (14) contain the solution of the problem we have set, namely, assuming the focal length of the combination, types of glasses available, and position of the object, to find the shapes of lenses which in contact will give images chromatically corrected and free from axial spherical aberration.

The author was interested primarily in telescopic lenses at the time the investigation was begun, and the solutions so far carried through are all for telescopic objectives. As an example, the case of a telescope objective to be made of ordinary crown and medium dense flint will be carried through.

VII. GRAPHICAL SOLUTION FOR A TELESCOPIC OBJECTIVE.

Let us assume that we have glass with the following constants:

	Index (n_D)	Dispersion constant (ν)
Crown	1.520	60
Flint	1.620	36

and that we elect to have the crown lens nearer the object. Here, then,

$$k = -\frac{f_1}{f_2} = \frac{\nu_2}{\nu_1} = 0.6$$

and

$$p_1 = -1 \quad p_2 = +2.333$$

From the values of S given in the tables, or the plotted values shown in the graphs, the values of S_1, for a glass of index 1.520 and for a position factor of −1, are obtained and are shown plotted in Figure 7, curve *I*. Similarly the values of S_2 for $p_2 = 2.33$ were read from the curves and these values are shown in curve *II* of the same figure, although for the purpose of the calculation curve *II* needs to be modified. The modification

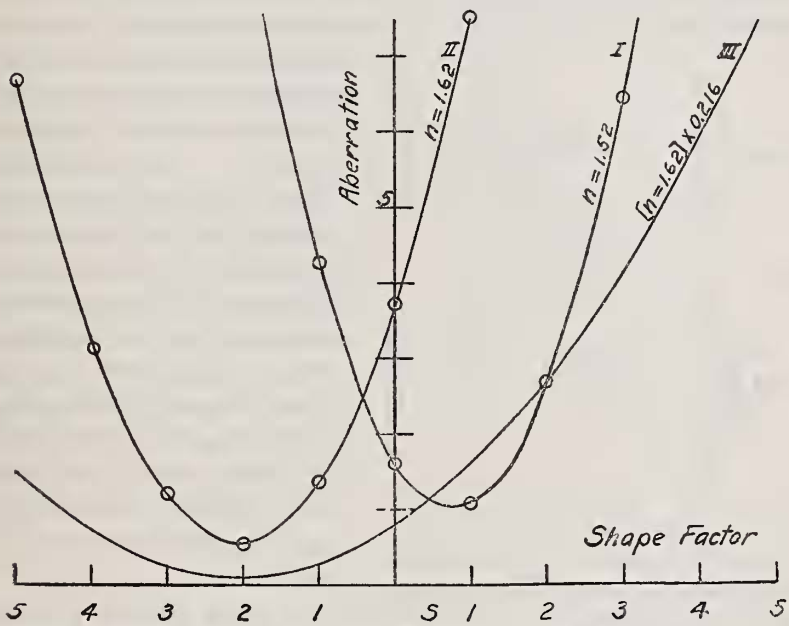

FIG. 7.—*The curves for the graphical calculation.*

Curve *I* represents the values of the aberration constant (S) as a function of the shape factor (s) for a crown glass lens (index 1.520), with the object at infinity ($p = -1$). Curve *II* represents the values of S for a flint glass lens (index 1.620), with the object virtual and located at $0.6 \times f_2$; that is, at a distance equal to the focal length of the crown lens ($p = 2.333$). Curve *III* is curve *II* multiplied by $(f_1/f_2)^3$, so that the aberrations of the two lenses may be directly compared.

It is obvious that $\triangle\frac{1}{v}$ for the first lens (curve *I*) is positive and that $\triangle\frac{1}{v}$ for the second lens (curve *II*) is negative, as f_2 is negative. Consult section VII.

required is shown in equation (11), from which it appears that the values of S_2 should be multiplied by k^3. Curve *III* shows k^3S_2 or .216 S_2. Curves *I* and *III* are the essential curves.

If any point on curve *I* is chosen, another point on curve *I* and two points on curve *III* may be found with the same "aberrations" as the first point selected. The shape factors corresponding to these four points give four pairs of crown and flint lenses which may be placed together to obtain a two-piece lens free from axial spherical aberration. For example, $s_1 = o$ or $s_1 = 1.5$ and $s_2 = 0.98$

or $s_2 = -5.18$ give the four pairs (0, 0.98), (0 − 5.18), (1.5, 0.98), (1.5, − 5.18), any one of which will be aberration free. The four lenses are shown in Figure 8. Either of the flint lenses placed back of either of the crown lenses will give an aberration free combination. The first of these is the common form of small telescope objective (0, 1.0), made up of a double convex crown and a plano-concave flint lens of equal radii.

A series of such sets of values for s_1 and s_2 were obtained from curves *I* and *III*, giving thus a number of examples of aberration free pairs of thin lenses. The values of s_1 and s_2 so obtained are represented by a graph in Figure 9. This graph shows all the possible pairs of thin lenses which are corrected for chromatic and spherical aberration under the conditions assumed for our problem. These conditions are that the lenses be made of the glasses specified at the beginning of this section, that the lenses be in contact, and that the crown lens be toward the object.

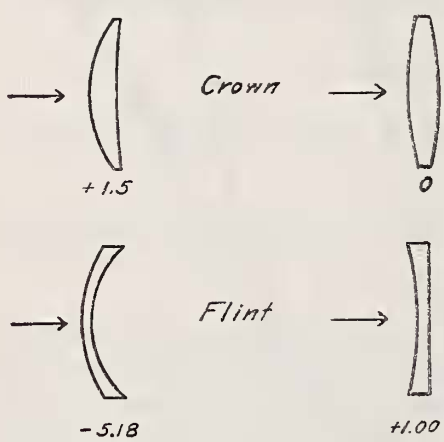

Fig. 8.—*Four lenses which may be combined to give a pair free from spherical aberration.*

Either of the crown lenses followed by either of the flint lenses will form such a pair, the direction of the light being as indicated by the arrows.

It is, of course, perfectly possible to put the flint lens in front in a telescope objective (though it is probably not advisable), and the combinations so resulting have been calculated and are shown in Figure 10.

In comparing two figures, 9 and 10, it should be borne in mind that when the direction of the light through the lens is reversed (that is, r_1 and r_2 interchanged) there is a change in the sign of the shape factor, s. For example, the lens (−0.60, 2.15) of Figure 9 is the same lens as (0.60, −2.15) of Figure 10, and this lens would be nearly free from spherical aberration when used as a telescopic lens with either the crown or the flint leading.

A curve of the type of those in Figure 9 and Figure 10 can be constructed for any pair of glasses with a few hours work of figuring and sketching. Mr. Eckel and the author have drawn such curves for a number of combinations of glasses, all of which show the same general characteristics as those shown in Figures 9 and 10. One other of these curves is reproduced as Figure 11, for the combination of a light barium crown and a medium dense flint. The indices are 1.570 and 1.620 with a ratio between the focal lengths of 0.65, which is a possible value. This lens is of interest because these indices are nearly the ones used in the manufacture of prism field-glass objectives.

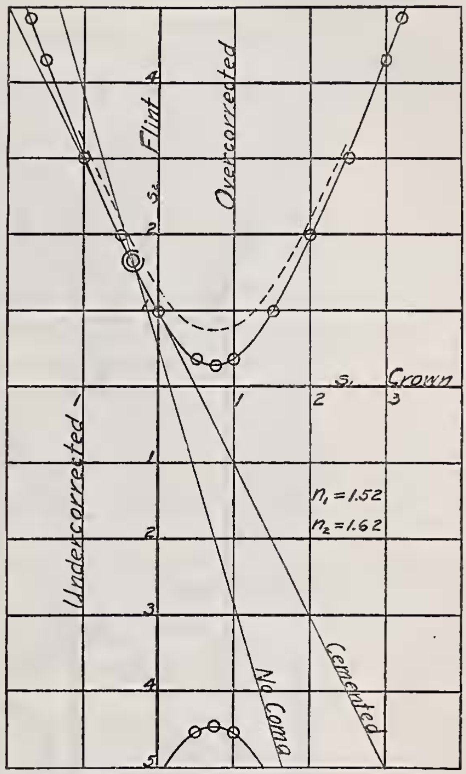

FIG. 9.—*Aberration free pairs of "ordinary" crown and medium dense flint glasses, the crown lens leading.*

The shape factors (cf. Sec. III) for the two lenses are the coordinates. The full line hyperbola gives the shapes of pairs of thin lenses with no spherical aberration. The broken line shows five times the change necessary in s_2 to produce a longitudinal aberration of $f/100$ for a value of h equal to $f/10$.

The straight lines labeled "No coma" and "Cemented" represent the conditions that the lens be free from coma and that the radii of the surfaces in contact be the same to permit of cementing the two lenses together.

Attention should, perhaps, be called to the fact that there is no great increase in labor involved in obtaining these curves for glasses with indices which are not quite the even numbers used in the tables. To obtain the curves corresponding to curve *I*, or curve *III* of Figure 7, it is necessary to draw two such curves for neighboring indices and then to extrapolate or interpolate in order to get the aberration for the desired index. The two auxiliary curves are similar in shape and should be close enough together that the errors of the interpolation will be probably well within the range of the accuracy which is possible with graphical work of this character. If one had much of this work to do, it would be desirable to tabulate and plot the change produced in the aberration by small changes in the index, and to use these differences in modifying the available curves.

VIII. CONDITION THAT THE LENS BE CEMENTED.

The additional condition to be satisfied, if it is desired to cement the two lenses, may be readily obtained. To satisfy this condition the back surface of the leading lens must fit the front surface of the second lens, or

$$r_2 = -r_1' \tag{15}$$

where the primes (′) refer to the second lens.

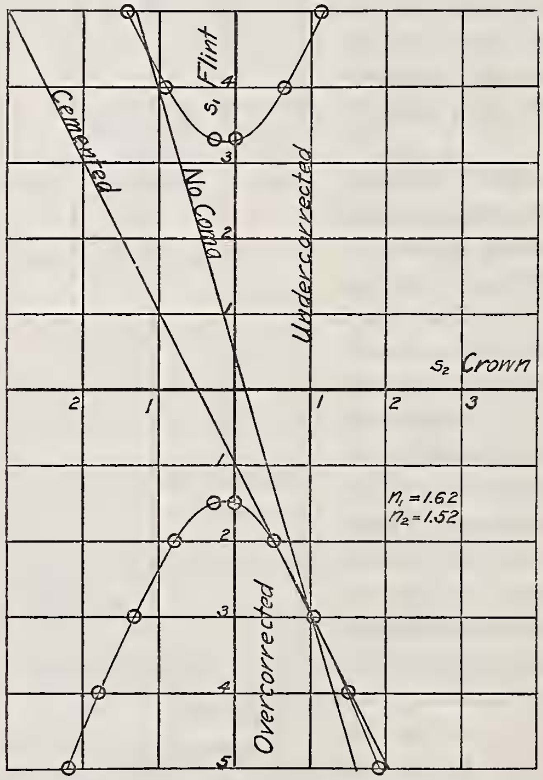

FIG. 10.—*Aberration free pairs of medium dense flint and "ordinary" crown glasses, the flint lens leading.*

In comparing Figure 10 with Figure 9 it should be borne in mind that the shape factor for a lens changes sign if the lens is turned over. Hence the first quadrant in Figure 9 is comparable with the third quadrant in Figure 10.

In terms of the shape factor, s, (see sec. 3)

$$\frac{1}{r_2} = \frac{1 - s_1}{2\rho_1} \tag{16}$$

$$\frac{1}{r_1'} = \frac{1 + s_2}{2\rho_2}$$

where

$$\frac{1}{\rho}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{(n-1)f}$$

and the desired relation is, therefore

$$s_1-1=\frac{\rho_1}{\rho_2}(s_2+1)\frac{f_1(n_1-1)}{f_2(n_2-1)}(s_2+1)=-\frac{n_1-1}{n_2-1}.k.(s_2+1) \quad (17)$$

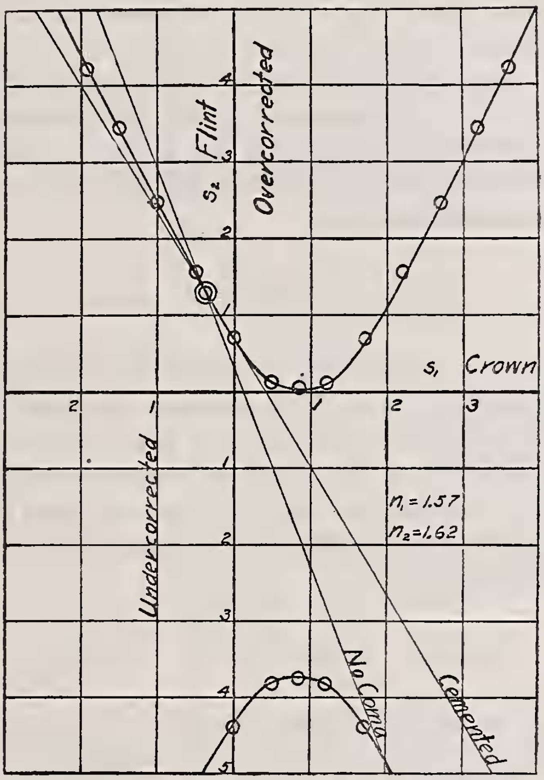

FIG. 11.—*Aberration free pairs of light barium crown and medium dense flint glasses, the crown lens leading.*

Consult Figure 9.

This relation between s_1 and s_2 is a linear one and is shown by straight lines in Figures 9, 10, and 11. The intersections of this line with the hyperbola give a cemented, achromatic, aberration free lens. These intersections may be imaginary, in which case the stipulation that the lenses are to be cemented will need to be omitted.

IX. CONDITION THAT THE LENS BE FREE FROM COMA.

There is a fourth condition which it is sometimes desired to impose upon a telescope objective, namely, that there shall be no coma. This is equivalent to saying that not only is a distant point on the axis to have a sharp image, but that points a little distance from the axis shall likewise be sharply defined. Dennis Taylor[4] has shown that this condition imposes a second linear relation between the shapes of the two lenses, and this relation is likewise represented in Figures 9, 10, and 11.

When the object lies slightly off the axis, there will be terms in the expression for the aberration which will involve the angle between the chief ray of the pencil and the axis of the lens. The term which involves the first power of the angle between axis and chief ray, ψ, assumes the form

$$\Delta\left(\frac{1}{v}\right)=\frac{h\tan\psi}{f^2}(P.p+Q.s) \qquad (18)$$

in which p and Q involve only the index of refraction. The approximate values for P and Q are sketched in Figure 12.

Here, as was the case with the axial spherical aberration, the reciprocal aberration for two lenses in contact may be added directly, and the condition for having a lens free from coma will be that in such a case the coefficient of $h \tan \psi$ will vanish. This may be expressed as

$$\frac{1}{f_1^2}(P_1.p_1+Q_1.s_1)+\frac{1}{f_2^2}(P_2.p_2+Q_2s_2)=O \qquad (19)$$

For the combination of ordinary crown ($n = 1.52$, $\nu = 60$) and of medium flint ($n = 1.62$, $\nu = 36$), crown leading, this equation becomes

$$s_2 = -3.39 \cdot s_1 + 0.495. \qquad (20)$$

The position factors, p_1 and p_2, do not appear in (20), because we have assumed a telescope objective, for which $p_1 = -1.00$ and $p_2 = +2.33$, for the glasses chosen. The straight line marked "No coma" in Figure 9 is the graph of equation (20). The corresponding straight lines in Figures 10 and 11 are similarly drawn.

[4] System of Applied Optics, p. 195; equation (4).

The interesting part of Figures 9, 10, and 11 is the region in the neighborhood of the crossing of the "No coma" and "Cemented" lines, and this portion for Figure 9 and Figure 11 has been redrawn to a larger scale. Figure 13 represents this portion of Figure 9 to ten times the scale of that figure. It is evident that the crossing of the two straight lines in Figure 13 is appreciably off the "No spherical" hyperbola. A trigonometric calculation veri-

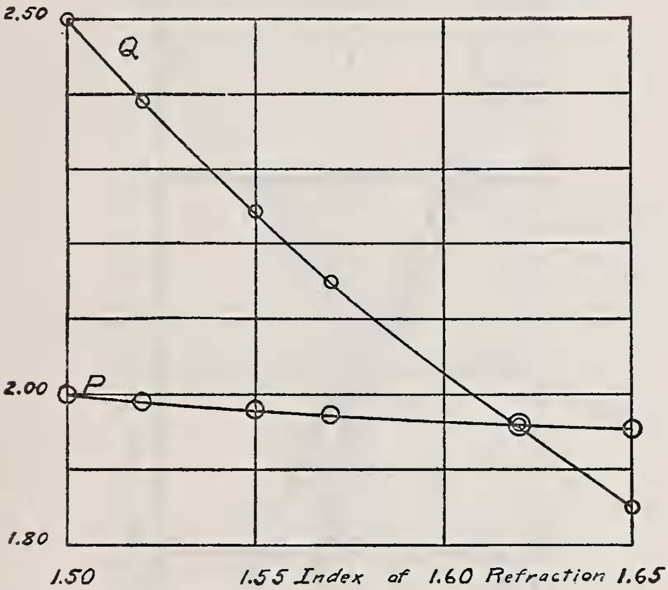

FIG. 12.—*The condition for the absence of coma.*

Plot of the values of P and Q of equation (18).

$P=\frac{3}{4}\frac{2n+1}{n}$ $\qquad Q=\frac{3}{4}\frac{n+1}{n(n-1)}$

fies the prediction, which one could make from the curves, that such a lens would not be a good one, the lens pair being markedly overcorrected.

It is possible to get a lens with the use of the two glasses in question which shall be free from spherical aberration and also free from coma. For this lens the radii of the surfaces will be, for a focal length of 100,

For the crown lens..................................	60.916	31.584
For the flint lens..................................	−32.219	142.80

On the graph this lens is indicated by a double circle.

Figure 14 represents similarly a portion of Figure 11, again to ten times the scale. It is evident that the cemented, coma-free pair will be better corrected than was the case with the ordinary crown combination of Figure 13. The lens would, however, still not be a good one, though at moderate apertures the spherical aberration might not prove troublesome. With glasses of these

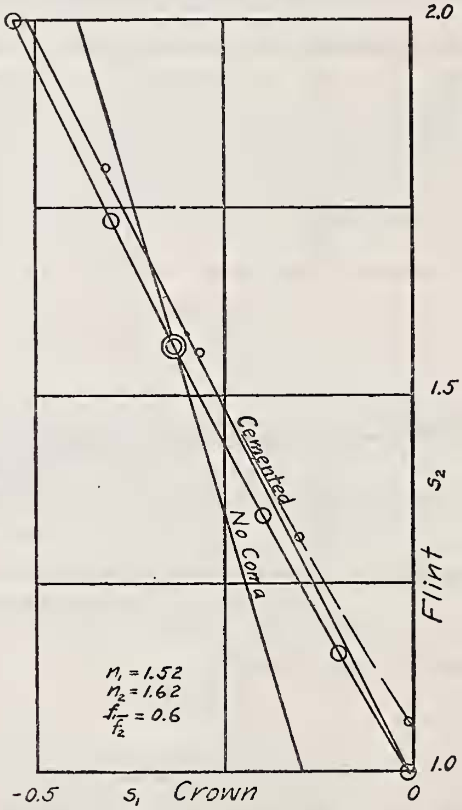

FIG. 13.—*A portion of Figure 9 to a larger scale.*

The broken line indicates the change in s_2 necessary to produce a longitudinal aberration of $f/100$ for a value of h equal to $f/10$.

indices (1.570 for the crown and 1.620 for the flint) it would be possible to get a cemented objective, free from coma, if the ratio of the focal lengths be varied from that assumed. If, instead of a ratio of 0.65, a somewhat smaller value for the ratio of the dispersion constants be assumed, such a lens can be calculated.

X. EFFECT OF SLIGHT VARIATIONS IN SHAPE.

The effect of a slight change in the shape of a lens upon the axial aberration is easily determined by differentiating the expression for S with respect to s. This gives, see equation (6)

$$\frac{\partial S}{\partial s} = \left[2As + Bp\right] \tag{21}$$

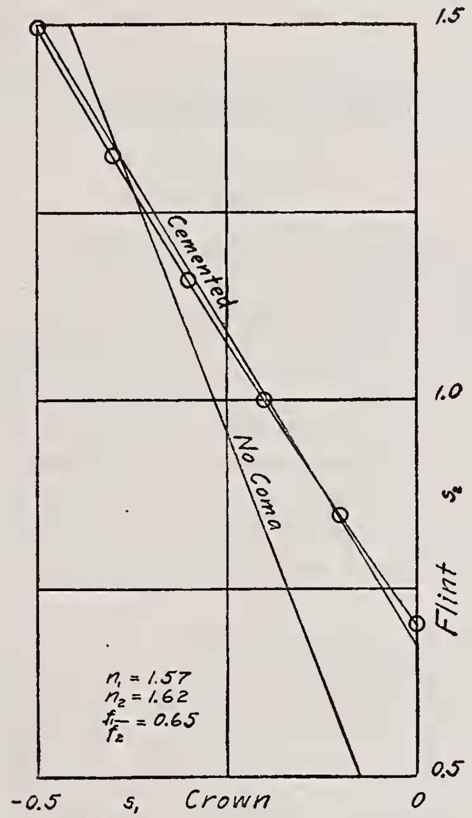

Fig. 14.—*A portion of Figure 11 to a larger scale.*

This expression may be used to determine the effect of slight changes in s upon the sperical aberration of the combination.

As an example, with the aid of (21) and (6), I have calculated the change which would be required in the shape of the second lens to produce a longitudinal aberration of $f/100$ in the case of an objective made from the ordinary crown and the medium dense flint glasses. The result of the calculation is indicated by the broken line, which lies above the upper branch of the hyperbola in Figure 9. The distance between the hyperbola and the broken line, measured parallel to the axis of s_2, gives five times the change in s_2 which would be required to produce a longitudinal aberration of 1 for an assumed focal length of 100 and an assumed h of 10.

The expression used in this calculation was

$$\Delta s_2 = \frac{\Delta S_2}{2A_2 s_2 + B_2 p_2} = \frac{f_2^3}{h^2 v_2^2} \cdot \frac{\Delta v_2}{2A_2 s_2 + B_2 p_2} \qquad (22)$$

where

$$h = 10 \qquad f_2 = 66.67 \qquad v_2 = 100 \qquad \Delta v_2 = 1$$

and A_2, B_2, and p_2 are also known. The formula for the actual calculation is, substituting numerical values in (22),

$$\Delta s_2 = \frac{\Delta v_2}{4.89 s_2 + 10.28}$$

Five times the variation was plotted in Figure 9 in order to prevent crowding of the lines.

In like manner there is indicated in Figure 13 the variation in s_2 which would be necessary to produce a like aberration, $f/100$. Here s_2 itself is indicated, for the larger scale of Figure 13 allows this to be done without crowding. The broken line is drawn for only a portion of the range covered by the figure, so as to prevent confusion with the "cemented" line. However, the calculated values for s_2 are indicated by small circles, so that an estimate of the variation can readily be made with the aid of the "cemented" line.

The not very exhaustive tests which I have made indicate that, at least for a longitudinal aberration no larger than the one considered above, the aberration may be assumed proportional to the change in s_2, and that the approximate aberration for a nearly corrected combination may be scaled from the figure.

Obviously the effect of a change of $-\Delta s_2$ will be to under correct the spherical aberration to the same extent that $+\Delta s_2$ over corrects it. It is further obvious that, having obtained the curve in this particular way, the curve represents the locus of points corresponding to the given aberration. The change in s_1 required to produce the same aberration should, therefore, be obtainable from the curve, except deep down in the bowl of the curve in Figure 9, where the graphical representation is too crude.

Similar data could be obtained for the effect of a slight change in n_1 or n_2, or of p_1 or of k. In determiming the change produced by a change in n_1 or n_2 one would need to proceed carefully, for one may not change either of these without affecting both p_2 and k, both of which enter into the calculations. The example given above is probably the simplest example of the effect of small variations in the lens constants upon the spherical aberration of a lens combination.

XI. IMPORTANCE OF THIN LENS CALCULATIONS.

Algebraic calculations in general give only a near approximation to the performance of a lens. If one wishes to know definitely and completely just what sort of an image will be given by a specified lens, one needs either to carry through extensive trigonometric calculations or to make the lens and test it experimentally.

T. Smith, of the National Physical Laboratory, speaks [5] with the authority of one who has checked many calculations. He gives it as his opinion that, for lenses of the character considered in this paper, the trigonometric calculation is superfluous. He further adduces certain reasons for expecting this result, and shows that the aberration due to the thickness of the lenses and the aberrations of order higher than the ones here considered tend to neutralize one another. As an illustration of this tendency, Figure 15 shows the result of a trigonometric calculation of the linear spherical aberration in the case of a telescope objective, for which the aberration, figured algebraically, is almost exactly zero. This case is probably typical. It is the ordinary cemented combination used in small telescopes, a double-convex crown ($n = 1.520$, $s = 0$) followed by a plano-concave flint ($n = 1.620$, $s = 1$).

Two interesting examples of the closeness of the approximation obtained with algebraic calculations are the lenses represented by the crossing of the two straight lines of Figure 13 and of Figure 14. For a lens nearly the one of Figure 13 ($s_1 = -.338$, $s_2 = 1.658$) the longitudinal aberration for a focal length of 100 was calculated with the aid of equation (22), the variation in s_2 being about 0.050, and the aberration for the same lens was obtained by a trigonometric calculation. A similar calculation, likewise for a focal length of 100, was made for the lens pair of Figure 14 ($s_1 = -.391$, $s_2 = 1.328$), the variation in s_2 being here about 0.018. The results of the two calculations are as follows:

[5] Proc. London Physical Soc., 30, p. 119; 1917–18.

h	Calculated longitudinal aberration.			
	Lens of Figure 12.		Lens of Figure 13.	
	Algebraic.	Trigonometric.	Algebraic.	Trigonometric.
4	0.15	0.16	0.085	0.09
6	.33	....	.19	.22
8	.58	.63	.34	.38
10	.91	1.11	.53	.62
12	1.31	1.73	.76	1.11

For a first approximation the agreement between the algebraic and the trigonometric calculations is good, sufficiently good to indicate that the approximation of the algebraic work is close.

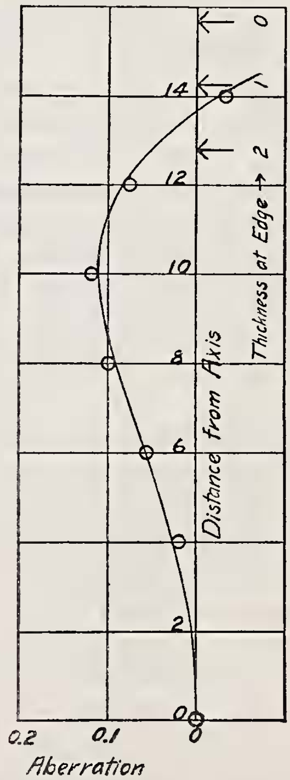

FIG. 15.—*Aberration for a common type of telescope objective.*

The points indicated are the result of a trigonometric calculation. The curve is drawn to fit the algebraic expression for the aberration, $-.0017h^2+.000,00h^4+.000,000,04h^6$. The lens combination is the equi-convex, plano-concave pair of Figure 8, for a focal length of about 100, indices of 1.520 and 1.620, and lens thicknesses at the axis of 6 and 3. A thin lens calculation gives a value for the aberration of this combination which is almost exactly zero. "Thickness at edge" refers to the crown lens.

For any study of the general behavior of lenses and combinations of lenses, the algebraic and graphical method exemplified in the present article, seems to the author to be the only feasible method. It is not thought that the limited field to which this article is confined represents by any means the limits within which this method of attack will give useful results. Probably the effect of separating the components of a lens system upon the axial aberration, coma, astigmatism, etc., could be studied systematically, and general results obtained. Telescopic eyepieces, some of the simpler forms of camera lenses (cf. Taylor, p. 182–188), and perhaps even some of the microscope objectives of moderate aperture might be so studied.

XII. TABULATED VALUES OF *S* FOR TYPICAL THIN LENSES.

Tables giving the two values of S, equation (6), for given values of index of refraction, s, and p—the upper values of S corresponding to cases in which s and p have like signs, the lower values to cases in which s and p have unlike signs.

Index 1.50.

A 1.16670 C 0.54166
B 1.66670 D 1.1250

s	p=0	1	2	3	4	5	6	7	8
0	1.12500	1.66666	3.29164	5.99994	9.79156	14.66650	20.62476	27.66634	35.79124
1	2.29170	4.50006 1.16666	7.79174 1.12494	12.46665 1.86663	17.62506 4.29146	24.16670 7.49970	30.79166 10.79126	40.49994 17.16614	50.29154 23.62434
2	5.79180	9.66686 3.00006	14.62524 1.29164	20.66694 0.66654	27.79196 1.12476	36.00030 2.66630	45.29196 5.29116	55.66694 8.99934	67.12524 13.79084
3	11.62530	17.46697 6.86695	23.79214 3.79174	31.50054 1.49994	40.29226 0.29146	50.16730 0.16630	61.12566 1.12446	73.16734 3.16594	86.64575 5.93734
4	19.79220	27.00066 3.66706	35.89244 9.22524	44.66754 4.66674	55.12596 1.79156	66.66770 0.00030n	79.64616 1.06224n	93.00114 0.33406n	107.79284 1.12404
5	30.29250	39.16766 22.50066	49.12614 15.79214	60.16794 10.16694	72.29306 5.62506	85.50150 2.16650	99.79326 0.20874n	115.16484 1.49716n	131.59874 1.68126n
6	43.12620	53.66806 33.66766	65.29324 25.29244	78.00174 18.00054	92.14696 11.43856	106.64770 6.68770	122.60196 2.64996	139.63954 0.30446n	157.76044 2.17556n
7	58.29330	70.50186 47.16806	83.79374 37.12914	98.16894 28.16754	113.62746 20.29226	130.16580 13.50380	147.76506 7.82106	166.46864 3.20064	186.25554 0.33646n
8	75.79380	89.66906 63.00186	104.62764 51.29324	121.02294 40.31454	137.79476 31.12596	155.97530 22.69530	175.26156 15.32556	195.67034 8.99994	217.12884 3.79124

Index 1.52.

A 1.07060 C 0.53948
B 1.59420 D 1.0680

s	$p=0$	1	2	3	4	5	6	7	8
0	1.06800	1.60748	3.22592	5.92332	9.69968	14.55500	20.48928	27.50252	35.59472
1	2.13860	4.27228 1.08388	7.48492 1.10812	11.77652 2.21132	17.14708 4.39348	23.59660 7.65460	31.12508 11.99468	39.73252 17.41372	49.41892 23.91172
2	5.35040	9.07828 2.70148	13.88512 1.13152	19.77092 0.64052	26.73568 1.22848	34.77940 2.81540	43.90208 5.64128	54.10372 9.46612	65.38432 14.36992
3	10.70340	16.02548 6.46028	22.42652 3.29612	29.90652 1.21092	38.46548 0.20468	48.10340 0.27740	58.82028 1.42908	70.61612 3.65972	83.49092 6.96932
4	18.19760	25.11388 12.36028	33.10912 7.60192	42.18332 3.92252	52.33648 1.32208	63.56860 0.19940n	75.87968 0.64192n	89.26972 0.00548n	103.73872 1.70992+
5	27.83300	36.34348 20.40148	45.93292 14.05892	56.60132 8.78532	69.34868 5.58068	81.17500 1.46500	95.08028 0.57172n	110.06452 1.52948n	126.11972 1.41128n
6	39.60960	49.71428 30.58378	60.89792 22.63712	73.16052 15.76932	86.50208 9.98048	100.92260 5.27060	116.42208 1.63968	133.00052 0.91228n	150.65782 2.38528n
7	53.52740	65.22628 42.90748	78.00412 33.36652	91.86092 24.90452	106.79668 17.52148	122.81140 11.21740	139.90508 5.99228	158.07772 .84612	177.32932 1.22108n
8	69.58640	82.87948 57.37228	97.25152 46.23712	112.70252 36.18092	129.23248 27.20368	146.84140 19.30540	165.52928 12.48608	185.29612 6.74572	206.14192 2.08432

Index 1.55.

A 0.94644 C 0.53625
B 1.49560 D .99278

s	$p=0$	1	2	3	4	5	6	7	8
0	0.99278	1.52906	3.13790	5.81930	9.57326	16.39978	20.29886	27.27050	35.31470
1	1.93922	3.97110 0.97990	7.07554 1.09314	11.25254 2.30894	16.50210 4.53730	22.82422 7.86822	30.21890 12.27170	38.68614 17.74774	48.22594 24.29634
2	4.77854	8.30602 2.32362	12.90606 0.94126	18.57866 0.63146	25.32382 1.39422	33.14154 3.22954	42.03182 6.13742	51.99466 10.11786	63.03006 15.17086
3	9.51074	14.53382 5.56022	20.62946 2.68226	27.79766 0.87686	36.03842 0.14402	45.35174 0.48374	55.73762 1.89602	67.19606 4.38086	79.72706 7.93826
4	16.13582	22.65450 10.68970	30.34574 6.41614	38.90954 3.01514	48.64590 0.78670	59.45482 0.36918n	71.33630 0.45250n	84.29134 0.53774+	98.31694 2.59854+
5	24.65378	32.66806 17.71206	41.75490 11.84290	51.91430 7.04630	63.14626 3.32226	75.45078 0.67078	88.82786 0.90814n	103.27750 1.41450n	118.79970 0.84830n
6	35.06462	44.57450 26.62730	85.25694 19.36254	66.81194 12.97034	79.53950 7.75070	93.33962 3.60362	108.21170 0.52850	124.15754 1.57286n	141.17534 2.40226n
7	47.36834	58.37382 37.43542	70.55186 28.67506	83.60246 20.78726	97.82562 14.07202	113.12134 8.42934	129.48962 3.85922	146.93046 0.36166+	165.44386 2.06334n
8	61.56494	74.06602 50.13642	87.73966 39.88046	102.28586 30.49706	118.00462 22.28622	134.79594 15.14794	152.65982 09.08222	171.59626 4.08906	191.60526 0.16846

Index 1.57.

A 0.87466 C 0.53411
B 1.4356 D .94880

s	p=0	1	2	3	4	5	6	7	8
0	0.94880	1.48291	3.08524	5.75579	9.49456	14.30155	20.17676	27.12019	35.13184
1	1.82346	3.81317 0.92197	6.83110 1.08870	10.93725 2.32365	16.11162 4.62682	22.35421 7.99821	29.66502 122.43782	38.04405 17.94565	47.49130 24.52170
2	4.44744	7.85275 2.11035	12.32628 .84148	17.85803 .64083	24.47800 1.50840	32.15619 3.44419	40.90260 6.44820	50.62867 10.60859	61.60008 15.66088
3	8.82074	13.66165 5.04805	19.57078 2.34358	26.54813 .70733	34.59370 .13930	43.70749 .63949	53.88950 2.20790	65.13973 4.84453	77.45818 8.54938
4	14.94336	21.21987 9.73507	28.56460 5.59500	36.97755 2.52315	46.45872 .51952	57.00811 .41589n	68.62572 .28308	81.31155 .91795	95.06560 3.18720
5	22.81530	30.52741 16.17141	39.30774 10.59574	49.15629 6.08829	60.07306 2.64906	72.05805 .27805	85.11126 1.02474n	99.23269 1.25931n	114.42234 .42566n
6	32.43656	41.58427 24.35707	51.80020 17.34580	63.08435 11.40275	75.43672 6.52792	88.85731 2.72131	103.34612 .01708n	118.90315 1.68725n	135.52840 2.28920n
7	43.80714	54.39045 34.29205	65.95342 25.93374	78.76173 18.46653	92.54970 12.15610	107.40589 6.91389	123.33030 .73990	140.32293 .30587n	158.38378 2.40342n
8	56.92704	68.94595 45.97635	82.03308 36.09388	96.18843 27.27963	111.41200 19.53360	127.70379 12.85579	145.06380 7.24620	163.49203 2.70483	182.98848 .76832n

Index 1.60.

A 0.78125 C 0.53125
B 1.35100 D .88890

s	p=0	1	2	3	4	5	6	7	8
0	0.88890	1.42015	3.01390	5.67015	9.38890	14.17015	20.01390	26.92015	34.88890
1	1.67015	3.55240 0.85040	6.49715 1.09315	10.50440 2.39840	15.57415 4.76615	21.70640 8.19640	28.90115 12.68915	37.15840 18.24440	46.47815 24.86215
2	4.01390	7.24715 1.84315	11.54290 0.73490	16.90115 0.68915	23.32190 1.70590	30.80515 3.78515	39.35090 6.92690	48.95915 11.13115	59.62990 16.39790
3	7.92015	12.50440 4.39840	18.15115 1.93915	24.86040 0.54240	32.63215 0.20815	41.46640 0.93640	51.36315 2.72715	62.32240 5.58040	74.34415 9.59615
4	13.38890	19.32415 8.51615	26.32190 4.70590	34.38215 1.95815	43.50490 0.27290	53.69015 0.34985n	64.93790 0.08990+	77.24815 1.59215	90.62090 4.15690
5	20.42015	27.70640 14.19640	36.05515 9.03515	45.46640 4.93640	55.94015 11.90015	67.47640 0.07360	80.07515 0.98485n	93.73640 0.83360n	108.46015 0.38015
6	29.01390	37.65115 21.43915	47.35090 14.92690	58.11315 9.47715	69.93790 5.08990	82.82515 1.76515	96.77490 0.49710n	111.78715 1.69685n	127.86190 1.83310n
7	39.17015	49.15840 30.24440	60.20915 22.38115	72.32240 15.58040	85.49815 9.84215	99.73640 5.16640	115.03715 1.55315	131.40040 0.99760n	148.82615 2.48585n
8	50.88890	62.22815 40.61215	74.62990 31.39790	88.09415 23.24615	102.62090 16.15690	118.21015 10.13015	134.86190 5.16590	152.57615 1.26415	171.35290 1.57510n

Index 1.62.

A 0.72663 C 0.52931
B 1.30420 D .85344

s	p=0	1	2	3	4	5	6	7	8
0	0.85344	1.38275	2.97098	5.61723	9.32240	14.08619	19.90860	26.78963	34.72928
1	1.58007	3.41358 0.80518	6.30571 1.08891	10.25646 2.43126	15.26583 4.83223	21.33382 8.29182	28.46043 12.80503	36.64566 18 38686	45.88951 25 02231
2	3.75996	6.89767 1.68087	11.09400 0.66040	16.34895 0.69855	22.66252 1.79532	30.13471 4.05071	38.46552 7.16472	47.95495 11.43735	58.50300 16.76860
3	7.39311	11.83502 4.00982	17.33555 1.68515	23.89470 0.41910	31.51247 0.21167	40.18886 1.06286	49.92387 2.97267	60.71750 5.94110	72.56975 9.96815
4	12.47952	18.22563 7.79203	25.03036 4.16316	32.89371 1.59291	41.81568 0.08128	51.79637 0.36163n	62.83548 0.23388+	74.93331 1.89811	88.08976 4.62096
5	19.01919	26.06950 13.02750	34.17843 8.09443	43.34598 4.21998	53.57215 1.40415	64.85694 0.35306n	77.20035 1.05165n	90.60238 0.69162n	105.06303 0.72703+
6	27.01212	35.36663 19.71623	44.77976 13.47896	55.25171 8.30051	66.78188 4.18028	79.37687 1.12487	93.01848 0.88392n	107.72471 1.82809n	123.48956 1.71364n
7	36.45831	46.11702 27.85822	56.83435 20.31675	68.61030 13.83390	81.44487 8.40967	95.33806 4.04406	110.28987 0.73707	126.30030 1.51130n	143.36935 2.70105n
8	47.35776	58.32067 37.40347	70.34220 28.60780	83.42235 20.82075	97.56112 14.09232	112.75851 8.42251	129.01452 3.81132	146.32915 0.24875	164.70240 2.23520n

Index 1.65.

A 0.65449 C 0.52651
B 1.23540 D .80547

s	p=0	1	2	3	4	5	6	7	8
0	0.80547	1.33198	2.91151	5.54406	9.22963	13.96822	19.75983	26.60446	34.50211
1	1.45996	3.22187 0.75107	6.03680 1.09520	9.90875 2.49635	14.82572 4.94252	20.79971 8.44571	27.82680 13.00184	35.90675 18.61115	45.03980 25.27340
2	3.42343	6.42074 1.47914	10.47107 0.58787	15.57450 0.74954	21.73079 1.96439	28.94018 4.23218	37.20259 7.55299	46.51802 11.92682	56.88647 17.35367
3	6.69588	10.92859 3.51619	16.21440 1.38944	22.55307 .31587	29.94484 0.29524	38.38963 0.67237n	47.88744 3.41000	58.43827 6.55147	65.04212 5.74292
4	11.27731	16.74542 6.86222	23.26655 3.50015	30.84070 1.19110	39.46787 0.06493n	49.14816 0.26784n	59.88127 0.58207	71.66750 2.48510	84.50675 5.44115
5	17.16772	23.87123 11.51723	31.62776 6.91976	40.43731 3.37531	50.29988 0.88388	61.21557 0.55443n	73.18408 1.03992n	86.20571 0.27229n	100.28036 1.44836
6	24.36711	32.30610 17.48114	41.29795 11.64835	51.34290 06.86850	62.44087 3.14167	74.59186 0.46786	87.79587 1.15293n	102.05290 1.42070n	117.86298 0.73545n
7	32.87548	42.04979 24.75419	52.27712 17.68592	63.55747 11.67067	75.89084 6.70844	89.27723 2.79923	103.71664 0.05696	119.20907 1.86013n	135.75452 2.61028n
8	42.69283	53.10254 33.33614	64.56527 25.03247	77.08102 17.78182	90.64979 11.58419	105.27168 6.43968	120.94639 2.34799	136.67422 1.69058n	155.45507 2.67613n

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