SPHERICAL DESIGNS AND MODULAR FORMS OF THE D_4 LATTICE

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Abstract. In this paper, we study shells of the D_4 lattice with a slightly general concept of spherical t-designs due to Delsarte-Goethals-Seidel, namely, the spherical design of harmonic index T (spherical T-design for short) introduced by Delsarte-Seidel. We first observe that the $2m$-shell of D_4 is an antipodal spherical $\{10, 4, 2\}$-design on the three dimensional sphere. We then prove that the 2-shell, which is the D_4 root system, is a tight $\{10, 4, 2\}$-design, using the linear programming method. The uniqueness of the D_4 root system as an antipodal spherical $\{10, 4, 2\}$-design with 24 points is shown. We give two applications of the uniqueness: a decomposition of the shells of the D_4 lattice in terms of orthogonal transformations of the D_4 root system: and the uniqueness of the D_4 lattice as an even integral lattice of level 2 in the four dimensional Euclidean space. We also reveal a connection between the harmonic strength of the shells of the D_4 lattice and non-vanishing of the Fourier coefficients of a certain newform of level 2. Motivated by this, congruence relations for the Fourier coefficients are discussed.

1. Introduction

The m-shell of a lattice is the set of lattice points on the sphere with $\sqrt{m}$ radius. These finite sets have been studied from the design theoretical viewpoint in connection with modular forms, in particular, weighted theta functions. In this paper, we wish to explicate shells of the D_4 lattice, an even integral lattice in the four dimensional Euclidean space, with a slightly general concept of spherical t-designs due to Delsarte-Goethals-Seidel [21], namely, the spherical design of harmonic index T (spherical T-design for short) introduced by Delsarte-Seidel [22] as a spherical analogue of the design in association schemes [16, §3.4]. A prototype of our work is due to Venkov [40]; one of his results shows that any non-empty (normalized) $2m$-shell of an extremal even unimodular lattice in $\mathbb{R}^{24n}$ $(n \geq 1)$, including the Leech lattice, is a spherical 11-design. In his study, the theory of modular forms for the full modular group plays an important role. Similar investigations are then made for several types of lattices (see e.g., [2, 18, 19, 35]).

The $2m$-shell of the D_4 lattice, denoted by $(D_4)_{2m}$, becomes the set of integer solutions to the equation $x_1^2 + \cdots + x_4^2 = 2m$. A starting point of our study is to prove that the normalized set $\frac{1}{\sqrt{2m}}(D_4)_{2m}$ on the unit sphere $\mathbb{S}^3$ is a spherical $\{10, 4, 2\}$-design for all $m \geq 1$ (Proposition 4.2). We indicate two proofs; the first proof is based on the fact that the Weyl group $W(F_4)$ of the F_4 root system acts on the D_4 lattice, together with the formula for the harmonic Molien series of $W(F_4)$; the second proof uses the theory of modular forms of level 2 with weighted theta functions of the D_4 lattice. As a special case, we see that the D_4 root system, which is the 2-shell $(D_4)_2$, is an antipodal spherical $\{10, 4, 2\}$-design of $\mathbb{S}^3$ with 24 points. A crucial discovery due to linear programming method is that the lower bound of

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the cardinality of such design is 24 (Theorem 3.2). Namely, the D_4 root system gives a tight antipodal spherical $\{10, 4, 2\}$-design, while it is not a tight spherical 5-design on $\mathbb{S}^3$.

The classification of tight spherical T-designs has only recently begun (see e.g., [8, 34, 41]). In this direction, we prove the uniqueness of the D_4 root system (Theorem 5.1). It means that every antipodal spherical $\{10, 4, 2\}$-design of $\mathbb{S}^3$ with 24 points is an orthogonal transformation of the D_4 root system. It seems that the normalized D_4 root system is the first example such that it is not unique as a spherical t-design, but unique as an antipodal spherical T-design (see Remark 5.2). This result not only contributes to the study of classification of spherical designs, but also has two striking applications: a decomposition of the shell of the D_4 lattice in terms of the disjoint union of orthogonal transformations of the D_4 root system (Theorem 6.1), and the uniqueness of the D_4 lattice as an even integral lattice of level 2 in $\mathbb{R}^4$ (Theorem 7.2).

In connection with modular forms, we reveal a relationship between the harmonic strength of the shells of the D_4 lattice and non-vanishing of the τ_2-function, where $\sum_{m\geq 1} \tau_2(m)q^m = \eta(z)^8\eta(2z)^8$ is the unique cusp form of weight 8 and level 2 (Theorem 8.1). This is analogy to the study of Venkov, Pache and de la Harpe [18, 19] in the case of the E_8 lattice, where they pointed out that the Ramanujan τ-function $\tau(m)$ vanishes if and only if the $2m$-shell of E_8 is a spherical 8-design (see also [3, §3.2]). In our case, we may believe that $\tau_2(m)$ would never be 0 (similar to Lehmer's conjecture), namely, that the harmonic strength of the $2m$-shell of the D_4 lattice is $\{10, 4, 2\}$ for all $m \geq 1$. For evidence, we prove congruence relations $\tau_2(p) \equiv p(p + 1) \bmod \ell$ for $\ell \in \{3, 5\}$ (Theorem 8.2) which shows $\tau_2(p) \neq 0$ for all prime $p \not\equiv -1 \bmod 15$. This congruence might not be new and can be deduced from results in the literature, e.g., [10, 17, 26, 28, 32], but our proof may shed new light on this study.

2. Spherical code and design

The concepts of spherical codes and spherical designs introduced by Delsarte-Goethals-Seidel [21] apply for finite subsets of the unit sphere $\mathbb{S}^{d-1} = \{\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid \|\boldsymbol{x}\| = 1\}$ in d-dimensional Euclidean space $\mathbb{R}^d$, where $\|\boldsymbol{x}\|^2 = \langle \boldsymbol{x}, \boldsymbol{x}\rangle = \sum_{i=1}^d x_i^2$. We recall their definitions, thereby also fixing some of our notation.

For a subset X of $\mathbb{S}^{d-1}$, let us denote the set of inner products of two distinct points in X by

$$A(X) = \{\langle \boldsymbol{x}, \boldsymbol{y}\rangle \mid \boldsymbol{x}, \boldsymbol{y} \in X,\ \boldsymbol{x} \neq \boldsymbol{y}\} \subset [-1, 1).$$

We denote by $\mathrm{Harm}_\ell(\mathbb{R}^d)$ the $\mathbb{R}$-vector space of real homogeneous harmonic polynomials of degree exactly ℓ in d variables, namely, a polynomial in $\mathbb{R}[x_1, \ldots, x_d]$ of homogeneous degree ℓ annihilated by the Laplace operator $\sum_{j=1}^d \partial^2/\partial x_j^2$.

Definition 2.1. *1) A set X of N points on $\mathbb{S}^{d-1}$ is called a (d, N, a) spherical code if every element in $A(X)$ is less than or equal to $a \in \mathbb{R}$.*
2) Let T be a subset of $\mathbb{N}$. A non-empty finite subset X of $\mathbb{S}^{d-1}$ is called a spherical design of harmonic index T (spherical T-design for short) if it holds that

$$\sum_{\boldsymbol{x}\in X} P(\boldsymbol{x}) = 0, \quad \forall P \in \mathrm{Harm}_\ell(\mathbb{R}^d), \quad \forall \ell \in T.$$

For $t \in \mathbb{N}$, a spherical $\{t, t-1, \ldots, 2, 1\}$-design is called a *spherical t-design* (see [21] for the original definition and its equivalence [21, Theorem 5.2]). The spherical T-design, which is a generalization of spherical t-designs, is first introduced by Delsarte-Seidel [22] and its classification has recently been studied by Bannai-Okuda-Tagami [8].

For a subset X of $\mathbb{R}^d$ and a scalar $c \in \mathbb{R}$, we write $cX = \{c\boldsymbol{x} \in \mathbb{R}^d \mid \boldsymbol{x} \in X\}$. A set X is said to be *antipodal*, if we have $-X = X$. For an antipodal subset X of $\mathbb{R}^d$, a subset $X' \subset X$ is said to be *a half set of* X if X is a disjoint union of X' and $-X'$; $X' \sqcup (-X') = X$. For any antipodal set, its half set always exists, but not unique.

Lemma 2.2. *Let X' be a half set of an antipodal subset $X \subset \mathbb{S}^{d-1}$. If X' is a spherical T-design, then X is an antipodal spherical T-design. On the other hand, if X is an antipodal spherical T-design, then X' is a spherical T'-design with $T' = \{2\ell \in \mathbb{Z}_{>0} \mid 2\ell \in T\}$.*

Proof. Suppose that X' is a spherical T-design. Then, for $\ell \in T$ and $P \in \mathrm{Harm}_\ell(\mathbb{R}^d)$, one has

$$\sum_{\boldsymbol{x}\in X} P(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in X'} P(\boldsymbol{x}) + \sum_{\boldsymbol{x}\in -X'} P(\boldsymbol{x}) = (1+(-1)^\ell)\sum_{\boldsymbol{x}\in X'} P(\boldsymbol{x}) = 0.$$

Hence, X is an antipodal spherical T-design. Now suppose that X is an antipodal spherical T-design. Then, for $\ell \in T$ even and $P \in \mathrm{Harm}_\ell(\mathbb{R}^d)$, we have $0 = \sum_{\boldsymbol{x}\in X} P(\boldsymbol{x}) = 2\sum_{\boldsymbol{x}\in X'} P(\boldsymbol{x})$, so X' is a spherical T'-design. We complete the proof. □

We also notice that if X is an antipodal spherical T-design, then T contains all positive odd integers. Since in this paper we only consider antipodal spherical T-designs, we may omit to write positive odd integers lying in T.

3. Linear programming bounds

The principal problem on a (d, N, a) spherical code (resp. a spherical T-design) is to maximize the cardinality N for a given value of a (resp. to minimize the cardinality for a given set T). The linear programming method, established by Delsarte-Goethals-Seidel [21], is a useful tool to provide upper (resp. lower) bounds on the cardinality of a spherical code (resp. design). In this section, we describe and apply it for our cases: a spherical $\{10, 4, 2\}$-design on $\mathbb{S}^3$ and a $(4, N, 1/2)$ spherical code.

Let $Q_\ell(x) := Q_{d,\ell}(x) = \frac{d+2\ell-2}{d-2} C_\ell^{((d-2)/2)}(x)$ be the (scaled) Gegenbauer polynomial of degree ℓ in one variable x as introduced in [21, Definition 2.1]: The Gegenbauer polynomials $Q_\ell(x)$ are orthogonal polynomials on the closed interval $[-1, 1]$ with respect to the inner product of the weight function $(1-x^2)^{(d-3)/2}$, and to any real polynomial $F(x) \in \mathbb{R}[x]$ of degree r one can associate its *Gegenbauer expansion*

$$F(x) = \sum_{\ell=0}^{r} f_\ell Q_\ell(x). \tag{1}$$

Let $\{\varphi_{\ell,i}\}_{i=1}^{N_\ell}$ be an orthonormal basis of $\mathrm{Harm}_\ell(\mathbb{S}^{d-1})$ which is the restriction of $\mathrm{Harm}_\ell(\mathbb{R}^d)$ to $\mathbb{S}^{d-1}$, where $N_\ell := N_{d,\ell} = \dim \mathrm{Harm}_\ell(\mathbb{S}^{d-1}) = \binom{d+\ell-1}{\ell} - \binom{d+\ell-3}{\ell-2}$. For a finite subset X of $\mathbb{S}^{d-1}$, we write

$$H_\ell = H_\ell(X) = \big(\varphi_{\ell,i}(\xi)\big)_{\substack{\xi\in X \\ 1\le i\le N_\ell}}$$

for the $|X| \times N_\ell$ matrix whose rows and columns are indexed by $\xi \in X$ and $1 \le i \le N_\ell$, respectivelyi. H_0 is of size $|X| \times 1$ whose entries are all 1. For $\ell \ge 1$, one has ${}^tH_\ell H_0 = \big(\sum_{\xi\in X} \varphi_{\ell,i}(\xi)\big)_{1\le i\le N_\ell}$. From this, we see that X is a spherical T-design if and only if $\|{}^tH_\ell H_0\| = 0$ holds for all $\ell \in T$, where for a real matrix $M = (a_{ij})$, we write $\|M\| = \sum_{i,j} a_{ij}^2$.

A key lemma for the linear programming method is as follows (cf. [21, Corollary 3.8]).

Lemma 3.1. *For a real polynomial $F(x) \in \mathbb{R}[x]$ with the Gegenbauer expansion* (1)*, we have*

$$f_0|X|^2 + \sum_{\ell=1}^{r} f_\ell \|{}^t H_\ell H_0\| = F(1)|X| + \sum_{\alpha \in A(X)} F(\alpha) d_\alpha. \tag{2}$$

where $d_\alpha = \sharp\{(\xi, \eta) \in X \times X \mid \langle \xi, \eta \rangle = \alpha\}$.

Proof. Recall the additive formula (cf. [21, Theorem 3.3]); for any $\xi, \eta \in \mathbb{S}^{d-1}$ we have

$$\sum_{i=1}^{N_\ell} \varphi_{\ell,i}(\xi)\varphi_{\ell,i}(\eta) = Q_\ell(\langle \xi, \eta \rangle).$$

Using this, one computes

$$\|{}^t H_\ell H_0\| = \sum_{1 \le i \le N_\ell} \left(\sum_{\xi \in X} \varphi_{\ell,i}(\xi) \right)^2 = \sum_{\xi, \eta \in X} Q_\ell(\langle \xi, \eta \rangle) = \sum_{\alpha \in A(X) \cup \{1\}} Q_\ell(\alpha) d_\alpha.$$

By linearity, it holds that

$$\sum_{\ell=0}^{r} f_\ell \|{}^t H_\ell H_0\| = \sum_{\alpha \in A(X) \cup \{1\}} F(\alpha) d_\alpha.$$

Now the desired result follows from $\|{}^t H_0 H_0\| = |X|^2$ and $d_1 = |X|$. □

We now apply the formula (2) to the lower bound of the cardinality of a spherical $\{10, 4, 2\}$-design on $\mathbb{S}^3$.

Theorem 3.2. *Let X be a spherical $\{10, 4, 2\}$-design on $\mathbb{S}^3$. Then $|X| \geq 12$. Moreover, X attains the lower bound if and only if X is a $(4, 12, 1/2)$ spherical code with $A(X) \subset \{-1/2, 0, 1/2\}$.*

Proof. Consider the real polynomial

$$\begin{aligned} F_T(x) &:= \frac{1}{11264} Q_{10}(x) + \frac{1}{2560} Q_4(x) + \frac{1}{768} Q_2(x) + \frac{3}{1024} \\ &= \frac{1}{16} x^2 (x + \frac{1}{2})^2 (x - \frac{1}{2})^2 (16x^4 - 28x^2 + 13). \end{aligned} \tag{3}$$

We write $F_T(x) = \sum_{\ell=0}^{10} f_\ell Q_\ell(x)$. One can easily check the inequality $F_T(x) \geq 0$ for all $x \in [-1, 1)$, and hence, by (2), we get the inequality

$$f_0|X|^2 - F_T(1)|X| = \sum_{\alpha \in A(X)} F_T(\alpha) d_\alpha \geq 0. \tag{4}$$

Since $F_T(1) = \frac{9}{256}$, the desired inequality $|X| \geq F_T(1)/f_0 = 12$ follows. The equality holds if $F_T(\alpha) = 0$ $(\forall \alpha \in A(X))$. We complete the proof, because $\{\alpha \in \mathbb{R} \mid F_T(\alpha) = 0\} = \{-1/2, 0, 1/2\}$. □

An antipodal spherical $\{10, 4, 2\}$-design $X \subset \mathbb{S}^3$ is said to be *tight* when $|X| = 24$. It should be noted that our definition of the tightness depends on the test function (3) and its induced linear programming bound, so is different from the classical definition of tight spherical t-designs (see e.g., [3, Definition 2.13] for the definition). A similar context can be found in [8], where the existence and non-existence of tight spherical T-designs are studied. Several investigations have been conducted in this direction; see e.g., [34, 41].

Theorem 3.2 says that every half set of a tight antipodal spherical $\{10,4,2\}$-design on $\mathbb{S}^3$ is a $(4,12,1/2)$ spherical code. The natural question to ask is the upper bound of N for a $(4,N,1/2)$ spherical code.

Theorem 3.3. *Let X be a $(4,N,1/2)$ spherical code with $A(X)\subset[-1/2,1/2]$. Then $N\le 12$. Furthermore, X attains the upper bound if and only if X is a spherical $\{10,4,2\}$-design and $A(X)\subseteq\{-\frac{1}{2},0,\frac{1}{2}\}$.*

Proof. For $a_1\ge 0$, let us consider the function

$$
\begin{aligned}
F_C(x) &:= \frac{1}{11264}Q_{10}(x)+\frac{64a_1+15}{5120}Q_4(x)+\frac{64a_1+15}{1536}Q_2(x)+\frac{4a_1+1}{64}\\
&= x^2\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)\left(x^6-2x^4+\frac{5}{4}x^2+a_1\right).
\end{aligned}\tag{5}
$$

For this, one can show the inequality $F_C(\alpha)\le 0$ ($\forall\alpha\in[-1/2,1/2]$). From (2) and the assumption $A(X)\subset[-1/2,1/2]$, we get the inequality

$$
F_C(1)|X|-f_0|X|^2=-\sum_{\alpha\in A(X)}F_C(\alpha)d_\alpha+\sum_{\ell=1}^{10}f_\ell\|{}^tH_\ell H_0\|\ge 0,\tag{6}
$$

where f_ℓ denotes the coefficient of F_C in Q_ℓ. Since $F_C(1)=\frac{3(4a_1+1)}{16}>0$, we obtain $F_C(1)/f_0=12\ge|X|=N$. The equality in (6) holds if and only if $F_C(\alpha)=0$ ($\forall\alpha\in A(X)$) and $\|{}^tH_\ell H_0\|=0$ for all $\ell\in\{10,4,2\}$. The desired result then follows from $\{\alpha\in\mathbb{R}\mid F_C(\alpha)=0\}=\{-1/2,0,1/2\}$. □

4. The D_4 lattice and spherical $\{10,4,2\}$-designs

This section gives the construction of a tight antipodal spherical $\{10,4,2\}$-design on $\mathbb{S}^3$ from the shells of the D_4 lattice.

Following [24, §1.4], we define *the D_4 lattice* by

$$D_4=\{\boldsymbol{x}=(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4\mid x_1+x_2+x_3+x_4\equiv 0 \mod 2\}.$$

For $m\in\mathbb{Z}_{\ge 0}$, *the m-shell* of the D_4 lattice is denoted by

$$\left(D_4\right)_m=\{\boldsymbol{x}\in D_4\mid x_1^2+x_2^2+x_3^2+x_4^2=m\}.$$

It follows that $\left(D_4\right)_m=\varnothing$, if m is odd. When m is even, $\left(D_4\right)_m$ is not the empty set because of Jacobi's four-square theorem

$$|(D_4)_{2m}|=24\sum_{\substack{d\mid 2m\\ d:\text{odd}}}d.\tag{7}$$

For instance, the 2-shell $(D_4)_2$ (the set of minimal vectors of D_4) consists of 24 points; all permutations of $(\pm1,\pm1,0,0)$. Note that the 2-shell $(D_4)_2$, which is called *the D_4 root system*, generates the D_4 lattice.

We now prove that the normalized set

$$\frac{1}{\sqrt{2m}}\left(D_4\right)_{2m}=\left\{\frac{1}{\sqrt{2m}}\boldsymbol{x}\;\middle|\;\boldsymbol{x}\in\left(D_4\right)_{2m}\right\}$$

on $\mathbb{S}^3$ is an example of antipodal spherical $\{10,4,2\}$-designs. There are at least two proofs of this. One is based on some spherical design properties on group orbits. The other uses the theory of modular forms, which will be mentioned in Remark 7.3. Here we give a proof of the former.

We recall that the orthogonal transformation group

$$O(\mathbb{R}^d) = \{\sigma : \mathbb{R}^d \to \mathbb{R}^d \mid \langle \sigma(\boldsymbol{x}), \sigma(\boldsymbol{y})\rangle = \langle \boldsymbol{x}, \boldsymbol{y}\rangle \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d\}$$

of $\mathbb{R}^d$ acts on $\mathrm{Harm}_\ell(\mathbb{R}^d)$ by $\big(\sigma^* P\big)(\boldsymbol{x}) = P(\sigma(\boldsymbol{x}))$ for $P \in \mathrm{Harm}_\ell(\mathbb{R}^d)$ and $\sigma \in O(\mathbb{R}^d)$. For a subgroup G of $O(\mathbb{R}^d)$, the G-invariant subspace of $\mathrm{Harm}_\ell(\mathbb{R}^d)$ is denoted by $\mathrm{Harm}_\ell(\mathbb{R}^d)^G = \{P \in \mathrm{Harm}_\ell(\mathbb{R}^d) \mid \sigma^* P = P \text{ for all } \sigma \in G\}$.

Lemma 4.1. *For any finite subgroup G of $O(\mathbb{R}^d)$ and $\boldsymbol{x} \in \mathbb{S}^{d-1}$, the G-orbit $\boldsymbol{x}^G = \{\sigma(\boldsymbol{x}) \in \mathbb{S}^{d-1} \mid \sigma \in G\}$ is a spherical T-design with $T = \{\ell \in \mathbb{N} \mid \dim \mathrm{Harm}_\ell(\mathbb{R}^d)^G = 0\}$. Moreover, if G has $-I$, which sends $\boldsymbol{y}$ to $-\boldsymbol{y}$ for $\boldsymbol{y} \in \mathbb{R}^d$, then $\boldsymbol{x}^G$ is antipodal, and every its half set is a spherical T'-design with $T' = \{2\ell \in 2\mathbb{Z}_{>0} \mid \dim \mathrm{Harm}_{2\ell}(\mathbb{R}^d)^G = 0\}$.*

Proof. Let $G_{\boldsymbol{x}}$ denote the stabilizer subgroup of $\boldsymbol{x}$. For $P \in \mathrm{Harm}_\ell(\mathbb{R}^d)$, we have

$$\sum_{\boldsymbol{y} \in \boldsymbol{x}^G} P(\boldsymbol{y}) = \frac{1}{|G_{\boldsymbol{x}}|} \sum_{\sigma \in G} (\sigma^* P)(\boldsymbol{x}).$$

The first statement follows from the fact that the map $\mathrm{Harm}_\ell(\mathbb{R}^d) \to \mathrm{Harm}_\ell(\mathbb{R}^d)^G, P \mapsto \sum_{\sigma \in G}(\sigma^* P)$ is surjective.

Suppose that $-I \in G$. Then, $-\boldsymbol{y} \in \boldsymbol{x}^G$ for any $\boldsymbol{y} \in \boldsymbol{x}^G$, so $\boldsymbol{x}^G$ is antipodal. Therefore, the latter statement follows from Lemma 2.2. $\square$

We note that for spherical T-designs X_1 and X_2 on $\mathbb{S}^{d-1}$, the union $X_1 \cup X_2$ is a spherical T-design if $X_1 \cap X_2 = \varnothing$.

Proposition 4.2. *For any $m \geq 1$, the subset $\frac{1}{\sqrt{2m}}\big(D_4\big)_{2m}$ of $\mathbb{S}^3$ is an antipodal spherical $\{10,4,2\}$-design. In particular, for any $n \geq 1$ the set $\frac{1}{\sqrt{2^{n+1}}}\big(D_4\big)_{2^n}$ is a tight antipodal spherical $\{10,4,2\}$-design on $\mathbb{S}^3$.*

Proof. We use the fact that the D_4 root system $\mathbf{D}_4 = (D_4)_2$ is invariant under the action of the Weyl group $W(F_4)$, a discrete subgroup of $O(\mathbb{R}^4)$ (this fact is already pointed out in [35, Proposition 2.7]). Since the D_4 lattice is generated by the set $\mathbf{D}_4$, the set $\frac{1}{\sqrt{2m}}\big(D_4\big)_{2m}$ is also invariant under the action of $W(F_4)$, so it has a $W(F_4)$-orbit decomposition. The harmonic Molien series for $W(F_4)$ (see e.g., [14]) is given by

$$\begin{aligned}\sum_{\ell \geq 0} \dim \mathrm{Harm}_\ell(\mathbb{R}^4)^{W(F_4)} t^\ell &= \frac{1}{(1-t^6)(1-t^8)(1-t^{12})} \\ &= 1 + t^6 + t^8 + 2t^{12} + t^{14} + t^{16} + 2t^{18} + \cdots .\end{aligned} \tag{8}$$

With this, the result follows from Lemma 4.1. The "in particular" part follows from (7), namely, $|(D_4)_{2^n}| = 24$. $\square$

Combining Proposition 4.2 with Lemma 2.2, we see that every half set of $\frac{1}{\sqrt{2m}}\big(D_4\big)_{2m}$ is a spherical $\{10,4,2\}$-design. In particular, it follows from Theorem 3.2 that every half set X

of $\frac{1}{\sqrt{2^{n+1}}}(D_4)_{2^n}$ is a $(4,12,1/2)$ spherical code with $A(X) \subset \{-1/2,0,1/2\}$. Indeed, one can check that the inner product set of the normalized D_4 root system $\frac{1}{\sqrt{2}}\mathbf{D}_4$ is given by

$$A\left(\frac{1}{\sqrt{2}}\mathbf{D}_4\right) = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}\right\}.$$

Remark 4.3. According to [7, Proposition 2], there exists a half set of $\frac{1}{\sqrt{2}}\mathbf{D}_4$ such that it is a spherical $\{10,4,2,1\}$-design.

5. Uniqueness of the antipodal spherical $\{10,4,2\}$-design

For a spherical T-design X on $\mathbb{S}^{d-1}$, the orthogonal transformation $\sigma(X) = \{\sigma(\boldsymbol{x}) \mid \boldsymbol{x} \in X\}$ of X is a spherical T-design for any $\sigma \in O(\mathbb{R}^d)$. Thus, the orthogonal transformation of $\frac{1}{\sqrt{2}}\mathbf{D}_4$ is still a tight antipodal spherical $\{10,4,2\}$-design on $\mathbb{S}^3$. The goal of this section is to prove the opposite statement, namely, any antipodal spherical $\{10,4,2\}$-design with 24 points is obtained from an orthogonal transformation of $\frac{1}{\sqrt{2}}\mathbf{D}_4$, which is referred as the uniqueness theorem in the study of the classification of spherical designs.

Our proof is along the line of the proof of the uniqueness of the 600-cell $C_{600} \subset \mathbb{S}^3$ as a spherical 11-design with 120 points, given by Boyvalenkov-Danev [13]. Let us first recall some relevant materials from it.

For $\boldsymbol{y} \in \mathbb{S}^{d-1}$ and a finite set $X \subset \mathbb{S}^{d-1}$, we let

$$A^{\boldsymbol{y}}(X) = \{\alpha \in [-1,1] \mid \text{there exists } \boldsymbol{x} \in X \text{ such that } \langle \boldsymbol{x}, \boldsymbol{y}\rangle = \alpha\},$$

and for $\alpha \in [-1,1]$, we write $\widetilde{X}^{\boldsymbol{y}}_\alpha = \{\boldsymbol{x} \in X \mid \langle \boldsymbol{x}, \boldsymbol{y}\rangle = \alpha\}$. Note that if $\boldsymbol{y} \in X$, then $1 \in A^{\boldsymbol{y}}(X) \subset A(X) \cup \{1\}$. The sequence of positive integers $(A^{\boldsymbol{y}}_\alpha(X))_{\alpha \in A^{\boldsymbol{y}}(X)}$ with $A^{\boldsymbol{y}}_\alpha(X) = |\widetilde{X}^{\boldsymbol{y}}_\alpha|$ is called the *distance distribution* of X with respect to $\boldsymbol{y}$. When X is a spherical t-design on $\mathbb{S}^{d-1}$ with $|A^{\boldsymbol{y}}(X)| \leq t+1$, the distance distribution of X with respect to $\boldsymbol{y} \in \mathbb{S}^{d-1}$ is obtained as the unique solution to the Vandermonde system

$$\sum_{\alpha \in A^{\boldsymbol{y}}(X)} A^{\boldsymbol{y}}_\alpha(X)\alpha^j = a_j|X|, \quad j = 0,1,\ldots,t, \tag{9}$$

where we set $a_0 = 1, a_{2j} = \frac{(2j-1)!!}{d(d+2)\cdots(d+2j-2)}$ and $a_{2j+1} = 0$ for $j \geq 1$ (the proof can be done by taking $F(x) = x^j,\ j = 0,1,\ldots,t$, in the following equivalent definition of a spherical t-design [21, Corollary 3.8, Theorem 5.5]; for a finite set $X \subset \mathbb{S}^{d-1}$, X is a spherical t-design if and only if for any $\boldsymbol{y} \in \mathbb{S}^{d-1}$ the equality $\sum_{\boldsymbol{x} \in X} F(\langle \boldsymbol{x}, \boldsymbol{y}\rangle) = |X| f_0$ holds for all $F(x) \in \mathbb{R}[x]$ of degree at most t, where f_0 is the constant term of the Gegenbauer expansion of F as in (1): See also [12, §2]).

An N points set X on $\mathbb{S}^{d-1}$ is called a (d,N,s,t) configuration [21], if X is a spherical t-design such that $s = |A(X)|$. It follows that for $\boldsymbol{y} \in X$ and a (d,N,s,t) configuration X with $s \leq t+1$, the Vandermonde system (9) has the unique solution, because of $|A^{\boldsymbol{y}}(X)| \leq s+1$. In this case, $A^{\boldsymbol{y}}_\alpha(X)$ does not depend on the choices of $\boldsymbol{y} \in X$ and we write $A_\alpha(X) = A^{\boldsymbol{y}}_\alpha(X)$.

Theorem 5.1. *For any antipodal spherical* $\{10,4,2\}$*-design* X *with* 24 *points, there exists an orthogonal transformation* $\sigma \in O(\mathbb{R}^4)$ *such that* $X = \sigma\big(\frac{1}{\sqrt{2}}\mathbf{D}_4\big)$.

Proof. By Lemma 2.2 and Theorem 3.2, a half set X' of X is a $(4,12,1/2)$ spherical code with $A(X') \subset \{-1/2,0,1/2\}$, so $A(X) \subset \{-1,-1/2,0,1/2\}$. Since X is a $(4,24,s,5)$ configuration with $s \leq 4$, the distance distribution $(A^{\boldsymbol{y}}_\alpha(X))_{\alpha\in A^{\boldsymbol{y}}(X)}$ of X does not depend on choices of $\boldsymbol{y} \in X$. Solving the equations (9), we get

$$A_{-1}(X) = 1, \quad A_{-\frac{1}{2}}(X) = A_{\frac{1}{2}}(X) = 8, \quad A_0(X) = 6,$$

which implies that X is a $(4, 24, 4, 5)$ configuration.

For each $\alpha \in A(X) \setminus \{-1\}$, we now recall a derived code $X_\alpha \subset \mathbb{S}^2$ of X introduced in [21, §8]. We may assume $\boldsymbol{e} = (0, 0, 0, 1) \in X$ (if not, one can take $\sigma \in O(\mathbb{R}^4)$ such that $\boldsymbol{e} \in \sigma(X)$). For any $\boldsymbol{x} \in \widetilde{X}^{\boldsymbol{e}}_\alpha$, it holds that

$$\frac{1}{\sqrt{1-\alpha^2}}(\boldsymbol{x} - \alpha\boldsymbol{e}) \in \{\boldsymbol{y} \in \mathbb{S}^3 \mid \langle \boldsymbol{y}, \boldsymbol{e}\rangle = 0\}.$$

Thus, the image of $\widetilde{X}^{\boldsymbol{e}}_\alpha$ under the composition map

$$\begin{array}{cccccc} p_\alpha: & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^3, \\ & \boldsymbol{x} & \longmapsto & \frac{1}{\sqrt{1-\alpha^2}}(\boldsymbol{x} - \alpha\boldsymbol{e}) = (z_1, z_2, z_3, z_4) & \longmapsto & (z_1, z_2, z_3). \end{array}$$

lies in $\mathbb{S}^2$. The image $X_\alpha = p_\alpha\big(\widetilde{X}^{\boldsymbol{e}}_\alpha\big) \subset \mathbb{S}^2$, called the derived code, is also a spherical design with the strength weakened (see [21, Theorem 8.2] for more details). In our case, X_α becomes a spherical 3-design on $\mathbb{S}^2$.

Let us consider the inner product set $A(X_\alpha)$ for each X_α. By definition, one easily finds that $A(X_\alpha) \subset \left\{\frac{\beta-\alpha^2}{1-\alpha^2} \,\middle|\, \beta \in A(X)\right\}$. Computing the terms $\frac{\beta-\alpha^2}{1-\alpha^2}$, we get

$$A(X_{\pm\frac{1}{2}}) \subset \left\{-1, -\frac{1}{3}, \frac{1}{3}\right\} \quad \text{and} \quad A(X_0) \subset \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}\right\}.$$

Namely, the sets $X_{\pm\frac{1}{2}}$ and X_0 are $(3, 8, s_1, 3)$ and $(3, 6, s_2, 3)$ configurations with $s_1 \leq 3$ and $s_2 \leq 4$, respectively. For each X_α, one can compute the unique solution to the Vandermonde system (9). Indeed,

$$\begin{aligned} A_{-1}\big(X_{\pm\frac{1}{2}}\big) = 1, \quad & A_{\pm\frac{1}{3}}\big(X_{\pm\frac{1}{2}}\big) = 3, \\ A_{-1}\big(X_0\big) = 1, \quad & A_{\pm\frac{1}{2}}\big(X_0\big) = 0, \quad A_0\big(X_0\big) = 4, \end{aligned}$$

so the sets $X_{\pm\frac{1}{2}}$ and X_0 are $(3, 8, 3, 3)$ and $(3, 6, 2, 3)$ configurations, respectively. Both $A_{-1}(X_\alpha) = 1$ and its independence of the choices of $\boldsymbol{y} \in X_\alpha$ imply X_α being antipodal. Remark that the antipodal $(3, 6, 2, 3)$ configuration X_0, which by [21, Theorem 6.8] is an antipodal tight spherical 3-design on $\mathbb{S}^2$, is an orthogonal transformation of the set $C_6 = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$ of vertices of the regular octahedron.

We now prove $X_{-\frac{1}{2}} = X_{\frac{1}{2}}$. It can be checked that the distance distribution of $X_{-\frac{1}{2}}$ with respect to $\boldsymbol{y} \in X_{\frac{1}{2}}$ satisfies

$$A^{\boldsymbol{y}}\big(X_{-\frac{1}{2}}\big) \subset \left\{-1, -\frac{1}{3}, \frac{1}{3}, 1\right\},$$

because, by definition of the derived code, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle \in \left\{\frac{\alpha+\frac{1}{4}}{1-\frac{1}{4}} \,\middle|\, \alpha \in A(X)\right\}$ holds for all $\boldsymbol{x} \in X_{-\frac{1}{2}}$. Thus, $|A^{\boldsymbol{y}}\big(X_{-\frac{1}{2}}\big)| \leq 4$, and hence, one can solve the Vandermonde system (9) to get

$$A^{\boldsymbol{y}}_{-1}\big(X_{-\frac{1}{2}}\big) = 1, \quad A^{\boldsymbol{y}}_{-\frac{1}{3}}\big(X_{-\frac{1}{2}}\big) = 3, \quad A^{\boldsymbol{y}}_{\frac{1}{3}}\big(X_{-\frac{1}{2}}\big) = 3, \quad A^{\boldsymbol{y}}_{1}\big(X_{-\frac{1}{2}}\big) = 1.$$

The last equality implies $\boldsymbol{y} \in X_{-\frac{1}{2}}$. Since the above equation holds for any $\boldsymbol{y} \in X_{\frac{1}{2}}$, one finds $X_{\frac{1}{2}} \subset X_{-\frac{1}{2}}$ and the desired equality $X_{\frac{1}{2}} = X_{-\frac{1}{2}}$ follows.

What is left is to show that $X_{\pm\frac{1}{2}}$ is (up to orthogonal transformations) uniquely determined from X_0. For this, take $\sigma \in O(\mathbb{R}^4)$ such that $X_0 = \sigma(C_6)$ and set $C_8 = \sigma^{-1}\big(X_{\frac{1}{2}}\big)$. We prove $C_8 = \big\{\big(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\big)\big\}$. Again one can compute the distance distribution of C_6 with respect to $\boldsymbol{y} \in C_8$ by the Vandermonde system (9) and it holds that

$$A^{\boldsymbol{y}}(C_6) = \left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\} \quad \text{and} \quad A^{\boldsymbol{y}}_{\pm\frac{1}{\sqrt{3}}}(C_6) = 3.$$

Namely, $\boldsymbol{y} \in C_8$ satisfies $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \pm\frac{1}{\sqrt{3}}$ for all $\boldsymbol{x} \in C_6$. This shows that $C_8 \subset \big\{\big(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\big)\big\}$. Since $|C_8| = 8$, the equality holds.

As a result, the whole set X is constructed with the help of only C_6, which is unique up to orthogonal transformations. Hence, we complete the proof. □

For comparison, we mention the other combinatorial structures on the D_4 root system $\mathbf{D}_4$ without going into details. The set $\mathbf{D}_4$ has the structure of a Q-polynomial association scheme [4] (this is verified because the inequality $t \geq 2s - 3$ holds for $\mathbf{D}_4$, where s is the size of the set of inner products between two distinct points and t is the strength). The set $\mathbf{D}_4$ also has the structure of a kissing number configuration on $\mathbb{S}^3$ [1, 30]. The positive semidefinite programming method is directly applicable for a proof of this kissing number [1]. On the other hand, the set $\mathbf{D}_4$ is not universally optimal code [15]. Any set satisfying $t \geq 2s - 1$ is universally optimal, so the strength of $\mathbf{D}_4$ is not strong enough to give the optimality by itself. Compared to these results, our main result provides a new characterization of $\mathbf{D}_4$ for the design aspect.

Remark 5.2. We briefly mention some of known uniqueness results. Each of the 600-cell $C_{600} \subset \mathbb{S}^3$ [13], the normalized E_8 root system $\frac{1}{\sqrt{2}}\mathbf{E}_8 \subset \mathbb{S}^7$ [6] and the set of minimal vectors of the Leech lattice $\frac{1}{2}\Lambda_{24} \subset \mathbb{S}^{23}$ [6] is known to be unique as a spherical t-design, where the strength $t \in \mathbb{N}$ is indicated as follows.

X	$\lvert X\rvert$	t	T
C_{600}	120	11	$\{18, 16, 14, 10, 8, 6, 4, 2\}$
$\frac{1}{\sqrt{2}}\mathbf{E}_8$	240	7	$\{10, 6, 4, 2\}$
$\frac{1}{2}\Lambda_{24}$	196560	11	$\{14, 10, 8, 6, 4, 2\}$

They are also unique as an antipodal spherical T-design for the above $T \subset \mathbb{N}$. In contrast, our case, the D_4 root system, is not unique as a spherical 5-design (which is a consequence of the result from [15]) and is unique as an antipodal spherical $\{10, 4, 2\}$-design. The D_4 root system is the first example such that it is not unique as a spherical t-design, but unique as an antipodal spherical T-design.

6. Application: orthogonal decompositions of shells

As an application of the uniqueness of the antipodal spherical $\{10, 4, 2\}$-design on $\mathbb{S}^3$ with 24 points, we now prove that every normalized shell of the D_4 lattice is a disjoint union of certain orthogonal transformations of the normalized D_4 root system $\frac{1}{\sqrt{2}}\mathbf{D}_4$.

Theorem 6.1. *For any $m \geq 1$, there exists a finite subset $S_m \subset O(\mathbb{R}^4)$ such that*

$$\frac{1}{\sqrt{2m}}(D_4)_{2m} = \bigsqcup_{\sigma \in S_m} \sigma\left(\frac{1}{\sqrt{2}}\mathbf{D}_4\right).$$

Proof. Since the Weyl group $W(F_4)$ acts on each shells of the D_4 lattice, we have a $W(F_4)$-orbit decomposition of $(D_4)_{2m}$. Thus, it suffices to show that each orbit $\boldsymbol{x}^{W(F_4)}$ of $\boldsymbol{x} \in \frac{1}{\sqrt{2m}}(D_4)_{2m}$ is a disjoint union of certain orthogonal transformations of $\frac{1}{\sqrt{2}}\mathbf{D}_4$. For this, using Magma system [11], one can check that there exists a subgroup N of $W(F_4)$ such that

- $|N| = 24$;
- $-I \in N$;
- the harmonic Molien series of N is given by

$$\sum_{\ell \geq 0} \dim \mathrm{Harm}_\ell(\mathbb{R}^4)^N t^\ell = 1 + 7t^6 + 9t^8 + 26t^{12} + \cdots .$$

Note that every $W(F_4)$-orbit has an N-orbit decomposition. It follows from the above data and Lemma 4.1 that every half set X of the N-orbit $\boldsymbol{x}^N$ is a spherical $\{10, 4, 2\}$-design on $\mathbb{S}^3$ with $|X| \leq 12$. In particular, we see from Theorem 3.2 that $|X| = 12$, and hence that $|\boldsymbol{x}^N| = 24$. Thus, by Theorem 5.1, the N-orbit $\boldsymbol{x}^N$ is an orthogonal transformation of the normalized D_4 root system $\frac{1}{\sqrt{2}}\mathbf{D}_4$. We complete the proof. □

Remark that we have $|(D_4)_{2m}| = 24|S_m|$. Thus, the cardinality of S_m can be deduced from Jacobi's four-square theorem (7);

$$|S_m| = \sum_{\substack{d|2m \\ d:\text{odd}}} d. \tag{10}$$

It might be interesting to ask if there is a similar decomposition of shells of other lattices.

7. Application: the uniqueness of the D_4 lattice

The goal of this section is to give a new proof of the uniqueness of the D_4 lattice as an even integral lattice of level 2, which is also another application of the uniqueness of the antipodal spherical $\{10, 4, 2\}$-design on $\mathbb{S}^3$ with 24 points. Since the theory of weighted theta functions of a lattice is our key ingredient, we begin with some basic terminology for lattices and weighted theta functions used in [24].

Let $\Lambda \subset \mathbb{R}^d$ be a full-ranked lattice. The lattice Λ is said to be *integral* (resp. *even*) if Λ is a subset of the dual lattice $\Lambda^* = \{\boldsymbol{y} \in \mathbb{R}^d \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \mathbb{Z} \text{ for all } \boldsymbol{x} \in \Lambda\}$ (resp. $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \in 2\mathbb{Z}$ for all $\boldsymbol{x} \in \Lambda$). Let B denote a basis matrix of Λ; $\Lambda = \{\boldsymbol{m}B \mid \boldsymbol{m} \in \mathbb{Z}^d\}$. The determinant $\mathrm{disc}(\Lambda) = |\det B|$, which does not depend on the choices of a basis matrix, is called the *discriminant* of the lattice Λ. The minimum of all $N \in \mathbb{N}$ with $N\langle \boldsymbol{x}, \boldsymbol{x} \rangle \in 2\mathbb{Z}$ for all $\boldsymbol{x} \in \Lambda^*$ is called the *level* of Λ.

Let Λ be an even lattice in $\mathbb{R}^d$ and $\Lambda_{2m} = \{\boldsymbol{x} \in \Lambda \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 2m\}$ the $2m$-shell of Λ. For $P \in \mathrm{Harm}_\ell(\mathbb{R}^d)$ and $m \geq 0$, we write $a_{\Lambda,P}(m) = \sum_{\boldsymbol{x} \in \Lambda_{2m}} P(\boldsymbol{x})$ and define the *weighted theta function* $\theta_{\Lambda,P}(z)$ by

$$\theta_{\Lambda,P}(z) = \sum_{m \geq 0} a_{\Lambda,P}(m) q^m \quad (q = e^{2\pi i z}),$$

which is a holomorphic function on the complex upper half plane $z \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. In particular, if $P = 1$ of degree 0, one gets the generating series of the cardinality of each $2m$-shells of Λ. Namely $\theta_{\Lambda,1}(z) = \sum_{m\geq 0} |\Lambda_{2m}| q^m$.

By Hecke and Schoenberg, for an even integral lattice Λ of level N in $\mathbb{R}^d$, the function $\theta_{\Lambda,P}(z)$ is known to be a modular form of weight $d/2 + \ell$ for $\Gamma_1(N)$ (see e.g., [24, Chap.3]), where $\Gamma_1(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \bmod N\}$ of $\Gamma_1(1) = \mathrm{SL}_2(\mathbb{Z})$. Let $M_k(\Gamma_1(N))$ denote the $\mathbb{C}$-vector space of modular forms of weight k for the congruence subgroup $\Gamma_1(N)$. Then we have the $\mathbb{C}$-linear map

$$\vartheta_{\Lambda,\ell} : \mathrm{Harm}_\ell(\mathbb{R}^d) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow M_{d/2+\ell}(\Gamma_1(N)), \qquad P \longmapsto \theta_{\Lambda,P}(z),$$

where $\mathrm{Harm}_\ell(\mathbb{R}^d) \otimes_{\mathbb{R}} \mathbb{C}$ is the $\mathbb{C}$-vector space spanned by real harmonic polynomials. When $\ell \geq 1$, the image

$$\operatorname{Im} \vartheta_{\Lambda,\ell} = \langle \theta_{\Lambda,P}(z) \mid P \in \mathrm{Harm}_\ell(\mathbb{R}^d) \rangle_{\mathbb{C}}$$

is a subspace of the $\mathbb{C}$-vector space $S_{d/2+\ell}(\Gamma_1(N))$ of cusp forms of weight $d/2 + \ell$ for $\Gamma_1(N)$.

Fundamental results on the weighted theta functions for the D_4 lattice are summarized as follows.

Proposition 7.1. *For $\ell \geq 1$, one has $\operatorname{Im} \vartheta_{D_4,\ell} \subset S_{2+\ell}(\Gamma_1(2))$. When $\ell = 0$, we find that $\theta_{D_4,1}(z) = 2E_2(2z) - E_2(z) = 1 + 24q + 24q^2 + 96q^3 + 24q^4 + \cdots$, where*

$$E_2(z) = 1 - 24 \sum_{m\geq 1} \left(\sum_{d|m} d \right) q^m = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^5 + \cdots .$$

Proof. The D_4 lattice is of level 2, so the first statement is a consequence of the classical results by Hecke and Schoenberg. For the last statement, we note that the space $M_2(\Gamma_1(2))$ is 1-dimensional spanned by $2E_2(2z) - E_2(z)$. Since $\operatorname{Im} \vartheta_{D_4,0} \subset M_2(\Gamma_1(2))$, $\theta_{D_4,1}$ is a constant multiple of $2E_2(2z) - E_2(z)$. Comparing the constant term, we get the desired result. □

Let us prove the uniqueness of the D_4 lattice.

Theorem 7.2. *For any even integral lattice $\Lambda \subset \mathbb{R}^4$ of level 2, there exists an orthogonal transformation $\sigma \in O(\mathbb{R}^4)$ such that $\Lambda = \sigma\big(D_4\big)$.*

Proof. Since $\operatorname{Im} \vartheta_{\Lambda,0} \subset M_2(\Gamma_1(2)) = \langle 2E_2(2z) - E_2(z) \rangle_{\mathbb{C}}$, we have $\theta_{\Lambda,1}(z) = 2E_2(2z) - E_2(z)$. This implies $|\Lambda_{2m}| = |(D_4)_{2m}|$ for all $m \geq 0$. We first consider the case Λ_2. Since Λ is integral, using the Cauchy-Schwarz inequality, we see that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \{0, \pm 1, \pm 2\}$ holds for any $\boldsymbol{x}, \boldsymbol{y} \in \Lambda_2$. Hence

$$A\left(\frac{1}{\sqrt{2}} \Lambda_2 \right) \subset \left\{ -1, -\frac{1}{2}, 0, \frac{1}{2} \right\}.$$

Since a half set X' of $\frac{1}{\sqrt{2}}\Lambda_2$ is a $(4, 12, 1/2)$ spherical code with $A(X') \subset \{-1/2, 0, 1/2\}$, by Theorem 3.2 and Lemma 2.2, the normalized set $\frac{1}{\sqrt{2}}\Lambda_2$ is an antipodal spherical $\{10, 4, 2\}$-design on $\mathbb{S}^3$ with 24 points. By Theorem 5.1, there exists $\sigma \in O(\mathbb{R}^4)$ such that $\Lambda_2 = \sigma\big(\mathbf{D}_4\big)$. Now let us consider the sublattice Λ' of Λ generated by Λ_2. Since the D_4 lattice is generated by $\mathbf{D}_4$, we have $\Lambda' = \sigma\big(D_4\big)$. The orthogonal transformation σ preserves the inner product, so we get

$$|(D_4)_{2m}| = |\Lambda'_{2m}| \leq |\Lambda_{2m}| = |(D_4)_{2m}|$$

for all $m \geq 0$. Thus, $\Lambda'_{2m} = \Lambda_{2m}$ and hence $\Lambda' = \Lambda$, from which the desired result follows. □

It should be noted that Theorem 7.2 can be shown by the same method with the one described in Serre's book [38, Chap. V]. In this direction, we shall use $\mathrm{Aut}(D_4) = W(F_4)$ and a version of the Siegel mass formula [39].

Remark 7.3. We briefly mention another proof of Proposition 4.2, using weighted theta function $\theta_{D_4,P}(z)$. For this, we first notice that if $\theta_{D_4,P}(z) = 0$ for all $P \in \mathrm{Harm}_\ell(\mathbb{R}^4)$, then every normalized $2m$-shell $\frac{1}{\sqrt{2m}}(D_4)_{2m}$ is a spherical $\{\ell\}$-design (this criterion was first used by Venkov [40] in his design theoretical study on even unimodular lattices). Therefore, it suffices to show that $\mathrm{Im}\,\vartheta_{\Lambda,\ell} = 0$ for $\ell \in \{10, 4, 2\}$, but this can be checked by a computer due to the fact that modular forms are determined by first several Fourier coefficients (also, we need a list of harmonic polynomials of these degrees and the simple expression of the $2m$-shell of D_4). Alternatively, the result would follow from the dimension formula for the space $S_k^{\mathrm{new}}(\Gamma_1(2))$ of newforms (see [23]), since we may have the equality $\mathrm{Im}\,\vartheta_{D_4,\ell} = S_{2+\ell}^{\mathrm{new}}(\Gamma_1(2))$ (this equality is a folklore, but well known for the experts; consult [9, 25, 27] for relevant materials).

Combining the uniqueness of level 2 lattices (Theorem 7.2) and Waldspurger's result [42, Théorèm 2'], we can at least make sure that the inclusion $\mathrm{Im}\,\vartheta_{D_4,\ell} \supset S_{2+\ell}^{\mathrm{new}}(\Gamma_1(2))$ holds for any $\ell \geq 1$. The first example of newforms for $\Gamma_1(2)$ exists in weight 8 of the form

$$\eta(z)^8\eta(2z)^8 = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 + \cdots ,$$

where $\eta(z) = q^{1/24}\prod_{n\geq 1}(1 - q^n)$ is the Dedekind eta function. The above inclusion implies that there exists a harmonic polynomial $P \in \mathrm{Harm}_6(\mathbb{R}^4)$ such that $\theta_{D_4,P}(z) = \eta(z)^8\eta(2z)^8$. We will give applications of this expression in the next section.

8. Strength of spherical design

For a finite set $X \subset \mathbb{S}^{d-1}$, we say $T \subset \mathbb{N}$ is *the harmonic strength* of X if X is not a spherical T'-design for any $T \subsetneq T' \subset \mathbb{N}$. In this section, we will be interested in the harmonic strength of the $2m$-shell of the D_4 lattice. We first indicate that this problem is intimately connected to non-vanishing of the Fourier coefficients of the newform $\eta(z)^8\eta(2z)^8$.

Theorem 8.1. *For $m \geq 1$, the normalized $2m$-shell $\frac{1}{\sqrt{2m}}\big(D_4\big)_{2m}$ is an antipodal spherical $\{10, 6, 4, 2\}$-design if and only if $\tau_2(m) = 0$, where $\sum_{m\geq 1}\tau_2(m)q^m = \eta(z)^8\eta(2z)^8$.*

Proof. We have proved in Proposition 4.2 that the set $\frac{1}{\sqrt{2m}}\big(D_4\big)_{2m}$ is an antipodal spherical $\{10, 4, 2\}$-design. We only deal with whether $\frac{1}{\sqrt{2m}}\big(D_4\big)_{2m}$ is a $\{6\}$-design or not. For this, we first notice that by the representation theory, we have

$$\mathrm{Harm}_\ell(\mathbb{R}^4) = \mathrm{Harm}_\ell(\mathbb{R}^4)^{W(F_4)} \oplus \{(1 - \sigma^*)P \mid P \in \mathrm{Harm}_\ell(\mathbb{R}^4), \sigma \in W(F_4)\}.$$

Since $\theta_{D_4,P}(z) = \theta_{D_4,\sigma^*P}(z)$ holds for all $P \in \mathrm{Harm}_\ell(\mathbb{R}^4)$ and $\sigma \in W(F_4)$, the latter space is a subspace of $\ker\vartheta_{D_4,\ell}$, and hence, $\mathrm{Im}\,\vartheta_{D_4,\ell} = \mathrm{Im}\,\vartheta_{D_4,\ell}\big|_{\mathrm{Harm}_\ell(\mathbb{R}^4)^{W(F_4)}}$.

By (8), the space $\mathrm{Harm}_6(\mathbb{R}^4)^{W(F_4)}$ is the 1-dimensional subspace of $\mathrm{Harm}_6(\mathbb{R}^4)$ and its basis is given (see e.g., [33, §5.1]) by

(11)

$$\begin{aligned} P_6(\boldsymbol{x}) :=& p_6(x_1, x_2, x_3, x_4) \\ & - 5\{x_1^4p_2(x_2, x_3, x_4) + x_1^2p_4(x_2, x_3, x_4) + (x_2^4 + x_3^2x_4^2)p_2(x_3, x_4) + x_2^2p_4(x_3, x_4)\} \\ & + 30\{x_1^2(x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2) + x_2^2x_3^2x_4^2\}, \end{aligned}$$

where $p_k(x_1, \ldots, x_d) = x_1^k + \cdots + x_d^k$. From the above argument, we see that $\frac{1}{\sqrt{2m}}(D_4)_{2m}$ is a $\{6\}$-design if and only if $\sum_{\boldsymbol{x}\in(D_4)_{2m}} P_6(\boldsymbol{x}) = 0$. Then, the result follows from the easily checked identity

$$\theta_{D_4,P_6}(z) = -192\eta(z)^8\eta(2z)^8, \tag{12}$$

where again, we have used the fact that the modular forms are determined by first several Fourier coefficients. □

We remark that Theorem 8.1 is an analogue to the one given by Venkov, Pache and de la Harpe [18, 19]; They observed that the normalized $2m$-shell of the E_8 lattice is an antipodal spherical 8-design if and only if $\tau(m) = 0$, where $\tau(m)$ is the mth Fourier coefficient of the discriminant function $\Delta(z) = \eta(z)^{24} = \sum_{m\geq 0} \tau(m)q^m \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$. The question of whether $\tau(m) \neq 0$ holds for all $m \geq 1$, posed by Lehmer [29], is still far from being solved, so it is a common understanding that determining the (harmonic) strength for all shells of a given lattice is a hard problem. A similar attempt for other lattices can be found in [5, 35]. In particular, Miezaki [31] obtained the harmonic strength for any shells of the square lattice $\mathbb{Z}^2$. His result is extended by Pandey [36] to rings of integers of imaginary quadratic fields over $\mathbb{Q}$ with class number 1.

Using Pari-GP [37], we have checked that $\tau_2(m)$ is non-zero up to $m \leq 10^8$. One would expect that the harmonic strength of the $2m$-shell of D_4 is given by $\{10, 4, 2\}$ for all $m \geq 1$. To give partial evidence, we consider the congruences of $\tau_2(m)$.

Theorem 8.2. *Let $\ell \in \{3, 5\}$. For any prime $p \geq 3$, we have*

$$\tau_2(p) \equiv p(p+1) \mod \ell. \tag{13}$$

Proof. We use the harmonic polynomial P_6 defined in (11). For the case $\ell = 3$, using $x^4 \equiv x^2 \mod 3$ for all $x \in \mathbb{Z}$, we get

$$\begin{aligned}
P_6(\boldsymbol{x}) &\equiv x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_1^2 p_2(x_2, x_3, x_4) + x_1^2 p_2(x_2, x_3, x_4) \\
&\quad + (x_2^2 + x_3^2 x_4^2) p_2(x_3, x_4) + x_2^2 p_2(x_3, x_4) \\
&\equiv (x_1^2 + \cdots + x_4^2)^2 \mod 3.
\end{aligned}$$

This shows that $P_6(\boldsymbol{x}) \equiv (2p)^2 \mod 3$ for all $\boldsymbol{x} \in (D_4)_{2p}$. Since $|(D_4)_{2p}| = 24(1+p)$ (see (7)) is divisible by 3, from (12) one obtains

$$-64\tau_2(p) = \frac{1}{3}\sum_{\boldsymbol{x}\in(D_4)_{2p}} P_6(\boldsymbol{x}) \equiv \frac{1}{3}(2p)^2|(D_4)_{2p}| = 32p^2(1+p) \mod 3,$$

from which the case $\ell = 3$ follows. For the case $\ell = 5$, notice that $x^6 \equiv x^2 \mod 5$ holds for any $x \in \mathbb{Z}$. We get

$$P_6(\boldsymbol{x}) \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 \mod 5,$$

and hence,

$$-192\tau_2(p) = \sum_{\boldsymbol{x}\in(D_4)_{2p}} P_6(\boldsymbol{x}) \equiv 2p|(D_4)_{2p}| = 48p(1+p) \mod 5.$$

So we are done. □

As a consequence of (13), we see that for any prime $p \not\equiv -1 \bmod 15$, we have $\tau_2(p) \neq 0$.

Apart from non-vanishing of the τ_2-function, we should mention that similar congruences to (13) have been established by many people since the time of Ramanujan (see e.g., [10, 17, 26, 28, 32]). Our congruences could be a special case of them, but our proof is new.

Remark 8.3. In much the same way as [29, Theorem 2], we can prove the following statement: If there exists the least value m_0 of m for which $\tau_2(m) = 0$, then m_0 is an odd prime number. Deligne's bound $|\tau_2(p)| \leq 2p^{\frac{7}{2}}$ (see [20, Theorem 8.2]) is one of key ingredients of the proof.

The data that support the findings of this study are available from the corresponding author, K.T. upon reasonable request.

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Statements and Declarations

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